Weller, G. P.
Osaka J. Math.
10 (1973), 507–510

EQUIVALENT SIZES OF LIPSCHITZ MANIFOLDS AND THE SMOOTHING PROBLEM

GLENN P. WELLER*

(Received September 2, 1972)

1. Introduction

A criterion for smoothing a closed topological manifold M was given by Y. Shikata in [3]. His criterion was based on the notion of the *size* of M, which will be denoted by $|M|_S$.

Shikata's Theorem. *A closed topological manifold M is smoothable if and only if* $|M|_S=0$.

In this paper we introduce another "size" of M, denoted $||M||$, which is conceptually somewhat simpler and which is definable for all open or closed Lipschitz manifolds. We then can prove the following result.

Theorem. *A closed Lipschitz manifold M is smoothable if and only if* $||M||=0$.

This theorem follows directly from Shikata's theorem and the following proposition, which shows that the two "sizes" are equivalent.

Proposition. *For each positive integer n there are positive numbers* $\alpha(n)$ *and* $\beta(n)$ *such that if M is a closed n-dimensional Lipschitz manifold then*

$$\alpha(n)||M|| \leq |M|_S \leq \beta(n)||M|| .$$

REMARK. Neither of these sizes is trivial, for L. Siebenmann has given examples in [4, pp. 135–137] of Lipschitz manifolds X^n, $n \geq 6$, having no piecewise linear manifold structure. In particular, X^n is not smoothable, and so $0<||X^n||<\infty$.

2. Lipschitz manifolds and their sizes

We recall from [2] that if (X_1, d_1) and (X_2, d_2) are metric spaces and $f:X_1 \to X_2$ is a map then the *Lipschitz size of f* (relative to d_1 and d_2) is

* This research is partially supported by NSF Grant GP 28487.

$$l(f)=\begin{cases}\text{Inf }\{k\geq 1\,|\,k^{-1}d_1(x,y)\leq d_2(fx,fy)\leq kd_1(x,y)\text{ for all }x,y\in X_1\}\\ \infty,\text{ if the above set is empty.}\end{cases}$$

If $l(f)<\infty$, f is a *regular Lipschitz map*. It is clearly seen that relative to the obvious distances $l(f)=l(f^{-1})$ and $l(gf)\leq l(f)\cdot l(g)$.

We further recall from [6, p. 165] that a topological n-manifold M is a *Lipschitz manifold* if there is a coordinate cover $C=\{(U_i, h_i)\}_{i\in I}$ such that U_i is an open subset of M, h_i is a homeomorphism of an open subset of Euclidean n-space onto U_i, and
$h_i^{-1}h_j$: $h_j^{-1}(U_i\cap U_j)\rightarrow h_i^{-1}(U_i\cap U_j)$ is a regular Lipschitz homeomorphism for all i, j. We may always assume C is finite when M is compact.

For any metric d on M denote

$$l_d(C)=\text{Sup}_{i\in I}\, l(h_i)\text{ relative to Euclidean metric and } d,$$
$$l(C)=\text{Sup}_{i,j\in I}\, l(h_i^{-1}h_j)\text{ relative to Euclidean metric.}$$

Then we may define two "sizes" for M:

$$|M|=\text{Inf}\log l_d(C)\quad\text{and}$$
$$||M||=\text{Inf}\log l(C)\,,$$

where both infima are taken over all coordinate covers C and all metrics d on M.

Lemma 1. *For all closed Lipschitz manifolds* M,

$$|M|\leq||M||\leq 2\cdot|M|\ .$$

Proof. For the first inequality let $\varepsilon^*>0$ be arbitrary and pick C as above so that $\log l(C)\leq||M||+\varepsilon^*$. Denote $L=l(C)$. Let $\{f_i\}$ be a partition of unity subordinate to $\{U_i\}$, $V_i=$Carrier (f_i) and $W=\cup_i V_i\times V_i$. For $(x,y)\in W$, the functions $\varphi_i(x,y)=f_i(x)f_i(y)[\sum_j f_j(x)f_j(y)]^{-1}$ define a partition of unity on W so that $\varphi_i(x,y)=\varphi_i(y,x)$. For $(x,y)\in W$ define $\rho(x,y)=\sum_i\varphi_i(x,y)||h_i^{-1}x-h_i^{-1}y||$.

We can choose sequences $S=\{x=x_0, x_1,\cdots, x_t=y\}$ so that for all i, $(x_{i-1}, x_i)\in W$ and define $[S]=\sum_i\rho(x_{i-1}, x_i)$ and $d(x,y)=\text{Inf }_S[S]$. It is easily verified that d is a pseudo-metric on M giving the original topology on M. By [1, Theorem 5.26, p. 154] there is a Lebesgue number η for the cover $\{V_i\}$ with respect to d.

Assertion. d is a metric, i.e., $d(x,y)=0$ implies $x=y$.

Choose $\varepsilon>0$ with $\varepsilon<\eta$. Then pick $[S]<\varepsilon$. Since $d(x_i, x_j)\leq[\{x_i, x_{i+1},\cdots, x_j\}]\leq[S]<\varepsilon$, diameter $(S)<\eta$ and $S\subset V_j$ for some j. Now

$$L^{-1}||h_j^{-1}x_{i-1}-h_j^{-1}x_i||\leq||h_k^{-1}h_jh_j^{-1}x_{i-1}-h_k^{-1}h_jh_j^{-1}x_i||\quad\text{and}$$
$$\begin{aligned}L^{-1}||h_j^{-1}x_{i-1}-h_j^{-1}x_i|| &= L^{-1}\textstyle\sum_k\varphi_k(x_{i-1}, x_i)\cdot||h_j^{-1}x_{i-1}-h_j^{-1}x_i||\\ &\leq\textstyle\sum_k\varphi_k(x_{i-1}, x_i)\cdot||h_k^{-1}x_{i-1}-h_k^{-1}x_i||\\ &= \rho(x_{i-1}, x_i)\end{aligned}$$

and

$$L^{-1}\cdot\|h_j^{-1}x-h_j^{-1}y\|\leq L^{-1}\sum_i\|h_j^{-1}x_{i-1}-h_j^{-1}x_i\|$$
$$\leq\sum_i\rho(x_{i-1},x_i)=[S]<\varepsilon\,.$$

Thus, $x=y$ since ε is arbitrary.

Now let $\{W_i\}$ be a cover of M so that diameter $(W_i)<\eta$. Then $W_i\subset V_j$ for some j. Let $k_i^{-1}=h_j^{-1}\,|\,W_i$ and $C^*=\{(W_i, k_i)\}$.

Assertion. k_i is regular Lipschitz relative to Euclidean metric and d.

If $h_jx, h_jy\in W_i\subset V_j$ and $d(h_jx, h_jy)<\eta$, then $d(h_jx, h_jy)\leq\rho(h_jx, h_jy)=\sum_k\varphi_k(h_jx, h_jy)\cdot\|h_k^{-1}h_jx-h_k^{-1}h_jy\|\leq\sum_k\varphi_k(h_jx, h_jy)\cdot L\cdot\|x-y\|=L\cdot\|x-y\|$.

Also we may pick $S=\{h_jx=x_0, x_1, \cdots, x_p=h_jy\}$ with $d(h_jx, h_jy)\leq[S]<\eta$. Then for all t, $d(h_jx, x_t)\leq\sum_{k=1}^t\rho(x_{k-1}, x_k)\leq[S]<\eta$ and $x_t\in V_j$. Hence

$$\begin{aligned}[S] &= \sum_{l,k}\varphi_k(x_{l-1},x_l)\cdot\|h_k^{-1}h_jh_j^{-1}x_{l-1}-h_k^{-1}h_jh_j^{-1}x_l\|\\ &\geq L^{-1}\cdot\sum_{l,k}\varphi_k(x_{l-1},x_l)\cdot\|h_j^{-1}x_{l-1}-h_j^{-1}x_l\|\\ &= L^{-1}\cdot\sum_l\|h_j^{-1}x_{l-1}-h_j^{-1}x_l\|\\ &\geq L^{-1}\cdot\|x-y\|\,.\end{aligned}$$

Thus, $L^{-1}\cdot\|x-y\|\leq d(k_ix, k_iy)\leq L\cdot\|x-y\|$ and $l(k_i)\leq L=l(C)$. Hence, $l_d(C^*)\leq l(C)$ and $|M|\leq\|M\|+\varepsilon^*$. Since ε^* is arbitrary, $|M|\leq\|M\|$.

For the second inequality let $\varepsilon>0$ and choose C and d so that $2\log l_d(C)\leq 2\cdot|M|+\varepsilon$. Then $l(h_i^{-1}h_j)\leq l(h_i)\cdot l(h_j)\leq l_d(C)^2$ and $l(C)\leq l_d(C^*)^2$. Thus $\|M\|\leq 2\cdot|M|+\varepsilon$ and $\|M\|\leq 2\cdot|M|$.

The above proof also shows an equivalent way of defining Lipschitz manifolds. We state this as a corollary.

Corollary. *A closed manifold M is Lipschitz manifold if and only if there is a coordinate cover $C=\{(U_i, h_i)\}_{i\in I}$ such that each homeomorphism h_i is regular Lipschitz with respect to the usual metric on Euclidean space and some metric d on M.*

3. A reformulation of Shikata's criterion

Shikata [3] defined the "size" of a compact topological n-manifold M to be

$$|M|_S=\operatorname{Inf}_{C,d}(8\gamma)^{m(C)}\log l_d(C)\,,$$

where $\gamma>0$ depends only on n and $m(C)$ is the maximum number of U_j that any U_i can intersect. It is clear that any C can be replaced by a coordinate cover C' refining it, provided we use the restrictions of the appropriate homeomorphisms h_i from C in computing $l_d(C')$. In [5] we show that there is a positive integer $\mu(n)$ depending only on n such that any C has a refinement C' as above with $m(C')\leq\mu(n)$. Let $\alpha'(n)=(8\gamma)^{\mu(n)}$ and $\beta'(n)=1$ if $8\gamma\leq 1$, and vice versa if $8\gamma \geqq 1$. Then noting the definitions of $|M|$ and $|M|_S$, we have proved the following lemma.

Lemma 2. *There are positive numbers $\alpha'(n)$ and $\beta'(n)$ depending only on n such that for all closed manifolds M*

$$\alpha'(n)|M| \leq |M|_S \leq \beta'(n)|M| .$$

Taken together, Lemmas 1 and 2 clearly yield the proposition. Hence, the theorem is established.

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