<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>On the zeta functions of the varieties $X(w)$ of the split classical groups and the unitary groups</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Asai, Teruaki</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 20(1) P.21-P.32</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1983</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/6361">https://doi.org/10.18910/6361</a></td>
</tr>
<tr>
<td><strong>DOI</strong></td>
<td>10.18910/6361</td>
</tr>
<tr>
<td><strong>rights</strong></td>
<td></td>
</tr>
</tbody>
</table>
ON THE ZETA FUNCTIONS OF THE VARIETIES
X(w) OF THE SPLIT CLASSICAL GROUPS AND
THE UNITARY GROUPS

TERUAKI ASAI

(Received April 30, 1981)

0. Introduction

Let G be one of the split classical groups SO_{2n}, SO_{2n+1}, Sp_{2n} or a unitary group defined over the finite field \( F_q \) of q elements. Let F be the Frobenius mapping, \( G^F \) the subgroup of F-stable elements, W the Weyl group of G and let \( \delta \) be the smallest positive integer such that \( F^\delta \) acts trivially on \( W \). For \( w \in W \), Deligne-Lusztig [3] has defined the \( F^\delta \)-stable variety \( X(w) \) for any connected reductive group. If \( w \) is a Coxeter element of \( W \), the zeta function of \( X(w) \) was obtained by Lusztig [9] as a by-product when he determined the Green polynomial associated with \( w \). In this paper we shall determine the zeta function of \( X(w) \) for any \( w \in W \).

To state our result more explicitly, let \( B \) be a fixed \( F \)-stable Borel subgroup of \( G \), \( \mathbb{H}(W) \) the Hecke algebra of the representation of \( G^F \) induced from the trivial representation of \( B^F \) and let \( \{ a_i^w; w \in W \} \) be the natural basis of \( \mathbb{H}(W) \). When \( \delta \) divides \( m \) the number of \( F^m \)-stable points of \( X(w) \) is expressed in terms of the dimensions of the unipotent representations of \( G^F \) and the trace of \( a_i^w \) on each irreducible representation of \( \mathbb{H}(W) \).

The crucial point of our arguments depends on the lifting theory due to Shintani-Kawanaka ([15], [7], [8]) and a result of Lusztig ([12], Corollary 3.9), which says that for any unipotent representation \( \rho \) of \( G^F \), the eigenvalues of \( F^\delta \) on the \( \rho \)-isotypic component of \( H_i(X(w)) \) are independent of \( i \) and \( w \) up to a multiple factor of the form \( q^{i\delta}, i \in \mathbb{Z} \).

Finally the author expresses his heartfelt gratitude to Professor N. Kawanaka for his valuable suggestions and kind encouragement during the preparation of this paper.

1. General results

1.1. First we summarize the known results (Shintani [14], Kawanaka [7], [8]) to apply for our use.
Let \( m \) be a positive integer (maybe 1), \( k=\mathbb{F}_q, K=\mathbb{F}_{q^m}, G \) a connected algebraic
group defined over $k$, $F$ the Frobenius over $k$, $\sigma=F|_{\mathbb{F}^m}$ and $A$ the cyclic group (of order $m$) generated by $\sigma$. Let $x_1, x_2 \in G_{\mathbb{F}^m}$. $x_1\sigma$ and $x_2\sigma$ are conjugate in $G_{\mathbb{F}^m}A$ (semi-direct) if and only if there exists $h \in G_{\mathbb{F}^m}$ such that $x_1 = h^{-1}x_2^\sigma h$. If this is the case, we say $x_1$ and $x_2$ are $\sigma$-conjugate and we write $x_1 \sim_{\sigma} x_2$. If $m=1$, we simply write $x_1 \sim x_2$ instead of $x_1 \sim_{\sigma} x_2$. The following lemma is proved in [7].

**Lemma 1.1.1.** For $x \in G_{\mathbb{F}^m}$, take $a \in G$ such that $x = a^{-1}F a$. Let $y = F^m a a^{-1}$. Then $y \in G_{\mathbb{F}^m}$, and the conjugacy class of $y$ in $G_{\mathbb{F}^m}$ is uniquely determined by the $\sigma$-conjugacy class of $x$ in $G_{\mathbb{F}^m}$. And the mapping $x \mapsto y$ defines a bijection: $G_{\mathbb{F}^m}/\sim \to G_{\mathbb{F}^m}/\sim$.

**Definition 1.1.2.** We denote the bijection $G_{\mathbb{F}^m}/\sim \to G_{\mathbb{F}^m}/\sim$ in the above lemma by $n_{\mathbb{F}^m/k}$. (Notice $n_{\mathbb{F}^m/k}$ is defined even if $m=1$.) Define $\mathcal{R}_{K/k} = n_{\mathbb{F}^m/k}^{-1} n_{\mathbb{F}^m/k}$. This also is a bijection from $G_{\mathbb{F}^m}/\sim$ onto $G_{\mathbb{F}^m}/\sim$.

**Remark 1.1.3.** The reader should refer Kawanaka [8] for the relation between the norm mapping in [loc. cit.] and our norm mapping $\mathcal{R}_{K/k}$.

The following lemma features some property of the mapping $\mathcal{R}_{K/k}$, which is not used in this paper. The proof is omitted.

**Lemma 1.1.4.** Let $G$ be a connected reductive group and $Z(G)$ the center of $G$. Let $s \in Z(G)_{\mathbb{F}^m}$ and $u \in G_{\mathbb{F}^m}$. Let $r$ be the order of $s$. Assume $m \equiv 1 \pmod{r}$. Then $\mathcal{R}_{K/k}(su) = s \mathcal{R}_{K/k}(u)$.

For $\chi_K \in \hat{G}_{\mathbb{F}^m}$ (= the set of $\sigma$-invariant irreducible characters of $\hat{G}_{\mathbb{F}^m}$), there exists $\tilde{\chi}_K \in \hat{G}_{\mathbb{F}^m}A$ such that $\tilde{\chi}_K|_{\mathbb{F}^m} = \chi_K$. Let $\chi_\delta \in \hat{G}_{\mathbb{F}^m}$.

**Definition 1.1.5.** Let $m > 1$. We say $\chi_K$ is the lifting of $\chi_\delta$ in $\hat{G}_{\mathbb{F}^m}$ if there exists a constant $c$ such that $\tilde{\chi}_K(y) = c \chi_\delta(\mathcal{R}_{K/k} y)$ for any $y \in G_{\mathbb{F}^m}$. (The lifting of $\chi_\delta$ is uniquely determined by $\chi_\delta$ if it exists. See [7].)

**Theorem 1.1.6 ([7], [8], [15]).**

Let $m > 1$. Assume one of the following.

1. $G = GL_n$.
2. $G = U_{m+p}, (m, p) = 1$.
3. $G = SO_{2m+1}, Sp_{2n},$ or $SO_{2n}, (m, 2p) = 1$.

Then any $\chi_\delta \in \hat{G}_{\mathbb{F}^m}$ has the lifting $\chi_K \in \hat{G}_{\mathbb{F}^m}$. And the mapping $\chi_\delta \mapsto \chi_K$ defines a bijection between $\hat{G}_{\mathbb{F}^m}$ and $\hat{G}_{\mathbb{F}^m}$.


The following lemmas can be extracted from [7].
Lemma 1.1.8. Let $f_1$ and $f_2$ be class functions on $G/F$. Define class functions $g_1$ and $g_2$ on $G$ by: $g_i(y\sigma) = f_i(y\sigma)$ for any $y \in G/F$. Then

$$|G/F|^{-1} \sum_{y \in G/F} f(y\sigma) f_2(y\sigma) = |G/F|^{-1} \sum_{x \in G} g_1(x) g_2(x)$$

Lemma 1.1.9. Let $H$ be an $F$-stable closed subgroup. Let $f$ and $g$ be class functions on $H/F$ and $H/F$ respectively. If $g(y\sigma) = f(y\sigma)$ for any $y \in H/F$, then

$$(\text{Ind}_{H/F}^G g) (y\sigma) = (\text{Ind}_{H/F}^G f) (y\sigma)$$

for any $y \in G/F$.

1.2. Henceforth $G$ is a connected reductive group defined over $k = F_q$, $B$ is an $F$-stable Borel subgroup, $U$ is the unipotent radical of $B$, $T$ is an $F$-stable maximal torus of $B$ and $W = N_G(T)/T$.

Let $w \in W^F$ and $\check{b}$ its representative in $N_G(T)^F$. Let $X(w), S_w, T(w)$ be as in [3]. They are as follows.

$$S_w = \{g \in G; g^{-1} F^U g \in \check{b} U\}, \quad T(w) = \{t \in T; \check{b} t \check{b}^{-1} = t\},$$

$$X(w) = S_w / T(w) F^U \cap \check{b} U \check{b}^{-1}$$

and $1_{w}^F$ is the virtual character of $G/F$ such that $\text{Tr}(x, 1_{w}^F) = \text{Tr}(x^{*^{-1}}, \sum_{w \in W^F} (-1)^{i} H_i'(X(w)))$.

Then we have

Lemma 1.2.1 (cf. Remark 1.4.2). Let $x \in G/F$. Take $a \in G$ such that $x = F^m a^{-1} a$. Let $y = a^F a^{-1} \in G/F$ (cf. Lemma 1.1.1). Then

$$\text{Tr}((x^{-1} F^m)^*, \sum_{w \in W^F} (-1)^i H_i'(X(w))) = \left( |T/F|^{d} |q^m|^{-1} \right)^7 \{h \in G/F; h^{-1} y^\sigma h \in \check{b} B\},$$

where $d = \dim U \cap \check{b} U \check{b}^{-1}$.

1.3. Let $Z^F = \text{Ind}_{B/F}^G 1 (= \text{the representation of } G/F \text{ induced from the trivial representation of } B/F)$. Then $Z^F = \sum_{g \in G/F} Q_i g^v$ as vector spaces with $B^F$ acting trivially on $Q_i v$. As is known, $\text{End}_{B/F} Z^F = \sum_{w \in W^F} Q_i a_w^F$, where $a_w^F$ is defined by: $a_w^F v = \sum_{w \in U^w_{w}} \check{b} U \check{b}^{-1} v$ with $U_w = U_{w} \cap \check{b} U \check{b}^{-1}$ ($U_{w}$ is the maximal unipotent subgroup opposite to $U$). Define the linear mapping $I_\sigma$ on $Z^F$ by:

$I_\sigma: \sum_{g \in G/F} c_{g} g^v \mapsto \sum_{g \in B^F} c_{g} g^v$ ($c_{g} \in Q_i$).

Then for any $g \in G/F$ and $z \in Z$, $I_\sigma(gz) = z I_\sigma(gz)$.

Then we have

Lemma 1.3.1 (cf. Remark 1.4.2). For $g \in G/F$ and $w \in W^F$, $\text{Tr}(ya^F_\sigma I_\sigma, Z^F) = (q^m |T/F|^{d})^{-1} \left\{ \sum_{g \in G/F} g^{-1} y^\sigma g \in \check{b} B\right\}$, where $d = \dim U \cap \check{b} U \check{b}^{-1}$.

1.4. For any $x \in G/F$, write $x = F^m a^{-1} a$ with $a \in G$ and let $y = a^F a^{-1} \in G/F$.

By Lemma 1.2.1 and 1.3.1, $\text{Tr}((x^{-1} F^m)^*, \sum_{w \in W^F} (-1)^i H_i'(X(w))) = \text{Tr}(ya^F_\sigma I_\sigma, Z^F)$.
Since $\text{Tr}(ya^p I_\sigma, Z^K)$ does not depend on the $\sigma$-conjugacy class of $y$, we have

**Theorem 1.4.1.** For any $y \in G^F$, $\text{Tr}((n^K I_\sigma(y)^{-1} F^w)^*, \sum (-1)^i H^i(X(w))) = \text{Tr}(ya^p I_\sigma, Z^K)$.

**Remark 1.4.2.** (i) The above formula (and also Lemma 1.2.1, 1.3.1) were first appeared in [2]. This was informed to the author by Kawanaka.

(ii) It should be noted here that there are similar formulae to that of the theorem. If $F^m$ acts canonically on $R_\tau$, the analogy of the theorem is also true as is easily checked.

1.5. Let $\delta$ be the smallest integer $\geq 1$ such that $F^\delta$ acts trivially on $W$. Let $\rho \in \mathcal{E}(G^F, \{1\}) (=\text{the set of all (equivalence classes of) unipotent representations of } G^F)$. By Lusztig [12], Coro. 3.9, if $\rho \in H^i_\tau(X(w))_\mu (=\text{the generalized } \mu\text{-eigenspace of } F^\delta\text{ on } H^i_\tau(X(w)))$, then $\mu$ is uniquely determined (up to an integral power of $q^\delta$) by $\rho$ (not depending on $i$ or $w$).

**Definition 1.5.1.** For $\rho \in \mathcal{E}(G^F, \{1\})$, let $\mu$ be as above. Define $\lambda_\rho$ to be the constant such that $x^\rho = \mu q^r \rho$ for some $r \in \mathbb{Z}$ and $1 \leq |\lambda_\rho| < q^\delta$.

For $\rho \in \mathcal{E}(G^F, \{1\})$, let $H^i_\tau(X(w))_\rho$ be the largest subspace of $H^i_\tau(X(w))$ on which $G^F$ acts by a multiple of $\rho$. Then

**Lemma 1.5.2.** For any $\rho \in \mathcal{E}(G^F, \{1\})$ and $w \in W$, there exists $f_{\rho, w}(X) \in Z[X, X^{-1}]$ such that if $\delta$ divides $m,$

\[
\text{Tr}(x^{-1} F^w)^*, \sum (-1)^i H^i_\tau(X(w))_\rho = f_{\rho, w}(q^m) \lambda_\rho^{m/\delta} \rho(x) \text{ for any } x \in G^F \text{ and } f_{\rho, w}(1) = \langle \rho, R_{\tau} \rangle.
\]

2. Split case

2.1. In introducing the notation we only assume that $G$ splits over $K$. Let $\mathfrak{X}(W) = \text{End}_{g^{\mathbb{Z}}} Z^K$ and $S$ the set of simple reflections of $W$ (corresponding to $B$). Let $\mathfrak{A}(W)$ be the generic algebra of $\mathfrak{X}(W)$ over the extension field of $Q(X)$ ($X$: indeterminate) and $\{a_w; w \in W\}$ be its basis. ($\mathfrak{X}(W)$ is obtained from $\mathfrak{A}(W)$ by the specialization $X \mapsto q^m$ or more precisely by the homomorphism from the integral closure of $Q[X]$ to $Q$ which maps $X$ to $q^m$.) Let $\tilde{W}$ be the set of equivalence classes of the irreducible representation of $W$. For any $X \in \tilde{W}$, let $\nu_X, \rho_X, \nu^X, \rho^X$ be the corresponding irreducible representation (or its character) of $\mathfrak{A}(W), \mathfrak{X}(W), G^F$ respectively. Then $Z^K$ can be written in the form: $Z^K = \bigoplus \nu_X \otimes \rho^X$. For an $F$-stable subset $J \subseteq S$, let $W_J$ be the subgroup of $W$ generated by $J, P_J$ the corresponding standard parabolic subgroup of $G$, $L_J$ its standard Levi subgroup and $Z^J = \text{Ind}_{B^F}^{G^F} 1 (= \text{Ind}_{(B \cap L_J)}^{L_J} 1$ as $L^J$-modules). $Z^J$ is cano-
nically regarded as a subspace of $Z^\kappa$ and $\text{End}_{P^m}Z^\kappa_f = \sum_{w \in W_f} Q_w a_w |_{Z^\kappa}$. The following are also defined: $\mathfrak{U}(W_f), \mathfrak{U}(W), \{\nu_x, \nu_x^\kappa, \rho_x^\kappa; \chi \in \hat{W}\}$. Since $W_f$ is a parabolic subgroup of $W$, $\mathfrak{U}(W_f)$ (resp. $\mathfrak{U}(W)$) is regarded as a subalgebra of $\mathfrak{U}(W)$ (resp. $\mathfrak{U}(W_f)$). For any $\chi' \in \hat{W}_f$ and $\chi \in \hat{W}$, define the non-negative integer $n_{x,x'}$ by: $\text{Ind}_{W_f}^W \chi' = \sum_{x \in \hat{W}_f} n_{x,x'} \chi$. For $\chi' \in \hat{W}_f$, let $Z^\kappa_{\chi'}$ (resp. $Z^\kappa_{\chi,x'}$) be the largest subspace of $Z^\kappa$ (resp. $Z^\kappa_{\chi}$) on which $\mathfrak{U}(W_f)$ acts by a multiple of $\nu_x^\kappa$. For $\chi \in \hat{W}$, $Z^\kappa_{\chi}$ is defined similarly. The following are checked easily: for $\chi' \in \hat{W}_f$, $\text{Ind}_{W_f}^W \chi'= \sum_{x \in \hat{W}_f} n_{x,x'} \chi$. For $\chi \in \hat{W}$, $\text{Ind}_{W_f}^W \chi_{\chi,x'} = n_{x,x'} \nu_x^\kappa \otimes p_x^\kappa$, and for $\chi' \in \hat{W}_f$ and $\chi \in \hat{W}$, $Z^\kappa_{\chi} \cap Z^\kappa_{\chi,x'} = n_{x,x'} \nu_x^\kappa \otimes p_x^\kappa$.

2.2. Henceforth in this section we assume $G$ to be split over $k$. Then the mapping $I_\sigma$ commutes with any $a_w^e (w \in W)$, thus with $\mathfrak{U}(W)$. Therefore each $\rho_x^\kappa$ is regarded as an irreducible $G^{\ell_m}$ $A$-modules which is denoted by $p_x^\kappa$. By Theorem 1.4.1, we have

**Lemma 2.2.1.** For any $y \in G^{\ell_m}$, 
$$\text{Tr}((n_{x,y})^{-1}I_{\mathfrak{K}}^{\ell_m}, \sum _{\gamma \in \hat{W}_f} (-1)^i H^i(X(\gamma))) = \sum _{\gamma \in \hat{W}_f} \nu_x^\kappa (a_w^e) \rho_x^\kappa (y \sigma).$$

Let $J \subset S$ be $F$-stable. $\rho_x^\kappa(\chi' \in \hat{W}_f)$ are similarly defined as $\rho_x^\kappa(\chi \in \hat{W})$. Now, for any $z \in Z^\kappa_f$ and $g \in G^{\ell_m}$, $I_\sigma(gz) = g I_\sigma(z)$. Thus for $\chi' \in \hat{W}_f$, $\text{Ind}_{P^m}^{G^{\ell_m}} \rho_{\chi,x'} = Z^\kappa_{\chi'}$ as $G^{\ell_m} A$-modules. Hence

**Lemma 2.2.2.** Assume $\text{Ind}_{P^m}^W \chi' = \sum_{\gamma \in \hat{W}_f} n_{x,x'} \chi(\chi' \in \hat{W}_f, n_{x,x'} > 0)$. Then $\text{Ind}_{P^m}^{G^{\ell_m}} \rho_{\chi,x'} = \sum_{x \in \hat{W}_f} n_{x,x'} \rho_x^\kappa$ and $\text{Ind}_{P^m}^{G^{\ell_m}} A \rho_{\chi,x'} = \sum_{x \in \hat{W}_f} n_{x,x'} \rho_x^\kappa$.

**Lemma 2.2.3.** Assume the Dynkin graph of $G$ does not have irreducible components of type $E_7$ or $E_8$. Assume that for any $J \subset S$ and $\chi' \in \hat{W}_f$, there exists the lifting of $\rho_{\chi'}$ in $L^\vee_{J^m}$. Then for any $\chi \in \hat{W}$ and $y \in G^{\ell_m}$, $\rho_x^\kappa(\mathfrak{K}_{J/k}(y)) = p_x^\kappa(\mathfrak{K}_{J/k}(y \sigma))$.

**Proof.** By Lemma 1.1.9, $(\text{Ind}_{B^m}^{G^{\ell_m}} 1) (\mathfrak{K}_{J/k}(y)) = (\text{Ind}_{B^m}^{G^{\ell_m}} A) 1 (y \sigma)$ for any $y \in G^{\ell_m}$. Thus

$$\sum_{x \in \hat{W}_f} \dim \chi_x^k (\mathfrak{K}_{J/k}(y)) = \sum_{x \in \hat{W}_f} \dim \chi_x^k (y \sigma)$$

for any $y \in G^{\ell_m}$. The existence of the lifting of each $\rho_x^k$ shows for each $\chi \in \hat{W}$ there exists $\chi' \in \hat{W}$ such that $\rho_x^k(\mathfrak{K}_{J/k}(y)) = c \rho_x^k(\mathfrak{K}_{J/k}(y \sigma))$ for any $y \in G^{\ell_m}$ and $c=1$. (This is checked by taking the inner product with the relation (a). See Lemma 1.1.8.) If $\chi = 1$, the statement of the lemma is obvious. If $\chi = St_w$ (= the sign character of $W$), it is also obvious. This proves the case when the semisimple rank of $G$ is 1. Assume the semisimple rank of $G \geq 2$ and the statement holds for any $L_J$ with $J \subset S$. 

Let \( J \subset S \). Then for any \( \chi' \in \hat{W}_J \) and \( y \in G^{F^m} \), \( \rho^J_k(\mathfrak{R}_{K/k}y) = \overline{\rho^J_k}(y\sigma) \). Write Ind\(^W_J \chi' = \sum_{x \in W} n_{x, \chi} \chi \). Then by Lemma 2.2.2, \( \sum_{x \in W} n_{x, \chi'} \overline{\rho^J_k}(\mathfrak{R}_{K/k}y) = \sum_{x \in W} n_{x, \chi'} \overline{\rho^J_k}(y\sigma) \) for any \( y \in G^{F^m} \). Thus the lemma is an easy consequence of the following well known result (cf. Benson-Curtis \[1\]):

Let \((W, S)\) be the Weyl group which does not have the irreducible factors of type \( G_2, E_7 \) or \( E_8 \) and assume rank \((W, S) \geq 2 \). For \( \chi_1, \chi_2 \in \hat{W} \), if \( \chi_1 \mid _{w_j} = \chi_2 \mid _{w_j} \) for any \( j \in S \), then \( \chi_1 = \chi_2 \).

By Lemma 2.2.1 and 2.2.3 we have

**Lemma 2.2.4.** Assume the assumption of Lemma 2.2.3. Then

\[ \text{Tr}( (X^{-1}F^m)^* , \sum (-1)^i H^i(\chi(\sigma)) = \sum_{x \in W} \nu^K(a^\nu_k) \rho^J_k(n_k^\nu_k \chi) \text{ for any } x \in G^{F^m}. \]

2.3. If \( G = GL_n \), we can easily check the following theorem, which is proved in \[2\] and also by Lusztig independently.

**Theorem 2.3.1.** Assume \( G = GL_n \). Then

(i) \( \rho^J_k(n_k^\nu_k \chi) = \overline{\rho^J_k}(y\sigma) \) for any \( \chi \in \hat{W} \) and \( y \in G^{F^m} \),

(ii) \( f_{\chi \circ \nu}(X) = \nu_{a^\nu_k} \) for any \( \chi \in \hat{W} \) and \( \nu \in W \),

(iii) \( \|X^F_m\| = \sum_{x \in W} \nu^K(a^\nu_k) \text{ dim } \rho^J_k \).

2.4. In 2.4 we assume \( G = Sp_{2n}, SO_{2n+1} \) or \( SO^*_n \).

**Lemma 2.4.1.** If \((m, 2p) = 1 \), then

(i) \( \sum_{x \in W} \rho^J_k(a^\nu_k) = \sum_{x \in W} \nu^K(a^\nu_k) \rho^J_k(n_k^\nu_k \chi) \),

(ii) \( \sum_{x \in W} f_{\chi \circ \nu}(q^m)^\nu_k \text{ dim } \rho = \sum_{x \in W} \nu^K(a^\nu_k) \text{ dim } \rho^J_k \),

where \( \rho \) ranges over \( \mathcal{E}(G^F, \{1\}) \).

Proof. By Lemma 1.5.2 and 2.2.1, \( \sum_{x \in W} f_{\chi \circ \nu}(q^m)^\nu_k \text{ dim } \rho = \sum_{x \in W} \nu^K(a^\nu_k) \rho^J_k(y\sigma) \) for any \( y \in G^{F^m} \). By Theorem 1.1.6 and Lemma 2.2.3, \( \rho^J_k(\mathfrak{R}_{K/k}y) = \rho^J_k(n_k^{-1}n_k^\nu_k \chi) \text{ for any } y \in G^{F^m}. \) Thus we have (i). Since \( n_k \in \{1\} = \{1\} \), we have (ii).

To proceed further we need some lemmas. The following one is obvious.

**Lemma 2.4.2.** Let \( e_1, \ldots, e_r \in Q^* \). Assume \( \sum_{i \leq t \leq r} c_i e_i = 0 \) for \( t = 1, \ldots, r \). Then there exist \( 1 \leq i \neq j \leq r \) such that \( x_i = x_j \).

**Lemma 2.4.3.** Let \( f(X), g(X) \in \mathbb{Q}[X] \), \( t \) a positive integer (maybe 1) and \( \lambda \in Q^*_t \). Assume \( f(q^m)^\lambda = g(q^m) \) for any positive integer \( m \) such that \((m, t) = 1\). Then \( \lambda = \zeta^t q^m \) with \( \zeta \) a \( t \)-th root of unity and \( \alpha \) an integer.
Proof. Write \( f(X) = \sum_{0 \leq i < r} a_i X^i \), \( g(X) = \sum_{0 \leq i < r} b_i X^i(a_i, b_i \in \mathbb{Q}) \). By the assumption, \( f(q^{mt+1})^m = g(q^{mt+1})^m \) for any \( m \in \mathbb{N} \). Thus \( \sum_{0 \leq i < r} a_i q^i (q^{mt+1})^m = \sum_{0 \leq i < r} b_i q^i (q^{mt+1})^m \) for any \( m \in \mathbb{N} \). If \( i \neq j \), \( q^i \neq q^j \) and \( q^{it} \lambda^i = q^{jt} \lambda^j \). Thus, by Lemma 2.4.2, \( q^{it} \lambda^i = q^{jt} \) for some \( 0 \leq i \leq r, 0 \leq j \leq s \). Therefore \( \lambda = \zeta^n \) with \( \zeta \) a \( t \)-th root of unity and \( n \) a positive integer.

The following proposition is known when \( q \) is larger than the Coxeter number of \( G \) (cf. Lusztig [12], p. 25, (d)).

**Proposition 2.4.4.** For any \( p \in \mathcal{E}(G^F, \{1\}) \), \( \lambda_p = 1 \) or \(-1\).

Proof. If \( p \) is not cuspidal, the computation of \( \lambda_p \) is reduced to the groups of smaller ranks. Thus it remains to check for the cuspidal \( \rho_0 \in \mathcal{E}(G^F, \{1\}) \). Take \( w \in W \) such that \( \langle \rho_0, R_w \rangle > 0 \). Then \( f_{\rho_0,w}(X) \neq 0 \) (cf. 1.5). If \( (m, 2p) = 1 \), \( \sum_{p} f_{\rho_0,w}(q^m) \lambda_{p}^m \dim \rho = \sum_{x \in W} v_x^p(a_x^w) \dim \rho_x^p \) by Lemma 2.4.1, (ii). We may assume if \( p \neq p_0 \), \( \lambda_p = 1 \) or \(-1\). Therefore \( \lambda_{p_0} = \zeta^n \) with \( \zeta \) a \( t \)-th root of unity and \( n \) a positive integer.

Since \( f_{\rho_0,w}(X) \neq 0 \), there exists an integer \( m_0 \) such that if \( m, 2p \neq 0 \), we have \( f_{\rho_0,w}(q^m) \lambda_{p_0}^m \dim \rho_0 + \sum_{p \neq p_0} f_{\rho_0,w}(q^m) \lambda_p^m \dim \rho_p^m \). Applying Lemma 2.4.3 we have \( \chi_{p_0}^m = 1 \) (since \( 0 \leq |\lambda_p| < q \)). Thus it suffices to prove \( \lambda_{p_0} \in \mathbb{Q} \).

But for any positive integer \( m \), \( f_{\rho_0,w}(q^m) \lambda_{p_0}^m \dim \rho_0 + \sum_{p \neq p_0} f_{\rho_0,w}(q^m) \lambda_p^m \dim \rho_p^m \). Since \( f_{\rho_0,w}(X) \neq 0 \), there exists an integer \( m_0 \) such that if \( m, 2p \neq 0 \), we have \( f_{\rho_0,w}(q^m) \lambda_{p_0}^m \dim \rho_0 + \sum_{p \neq p_0} f_{\rho_0,w}(q^m) \lambda_p^m \dim \rho_p^m \). Therefore \( \lambda_{p_0} = \chi_{p_0}^m \lambda_{p_0}^m + 1 \chi_{p_0}^m \lambda_{p_0}^m - \chi_{p_0}^m \lambda_{p_0}^m = \mathbb{Q} \).

**Lemma 2.4.5.** \( \sum_{p} f_{\rho_0,w}(X) \lambda_p = \sum_{x \in W} \nu_x(a_x^w) \rho_x^p \cdot n_{x/k}^p \) as \( \mathbb{Q}[X] \)-linear combinations of class functions of \( G^F \).

Proof. Fix \( y \in G^F \). By Lemma 2.4.1 and Proposition 2.4.4, if \( (m, 2p) = 1 \), then \( \sum_{p} f_{\rho_0,w}(q^m) \lambda_{p} \dim \rho_0 + \sum_{p \neq p_0} f_{\rho_0,w}(q^m) \lambda_p \dim \rho_p = \text{Tr}(R_{v}^m, X_{v}) \). This proves the lemma.
Lemma 2.4.7.  (i) For any \( w \in W \) and \( \rho \in \mathcal{E}(G^F, \{1\}) \), \( f_{p, w}(X) = \sum_{x \in W} \nu_x(a_w) \langle R_x, \rho \rangle \).
(ii) \( \sum_p f_{p, w}(X) \rho = \sum_{x \in W} \nu_x(a_w) R_x \).

Proof. (i) \( \lambda_x f_{p, w}(X) = \sum_{x \in W} \nu_x(a_w) \langle \rho^k_x, n_{k/\lambda} \rangle \). This proves (i). (ii) is an easy consequence of (i).

Theorem 2.4.8. Let \( w \in W \).
(i) If \( m \) is odd, \( \mu((w)^m) = \sum_{x \in W} \nu_x(a_w)^m \dim \rho_x \).
(ii) If \( m \) is even, \( \mu((w)^m) = \sum_{x \in W} \nu_x(a_w)^m \dim R_x \).

Proof. \( \mu((w)^m) = \sum_{p} f_{p, w}(q^m) \lambda_p \dim \rho \). Assume \( m \) is odd. Then \( \mu((w)^m) = \sum_{p} f_{p, w}(q^m) \lambda_p \dim \rho \) (since \( \lambda_p = 1 \) or \( -1 \)). Assume \( m \) is even. Then \( \mu((w)^m) = \sum_{p} f_{p, w}(q^m) \dim \rho \).

The following lemma is well known (cf. [4]).

Lemma 2.4.9. Let \( \mathfrak{U} \) be a semisimple and symmetric algebra over the algebraic closed field of characteristic 0. Let \( \{e_1, \ldots, e_r\} \) be a basis of \( \mathfrak{U} \) and \( \{e^1, \ldots, e^s\} \) be its dual basis. Let \( \chi_1, \chi_2 \) be the irreducible characters of \( \mathfrak{U} \). Then \( \sum_{i} \chi_i(e_i) \chi_2(e^i) = 0 \) if and only if \( \chi_1 \neq \chi_2 \).

Theorem 2.4.10.  (i) If \( m \) is odd, \( \rho^x_\kappa(y \sigma) = \rho^x_\kappa(\mathfrak{U}_{K/I}) \) for any \( \chi \in \hat{W} \) and \( y \in G^F \).
(ii) If \( m \) is even, \( \rho^x_\kappa(y \sigma) = R_x(\mathcal{R}_{K/I}) \) for any \( \chi \in \hat{W} \) and \( y \in G^F \).

Proof. For any \( y \in G^F \) and \( w \in W \), \( \sum_{x \in W} \nu_x(a_w) \rho^x_\kappa(y \sigma) = \text{Tr}((\mathcal{R}_{K/I})^{-1} F^m)^* \).

Remark 2.4.11. If \( \text{char } F_q \neq 2 \), then \( R_x = R_x \cdot n_{k/\lambda} \) by the following lemma. Therefore \( \text{char } F_q \neq 2 \), in (ii) of Theorem 2.4.10 can be replaced by \( \text{char } F_q \neq 2 \). This seems to be true even if \( \text{char } F_q \neq 2 \).
Lemma 2.4.12. Let $G$ be a connected reductive group over $F_q$. (We do not assume the assumption imposed on $G$ in 2.4.) Let $x \in G^F$ and $x = su$ be the Jordan decomposition ($s$: a semisimple element, $u$: a unipotent element). Assume $u$ is contained in the identity component of the centralizer of $u$ in $Z_G(s)^0$. (Notice $u \in Z_G(s)^0$ by [16], Corollary 4.4.) Then for any $F$-stable maximal torus $T$ of $G$ and linear character $\theta$ of $T^F$, $R_T^\theta(n_{s|a}(x)) = R_T^\theta(x)$.

Proof. Let $H = Z_G(s)^0$. Let $T'$ be an $F$-stable maximal torus of $H$. Take $a \in T'$ such that $s = a^{-1}F_a$. Take $b \in Z_H(u)^0$ such that $u = b^{-1}Fb$. Then $x = su = sb^{-1}Fb = b^{-1}Fb = b^{-1}a^{-1}Fb = (ab)^{-1}F(ab)$. Thus $n_{s|b}(x) = F(ab)(ab)^{-1} = Fb$ $b^{-1}a^{-1}$. $ab$ $b^{-1}a^{-1} = b^{-1}a^{-1}$ (by the assumption imposed on $G$ in 2.4.) Similarly, $R_T^\theta(su) = \sum_{1 \leq i < r} Q_{g_i, T_g, H(u)} \theta(g_i, \sigma_g, \tau_a)$. Similarly, $R_T^\theta(n_{s|b}(x)) = \sum_{1 \leq i < r} Q_{g_i, T_g, H(aau^{-1})} \theta(g_i, \sigma_g, \tau_a)$.

2.5. In 2.5, we wish to describe some conjectural statements flourishing from Lemma 2.4.6, if we assume Conjecture 4.3 of Lusztig [12]. To do this we need to recall some results of [11], [12]. For $\Lambda \in \Phi_\ast$ (resp. $\Phi^\ast$), let $\rho_\Lambda$ be the corresponding unipotent representations of $S_P^\ast$ or $SO_2^\ast$ (resp. $SO_2^\ast$).

For $\chi \in \hat{W}_\ast$ (resp. $\hat{W}$), let $\Lambda$ be the corresponding symbol class in $\Phi_\ast$ (resp. $\Phi^\ast$) and we put $R_\Lambda = R_x$. For $\Lambda \in \Phi_\ast$ (resp. $\Phi^\ast$), we write $\Lambda = (X \cup (Y - I), X \cup I)$, where $X$, $Y$ are finite subsets of $\{0, 1, 2, \ldots\}$, $X \cap Y = \emptyset$, $I$ is a subset of $Y$ such that $2 \mid |I| + 1 \equiv |Y| \pmod 4$ (resp. $2 \mid |I| \equiv |Y| \pmod 4$). Now, fix $X$ and $Y$. We put $|Y| = 2s$ or $2s + 1$, and assume $s > 0$ if $|Y| = 2s$. Let $Y = \{\lambda_0 < \lambda_1 < \lambda_2 < \ldots\}$, $Y^0 = \{\lambda_0, \lambda_2, \lambda_4, \ldots\}$ and $Y^1 = \{\lambda_1, \lambda_3, \lambda_5, \ldots\}$. Let $\mathcal{P}$ be the set of all subsets of $Y$ and $\mathcal{P}_s = \{I \in \mathcal{P} : |I| = s \pmod 2\}$. Then $\mathcal{P}$ is regarded as a vector space over $F_2$ by the addition: $I, J \in \mathcal{P} \rightarrow I + J \in \mathcal{P}$ and $I \cap J \in \mathcal{P}$ and $\mathcal{P}_0$ is regarded as a subspace. By the bijection $\mathcal{P}_s \rightarrow \mathcal{P}_0$ ($I \rightarrow IY^0$), we can regard $\mathcal{P}_s$ as a vector space over $F_2$. Define $Q : \mathcal{P}_s \rightarrow \{\pm 1\}$ ($I \rightarrow (-1)^{(|I| - s)/2}$). If we identify $F_2$ canonically with $\{1, 1\}$, the mapping $Q$ is regarded as a quadratic form on $\mathcal{P}_s$ whose associated bilinear form $B$ is: $I, J \in \mathcal{P}_s \rightarrow B(I, J) = (-1)^{\sum_{x \in I \cap J} x \cdot \lambda}$ for $I \in \mathcal{P}_s$. Thus the Fourier transform of Lusztig [11], [12] takes the form:

$$\rho(I \cup J, x \cup y) = 2^{-s} \sum_{j \in \mathcal{P}_s} B(I, J) \rho(I \cup J', x \cup y)$$

for $I \in \mathcal{P}_s$.

Definition 2.5.1. (i) For a class function $f$ on $G^F$, let $f^\Lambda = f \cdot n_{s|b}$. $\Delta^2 = 1$ and $\dim f^\Lambda = \dim f$. (Notice that for any connected algebraic group $G$ over $F_q$,}
if $x' \in Z_{G}(x)^{0}$, then $n_{r_{i}i}(\{x\}) = \{x\}$.

(ii) Let $R_{X,Y} = \sum_{\gamma \in G} Q(\rho_{(x_{1}',x_{2})})$. Define the linear automorphims $\Delta$ of $R_{X,Y}$ by the condition $\rho_{(x_{1}',x_{2})} = Q(I)\rho_{(x_{1}',x_{2})}$ for $I \in \mathcal{O}_{s}$. Since $\dim \rho_{(x_{1}',x_{2})} = 0$ if $|I| \neq s$, we have $\dim f^{\Delta} = \dim f$ for any $f \in R_{X,Y}$.

It can easily be checked the following

**Lemma 2.5.2.** For any $J \in \mathcal{O}_{s}$,

$$Q(I)\rho_{(x_{1}',x_{2})} = 2^{-s} \sum_{J \in \mathcal{O}_{s}} B(I, J)Q(J)\rho_{(x_{1}',x_{2})}.$$  

**Theorem 2.5.3.** Assume Conjecture 4.3 in [12] is true. Then for $I \in \mathcal{O}_{s}$, $\lambda_{\rho_{(x_{1}',x_{2})}}(I) = (-1)^{(1-s)/2}R(XV_{I}, XV_{I}) = \beta_{(x_{1}',x_{2})}$ if $|I| = s$. Thus by Lemma 2.4.6, $R_{(x_{1}',x_{2})} = 2^{-s} \sum_{J \in \mathcal{O}_{s}} B(I, J)Q(J)\rho_{(x_{1}',x_{2})}$. But $\dim R_{(x_{1}',x_{2})} = \dim R_{(x_{1}',x_{2})} = \dim R_{(x_{1}',x_{2})}$. Thus $2^{-s} \sum_{J \in \mathcal{O}_{s}} B(I, J)Q(J)\rho_{(x_{1}',x_{2})} = 2^{-s} \sum_{J \in \mathcal{O}_{s}} B(I, J)\lambda_{\rho_{(x_{1}',x_{2})}}(I) = \dim \rho_{(x_{1}',x_{2})}$ by Lemma 2.5.2. This relation shows our statement.

**Remark 2.5.4.** (i) The statement of Theorem 2.5.3 is a counterpart of the statements for some families of the unipotent representations of the exceptional groups given in Lusztig [12], p. 45 and [13], p. 335.

(ii) Assume $\text{char } F_{q} \neq 2$. Lemma 2.4.12 and the proof of Theorem 2.5.3 show that $\Delta$ and $\Delta$ coincide on the subspace $\mathfrak{S}$ of $R_{X,Y}$ which is spanned by $\{\rho_{(x_{1}',x_{2})}; I \in \mathcal{O}_{s}, |I| = s\}$ and $\{R_{(x_{1}',x_{2})}; I \in \mathcal{O}_{s}, |I| = s\}$ under the assumption of Theorem 2.5.3. If $|Y| = 1, 3$ or 4, $\mathfrak{S} = R_{X,Y}$. If $|Y| = 5$ or 6, $\dim \mathfrak{S} = \dim R_{X,Y} - 1$. We may ask if the following is true (cf. Remark 2.4.11).

**Conjecture 2.5.5.** $\Delta = \Delta$.

### 3. Unitary case

The method which we applied in the case of split classical groups is also effective for the unitary groups. Let $G$ be the unitary group $U_{n}$ over $F_{q}$ and we assume $m$ is an even integer. The Weyl group $W$ is canonically identified with the symmetric group $S_{n}$ and we assume the generic algebra $\mathfrak{V}(W)$ (cf. 2.1) is over the extension field of $\mathfrak{V}(X)$ which contains $X^{1/2}$ ($X^{1/2}$ being fixed). Let $\mathcal{O}(n)$ be the set of all partitions of $n$. For $\alpha \in \mathcal{O}(n)$, let $\chi_{\alpha}$ (resp. $\nu_{\alpha}$) be the corresponding irreducible representation (or its character) of $W$ (resp. $\mathfrak{V}(W)$). The following lemma is easily checked by the induction on $n$. 

Lemma 3.1. Let \( w_0 \) be the longest element of \( W \) and \( \alpha=(\alpha_1 \geq \cdots \geq \alpha_s > 0) \in \mathfrak{A}(n) \). Define \( C_\alpha = (\frac{N}{2}) + \sum_{1 \leq i < j \leq s} (j-i) \). Then \( a_{w_0}^2 \) acts as a scalar \( X^{C_\alpha} \) on the representation \( \nu_\alpha \) of \( \mathfrak{A}(W) \).

Let the notation be as in 2.1. We write \( \rho^K_\alpha \) instead of \( \rho^K_{1,\alpha} \) for \( \alpha \in \mathfrak{A}(n) \) to simplify the notation. Since \( a_{w_0}^2 I_\alpha \) commutes with \( \mathfrak{A}(W) \), each irreducible component \( \rho^K_\alpha \) of \( Z^K \) is regarded as a \( GF^n \) \( A \)-module \( \rho^K_\alpha \) by the mapping \( \sigma_\rightarrow (q^{-\frac{C_\alpha}{2}})^{a_{w_0}} I_\rho \).

For \( \alpha \in \mathfrak{A}(n) \), let \( \rho^K_\alpha = |W|^{-1} \sum_{\omega \in \mathfrak{W}} \chi_{\alpha}(w\omega_0) R^K_{\omega,\alpha} \). If we put \( \eta_\alpha \) = the signature of \( \lim \rho^K_\alpha \), then by [14], \( \eta_\alpha \rho^K_\alpha \) is the irreducible representation of \( GF \) and all the unipotent representations of \( GF \) are of this form. For the simplification of the notation we let \( f_{\alpha,\omega}(X) = \eta_\alpha f_{\alpha,\omega}(X) \) \( (\alpha \in \mathfrak{A}(n), \omega \in \mathfrak{W}) \), \( \lambda_\alpha = \lambda_{\eta_\alpha} \rho^K_\alpha \). Then

Theorem 3.2. Assume \( \text{char } F \neq 2 \). Let \( \alpha \in \mathfrak{A}(n) \) and \( \omega \in \mathfrak{W} \). Then

\[
\begin{align*}
(1) \quad & \rho^K_{\omega}(n_{\eta_\alpha,\omega}(y)) = (-1)^{mc_{w_0}/2} \rho^K_{\omega}(y) \\
(2) \quad & f_{\alpha,\omega}(q^m)\lambda_{\alpha}^{\frac{m}{2}} = v^K_{\omega}(a_{\alpha,\omega}^2) (-q)^{mc_{w_0}} \dim \rho^K_{\omega} \quad \text{for any even integer } m, \\
(3) \quad & |X(w)^F|^m = \sum_{\alpha \in \mathfrak{A}(\alpha)} v^K_{\alpha}(a_{\alpha,\omega}^2) (-q)^{-mc_{w_0}} \dim \rho^K_{\omega} \quad \text{for any even integer } m.
\end{align*}
\]

Our proof is based on Kawanaka [7] as is stated in the introduction. In this respect, (i) of the theorem for cuspidal or subcuspidal \( \rho^K_{\omega} \)'s is essential. The detailed arguments, which is slightly tedious, are omitted.

References


Department of Mathematics
Nara University of Education
Nara 630, Japan