Asai, T.
Osaka J. Math.
**20** (1983), 21–32

# ON THE ZETA FUNCTIONS OF THE VARIETIES $X(w)$ OF THE SPLIT CLASSICAL GROUPS AND THE UNITARY GROUPS

TERUAKI ASAI

(Received April 30, 1981)

## 0. Introduction

Let $G$ be one of the split classical groups $SO_{2n}^{+}$, $SO_{2n+1}$, $Sp_{2n}$ or a unitary group defined over the finite field $\boldsymbol{F}_q$ of $q$ elements. Let $F$ be the Frobenius mapping, $G^F$ the subgroup of $F$-stable elements, $W$ the Weyl group of $G$ and let $\delta$ be the smallest positive integer such that $F^\delta$ acts trivially on $W$. For $w \in W$, Deligne-Lusztig [3] has defined the $F^\delta$-stable variety $X(w)$ for any connected reductive group. If $w$ is a Coxeter element of $W$, the zeta function of $X(w)$ was obtained by Lusztig [9] as a by-product when he determined the Green polynomial associated with $w$. In this paper we shall determine the zeta function of $X(w)$ for any $w \in W$.

To state our result more explicitly, let $B$ be a fixed $F$-stable Borel subgroup of $G$, $\mathfrak{A}^K(W)$ the Hecke algebra of the representation of $G^{F^m}$ induced from the trivial representation of $B^{F^m}$ and let $\{a_w^K;\ w \in W\}$ be the natural basis of $\mathfrak{A}^K(W)$. When $\delta$ divides $m$ the number of $F^m$-stable points of $X(w)$ is expressed in terms of the dimensions of the unipotent representations of $G^F$ and the trace of $a_w^K$ on each irreducible representation of $\mathfrak{A}^K(W)$.

The crucial point of our arguments depends on the lifting theory due to Shintani-Kawanaka ([15], [7], [8]) and a result of Lusztig ([12], Corollary 3.9), which says that for any unipotent representation $\rho$ of $G^F$, the eigenvalues of $F^\delta$ on the $\rho$-isotypic component of $H_c^i(X(w))$ are independent of $i$ and $w$ up to a multiple factor of the form $q^{i\delta}$, $i \in \boldsymbol{Z}$.

Finally the author expresses his heartfelt gratitude to Professor N. Kawanaka for his valuable suggestions and kind encouragement during the preparation of this paper.

## 1. General results

1.1. First we summarize the known results (Shintani [14], Kawanaka [7], [8]) to apply for our use.

Let $m$ be a positive integer (maybe 1), $k=\boldsymbol{F}_q$, $K=\boldsymbol{F}_{q^m}$, $G$ a connected algebraic group defined over $k$, $F$ the Frobenius over $k$, $\sigma=F|_{G^{F^m}}$ and $A$ the cyclic group (of order $m$) generated by $\sigma$. Let $x_1, x_2 \in G^{F^m}$. $x_1\sigma$ and $x_2\sigma$ are conjugate in $G^{F^m}A$ (semi-direct) if and only if there exists $h \in G^{F^m}$ such that $x_1=h^{-1}x_2^{\sigma}h$. If this is the case, we say $x_1$ and $x_2$ are $\sigma$-conjugate and we write $x_1 \underset{\sigma}{\sim} x_2$. If $m=1$, we simply write $x_1 \sim x_2$ instead of $x_1 \underset{\sigma}{\sim} x_2$. The following lemma is proved in [7].

**Lemma 1.1.1.** *For $x \in G^{F^m}$, take $a \in G$ such that $x=a^{-1F}a$. Let $y={}^{F^m}a\, a^{-1}$. Then $y \in G^F$, and the conjugacy class of $y$ in $G^F$ is uniquely determined by the $\sigma$-conjugacy class of $x$ in $G^{F^m}$. And the mapping $x \mapsto y$ defines a bijection: $G^{F^m}/\underset{\sigma}{\sim} \to G^F/\sim$.*

Definition 1.1.2. We denote the bijection $G^{F^m}/\underset{\sigma}{\sim} \to G^F/\sim$ in the above lemma by $n_{K/k}$. (Notice $n_{K/k}$ is defined even if $m=1$.) Define $\mathfrak{N}_{K/k}=n_{k/k}^{-1}\, n_{K/k}$. This also is a bijection from $G^{F^m}/\underset{\sigma}{\sim}$ onto $G^F/\sim$.

Remark 1.1.3. The reader should refer Kawanaka [8] for the relation between the norm mapping in [loc. cit.] and our norm mapping $\mathfrak{N}_{K/k}$.

The following lemma features some property of the mapping $\mathfrak{N}_{K/k}$, which is not used in this paper. The proof is omitted.

**Lemma 1.1.4.** *Let $G$ be a connected reductive group and $Z(G)$ the center of $G$. Let $s \in Z(G)^F$ and $u \in G^F$. Let $r$ be the order of $s$. Assume $m \equiv 1 \bmod r$. Then $\mathfrak{N}_{K/k}^{-1}(su)=s\mathfrak{N}_{K/k}^{-1}(u)$.*

For $\chi_K \in \widehat{G^{F^m}}^{\sigma}$ (=the set of $\sigma$-invariant irreducible characters of $G^{F^m}$), there exists $\tilde{\chi}_K \in \widehat{G^{F^m}A}$ such that $\tilde{\chi}_K|_{G^{F^m}}=\chi_K$. Let $\chi_k \in \widehat{G^F}$.

Definition 1.1.5. Let $m>1$. We say $\chi_K$ is the lifting of $\chi_k$ in $\widehat{G^{F^m}}$ if there exists a constant $c$ such that $\tilde{\chi}_K(y\sigma)=c\chi_k(\mathfrak{N}_{K/k}\, y)$ for any $y \in G^{F^m}$. (The lifting of $\chi_k$ is uniquely determined by $\chi_k$ if it exists. See [7].)

**Theorem 1.1.6** ([7], [8], [15]).
*Let $m>1$. Assume one of the following.*

(1) $G=GL_n$.

(2) $G=U_n$, $(m, p)=1$.

(3) $G=SO_{2n+1}$, $Sp_{2n}$ *or* $SO_{2n}^{\pm}$, $(m, 2p)=1$.

*Then any $\chi_k \in \widehat{G^F}$ has the lifting $\chi_K \in \widehat{G^{F^m}}$. And the mapping $\chi_k \mapsto \chi_K$ defines a bijection between $\widehat{G^F}$ and $\widehat{G^{F^m}}^{\sigma}$.*

Remark 1.1.7. The theorem is proved by Shintani [15] in case (1), by Kawanaka [7] in case (2) and by Kawanaka [8] in case (3).

The following lemmas can be extracted from [7].

**Lemma 1.1.8.** *Let $f_1$ and $f_2$ be class functions on $G^{F^m} A$. Define class functions $g_1$ and $g_2$ on $G^F$ by: $g_i(\mathfrak{N}_{K/k}y)=f_i(y\sigma)$ for any $y\in G^{F^m}$. Then*

$$|G^{F^m}|^{-1}\sum_{y\in G^{F^m}} f_1(y\sigma)\overline{f_2(y\sigma)} = |G^F|^{-1}\sum_{x\in G^F} g_1(x)\overline{g_2(x)}$$

**Lemma 1.1.9.** *Let $H$ be an $F$-stable closed subgroup. Let $f$ and $g$ be class functions on $H^{F^m} A$ and $H^F$ respectively. If $g(\mathfrak{N}_{K/k}y)=f(y\sigma)$ for any $y\in H^{F^m}$, then* $(\mathrm{Ind}\,^{G^F}_{H^F} g)\,(\mathfrak{N}_{K/k}y)=(\mathrm{Ind}\,^{G^{F^m}A}_{H^{F^m}A} f)\,(y\sigma)$ *for any $y\in G^{F^m}$.*

1.2. Henceforth $G$ is a connected reductive group defined over $k=\boldsymbol{F}_q$, $B$ is an $F$-stable Borel subgroup, $U$ is the unipotent radical of $B$, $T$ is an $F$-stable maximal torus of $B$ and $W=N_G(T)/T$.
Let $w\in W^{F^m}$ and $\dot{w}$ its representative in $N_G(T)^{F^m}$.
Let $X(w)$, $S_{\dot{w}}$, $T(w)^F$ and $R^1_{T_w}$ be as in [3]. They are as follows.

$$S_{\dot{w}} = \{g\in G; g^{-1F}g\in \dot{w}U\},\ T(w)^F = \{t\in T;\ \dot{w}^F t\dot{w}^{-1} = t\}\ ,$$

$X(w)=S_{\dot{w}}/T(w)^F U\cap \dot{w}U\dot{w}^{-1}$ and $R^1_{T_w}$ is the virtual character of $G^F$ such that $\mathrm{Tr}(x, R^1_{T_w})=\mathrm{Tr}(x^{*-1}, \sum_{i\geqslant 0}(-1)^i H^i_c(X(w))$.
Then we have

**Lemma 1.2.1** (cf. Remark 1.4.2). *Let $x\in G^F$. Take $a\in G$ such that $x={}^{F^m}a^{-1}a$. Let $y=a^Fa^{-1}\in G^F$* (cf. Lemma 1.1.1). *Then*

$$\mathrm{Tr}((x^{-1}F^m)^*, \sum_{i\geqslant 0}(-1)^i H^i_c(X(w)) = (|T^{F^m}|q^{md})^{-1}\#\{h\in G^{F^m};\ h^{-1}y^\sigma h\in \dot{w}B\}\ ,$$

*where $d=\dim\, U\cap \dot{w}U\dot{w}^{-1}$.*

1.3. Let $Z^K=\mathrm{Ind}^{G^{F^m}}_{B^{F^m}}1$($=$the representation of $G^{F^m}$ induced from the trivial representation of $B^{F^m}$). Then $Z^K=\sum_{g\in G^{F^m}/B^{F^m}}\overline{\boldsymbol{Q}}_l gv$ as vector spaces with $B^{F^m}$ acting trivially on $\overline{\boldsymbol{Q}}_l v$. As is known, $\mathrm{End}_{G^{F^m}}Z^K=\sum_{w\in W^{F^m}}\overline{\boldsymbol{Q}}_l a^K_w$, where $a^K_w$ is defined by: $a^K_w v=\sum_{u\in U_w^{-,F^m}} u\dot{w}^{-1}v$ with $U^-_w=U\cap \dot{w}U^-\dot{w}^{-1}$ ($U^-$ is the maximal unipotent subgroup opposite to $U$). Define the linear mapping $I_\sigma$ on $Z^K$ by:
$I_\sigma$: $\sum_{g\in G^{F^m}/B^{F^m}} c_g gv\mapsto \sum_{g\in B^{F^m}/B^{F^m}} c^\sigma_g gv$ $(c_g\in\overline{\boldsymbol{Q}}_l)$. Then for any $g\in G^{F^m}$ and $z\in Z$, $I_\sigma(gz)={}^\sigma g I_\sigma z$.
Then we have

**Lemma 1.3.1** (cf. Remark 1.4.2). *For $g\in G^{F^m}$ and $w\in W^{F^m}$,* $\mathrm{Tr}(ya^K_w I_\sigma, Z^K)$ $=(q^{md}|T^{F^m}|)^{-1}\#\{g\in G^{F^m};\ g^{-1}y^\sigma g\in \dot{w}B\}$, *where $d=\dim\, U\cap \dot{w}U\dot{w}^{-1}$.*

1.4. For any $x\in G^F$, write $x={}^{F^m}a^{-1}a$ with $a\in G$ and let $y=a^Fa^{-1}\in G^{F^m}$. By Lemma 1.2.1 and 1.3.1, $\mathrm{Tr}((x^{-1}F^m)^*, \sum_i(-1)^i H^i_c(X(w)))=\mathrm{Tr}(ya^K_w I_\sigma, Z^K)$.

Since $\mathrm{Tr}(ya_w^K I_\sigma, Z^K)$ does not depend on the $\sigma$-conjugacy class of $y$, we have

**Theorem 1.4.1.** *For any* $y \in G^{F^m}$, $\mathrm{Tr}((n_{K/k}(y)^{-1}F^m)^*, \sum_i (-1)^i H_c^i(X(w)))$ $= \mathrm{Tr}(ya_w^K I_\sigma, Z^K)$.

REMARK 1.4.2. (i) The above formula (and also Lemma 1.2.1, 1.3.1) were first appeared in [2]. This was informed to the author by Kawanaka.

(ii) It should be noted here that there are similar formulae to that of the theorem. If $F^m$ acts canonically on $R_T^\theta$ or $R_{L\subset P}(\pi)$, the analogy of the theorem is also true as is easily checked.

1.5. Let $\delta$ be the smallest integer $\geqslant 1$ such that $F^\delta$ acts trivially on $W$. Let $\rho \in \mathcal{E}(G^F, \{1\})$ (=the set of all (equivalence classes of) unipotent representations of $G^F$). By Lusztig [12], Coro. 3.9, if $\rho \in H_c^i(X(w))_\mu$ (=the generalized $\mu$-eigenspace of $F^{\delta *}$ on $H_c^i(X(w))$), then $\mu$ is uniquely determined (up to an integral power of $q^\delta$) by $\rho$ (not depending on $i$ or $w$).

DEFINITION 1.5.1. For $\rho \in \mathcal{E}(G^F, \{1\})$, let $\mu$ be as above. Define $\lambda_\rho$ to be the constant such that $\lambda_\rho = \mu q^{\delta r}$ for some $r \in \boldsymbol{Z}$ and $1 \leqslant |\lambda_\rho| < q^\delta$.

For $\rho \in \mathcal{E}(G^F, \{1\})$, let $H_c^i(X(w))_\rho$ be the largest subspace of $H_c^i(X(w))$ on which $G^F$ acts by a multiple of $\rho$. Then

**Lemma 1.5.2.** *For any* $\rho \in \mathcal{E}(G^F, \{1\})$ *and* $w \in W$, *there exists* $f_{\rho,w}(X) \in \boldsymbol{Z}[X, X^{-1}]$ *such that if* $\delta$ *divides* $m$,
$\mathrm{Tr}((x^{-1}F^m)^*, \sum_i (-1)^i H_c^i(X(w))_\rho) = f_{\rho,w}(q^m)\lambda_\rho^{m/\delta}\rho(x)$ *for any* $x \in G^F$ *and* $f_{\rho,w}(1) = \langle \rho, R_{T_w}^1 \rangle$.

## 2. Split case

2.1. In introducing the notation we only assume that $G$ splits over $K$. Let $\mathfrak{A}^K(W) = \mathrm{End}_{G^{F^m}} Z^K$ and $S$ the set of simple reflections of $W$ (corresponding to $B$). Let $\mathfrak{A}(W)$ be the generic algebra of $\mathfrak{A}^K(W)$ over the extension field of $\boldsymbol{Q}(X)$ ($X$: indeterminate) and $\{a_w; w \in W\}$ be its basis. ($\mathfrak{A}^K(W)$ is obtained from $\mathfrak{A}(W)$ by the specialization $X \mapsto q^m$ or more precisely by the homomorphism from the integral closure of $\boldsymbol{Q}[X]$ to $\boldsymbol{Q}$ which maps $X$ to $q^m$.) Let $\hat{W}$ be the set of equivalence classes of the irreducible representation of $W$. For any $\chi \in \hat{W}$, let $\nu_\chi$, $\nu_\chi^K$, $\rho_\chi^K$ be the corresponding irreducible representation (or its character) of $\mathfrak{A}(W)$, $\mathfrak{A}^K(W)$, $G^{F^m}$ respectively. Then $Z^K$ can be written in the form: $Z^K = \bigoplus_{\chi \in \hat{W}} \nu_\chi^K \otimes \rho_\chi^K$. For an $F$-stable subset $J \subseteq S$, let $W_J$ be the subgroup of $W$ generated by $J$, $P_J$ the corresponding standard parabolic subgroup of $G$, $L_J$ its standard Levi subgroup and $Z_J^K = \mathrm{Ind}_{B^{F^m}}^{P_J^{F^m}} 1 (= \mathrm{Ind}_{(B \cap L_J)}^{L_J^{F^m}} 1$ as $L_J^{F^m}$-modules). $Z_J^K$ is canonically regarded as a subspace of $Z^K$ and $\mathrm{End}_{P_J^{F^m}} Z_J^K = \sum_{w \in W_J} \overline{\boldsymbol{Q}}_l\, a_w|_{Z_J^K}$. The following are also defined: $\mathfrak{A}^K(W_J)$, $\mathfrak{A}(W_J)$, $\{\nu_\chi, \nu_\chi^K, \rho_\chi^K;\ \chi \in \hat{W}_J\}$. Since $W_J$ is a parabolic subgroup of $W$, $\mathfrak{A}(W_J)$ (resp. $\mathfrak{A}^K(W_J)$) is regarded as a subalgebra of $\mathfrak{A}(W)$ (resp. $\mathfrak{A}^K(W)$). For any $\chi' \in \hat{W}_J$ and $\chi \in \hat{W}$, define the non-negative integer $n_{\chi,\chi'}$ by: $\mathrm{Ind}_{W_J}^W \chi' = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \chi$. For $\chi' \in \hat{W}_J$, let $Z_{\chi'}^K$ (resp. $Z_{J,\chi'}^K$) be the largest subspace of $Z^K$ (resp. $Z_J^K$) on which $\mathfrak{A}^K(W_J)$ acts by a multiple of $\nu_{\chi'}^K$. For $\chi \in \hat{W}$, $Z_\chi^K$ is defined similarly. The following are checked easily: for $\chi' \in \hat{W}_J$, $\mathrm{Ind}_{P_J^{F^m}}^{G^{F^m}} Z_{J,\chi'}^K = Z_{\chi'}^K$, $Z_{J,\chi'}^K = \nu_{\chi'} \otimes \rho_{\chi'}^K$, $Z_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \nu_{\chi'}^K \otimes \rho_\chi^K$, and for $\chi' \in \hat{W}_J$ and $\chi \in \hat{W}$, $Z_\chi^K \cap Z_{\chi'}^K = n_{\chi,\chi'} \nu_{\chi'}^K \otimes \rho_\chi^K$.

2.2. Henceforth in this section we assume $G$ to be split over $k$. Then the mapping $I_\sigma$ commutes with any $a_w^K (w \in W)$, thus with $\mathfrak{A}^K(W)$. Therefore each $\rho_\chi^K$ is regarded as an irreducible $G^{F^m}$ $A$-modules which is denoted by $\tilde{\rho}_\chi^K$. By Theorem 1.4.1, we have

**Lemma 2.2.1.** *For any $y \in G^{F^m}$,*
$\mathrm{Tr}((n_{K k/}(y)^{-1} F^m)^*, \sum_i (-1)^i H_c^i(X(w))) = \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K) \tilde{\rho}_\chi^K(y\sigma)$.

Let $J \subset S$ be $F$-stable. $\tilde{\rho}_{\chi'}^k (\chi' \in \hat{W}_J)$ are similiarly defined as $\tilde{\rho}_\chi^K (\chi \in \hat{W})$. Now, for any $z \in Z_J^K$ and $g \in G^{F^m}$, $I_\sigma(gz) = {}^\sigma g I_\sigma(z)$. Thus for $\chi' \in \hat{W}_J$, $\mathrm{Ind}_{P_J^{F^m} A}^{G^{F^m} A} Z_{J,\chi'}^K = Z_{\chi'}^K$ as $G^{F^m}$ $A$-modules. Hence

**Lemma 2.2.2.** *Assume* $\mathrm{Ind}_{W_J}^W \chi' = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \chi\ (\chi' \in \hat{W}_J,\ n_{\chi,\chi'} \geqslant 0)$. *Then* $\mathrm{Ind}_{P_J^{F^m}}^{G^{F^m}} \rho_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \rho_\chi^K$ and $\mathrm{Ind}_{P_J^{F^m} A}^{G^{F^m} A} \tilde{\rho}_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \tilde{\rho}_\chi^K$.

**Lemma 2.2.3.** *Assume the Dynkin graph of $G$ does not have irreducible components of type $E_7$ or $E_8$. Assume that for any $J \subseteqq S$ and $\chi' \in \hat{W}_J$, there exists the lifting of $\rho_{\chi'}^k$ in $\widehat{L_J^{F^m}}$. Then for any $\chi \in \hat{W}$ and $y \in G^{F^m}$, $\rho_\chi^k(\mathfrak{N}_{K/k}(y)) = \tilde{\rho}_\chi^K(y\sigma)$.*

Proof. By Lemma 1.1.9, $(\mathrm{Ind}_{B^F}^{G^F} 1)\ (\mathfrak{N}_{K/k} y) = (\mathrm{Ind}_{B^{F^m} A}^{G^{F^m} A} 1)\ (y\sigma)$ for any $y \in G^{F^m}$. Thus

$$\text{(a)} \qquad \sum_{\chi \in \hat{W}} \dim \chi_{\rho_\chi^k}(\mathfrak{N}_{K/k} y) = \sum_{\chi \in \hat{W}} \dim \chi_{\tilde{\rho}_\chi^K}(y\sigma) \quad \text{for any } y \in G^{F^m}.$$

The existence of the lifting of each $\rho_\chi^k$ shows for each $\chi \in \hat{W}$ there exists $\chi' \in \hat{W}$ such that $\rho_\chi^k(\mathfrak{N}_{K/k} y) = c\,\tilde{\rho}_{\chi'}^K(y\sigma)$ for any $y \in G^{F^m}$ and $c=1$. (This is checked by taking the inner product with the relation (a). See Lemma 1.1.8.) If $\chi = 1$, the statement of the lemma is obvious. If $\chi = St_W$ (=the sign character of $W$), it is also obvious. This proves the case when the semisimple rank of $G$ is 1. Assume the semisimple rank of $G \geqslant 2$ and the statement holds for any $L_J$ with $J \subsetneqq S$.

Let $J \subsetneqq S$. Then for any $\chi' \in \hat{W}_J$ and $y \in G^{F^m}$, $\rho^k_{\chi'}(\mathfrak{N}_{K/k}y) = \tilde{\rho}^K_{\chi'}(y\sigma)$. Write $\mathrm{Ind}^W_{W_J}\chi' = \sum_{\chi \in \hat{W}} n_{\chi,\chi'}\chi$. Then by Lemma 2.2.2, $\sum_{\chi \in \hat{W}} n_{\chi,\chi'}\tilde{\rho}^K_\chi(\mathfrak{N}_{K/k}y) = \sum_{\chi \in \hat{W}} n_{\chi,\chi'}\tilde{\rho}^K_\chi(y\sigma)$ for any $y \in G^{F^m}$. Thus the lemma is an easy consequence of the following well known result (cf. Benson-Curtis [1]):

Let $(W, S)$ be the Weyl group which does not have the irreducible factors of type $G_2$, $E_7$ or $E_8$ and assume rank $(W, S) \geqslant 2$. For $\chi_1, \chi_2 \in \hat{W}$, if $\chi_1|_{W_J} = \chi_2|_{W_J}$ for any $J \subsetneqq S$, then $\chi_1 = \chi_2$.

By Lemma 2.2.1 and 2.2.3 we have

**Lemma 2.2.4.** *Assume the assumption of* Lemma 2.2.3. *Then*

$$\mathrm{Tr}((x^{-1}F^m)^*, \sum_i (-1)^i H^i_c(X(w)) = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w)\rho^k_\chi(n^{-1}_{k/k}x) \quad \textit{for any } x \in G^F .$$

2.3. If $G = GL_n$, we can easily check the following theorem, which is proved in [2] and also by Lusztig independently.

**Theorem 2.3.1.** *Assume* $G = GL_n$. *Then*

( i ) $\rho^k_\chi(n_{K/k}y) = \tilde{\rho}^K_\chi(y\sigma)$ *for any* $\chi \in \hat{W}$ *and* $y \in G^{F^m}$,

(ii) $f_{\rho_\chi, w}(X) = \nu_\chi(a_w)$ *for any* $\chi \in \hat{W}$ *and* $w \in W$,

(iii) $|X^{F^m}_w| = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w) \dim \rho^k_\chi$ .

2.4. In 2.4 we assume $G = Sp_{2n}$, $SO_{2n+1}$ or $SO^+_{2n}$.

**Lemma 2.4.1.** *If* $(m, 2p) = 1$, *then*

( i ) $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho \rho = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w)\rho^k_\chi \cdot n^{-1}_{k/k}$ ,

(ii) $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho \dim \rho = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w) \dim \rho^k_\chi$ ,

*where* $\rho$ *ranges over* $\mathcal{E}(G^F, \{1\})$.

Proof. By Lemma 1.5.2 and 2.2.1, $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho \rho(n_{K/k}y) = \sum_{\chi \in \hat{W}} \nu^K_\chi(a^K_w)\tilde{\rho}^K_\chi(y\sigma)$ for any $y \in G^{F^m}$. By Theorem 1.1.6 and Lemma 2.2.3, $\tilde{\rho}^K_\chi(y\sigma) = \rho^k_\chi(\mathfrak{N}_{K/k}y) = \rho^k_\chi(n^{-1}_{k/k}n_{K/k}y)$ for any $y \in G^{F^m}$. Thus we have (i). Since $n_{k/k}(\{1\}) = \{1\}$, we have (ii).

To proceed further we need some lemmas. The following one is obvious.

**Lemma 2.4.2.** *Let* $c_1, \cdots, c_r, x_1, \cdots, x_r \in \bar{\boldsymbol{Q}}^\times_l$. *Assume* $\sum_{1 \leqslant i \leqslant r} c_i x^t_i = 0$ *for* $t = 1, \cdots, r$. *Then there exist* $1 \leqslant i \neq j \leqslant r$ *such that* $x_i = x_j$.

**Lemma 2.4.3.** *Let* $f(X), g(X) \neq 0 \in \bar{\boldsymbol{Q}}_l[X]$, *t a positive integer (maybe* 1) *and* $\lambda \in \boldsymbol{Q}^\times_l$. *Assume* $f(q^m)\lambda^m = g(q^m)$ *for any positive integer m such that* $(m, t) = 1$. *Then* $\lambda = \zeta q^\alpha$ *with* $\zeta$ *a t-th root of unity and* $\alpha$ *an integer.*

Proof. Write $f(X)=\sum_{0\leqslant i\leqslant r} a_i X^i$, $g(X)=\sum_{0\leqslant i\leqslant s} b_i X^i (a_i,\ b_i\in \bar{\boldsymbol{Q}}_l)$. By the assumption, $f(q^{mt+1})\lambda^{mt+1}=g(q^{mt+1})$ for any $m\in \boldsymbol{N}$. Thus $\sum_{0\leqslant i\leqslant r} a_i q^i \lambda (q^{ti}\lambda^t)^m=\sum_{0\leqslant i\leqslant s} b_i q^i (q^{ti})^m$ for any $m\in \boldsymbol{N}$. If $i\neq j$, $q^{ti}\neq q^{tj}$ and $q^{ti}\lambda^t\neq q^{tj}\lambda^t$. Thus, by Lemma 2.4.2, $q^{ti}\lambda^t=q^{tj}$ for some $0\leqslant i\leqslant r$, $0\leqslant j\leqslant s$. Therefore $\lambda=\zeta q^{\alpha}$ with $\zeta$ a $t$-th root of unity and $\alpha$ a positive integer.

The following proposition is known when $q$ is larger than the Coxeter number of $G$ (cf. Lusztig [12], p. 25, (d)).

**Proposition 2.4.4.** For any $\rho\in \mathcal{E}(G^F, \{1\})$, $\lambda_\rho=1$ or $-1$.

Proof. If $\rho$ is not cuspidal, the computation of $\lambda_\rho$ is reduced to the groups of smaller ranks. Thus it remains to check for the cuspidal $\rho_0\in\mathcal{E}(G^F, \{1\})$. Take $w\in W$ such that $\langle \rho_0, R^1_{T_w}\rangle\neq 0$. Then $f_{\rho_0,w}(X)\neq 0$ (cf. 1.5). If $(m, 2p)=1$, $\sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m \dim\rho=\sum_{\chi\in\hat{W}} \nu_\chi^K(a_w^K)\dim\rho_\chi^k$ by Lemma 2.4.1, (ii). We may assume if $\rho\neq\rho_0$, $\lambda_\rho=1$ or $-1$. Thus, for any positive integer $m$ such that $(m, 2p)=1$, we have $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\dim\rho_0+\sum_{\rho\neq\rho_0} f_{\rho,w}(q^m)\lambda_\rho \dim\rho=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim\rho_\chi^k$. Applying Lemma 2.4.3 we have $\lambda_{\rho_0}^{2p}=1$ (since $0\leqslant|\lambda_{\rho_0}|<q$). Thus it suffices to prove $\lambda_{\rho_0}\in\boldsymbol{Q}$. But for any positive integer $m$, $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\dim\rho_0+\sum_{\rho\neq\rho_0} f_{\rho,w}(q^m)\lambda_\rho\dim\rho=\mathrm{Tr}(F^{m*}, \sum_i(-1)^i H^i_c(X(w)))=|X(w)^{F^m}|$. Thus $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\in\boldsymbol{Q}$ for any positive integer $m$. Since $f_{\rho_0,w}(X)\neq 0$, there exists an integer $m_0$ such that if $m\geqslant m_0$, $f_{\rho_0,w}(q^m)\neq 0$. Thus if $m\geqslant m_0$, $\lambda^m_{\sigma_0}\in\boldsymbol{Q}$. Therefore $\lambda_{\rho_0}=(\lambda_{\rho_0})^{m_0+1}\lambda_{\rho_0}^{-m_0}\in\boldsymbol{Q}$.

**Lemma 2.4.5.** $\sum_\rho f_{\rho,w}(X)\lambda_\rho\rho=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\rho^k_\chi\cdot n^{-1}_{k/k}$ *as* $\boldsymbol{Q}[X]$*-linear combinations of class functions of* $G^F$.

Proof. Fix $y\in G^F$. By Lemma 2.4.1 and Proposition 2.4.4, if $(m, 2p)=1$, then $\sum_\rho f_{\rho,w}(q^m)\lambda_\rho\rho(y)=\sum_{\chi\in\hat{W}}\nu^K_\chi(a^K_w)\rho^k_\chi(n^{-1}_{k/k}y)$. Since there exist infinitely many positive integers $m$ such that $(m, 2p)=1$, $\sum_\rho f_{\rho,w}(X)\lambda_\rho\rho(y)=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\rho^k_\chi(n^{-1}_{k/k}y)$ as polynomials in $X$ (with $y\in G^F$ being fixed). This proves the lemma.

For $\chi\in\hat{W}$, let $R_\chi=|W|^{-1}\sum_{w\in W}\chi(w)\, R^1_{T_w}$. Then

**Lemma 2.4.6.** $\rho^k_\chi\cdot n^{-1}_{k/k}=\sum_\rho\langle R_\chi, \rho\rangle\lambda_\rho\rho$.

Proof. By the specialization $X\mapsto 1$, the relation in Lemma 2.4.5 is specialized to: $\sum_\rho\langle R^1_{T_w}, \rho\rangle\lambda_\rho\rho=\sum_{\chi\in\hat{W}}\chi(w)\rho^k_\chi\cdot n^{-1}_{k/k}$. Hence

$$\rho^k_\chi\cdot n^{-1}_{k/k}\,(=|W|^{-1}\sum_{w\in W}\chi(w)\sum_{\chi_1\in\hat{W}}\chi_1(w)\rho^k_{\chi_1}\cdot n^{-1}_{k/k})$$
$$=|W|^{-1}\sum_{w\in W}\chi(w)\sum_\rho\langle R^1_{T_w}, \rho\rangle\lambda_\rho\rho=\sum_\rho\langle R_\chi, \rho\rangle\lambda_\rho\rho\,.$$

**Lemma 2.4.7.** (i) *For any* $w \in W$ *and* $\rho \in \mathcal{E}(G^F, \{1\})$, $f_{\rho,w}(X) = \sum_{\chi \in \hat{W}} \nu_\chi(a_w) \langle R_\chi, \rho \rangle$.

(ii) $\sum_\rho f_{\rho,w}(X)\rho = \sum_{\chi \in \hat{W}} \nu_\chi(a_w) R_\chi$.

Proof. (i) $\lambda_\rho f_{\rho,W}(X) = \langle \sum_{\rho_1} f_{\rho_1,w}(X)\lambda_{\rho_1}\rho_1, \rho \rangle = \sum_{\chi \in \hat{W}} \nu_\chi(a_w) \langle \rho_\chi^k \cdot n_{k/k}^{-1}, \rho \rangle$ (by Lemma 2.4.5) $= \sum_{\chi \in \hat{W}} \nu_\chi(a_w) \langle R_\chi, \rho \rangle \lambda_\rho$ (by Lemma 2.4.6). This proves (i). (ii) is an easy consequence of (i).

**Theorem 2.4.8.** *Let* $w \in W$.

(i) *If* $m$ *is odd*, $|X(w)^{F^m}| = \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K) \dim \rho_\chi^k$.

(ii) *If* $m$ *is even*, $|X(w)^{F^m}| = \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K) \dim R_\chi$.

Proof. $|X(w)^{F^m}| = \sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m \dim \rho$. Assume $m$ is odd. Then $|X(w)^{F^m}| = \sum_\rho f_{\rho,w}(q^m)\lambda_\rho \dim \rho$ (since $\lambda_\rho = 1$ or $-1$) $= \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K)\, \rho_k^\chi(n_{k/k}^{-1}\{1\})$ (by Lemma 2.4.5) $= \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K) \dim \rho_\chi^k$. Assume $m$ is even. Then $|X(w)^{F^m}| = \sum_\rho f_{\rho,w}(q^m) \dim \rho$ $= \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K) \dim R_\chi$ (by Lemma 2.4.7, (ii)).

The following lemma is well known (cf. [4]).

**Lemma 2.4.9.** *Let* $\mathfrak{A}$ *be a semisimple and symmetric algebra over the algebraic closed field of characteristic* 0. *Let* $\{e_1, \cdots, e_r\}$ *be a basis of* $\mathfrak{A}$ *and* $\{e_1^*, \cdots, e_r^*\}$ *be its dual basis. Let* $\chi_1, \chi_2$ *be the irreducible characters of* $\mathfrak{A}$. *Then* $\sum_i \chi_1(e_i) \chi_2(e_i^*) = 0$ *if and only if* $\chi_1 \neq \chi_2$.

**Theorem 2.4.10.** (i) *If* $m$ *is odd*, $\tilde{\rho}_\chi^K(y\sigma) = \rho_\chi^k(\mathfrak{A}_{K/k}y)$ *for any* $\chi \in \hat{W}$ *and* $y \in G^{F^m}$.

(ii) *If* $m$ *is even*, $\tilde{\rho}_\chi^K(y\sigma) = R_\chi(n_{K/k}y)$ *for any* $\chi \in \hat{W}$ *and* $y \in G^{F^m}$.

Proof. For any $y \in G^{F^m}$ and $w \in W$, $\sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma) = \mathrm{Tr}(((n_{K/k}y)^{-1}F^m)^*, \sum_i (-1)^i H_c^i(X(w)))$ (by Lemma 2.2.1) $= \sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m \rho(n_{K/k}y)$. Assume $m$ is odd. Then $\sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma) = \sum_\rho f_{\rho,w}(q^m)\lambda_\rho \rho(n_{k/k}\mathfrak{N}_{K/k}y) = \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K)\rho_\chi^k(\mathfrak{N}_{K/k}y)$ (by Lemma 2.4.5). Thus $\sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma) = \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K)\rho_\chi^k(\mathfrak{N}_{K/k}y)$. Hence we have (i) by the orthogonality relations in Lemma 2.4.9. (ii) is proved similarly.

REMARK 2.4.11. If char $\boldsymbol{F}_q \neq 2$, then $R_\chi = R_\chi \cdot n_{k/k}^{-1}$ by the following lemma. Therefore "$n_{k/k}$" in (ii) of Theorem 2.4.10 can be replaced by "$\mathfrak{N}_{K/k}$" if char $\boldsymbol{F}_q \neq 2$. This seems to be true even if char $\boldsymbol{F}_q = 2$.

**Lemma 2.4.12.** *Let $G$ be a connected reductive group over $\boldsymbol{F}_q$. (We do not assume the assumption imposed on $G$ in 2.4.) Let $x \in G^F$ and $x=su$ be the Jordan decomposition ($s$: a semisimple element, $u$: a unipotent element). Assume $u$ is contained in the identity component of the centralizer of $u$ in $Z_G(s)^0$. (Notice $u \in Z_G(s)^0$ by* [16], Corollary 4.4.) *Then for any $F$-stable maximal torus $T$ of $G$ and linear character $\theta$ of $T^F$, $R_T^\theta(n_{k/k}(x))=R_T^\theta(x)$.*

Proof. Let $H=Z_G(s)^0$. Let $T'$ be an $F$-stable maximal torus of $H$. Take $a \in T'$ such that $s=a^{-1F}a$. Take $b \in Z_H(u)^0$ such that $u=b^{-1F}b$. Then $x=su=sb^{-1F}b=b^{-1}s^Fb=b^{-1}a^{-1F}a^Fb=(ab)^{-1F}(ab)$. Thus $n_{k/k}(x)={}^F(ab)\,(ab)^{-1}={}^Fa^Fbb^{-1}a^{-1}$ ${}^Fabub^{-1}a^{-1}={}^Faua^{-1}$ ($b$ commutes with $u$)$={}^Faa^{-1}aua^{-1}=saua^{-1}$. Therefore $n_{k/k}(x)=saua^{-1}$. Since $s$ commutes with $aua^{-1}$ and $aua^{-1}$ is a unipotent element, $n_{k/k}(x)=s(aua^{-1})$ is the Jordan decomposition of $n_{k/k}(x)$. Let $\{g_1,\cdots,g_r\}$ be the representatives of $H^F\backslash\{g \in G^F; g^{-1}sg \in T\}$. Then by [2], Theorem 4.2, we have $R_T^\theta(x)=\sum_{1\leqslant i\leqslant r} Q_{g_iTg_i^{-1},H}(u)\theta(g_i^{-1}sg_i)$. Similarly, $R_T^\theta(n_{k/k}(x))=\sum_{1\leqslant i\leqslant r} Q_{g_iTg_i^{-1},H}(aua^{-1})$ $\theta(g_i^{-1}sg_i)$. Let $H_{ad}$ be the adjoint group of $H$ and $\pi$: $H \to H_{ad}$ be the canonical mapping. Since $a^{-1F}a=s \in Z(H)$, $\pi(a) \in H_{ad}^F$. Thus $\pi(u)$ and $\pi(aua^{-1})$ are conjugate in $H_{ad}^F$. Therefore $Q_{g_iTg_i^{-1},H}(u)=Q_{\pi(g_iTg_i^{-1}),H_{ad}}(\pi(u))=Q_{g_iTg_i^{-1},H}(aua^{-1})$. Hence $R_T^\theta(x)=R_T^\theta(n_{k/k}(x))$.

2.5. In 2.5, we wish to describe some conjectural statements flourishing from Lemma 2.4.6, if we assume Conjecture 4.3 of Lusztig [12]. To do this we need to recall some results of [11], [12]. For $\Lambda \in \Phi_n$ (resp. $\Phi_n^+$), let $\rho_\Lambda$ be the corresponding unipotent representations of $Sp_{2n}^F$ or $SO_{2n+1}^F$ (resp. $SO_{2n}^{+,F}$). For $\chi \in \hat{W}_n$ (resp. $\hat{W}_n$), let $\Lambda$ be the corresponding symbol class in $\Phi_n$ (resp. $\Phi_n^+$) and we put $R_\Lambda=R_\chi$. For $\Lambda \in \Phi_n$ (resp. $\Phi_n^+$), write $\Lambda=(X \cup (Y-I), X \cup I)$, where $X$, $Y$ are finite subsets of $\{0, 1, 2, \cdots\}$, $X \cap Y=\phi$, $I$ is a subset of $Y$ such that $2\,|I|+1 \equiv |Y| \bmod 4$ (resp. $2\,|I| \equiv |Y| \bmod 4$). Now, fix $X$ and $Y$. We put $|Y|=2s$ or $2s+1$, and assume $s>0$ if $|Y|=2s$. Let $Y=\{\lambda_0<\lambda_1<\lambda_2\cdots\}$, $Y^0=\{\lambda_0, \lambda_2, \lambda_4, \cdots\}$ and $Y^1=\{\lambda_1, \lambda_3, \lambda_5, \cdots\}$. Let $\mathcal{P}$ be the set of all subsets of $Y$ and $\mathcal{P}_s=\{I \in \mathcal{P}: |I| \equiv s \bmod 2\}$. Then $\mathcal{P}$ is regarded as a vector space over $\boldsymbol{F}_2$ by the addition: $I, J \in \mathcal{P} \mapsto IJ=I \cup J-I \cap J$ and $\mathcal{P}_0$ is regarded as a subspace. By the bijection $\mathcal{P}_s \to \mathcal{P}_0$ $(I \mapsto IY^1)$, we can regard $\mathcal{P}_s$ as a vector space over $\boldsymbol{F}_2$. Define $Q$: $\mathcal{P}_s \to \{\pm 1\}$ $(I \mapsto (-1)^{(|I|-s)/2})$. If we identify $\boldsymbol{F}_2$ canonically with $\{\pm 1\}$, the mapping $Q$ is regarded as a quadratic form on $\mathcal{P}_s$ whose associated bilinear form $B$ is: $I, J \in \mathcal{P}_s \mapsto B(I, J)=(-1)^{|I \cap X^0|+|J \cap Y^1|+|I \cup J|}$. Thus the Fourier transform of Lusztig [11], [12] takes the form:

$$\hat{\rho}_{(X \cup I', X \cup I)} = 2^{-s} \sum_{J \in \mathcal{P}_s} B(I, J) \rho_{(X \cup J', X \cup J)} \quad \text{for } I \in \mathcal{P}_s\,.$$

DEFINITION 2.5.1. (i) For a class function $f$ on $G^F$, let $f^\Delta=f \cdot n_{k/k}$. $\Delta^2=1$ and $\dim f^\Delta=\dim f$. (Notice that for any connected algebraic group $G$ over $\boldsymbol{F}_q$, if $x^r \in Z_G(x)^0$, then $n^r_{k/k}(\{x\})=\{x\}$.)

(ii) Let $\mathfrak{R}_{X,Y}=\sum_{J\in\mathcal{P}_s}\bar{Q}_I\rho_{(X\cup J',X\cup J)}$. Define the linear automorphims $\tilde{\Delta}$ of $\mathfrak{R}_{X,Y}$ by the condition $\hat{\rho}^{\tilde{\Delta}}_{(X\cup I',X\cup I)}=Q(I)\hat{\rho}_{(X\cap I',X\cup I)}$ for $I\in\mathcal{P}_s$. Since $\dim\hat{\rho}_{(X\cup I',X\cup I)}=0$ if $|I|\neq s$, we have $\dim f^{\tilde{\Delta}}=\dim f$ for any $f\in\mathfrak{R}_{X,Y}$.

It can easily be checked the following

**Lemma 2.5.2.** *For any* $J\in\mathcal{P}_s$,

$$Q(I)\rho^{\tilde{\Delta}}_{(X\cup I',X\cup I)}=2^{-s}\sum_{J\in\mathcal{P}_s}B(I,J)Q(J)\rho_{(X\cup J',X\cup J)}\,.$$

**Theorem 2.5.3.** *Assume Conjecture* 4.3 *in* [12] *is true. Then for* $I\in\mathcal{P}_s$, $\lambda_{\rho(X\cup I',X\cup I)}=Q(I)\ (=(-1)^{(|I|-s)/2})$.

Proof. By the induction of the semisimple rank of $G$, it is needed to check only for the cuspidal $\rho_{(X\cup I_0',X\cup I_0)}(I_0$ or $I_0'=Y)$. Thus we may assume the statement is true if $I\neq I_0, I_0'$. Since we have assumed Conjecture 4.3 in [12], $R_{(X\cup I',X\cup I)}=\hat{\rho}_{(X\cup I',X\cup I)}$ if $|I|=s$. Thus by Lemma 2.4.6, $R_{(X\cup I',X\cup I)}=2^{-r}\sum_{J\in\mathcal{P}_s}B(I,J)Q(J)\,\rho_{(X\cup J',X\cup J)}$. But $\dim R^{\Delta}_{(X\cup I',X\cup I)}=\dim R_{(X\cup I',X\cup I)}=\dim R^{\tilde{\Delta}}_{(X\cup I',X\cup I)}$. Thus $2^{-s}\sum_{J\in\mathcal{P}_s}B(I,J)Q(J)\dim\rho_{(X\cup J',X\cup J)}=2^{-s}\sum_{J\in\mathcal{P}_s}B(I,J)\lambda_{\rho(X\cup J',X\cup J)}\dim\rho_{(X\cup J',X\cup J)}$ by Lemma 2.5.2. This relation shows our statement.

Remark 2.5.4. (i) The statement of Theorem 2.5.3 is a counterpart of the statements for some families of the unipotent representations of the exceptional groups given in Lusztig [12], p. 45 and [13], p. 335.

(ii) Assume char $\boldsymbol{F}_q\neq 2$. Lemma 2.4.12 and the proof of Theorem 2.5.3 show that $\Delta$ and $\tilde{\Delta}$ coincide on the subspace $\mathfrak{S}$ of $\mathfrak{R}_{X,Y}$ which is spanned by $\{\rho_{(X\cup I',X\cup I)},\rho^{\Delta}_{(X\cup I',X\cup I)};I\in\mathcal{P}_s,|I|=s\}$ and $\{R_{(X\cup I',X\cup I)};I\in\mathcal{P}_s,|I|=s\}$ under the assumption of Theorem 2.5.3. If $|Y|=1$, 3 or 4, $\mathfrak{S}=\mathfrak{R}_{X,Y}$. If $|Y|=5$ or 6, $\dim\mathfrak{S}=\dim\mathfrak{R}_{X,Y}-1$. We may ask if the following is true (cf. Remark 2.4.11).

Conjecture 2.5.5. $\tilde{\Delta}=\Delta$.

## 3. Unitary case

The method which we applied in the case of split classical groups is also effective for the unitary groups. Let $G$ be the unitary group $U_n$ over $\boldsymbol{F}_q$ and we assume $m$ is an even integer. The Weyl group $W$ is canonically identified with the symmetric group $S_n$ and we assume the generic algebra $\mathfrak{A}(W)$ (cf. 2.1) is over the extension field of $\mathfrak{A}(X)$ which contains $X^{1/2}$ ($X^{1/2}$ being fixed). Let $\mathcal{P}(n)$ be the set of all partitions of $n$. For $\alpha\in\mathcal{P}(n)$, let $\chi_\alpha$ (resp. $\nu_{\chi_\alpha}$) be the corresponding irreducible representation (or its character) of $W$ (resp. $\mathfrak{A}(W)$). The following lemma is easily checked by the induction on $n$.

**Lemma 3.1.** *Let $w_0$ be the longest element of $W$ and $\alpha=(\alpha_1\geqslant\cdots\geqslant\alpha_s>0)$ $\in\mathcal{P}(n)$. Define $C_\alpha=\binom{n}{2}+\sum_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant\alpha_i}}(j-i)$. Then $a_{w_0}^2$ acts as a scalar $X^{C_\alpha}$ on the representation $v_{\chi_\alpha}$ of $\mathfrak{A}(W)$.*

Let the notation be as in 2.1. We write $\rho_\alpha^K$ instead $\rho_{\chi_\alpha}^K$ of for $\alpha\in\mathcal{P}(n)$ to simplify the notation. Since $a_{w_0}^K I_\sigma$ commutes with $\mathfrak{A}^K(W)$, each irreducible component $\rho_\alpha^K$ of $Z^K$ is regarded as a $G^{F^m}$ $A$-module $\rho_\alpha^K$ by the mapping $\sigma\mapsto(q^m)^{-C_\alpha/m}$ $a_{w_0}^K I_\rho$.

For $\alpha\in\mathcal{P}(n)$, let $\rho_\alpha^k=|W|^{-1}\sum_{w\in W}\chi_\alpha(ww_0)R_{T_w}^1$. If we put $\eta_\alpha=$the signature of dim $\rho_\alpha^k$, then by [14], $\eta_\alpha\rho_\alpha^k$ is the irreducible representation of $G^F$ and all the unipotent representations of $G^F$ are of this form. For the simplification of the notation we let $f_{\alpha,w}(X)=\eta_\alpha f_{\eta_\alpha\rho_\alpha^k,w}(X)$ $(\alpha\in\mathcal{P}(n),\ w\in W)$, $\lambda_\alpha=\lambda_{\eta_\alpha\rho_\alpha^k}$. Then

**Theorem 3.2.** *Assume* char $\boldsymbol{F}_q\neq 2$. *Let $\alpha\in\mathcal{P}(n)$ and $w\in W$.* Then

(i) $\rho_\alpha^k(n_{K/k}y)=(-1)^{mC_\alpha/2}\tilde{\rho}_\alpha^K(y\sigma)$ *for any even integer $m$ and $y\in G^{F^m}$,*

(ii) $f_{\alpha,w}(q^m)\lambda_\alpha^{m/2}=v_{\chi_\alpha}^K(a_w^K a_{w_0}^K)(-q)$ *for any even integer $m$,*

(iii) $|X(w)^{F^m}|=\sum_{\alpha\in\mathcal{P}(n)}v_{\chi_\alpha}^K(a_w^K a_{w_0}^K)(-q)^{-mC_\alpha/2}$ dim $\rho_\alpha^k$ *for any even integer $m$.*

Our proof is based on Kawanaka [7] as is stated in the introduction. In this respect, (i) of the theorem for cuspidal or subcuspidal $\rho_\alpha^k$'s is essential. The detailed arguments, which is slightly tedious, are omitted.

---

## References


[1] C.T. Benson and C.W. Curtis: *On the degrees and rationality of certain characters of finite Chevalley groups,* Trans. Amer. Math. Soc. **165** (1972), 251–273 and **202** (1975), 405–406.

[2] F. Digne et J. Michel: *Nombre de points rationnels des varietes de Deligne-Lusztig et charactères de l'algebre de Hecke,* C. R. Acad. Sci. Paris **287**, Sér. A (1978), 811–814.

[3] P. Deligne and G. Lusztig: *Representations of reductive groups over finite fields,* Ann. of Math. (2) **103** (1976), 103–161.

[4] T.V. Fossum: *Characters and centers of symmetric algebras,* J. Algebra **16** (1970), 4–13.

[5] A. Gyoja: *Liftings of irreducible characters of finite reductive groups,* Osaka J. Math. **16** (1979), 1–30.

[6] P.N. Hoeffsmit: *Representations of Hecke algebras of finite groups with BN-pairs of classical type,* Ph. D. Dissertation, Univ. British Columbia, Vancouver, 1974.

[7] N. Kawanaka: *On the irreducible characters of finite unitary groups,* J. Math. Soc. Japan **29** (1977), 425–450.

[8] N. Kawanaka: *Lifting of irreducible characters of finite classical groups,* J. Fac. Sci. Univ. Tokyo Sect. IA Math.

[9] G. Lusztig: *On the Green polynomials of classical groups,* Proc. London Math. Soc. (3) **33** (1976), 443–475.

[10] G. Lusztig: *Coxeter orbits and eigenspaces of Frobenius,* Invent. Math. **38** (1976), 101–159.

[11] G. Lusztig: *Irreducible representations of finite classical groups,* Invent. Math. **43** (1977), 125–175.

[12] G. Lusztig: Representations of finite Chevalley groups, C.B.M.S. Regional conference series in math. 39, AMS, 1978.

[13] G. Lusztig: *Unipotent representations of a finite Chevalley group of type $E_8$,* Quart. J. Math. Oxford Ser. (2) **30** (1979), 315–338.

[14] G. Lusztig and B. Srinivasan: *The characters of the finite unitary groups,* J. Algebra **49** (1977), 167–171.

[15] T. Shintani: *Two remarks on irreducible characters of finite general linear groups,* J. Math. Soc. Japan **28** (1976), 396–414.

[16] T.A. Springer and R. Steinberg: *Conjugacy classes,* in Lecture Notes in Math. 131, Springer-Verlag, 1970.

Department of Mathematics
Nara University of Educatinn
Nara 630, Japan