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Author(s)	Weller, Glenn P.
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EQUIVALENT SIZES OF LIPSCHITZ MANIFOLDS AND THE SMOOTHING PROBLEM

GLENN P. WELLER*

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1. Introduction

A criterion for smoothing a closed topological manifold M was given by Y. Shikata in [3]. His criterion was based on the notion of the *size* of M, which will be denoted by $|M|_S$.

Shikata's Theorem. *A closed topological manifold M is smoothable if and only if* $|M|_S=0$.

In this paper we introduce another "size" of M, denoted $||M||$, which is conceptually somewhat simpler and which is definable for all open or closed Lipschitz manifolds. We then can prove the following result.

Theorem. *A closed Lipschitz manifold M is smoothable if and only if* $||M||=0$.

This theorem follows directly from Shikata's theorem and the following proposition, which shows that the two "sizes" are equivalent.

Proposition. *For each positive integer n there are positive numbers* $\alpha(n)$ *and* $\beta(n)$ *such that if M is a closed n-dimensional Lipschitz manifold then*

$$\alpha(n)||M|| \leq |M|_S \leq \beta(n)||M|| .$$

REMARK. Neither of these sizes is trivial, for L. Siebenmann has given examples in [4, pp. 135–137] of Lipschitz manifolds X^n, $n\geq 6$, having no piecewise linear manifold structure. In particular, X^n is not smoothable, and so $0<||X^n||<\infty$.

2. Lipschitz manifolds and their sizes

We recall from [2] that if (X_1, d_1) and (X_2, d_2) are metric spaces and $f:X_1\to X_2$ is a map then the *Lipschitz size of f* (relative to d_1 and d_2) is

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$$l(f) = \begin{cases} \text{Inf } \{k \geq 1 \mid k^{-1}d_1(x, y) \leq d_2(fx, fy) \leq kd_1(x, y) \text{ for all } x, y \in X_1\} \\ \infty, \text{ if the above set is empty.} \end{cases}$$

If $l(f) < \infty$, f is a *regular Lipschitz map*. It is clearly seen that relative to the obvious distances $l(f) = l(f^{-1})$ and $l(gf) \leq l(f) \cdot l(g)$.

We further recall from [6, p. 165] that a topological n-manifold M is a *Lipschitz manifold* if there is a coordinate cover $C = \{(U_i, h_i)\}_{i \in I}$ such that U_i is an open subset of M, h_i is a homeomorphism of an open subset of Euclidean n-space onto U_i, and
$h_i^{-1}h_j \colon h_j^{-1}(U_i \cap U_j) \rightarrow h_i^{-1}(U_i \cap U_j)$ is a regular Lipschitz homeomorphism for all i, j. We may always assume C is finite when M is compact.

For any metric d on M denote

$$l_d(C) = \text{Sup}_{i \in I}\, l(h_i) \text{ relative to Euclidean metric and } d,$$
$$l(C) = \text{Sup}_{i, j \in I}\, l(h_i^{-1}h_j) \text{ relative to Euclidean metric.}$$

Then we may define two "sizes" for M:

$$|M| = \text{Inf} \log l_d(C) \quad \text{and}$$
$$||M|| = \text{Inf} \log l(C)\,,$$

where both infima are taken over all coordinate covers C and all metrics d on M.

Lemma 1. *For all closed Lipschitz manifolds M,*

$$|M| \leq ||M|| \leq 2 \cdot |M|\,.$$

Proof. For the first inequality let $\varepsilon^* > 0$ be arbitrary and pick C as above so that $\log l(C) \leq ||M|| + \varepsilon^*$. Denote $L = l(C)$. Let $\{f_i\}$ be a partition of unity subordinate to $\{U_i\}$, $V_i = \text{Carrier}\,(f_i)$ and $W = \cup_i V_i \times V_i$. For $(x, y) \in W$, the functions $\varphi_i(x, y) = f_i(x)f_i(y)[\sum_j f_j(x)f_j(y)]^{-1}$ define a partition of unity on W so that $\varphi_i(x, y) = \varphi_i(y, x)$. For $(x, y) \in W$ define $\rho(x, y) = \sum_i \varphi_i(x, y) ||h_i^{-1}x - h_i^{-1}y||$.

We can choose sequences $S = \{x = x_0, x_1, \cdots, x_t = y\}$ so that for all i, $(x_{i-1}, x_i) \in W$ and define $[S] = \sum_i \rho(x_{i-1}, x_i)$ and $d(x, y) = \text{Inf}_S [S]$. It is easily verified that d is a pseudo-metric on M giving the original topology on M. By [1, Theorem 5.26, p. 154] there is a Lebesgue number η for the cover $\{V_i\}$ with respect to d.

Assertion. d is a metric, i.e., $d(x, y) = 0$ implies $x = y$.

Choose $\varepsilon > 0$ with $\varepsilon < \eta$. Then pick $[S] < \varepsilon$. Since $d(x_i, x_j) \leq [\{x_i, x_{i+1}, \cdots, x_j\}] \leq [S] < \varepsilon$, diameter $(S) < \eta$ and $S \subset V_j$ for some j. Now

$$L^{-1}||h_j^{-1}x_{i-1} - h_j^{-1}x_i|| \leq ||h_k^{-1}h_j h_j^{-1}x_{i-1} - h_k^{-1}h_j h_j^{-1}x_i|| \quad \text{and}$$
$$\begin{aligned} L^{-1}||h_j^{-1}x_{i-1} - h_j^{-1}x_i|| &= L^{-1}\textstyle\sum_k \varphi_k(x_{i-1}, x_i) \cdot ||h_j^{-1}x_{i-1} - h_j^{-1}x_i|| \\ &\leq \textstyle\sum_k \varphi_k(x_{i-1}, x_i) \cdot ||h_k^{-1}x_{i-1} - h_k^{-1}x_i|| \\ &= \rho(x_{i-1}, x_i) \end{aligned}$$

and

$$L^{-1}\cdot\|h_j^{-1}x-h_j^{-1}y\|\leq L^{-1}\textstyle\sum_i\|h_j^{-1}x_{i-1}-h_j^{-1}x_i\|$$
$$\leq\textstyle\sum_i\rho(x_{i-1},x_i)=[S]<\varepsilon\,.$$

Thus, $x=y$ since ε is arbitrary.

Now let $\{W_i\}$ be a cover of M so that diameter $(W_i)<\eta$. Then $W_i\subset V_j$ for some j. Let $k_i^{-1}=h_j^{-1}|W_i$ and $C^*=\{(W_i,k_i)\}$.

Assertion. k_i is regular Lipschitz relative to Euclidean metric and d.
If $h_jx, h_jy\in W_i\subset V_j$ and $d(h_jx,h_jy)<\eta$, then $d(h_jx,h_jy)\leq\rho(h_jx,h_jy)=\sum_k\varphi_k(h_jx,h_jy)\cdot\|h_k^{-1}h_jx-h_k^{-1}h_jy\|\leq\sum_k\varphi_k(h_jx,h_jy)\cdot L\cdot\|x-y\|=L\cdot\|x-y\|$.
Also we may pick $S=\{h_jx=x_0,x_1,\cdots,x_p=h_jy\}$ with $d(h_jx,h_jy)\leq[S]<\eta$. Then for all t, $d(h_jx,x_t)\leq\sum_{k=1}^t\rho(x_{k-1},x_k)\leq[S]<\eta$ and $x_t\in V_j$. Hence

$$\begin{aligned}[S]&=\textstyle\sum_{l,k}\varphi_k(x_{l-1},x_l)\cdot\|h_k^{-1}h_jh_j^{-1}x_{l-1}-h_k^{-1}h_jh_j^{-1}x_l\|\\&\geq L^{-1}\cdot\textstyle\sum_{l,k}\varphi_k(x_{l-1},x_l)\cdot\|h_j^{-1}x_{l-1}-h_j^{-1}x_l\|\\&=L^{-1}\cdot\textstyle\sum_l\|h_j^{-1}x_{l-1}-h_j^{-1}x_l\|\\&\geq L^{-1}\cdot\|x-y\|\,.\end{aligned}$$

Thus, $L^{-1}\cdot\|x-y\|\leq d(k_ix,k_iy)\leq L\cdot\|x-y\|$ and $l(k_i)\leq L=l(C)$. Hence, $l_d(C^*)\leq l(C)$ and $|M|\leq\|M\|+\varepsilon^*$. Since ε^* is arbitrary, $|M|\leq\|M\|$.
For the second inequality let $\varepsilon>0$ and choose C and d so that $2\log l_d(C)\leq 2\cdot|M|+\varepsilon$. Then $l(h_i^{-1}h_j)\leq l(h_i)\cdot l(h_j)\leq l_d(C)^2$ and $l(C)\leq l_d(C^*)^2$. Thus $\|M\|\leq 2\cdot|M|+\varepsilon$ and $\|M\|\leq 2\cdot|M|$.
The above proof also shows an equivalent way of defining Lipschitz manifolds. We state this as a corollary.

Corollary. *A closed manifold M is Lipschitz manifold if and only if there is a coordinate cover $C=\{(U_i,h_i)\}_{i\in I}$ such that each homeomorphism h_i is regular Lipschitz with respect to the usual metric on Euclidean space and some metric d on M.*

3. A reformulation of Shikata's criterion

Shikata [3] defined the "size" of a compact topological n-manifold M to be

$$|M|_S=\operatorname{Inf}_{C,d}(8\gamma)^{m(C)}\log l_d(C)\,,$$

where $\gamma>0$ depends only on n and $m(C)$ is the maximum number of U_j that any U_i can intersect. It is clear that any C can be replaced by a coordinate cover C' refining it, provided we use the restrictions of the appropriate homeomorphisms h_i from C in computing $l_d(C')$. In [5] we show that there is a positive integer $\mu(n)$ depending only on n such that any C has a refinement C' as above with $m(C')\leq\mu(n)$. Let $\alpha'(n)=(8\gamma)^{\mu(n)}$ and $\beta'(n)=1$ if $8\gamma\leq 1$, and vice versa if $8\gamma \geqq 1$. Then noting the definitions of $|M|$ and $|M|_S$, we have proved the following lemma.

Lemma 2. *There are positive numbers $\alpha'(n)$ and $\beta'(n)$ depending only on n such that for all closed manifolds M*

$$\alpha'(n)|M| \leq |M|_S \leq \beta'(n)|M| .$$

Taken together, Lemmas 1 and 2 clearly yield the proposition. Hence, the theorem is established.

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