Asai, T.
Osaka J. Math.
20 (1983), 21–32

ON THE ZETA FUNCTIONS OF THE VARIETIES $X(w)$ OF THE SPLIT CLASSICAL GROUPS AND THE UNITARY GROUPS

TERUAKI ASAI

(Received April 30, 1981)

0. Introduction

Let G be one of the split classical groups SO_{2n}^{+}, SO_{2n+1}, Sp_{2n} or a unitary group defined over the finite field $\boldsymbol{F}_q$ of q elements. Let F be the Frobenius mapping, G^F the subgroup of F-stable elements, W the Weyl group of G and let δ be the smallest positive integer such that F^δ acts trivially on W. For $w\in W$, Deligne-Lusztig [3] has defined the F^δ-stable variety $X(w)$ for any connected reductive group. If w is a Coxeter element of W, the zeta function of $X(w)$ was obtained by Lusztig [9] as a by-product when he determined the Green polynomial associated with w. In this paper we shall determine the zeta function of $X(w)$ for any $w\in W$.

To state our result more explicitly, let B be a fixed F-stable Borel subgroup of G, $\mathfrak{A}^K(W)$ the Hecke algebra of the representation of G^{F^m} induced from the trivial representation of B^{F^m} and let $\{a_w^K;\ w\in W\}$ be the natural basis of $\mathfrak{A}^K(W)$. When δ divides m the number of F^m-stable points of $X(w)$ is expressed in terms of the dimensions of the unipotent representations of G^F and the trace of a_w^K on each irreducible representation of $\mathfrak{A}^K(W)$.

The crucial point of our arguments depends on the lifting theory due to Shintani-Kawanaka ([15], [7], [8]) and a result of Lusztig ([12], Corollary 3.9), which says that for any unipotent representation ρ of G^F, the eigenvalues of F^δ on the ρ-isotypic component of $H_c^i(X(w))$ are independent of i and w up to a multiple factor of the form $q^{i\delta}$, $i\in\boldsymbol{Z}$.

Finally the author expresses his heartfelt gratitude to Professor N. Kawanaka for his valuable suggestions and kind encouragement during the preparation of this paper.

1. General results

1.1. First we summarize the known results (Shintani [14], Kawanaka [7], [8]) to apply for our use.
Let m be a positive integer (maybe 1), $k=\boldsymbol{F}_q$, $K=\boldsymbol{F}_{q^m}$, G a connected algebraic

group defined over k, F the Frobenius over k, $\sigma=F|_{G^{F^m}}$ and A the cyclic group (of order m) generated by σ. Let $x_1, x_2\in G^{F^m}$. $x_1\sigma$ and $x_2\sigma$ are conjugate in $G^{F^m}A$ (semi-direct) if and only if there exists $h\in G^{F^m}$ such that $x_1=h^{-1}x_2^{\sigma}h$. If this is the case, we say x_1 and x_2 are σ-conjugate and we write $x_1\underset{\sigma}{\sim}x_2$. If $m=1$, we simply write $x_1\sim x_2$ instead of $x_1\underset{\sigma}{\sim}x_2$. The following lemma is proved in [7].

Lemma 1.1.1. *For $x\in G^{F^m}$, take $a\in G$ such that $x=a^{-1F}a$. Let $y={}^{F^m}a\,a^{-1}$. Then $y\in G^F$, and the conjugacy class of y in G^F is uniquely determined by the σ-conjugacy class of x in G^{F^m}. And the mapping $x\mapsto y$ defines a bijection: $G^{F^m}/\underset{\sigma}{\sim}\to G^F/\sim$.*

Definition 1.1.2. We denote the bijection $G^{F^m}/\underset{\sigma}{\sim}\to G^F/\sim$ in the above lemma by $n_{K/k}$. (Notice $n_{K/k}$ is defined even if $m=1$.) Define $\mathfrak{N}_{K/k}=n_{k/k}^{-1}\,n_{K/k}$. This also is a bijection from $G^{F^m}/\underset{\sigma}{\sim}$ onto $G^F/\sim$.

Remark 1.1.3. The reader should refer Kawanaka [8] for the relation between the norm mapping in [loc. cit.] and our norm mapping $\mathfrak{N}_{K/k}$.

The following lemma features some property of the mapping $\mathfrak{N}_{K/k}$, which is not used in this paper. The proof is omitted.

Lemma 1.1.4. *Let G be a connected reductive group and $Z(G)$ the center of G. Let $s\in Z(G)^F$ and $u\in G^F$. Let r be the order of s. Assume $m\equiv 1 \bmod r$. Then $\mathfrak{N}_{K/k}^{-1}(su)=s\mathfrak{N}_{K/k}^{-1}(u)$.*

For $\chi_K\in \widehat{G^{F^m}}^{\sigma}$ (=the set of σ-invariant irreducible characters of G^{F^m}), there exists $\tilde{\chi}_K\in \widehat{G^{F^m}A}$ such that $\tilde{\chi}_K|_{G^{F^m}}=\chi_K$. Let $\chi_k\in\widehat{G^F}$.

Definition 1.1.5. Let $m>1$. We say χ_K is the lifting of χ_k in $\widehat{G^{F^m}}$ if there exists a constant c such that $\tilde{\chi}_K(y\sigma)=c\chi_k(\mathfrak{N}_{K/k}\,y)$ for any $y\in G^{F^m}$. (The lifting of χ_k is uniquely determined by χ_k if it exists. See [7].)

Theorem 1.1.6 ([7], [8], [15]).
Let $m>1$. Assume one of the following.

(1) $G=GL_n$.

(2) $G=U_n$, $(m,p)=1$.

(3) $G=SO_{2n+1}$, Sp_{2n} *or* $SO_{2n}^{\pm}$, $(m,2p)=1$.

Then any $\chi_k\in\widehat{G^F}$ has the lifting $\chi_K\in\widehat{G^{F^m}}$. And the mapping $\chi_k\mapsto\chi_K$ defines a bijection between $\widehat{G^F}$ and $\widehat{G^{F^m}}^{\sigma}$.

Remark 1.1.7. The theorem is proved by Shintani [15] in case (1), by Kawanaka [7] in case (2) and by Kawanaka [8] in case (3).

The following lemmas can be extracted from [7].

Lemma 1.1.8. *Let f_1 and f_2 be class functions on $G^{F^m}A$. Define class functions g_1 and g_2 on G^F by: $g_i(\mathfrak{N}_{K/k}y)=f_i(y\sigma)$ for any $y\in G^{F^m}$. Then*

$$|G^{F^m}|^{-1}\sum_{y\in G^{F^m}}f_1(y\sigma)\overline{f_2(y\sigma)} = |G^F|^{-1}\sum_{x\in G^F}g_1(x)\overline{g_2(x)}$$

Lemma 1.1.9. *Let H be an F-stable closed subgroup. Let f and g be class functions on $H^{F^m}A$ and H^F respectively. If $g(\mathfrak{N}_{K/k}y)=f(y\sigma)$ for any $y\in H^{F^m}$, then* $(\mathrm{Ind}\,{}^{G^F}_{H^F}g)(\mathfrak{N}_{K/k}y)=(\mathrm{Ind}\,{}^{G^{F^m}A}_{H^{F^m}A}f)(y\sigma)$ *for any $y\in G^{F^m}$.*

1.2. Henceforth G is a connected reductive group defined over $k=\boldsymbol{F}_q$, B is an F-stable Borel subgroup, U is the unipotent radical of B, T is an F-stable maximal torus of B and $W=N_G(T)/T$.
Let $w\in W^{F^m}$ and $\dot{w}$ its representative in $N_G(T)^{F^m}$.
Let $X(w)$, $S_{\dot{w}}$, $T(w)^F$ and $R^1_{T_w}$ be as in [3]. They are as follows.

$$S_{\dot{w}} = \{g\in G; g^{-1F}g\in\dot{w}U\},\ T(w)^F = \{t\in T;\ \dot{w}^Ft\dot{w}^{-1}=t\}\ ,$$

$X(w)=S_{\dot{w}}/T(w)^FU\cap\dot{w}U\dot{w}^{-1}$ and $R^1_{T_w}$ is the virtual character of G^F such that $\mathrm{Tr}(x, R^1_{T_w})=\mathrm{Tr}(x^{*-1}, \sum_{i\geqslant 0}(-1)^iH^i_c(X(w))$.
Then we have

Lemma 1.2.1 (cf. Remark 1.4.2). *Let $x\in G^F$. Take $a\in G$ such that $x={}^{F^m}a^{-1}a$. Let $y=a^Fa^{-1}\in G^F$* (cf. Lemma 1.1.1). *Then*

$$\mathrm{Tr}((x^{-1}F^m)^*, \sum_{i\geqslant 0}(-1)^iH^i_c(X(w)) = (|T^{F^m}|q^{md})^{-1}\#\{h\in G^{F^m};\ h^{-1}y^\sigma h\in\dot{w}B\}\ ,$$

where $d=\dim\ U\cap\dot{w}U\dot{w}^{-1}$.

1.3. Let $Z^K=\mathrm{Ind}^{G^{F^m}}_{B^{F^m}}1$(=the representation of G^{F^m} induced from the trivial representation of B^{F^m}). Then $Z^K=\sum_{g\in G^{F^m}/B^{F^m}}\overline{\boldsymbol{Q}}_lgv$ as vector spaces with B^{F^m} acting trivially on $\overline{\boldsymbol{Q}}_l v$. As is known, $\mathrm{End}_{G^{F^m}}Z^K=\sum_{w\in W^{F^m}}\overline{\boldsymbol{Q}}_la^K_w$, where a^K_w is defined by: $a^K_w v=\sum_{u\in U_w^{-,F^m}}u\dot{w}^{-1}v$ with $U^-_w=U\cap\dot{w}U^-\dot{w}^{-1}$ (U^- is the maximal unipotent subgroup opposite to U). Define the linear mapping I_σ on Z^K by:
I_σ: $\sum_{g\in G^{F^m}/B^{F^m}}c_ggv\mapsto\sum_{g\in B^{F^m}/B^{F^m}}c^\sigma_ggv$ $(c_g\in\overline{\boldsymbol{Q}}_l)$. Then for any $g\in G^{F^m}$ and $z\in Z$, $I_\sigma(gz)={}^\sigma gI_\sigma z$.
Then we have

Lemma 1.3.1 (cf. Remark 1.4.2). *For $g\in G^{F^m}$ and $w\in W^{F^m}$,* $\mathrm{Tr}(ya^K_wI_\sigma, Z^K)$ $=(q^{md}|T^{F^m}|)^{-1}\#\{g\in G^{F^m};\ g^{-1}y^\sigma g\in\dot{w}B\}$, *where $d=\dim\ U\cap\dot{w}U\dot{w}^{-1}$.*

1.4. For any $x\in G^F$, write $x={}^{F^m}a^{-1}a$ with $a\in G$ and let $y=a^Fa^{-1}\in G^{F^m}$. By Lemma 1.2.1 and 1.3.1, $\mathrm{Tr}((x^{-1}F^m)^*, \sum_i(-1)^iH^i_c(X(w)))=\mathrm{Tr}(ya^K_wI_\sigma, Z^K)$.

Since $\mathrm{Tr}(ya_w^K I_\sigma, Z^K)$ does not depend on the σ-conjugacy class of y, we have

Theorem 1.4.1. *For any* $y \in G^{F^m}$, $\mathrm{Tr}((n_{K/k}(y)^{-1}F^m)^*, \sum_i (-1)^i H_c^i(X(w)))$ $=\mathrm{Tr}(ya_w^K I_\sigma, Z^K)$.

REMARK 1.4.2. (i) The above formula (and also Lemma 1.2.1, 1.3.1) were first appeared in [2]. This was informed to the author by Kawanaka.

(ii) It should be noted here that there are similar formulae to that of the theorem. If F^m acts canonically on R_T^θ or $R_{L\subset P}(\pi)$, the analogy of the theorem is also true as is easily checked.

1.5. Let δ be the smallest integer $\geqslant 1$ such that F^δ acts trivially on W. Let $\rho \in \mathcal{E}(G^F, \{1\})$ (=the set of all (equivalence classes of) unipotent representations of G^F). By Lusztig [12], Coro. 3.9, if $\rho \in H_c^i(X(w))_\mu$ (=the generalized μ-eigenspace of $F^{\delta *}$ on $H_c^i(X(w))$), then μ is uniquely determined (up to an integral power of q^δ) by ρ (not depending on i or w).

DEFINITION 1.5.1. For $\rho \in \mathcal{E}(G^F, \{1\})$, let μ be as above. Define λ_ρ to be the constant such that $\lambda_\rho = \mu q^{\delta r}$ for some $r \in \boldsymbol{Z}$ and $1 \leqslant |\lambda_\rho| < q^\delta$.

For $\rho \in \mathcal{E}(G^F, \{1\})$, let $H_c^i(X(w))_\rho$ be the largest subspace of $H_c^i(X(w))$ on which G^F acts by a multiple of ρ. Then

Lemma 1.5.2. *For any* $\rho \in \mathcal{E}(G^F, \{1\})$ *and* $w \in W$, *there exists* $f_{\rho,w}(X) \in \boldsymbol{Z}[X, X^{-1}]$ *such that if* δ *divides* m,
$\mathrm{Tr}((x^{-1}F^m)^*, \sum_i (-1)^i H_c^i(X(w))_\rho) = f_{\rho,w}(q^m)\lambda_\rho^{m/\delta}\rho(x)$ *for any* $x \in G^F$ *and* $f_{\rho,w}(1) = \langle \rho, R_{T_w}^1 \rangle$.

2. Split case

2.1. In introducing the notation we only assume that G splits over K. Let $\mathfrak{A}^K(W) = \mathrm{End}_{G^{F^m}} Z^K$ and S the set of simple reflections of W (corresponding to B). Let $\mathfrak{A}(W)$ be the generic algebra of $\mathfrak{A}^K(W)$ over the extension field of $\boldsymbol{Q}(X)$ (X: indeterminate) and $\{a_w; w \in W\}$ be its basis. ($\mathfrak{A}^K(W)$ is obtained from $\mathfrak{A}(W)$ by the specialization $X \mapsto q^m$ or more precisely by the homomorphism from the integral closure of $\boldsymbol{Q}[X]$ to $\boldsymbol{Q}$ which maps X to q^m.) Let $\hat{W}$ be the set of equivalence classes of the irreducible representation of W. For any $\chi \in \hat{W}$, let ν_χ, ν_χ^K, ρ_χ^K be the corresponding irreducible representation (or its character) of $\mathfrak{A}(W)$, $\mathfrak{A}^K(W)$, G^{F^m} respectively. Then Z^K can be written in the form: $Z^K = \bigoplus_{\chi \in \hat{W}} \nu_\chi^K \otimes \rho_\chi^K$. For an F-stable subset $J \subseteq S$, let W_J be the subgroup of W generated by J, P_J the corresponding standard parabolic subgroup of G, L_J its standard Levi subgroup and $Z_J^K = \mathrm{Ind}_{B^{F^m}}^{P_J^{F^m}} 1 (= \mathrm{Ind}_{(B \cap L_J)}^{L_J^{F^m}} 1$ as $L_J^{F^m}$-modules). Z_J^K is canonically regarded as a subspace of Z^K and $\mathrm{End}_{P_J^{F^m}} Z_J^K = \sum_{w \in W_J} \bar{\boldsymbol{Q}}_l\, a_w|_{Z_J^K}$. The following are also defined: $\mathfrak{A}^K(W_J)$, $\mathfrak{A}(W_J)$, $\{\nu_\chi, \nu_\chi^K, \rho_\chi^K;\ \chi \in \hat{W}_J\}$. Since W_J is a parabolic subgroup of W, $\mathfrak{A}(W_J)$ (resp. $\mathfrak{A}^K(W_J)$) is regarded as a subalgebra of $\mathfrak{A}(W)$ (resp. $\mathfrak{A}^K(W)$). For any $\chi' \in \hat{W}_J$ and $\chi \in \hat{W}$, define the non-negative integer $n_{\chi,\chi'}$ by: $\mathrm{Ind}_{W_J}^W \chi' = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \chi$. For $\chi' \in \hat{W}_J$, let $Z_{\chi'}^K$ (resp. $Z_{J,\chi'}^K$) be the largest subspace of Z^K (resp. Z_J^K) on which $\mathfrak{A}^K(W_J)$ acts by a multiple of $\nu_{\chi'}^K$. For $\chi \in \hat{W}$, Z_χ^K is defined similarly. The following are checked easily: for $\chi' \in \hat{W}_J$, $\mathrm{Ind}_{P_J^{F^m}}^{G^{F^m}} Z_{J,\chi'}^K = Z_{\chi'}^K$, $Z_{J,\chi'}^K = \nu_{\chi'} \otimes \rho_{\chi'}^K$, $Z_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \nu_{\chi'}^K \otimes \rho_\chi^K$, and for $\chi' \in \hat{W}_J$ and $\chi \in \hat{W}$, $Z_\chi^K \cap Z_{\chi'}^K = n_{\chi,\chi'} \nu_{\chi'}^K \otimes \rho_\chi^K$.

2.2. Henceforth in this section we assume G to be split over k. Then the mapping I_σ commutes with any $a_w^K (w \in W)$, thus with $\mathfrak{A}^K(W)$. Therefore each ρ_χ^K is regarded as an irreducible G^{F^m} A-modules which is denoted by $\tilde{\rho}_\chi^K$. By Theorem 1.4.1, we have

Lemma 2.2.1. *For any* $y \in G^{F^m}$,

$$\mathrm{Tr}((n_{Kk/}(y)^{-1}F^m)^*, \sum_i (-1)^i H_c^i(X(w))) = \sum_{\chi \in \hat{W}} \nu_\chi^K(a_w^K) \tilde{\rho}_\chi^K(y\sigma).$$

Let $J \subset S$ be F-stable. $\tilde{\rho}_{\chi'}^k (\chi' \in \hat{W}_J)$ are similiarly defined as $\tilde{\rho}_\chi^K (\chi \in \hat{W})$. Now, for any $z \in Z_J^K$ and $g \in G^{F^m}$, $I_\sigma(gz) = {}^\sigma g I_\sigma(z)$. Thus for $\chi' \in \hat{W}_J$, $\mathrm{Ind}_{P_J^{F^m} A}^{G^{F^m} A} Z_{J,\chi'}^K = Z_{\chi'}^K$ as G^{F^m} A-modules. Hence

Lemma 2.2.2. *Assume* $\mathrm{Ind}_{W_J}^W \chi' = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \chi (\chi' \in \hat{W}_J,\ n_{\chi,\chi'} \geqslant 0)$. *Then* $\mathrm{Ind}_{P_J^{F^m}}^{G^{F^m}} \rho_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \rho_\chi^K$ *and* $\mathrm{Ind}_{P_J^{F^m} A}^{G^{F^m} A} \tilde{\rho}_{\chi'}^K = \sum_{\chi \in \hat{W}} n_{\chi,\chi'} \tilde{\rho}_\chi^K$.

Lemma 2.2.3. *Assume the Dynkin graph of* G *does not have irreducible components of type* E_7 *or* E_8. *Assume that for any* $J \subseteqq S$ *and* $\chi' \in \hat{W}_J$, *there exists the lifting of* $\rho_{\chi'}^k$ *in* $\widehat{L_J^{F^m}}$. *Then for any* $\chi \in \hat{W}$ *and* $y \in G^{F^m}$, $\rho_\chi^k(\mathfrak{N}_{K/k}(y)) = \tilde{\rho}_\chi^K(y\sigma)$.

Proof. By Lemma 1.1.9, $(\mathrm{Ind}_{B^F}^{G^F} 1)\ (\mathfrak{N}_{K/k} y) = (\mathrm{Ind}_{B^{F^m} A}^{G^{F^m} A} 1)\ (y\sigma)$ for any $y \in G^{F^m}$. Thus

(a) $$\sum_{\chi \in \hat{W}} \dim \chi_{\rho_\chi^k}(\mathfrak{N}_{K/k} y) = \sum_{\chi \in \hat{W}} \dim \chi_{\tilde{\rho}_\chi^K}(y\sigma) \quad \text{for any } y \in G^{F^m}.$$

The existence of the lifting of each ρ_χ^k shows for each $\chi \in \hat{W}$ there exists $\chi' \in \hat{W}$ such that $\rho_\chi^k(\mathfrak{N}_{K/k} y) = c\, \tilde{\rho}_{\chi'}^K(y\sigma)$ for any $y \in G^{F^m}$ and $c = 1$. (This is checked by taking the inner product with the relation (a). See Lemma 1.1.8.) If $\chi = 1$, the statement of the lemma is obvious. If $\chi = St_W$ (=the sign character of W), it is also obvious. This proves the case when the semisimple rank of G is 1. Assume the semisimple rank of $G \geqslant 2$ and the statement holds for any L_J with $J \subsetneqq S$.

Let $J\subsetneqq S$. Then for any $\chi'\in\hat{W}_J$ and $y\in G^{F^m}$, $\rho^k_{\chi'}(\mathfrak{N}_{K/k}y)=\tilde{\rho}^K_{\chi'}(y\sigma)$. Write $\mathrm{Ind}^W_{W_J}\chi'=\sum_{\chi\in\hat{W}} n_{\chi,\chi'}\chi$. Then by Lemma 2.2.2, $\sum_{\chi\in\hat{W}} n_{\chi,\chi'}\tilde{\rho}^K_\chi(\mathfrak{N}_{K/k}y)=\sum_{\chi\in\hat{W}} n_{\chi,\chi'}\tilde{\rho}^K_\chi(y\sigma)$ for any $y\in G^{F^m}$. Thus the lemma is an easy consequence of the following well known result (cf. Benson-Curtis [1]):

Let (W, S) be the Weyl group which does not have the irreducible factors of type G_2, E_7 or E_8 and assume rank $(W, S)\geqslant 2$. For $\chi_1, \chi_2\in\hat{W}$, if $\chi_1|_{W_J}=\chi_2|_{W_J}$ for any $J\subsetneqq S$, then $\chi_1=\chi_2$.

By Lemma 2.2.1 and 2.2.3 we have

Lemma 2.2.4. *Assume the assumption of* Lemma 2.2.3. *Then*

$$\mathrm{Tr}((x^{-1}F^m)^*, \sum_i(-1)^i H^i_c(X(w))=\sum_{\chi\in\hat{W}}\nu^K_\chi(a^K_w)\rho^k_\chi(n^{-1}_{k/k}x) \quad \textit{for any } x\in G^F .$$

2.3. If $G=GL_n$, we can easily check the following theorem, which is proved in [2] and also by Lusztig independently.

Theorem 2.3.1. *Assume $G=GL_n$. Then*

(i) $\rho^k_\chi(n_{K/k}y)=\tilde{\rho}^K_\chi(y\sigma)$ *for any $\chi\in\hat{W}$ and $y\in G^{F^m}$*,

(ii) $f_{\rho_\chi,w}(X)=\nu_\chi(a_w)$ *for any $\chi\in\hat{W}$ and $w\in W$*,

(iii) $|X^{F^m}_w|=\sum_{\chi\in\hat{W}}\nu^K_\chi(a^K_w)\dim\rho^k_\chi$.

2.4. In 2.4 we assume $G=Sp_{2n}$, SO_{2n+1} or SO^+_{2n}.

Lemma 2.4.1. *If $(m, 2p)=1$, then*

(i) $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho\rho=\sum_{\chi\in\hat{W}}\nu^K_\chi(a^K_w)\rho^k_\chi\cdot n^{-1}_{k/k}$,

(ii) $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho\dim\rho=\sum_{\chi\in\hat{W}}\nu^K_\chi(a^K_w)\dim\rho^k_\chi$,

where ρ ranges over $\mathcal{E}(G^F, \{1\})$.

Proof. By Lemma 1.5.2 and 2.2.1, $\sum_\rho f_{\rho,w}(q^m)\lambda^m_\rho\rho(n_{K/k}y)=\sum_{\chi\in\hat{W}}\nu^K_\chi(a^K_w)\tilde{\rho}^K_\chi(y\sigma)$ for any $y\in G^{F^m}$. By Theorem 1.1.6 and Lemma 2.2.3, $\tilde{\rho}^K_\chi(y\sigma)=\rho^k_\chi(\mathfrak{N}_{K/k}y)=\rho^k_\chi(n^{-1}_{k/k}n_{K/k}y)$ for any $y\in G^{F^m}$. Thus we have (i). Since $n_{k/k}(\{1\})=\{1\}$, we have (ii).

To proceed further we need some lemmas. The following one is obvious.

Lemma 2.4.2. *Let $c_1,\cdots,c_r,x_1,\cdots,x_r\in\bar{\boldsymbol{Q}}^\times_l$. Assume $\sum_{1\leqslant i\leqslant r}c_ix^t_i=0$ for $t=1,\cdots,r$. Then there exist $1\leqslant i\neq j\leqslant r$ such that $x_i=x_j$.*

Lemma 2.4.3. *Let $f(X), g(X)\neq 0\in\bar{\boldsymbol{Q}}_l[X]$, t a positive integer (maybe* 1) *and $\lambda\in\boldsymbol{Q}^\times_l$. Assume $f(q^m)\lambda^m=g(q^m)$ for any positive integer m such that $(m, t)=1$. Then $\lambda=\zeta q^\alpha$ with ζ a t-th root of unity and α an integer.*

Proof. Write $f(X)=\sum_{0\leqslant i\leqslant r} a_iX^i$, $g(X)=\sum_{0\leqslant i\leqslant s} b_iX^i (a_i, b_i\in\bar{\boldsymbol{Q}}_l)$. By the assumption, $f(q^{mt+1})\lambda^{mt+1}=g(q^{mt+1})$ for any $m\in\boldsymbol{N}$. Thus $\sum_{0\leqslant i\leqslant r} a_iq^i\lambda(q^{ti}\lambda^t)^m=\sum_{0\leqslant i\leqslant s} b_iq^i(q^{ti})^m$ for any $m\in\boldsymbol{N}$. If $i\neq j$, $q^{ti}\neq q^{tj}$ and $q^{ti}\lambda^t\neq q^{tj}\lambda^t$. Thus, by Lemma 2.4.2, $q^{ti}\lambda^t=q^{tj}$ for some $0\leqslant i\leqslant r$, $0\leqslant j\leqslant s$. Therefore $\lambda=\zeta q^{\alpha}$ with ζ a t-th root of unity and α a positive integer.

The following proposition is known when q is larger than the Coxeter number of G (cf. Lusztig [12], p. 25, (d)).

Proposition 2.4.4. For any $\rho\in\mathcal{E}(G^F, \{1\})$, $\lambda_\rho=1$ or -1.

Proof. If ρ is not cuspidal, the computation of λ_ρ is reduced to the groups of smaller ranks. Thus it remains to check for the cuspidal $\rho_0\in\mathcal{E}(G^F, \{1\})$. Take $w\in W$ such that $\langle\rho_0, R^1_{T_w}\rangle\neq 0$. Then $f_{\rho_0,w}(X)\neq 0$ (cf. 1.5). If $(m, 2p)=1$, $\sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m \dim\rho=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim\rho_\chi^k$ by Lemma 2.4.1, (ii). We may assume if $\rho\neq\rho_0$, $\lambda_\rho=1$ or -1. Thus, for any positive integer m such that $(m, 2p)=1$, we have $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\dim\rho_0+\sum_{\rho\neq\rho_0} f_{\rho,w}(q^m)\lambda_\rho\dim\rho=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim\rho_\chi^k$. Applying Lemma 2.4.3 we have $\lambda^{2p}_{p_0}=1$ (since $0\leqslant|\lambda_{\rho_0}|<q$). Thus it suffices to prove $\lambda_{\rho_0}\in\boldsymbol{Q}$. But for any positive integer m, $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\dim\rho_0+\sum_{\rho\neq\rho_0} f_{\rho,w}(q^m)\lambda_\rho\dim\rho=\mathrm{Tr}(F^{m*}, \sum_i(-1)^iH^i_c(X(w)))=|X(w)^{F^m}|$. Thus $f_{\rho_0,w}(q^m)\lambda^m_{\rho_0}\in\boldsymbol{Q}$ for any positive integer m. Since $f_{\rho_0,w}(X)\neq 0$, there exists an integer m_0 such that if $m\geqslant m_0$, $f_{\rho_0,w}(q^m)\neq 0$. Thus if $m\geqslant m_0$, $\lambda^m_{d_0}\in\boldsymbol{Q}$. Therefore $\lambda_{\rho_0}=(\lambda_{\rho_0})^{m_0+1}\lambda_{\rho_0}^{-m_0}\in\boldsymbol{Q}$.

Lemma 2.4.5. $\sum_\rho f_{\rho,w}(X)\lambda_\rho\rho=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\rho_\chi^k\cdot n^{-1}_{k/k}$ *as* $\boldsymbol{Q}[X]$*-linear combinations of class functions of* G^F.

Proof. Fix $y\in G^F$. By Lemma 2.4.1 and Proposition 2.4.4, if $(m, 2p)=1$, then $\sum_\rho f_{\rho,w}(q^m)\lambda_\rho\rho(y)=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\rho_\chi^k(n^{-1}_{k/k}y)$. Since there exist infinitely many positive integers m such that $(m, 2p)=1$, $\sum_\rho f_{\rho,w}(X)\lambda_\rho\rho(y)=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\rho_\chi^k(n^{-1}_{k/k}y)$ as polynomials in X (with $y\in G^F$ being fixed). This proves the lemma.

For $\chi\in\hat{W}$, let $R_\chi=|W|^{-1}\sum_{w\in W}\chi(w)R^1_{T_w}$. Then

Lemma 2.4.6. $\rho_\chi^k\cdot n^{-1}_{k/k}=\sum_\rho\langle R_\chi, \rho\rangle\lambda_\rho\rho$.

Proof. By the specialization $X\mapsto 1$, the relation in Lemma 2.4.5 is specialized to: $\sum_\rho\langle R^1_{T_w}, \rho\rangle\lambda_\rho\rho=\sum_{\chi\in\hat{W}}\chi(w)\rho_\chi^k\cdot n^{-1}_{k/k}$. Hence

$$\begin{aligned}\rho_\chi^k\cdot n^{-1}_{k/k}\,&(=|W|^{-1}\sum_{w\in W}\chi(w)\sum_{\chi_1\in\hat{W}}\chi_1(w)\rho^k_{\chi_1}\cdot n^{-1}_{k/k})\\&=|W|^{-1}\sum_{w\in W}\chi(w)\sum_\rho\langle R^1_{T_w}, \rho\rangle\lambda_\rho\rho=\sum_\rho\langle R_\chi, \rho\rangle\lambda_\rho\rho\,.\end{aligned}$$

Lemma 2.4.7. (i) *For any* $w\in W$ *and* $\rho\in\mathcal{E}(G^F,\{1\}), f_{\rho,w}(X)=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)$ $\langle R_\chi,\rho\rangle$.

(ii) $\sum_\rho f_{\rho,w}(X)\rho=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)R_\chi$.

Proof. (i) $\lambda_\rho f_{\rho,W}(X)=\langle\sum_{\rho_1}f_{\rho_1,w}(X)\lambda_{\rho_1}\rho_1,\rho\rangle=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\langle\rho_\chi^k\cdot n_{k/k}^{-1},\rho\rangle$ (by Lemma 2.4.5) $=\sum_{\chi\in\hat{W}}\nu_\chi(a_w)\langle R_\chi,\rho\rangle\lambda_\rho$ (by Lemma 2.4.6). This proves (i). (ii) is an easy consequence of (i).

Theorem 2.4.8. *Let* $w\in W$.

(i) *If m is odd,* $|X(w)^{F^m}|=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim\rho_\chi^k$.

(ii) *If m is even,* $|X(w)^{F^m}|=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim R_\chi$.

Proof. $|X(w)^{F^m}|=\sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m\dim\rho$. Assume m is odd. Then $|X(w)^{F^m}|$ $=\sum_\rho f_{\rho,w}(q^m)\lambda_\rho\dim\rho$ (since $\lambda_\rho=1$ or -1) $=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\,\rho_k^\chi(n_{k/k}^{-1}\{1\})$ (by Lemma 2.4.5) $=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim\rho_\chi^k$. Assume m is even. Then $|X(w)^{F^m}|=\sum_\rho f_{\rho,w}(q^m)\dim\rho$ $=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\dim R_\chi$ (by Lemma 2.4.7, (ii)).

The following lemma is well known (cf. [4]).

Lemma 2.4.9. *Let* $\mathfrak{A}$ *be a semisimple and symmetric algebra over the algebraic closed field of characteristic* 0. *Let* $\{e_1,\cdots,e_r\}$ *be a basis of* $\mathfrak{A}$ *and* $\{e_1^*,\cdots,e_r^*\}$ *be its dual basis. Let* χ_1,χ_2 *be the irreducible characters of* $\mathfrak{A}$. *Then* $\sum_i\chi_1(e_i)$ $\chi_2(e_i^*)=0$ *if and only if* $\chi_1\neq\chi_2$.

Theorem 2.4.10. (i) *If m is odd,* $\tilde{\rho}_\chi^K(y\sigma)=\rho_\chi^k(\mathfrak{A}_{K/k}y)$ *for any* $\chi\in\hat{W}$ *and* $y\in G^{F^m}$.

(ii) *If m is even,* $\tilde{\rho}_\chi^K(y\sigma)=R_\chi(n_{K/k}y)$ *for any* $\chi\in\hat{W}$ *and* $y\in G^{F^m}$.

Proof. For any $y\in G^{F^m}$ and $w\in W$, $\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma)=\mathrm{Tr}(((n_{K/k}y)^{-1}F^m)^*,$ $\sum_i(-1)^iH_c^i(X(w))$ (by Lemma 2.2.1) $=\sum_\rho f_{\rho,w}(q^m)\lambda_\rho^m\rho(n_{K/k}y)$. Assume m is odd. Then $\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma)=\sum_\rho f_{\rho,w}(q^m)\lambda_\rho\rho(n_{k/k}\mathfrak{N}_{K/k}y)=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\rho_\chi^k(\mathfrak{N}_{K/k}y)$ (by Lemma 2.4.5). Thus $\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\tilde{\rho}_\chi^K(y\sigma)=\sum_{\chi\in\hat{W}}\nu_\chi^K(a_w^K)\rho_\chi^k(\mathfrak{N}_{K/k}y)$. Hence we have (i) by the orthogonality relations in Lemma 2.4.9. (ii) is proved similarly.

Remark 2.4.11. If char $\boldsymbol{F}_q\neq 2$, then $R_\chi=R_\chi\cdot n_{k/k}^{-1}$ by the following lemma. Therefore "$n_{k/k}$" in (ii) of Theorem 2.4.10 can be replaced by "$\mathfrak{N}_{K/k}$" if char $\boldsymbol{F}_q$ $\neq 2$. This seems to be true even if char $\boldsymbol{F}_q=2$.

Lemma 2.4.12. *Let G be a connected reductive group over $\boldsymbol{F}_q$. (We do not assume the assumption imposed on G in 2.4.) Let $x \in G^F$ and $x=su$ be the Jordan decomposition (s: a semisimple element, u: a unipotent element). Assume u is contained in the identity component of the centralizer of u in $Z_G(s)^0$. (Notice $u \in Z_G(s)^0$ by* [16], Corollary 4.4.) *Then for any F-stable maximal torus T of G and linear character θ of T^F, $R_T^\theta(n_{k/k}(x))=R_T^\theta(x)$.*

Proof. Let $H=Z_G(s)^0$. Let T' be an F-stable maximal torus of H. Take $a \in T'$ such that $s=a^{-1F}a$. Take $b \in Z_H(u)^0$ such that $u=b^{-1F}b$. Then $x=su=sb^{-1F}b=b^{-1}s^Fb=b^{-1}a^{-1F}a^Fb=(ab)^{-1F}(ab)$. Thus $n_{k/k}(x)={}^F(ab)\,(ab)^{-1}={}^Fa^Fbb^{-1}a^{-1}$ ${}^Fabub^{-1}a^{-1}={}^Faua^{-1}$ (b commutes with u)$={}^Faa^{-1}aua^{-1}=saua^{-1}$. Therefore $n_{k/k}(x)$ $=saua^{-1}$. Since s commutes with aua^{-1} and aua^{-1} is a unipotent element, $n_{k/k}(x)$ $=s(aua^{-1})$ is the Jordan decomposition of $n_{k/k}(x)$. Let $\{g_1,\cdots,g_r\}$ be the representatives of $H^F\backslash\{g\in G^F; g^{-1}sg\in T\}$. Then by [2], Theorem 4.2, we have $R_T^\theta(x)=\sum_{1\leqslant i\leqslant r}Q_{g_iTg_i^{-1},H}(u)\theta(g_i^{-1}sg_i)$. Similarly, $R_T^\theta(n_{k/k}(x))=\sum_{1\leqslant i\leqslant r}Q_{g_iTg_i^{-1},H}(aua^{-1})$ $\theta(g_i^{-1}sg_i)$. Let H_{ad} be the adjoint group of H and $\pi\colon H\to H_{ad}$ be the canonical mapping. Since $a^{-1F}a=s\in Z(H)$, $\pi(a)\in H_{ad}^F$. Thus $\pi(u)$ and $\pi(aua^{-1})$ are conjugate in H_{ad}^F. Therefore $Q_{g_iTg_i^{-1},H}(u)=Q_{\pi(g_iTg_i^{-1}),H_{ad}}(\pi(u))=Q_{g_iTg_i^{-1},H}(aua^{-1})$. Hence $R_T^\theta(x)=R_T^\theta(n_{k/k}(x))$.

2.5. In 2.5, we wish to describe some conjectural statements flourishing from Lemma 2.4.6, if we assume Conjecture 4.3 of Lusztig [12]. To do this we need to recall some results of [11], [12]. For $\Lambda\in\Phi_n$ (resp. Φ_n^+), let ρ_Λ be the corresponding unipotent representations of Sp_{2n}^F or SO_{2n+1}^F (resp. $SO_{2n}^{+,F}$). For $\chi\in\hat{W}_n$ (resp. $\hat{W}_n$), let Λ be the corresponding symbol class in Φ_n (resp. Φ_n^+) and we put $R_\Lambda=R_\chi$. For $\Lambda\in\Phi_n$ (resp. Φ_n^+), write $\Lambda=(X\cup(Y-I), X\cup I)$, where X, Y are finite subsets of $\{0, 1, 2, \cdots\}$, $X\cap Y=\phi$, I is a subset of Y such that $2\,|I|+1\equiv|Y| \bmod 4$ (resp. $2\,|I|\equiv|Y| \bmod 4$). Now, fix X and Y. We put $|Y|=2s$ or $2s+1$, and assume $s>0$ if $|Y|=2s$. Let $Y=\{\lambda_0<\lambda_1<\lambda_2\cdots\}$, $Y^0=\{\lambda_0,\lambda_2,\lambda_4,\cdots\}$ and $Y^1=\{\lambda_1,\lambda_3,\lambda_5,\cdots\}$. Let $\mathcal{P}$ be the set of all subsets of Y and $\mathcal{P}_s=\{I\in\mathcal{P}:\ |I|\equiv s \bmod 2\}$. Then $\mathcal{P}$ is regarded as a vector space over $\boldsymbol{F}_2$ by the addition: $I, J\in\mathcal{P}\mapsto IJ=I\cup J-I\cap J$ and $\mathcal{P}_0$ is regarded as a subspace. By the bijection $\mathcal{P}_s\to\mathcal{P}_0$ ($I\mapsto IY^1$), we can regard $\mathcal{P}_s$ as a vector space over $\boldsymbol{F}_2$. Define $Q\colon\mathcal{P}_s\to\{\pm1\}$ ($I\mapsto(-1)^{(|I|-s)/2}$). If we identify $\boldsymbol{F}_2$ canonically with $\{\pm1\}$, the mapping Q is regarded as a quadratic form on $\mathcal{P}_s$ whose associated bilinear form B is: $I, J\in\mathcal{P}_s\mapsto B(I,J)=(-1)^{|I\cap X^0|+|J\cap Y^1|+|I\cup J|}$. Thus the Fourier transform of Lusztig [11], [12] takes the form:

$$\hat{\rho}_{(X\cup I',X\cup I)}=2^{-s}\sum_{J\in\mathcal{P}_s}B(I,J)\rho_{(X\cup J',X\cup J)}\quad\text{for } I\in\mathcal{P}_s\,.$$

DEFINITION 2.5.1. (i) For a class function f on G^F, let $f^\Delta=f\cdot n_{k/k}$. $\Delta^2=1$ and $\dim f^\Delta=\dim f$. (Notice that for any connected algebraic group G over $\boldsymbol{F}_q$, if $x^r \in Z_G(x)^0$, then $n^r_{k/k}(\{x\}) = \{x\}$.)

(ii) Let $\mathfrak{R}_{X,Y} = \sum_{J \in \mathcal{P}_s} \bar{\boldsymbol{Q}}_I \rho_{(X \cup J', X \cup J)}$. Define the linear automorphims $\tilde{\Delta}$ of $\mathfrak{R}_{X,Y}$ by the condition $\hat{\rho}^{\tilde{\Delta}}_{(X \cup I', X \cup I)} = Q(I)\hat{\rho}_{(X \cap I', X \cup I)}$ for $I \in \mathcal{P}_s$. Since $\dim \hat{\rho}_{(X \cup I', X \cup I)} = 0$ if $|I| \neq s$, we have $\dim f^{\tilde{\Delta}} = \dim f$ for any $f \in \mathfrak{R}_{X,Y}$.

It can easily be checked the following

Lemma 2.5.2. *For any* $J \in \mathcal{P}_s$,

$$Q(I)\rho^{\tilde{\Delta}}_{(X \cup I', X \cup I)} = 2^{-s} \sum_{J \in \mathcal{P}_s} B(I, J) Q(J) \rho_{(X \cup J', X \cup J)} .$$

Theorem 2.5.3. *Assume Conjecture* 4.3 *in* [12] *is true. Then for* $I \in \mathcal{P}_s$, $\lambda_{\rho(X \cup I', X \cup I)} = Q(I)\ (= (-1)^{(|I|-s)/2})$.

Proof. By the induction of the semisimple rank of G, it is needed to check only for the cuspidal $\rho_{(X \cup I'_0, X \cup I_0)}$ (I_0 or $I'_0 = Y$). Thus we may assume the statement is true if $I \neq I_0, I'_0$. Since we have assumed Conjecture 4.3 in [12], $R_{(X \cup I', X \cup I)} = \hat{\rho}_{(X \cup I', X \cup I)}$ if $|I| = s$. Thus by Lemma 2.4.6, $R_{(X \cup I', X \cup I)} = 2^{-r} \sum_{J \in \mathcal{P}_s} B(I, J) Q(J)\, \rho_{(X \cup J', X \cup J)}$. But $\dim R^{\Delta}_{(X \cup I', X \cup I)} = \dim R_{(X \cup I', X \cup I)} = \dim R^{\tilde{\Delta}}_{(X \cup I', X \cup I)}$. Thus $2^{-s} \sum_{J \in \mathcal{P}_s} B(I, J) Q(J) \dim \rho_{(X \cup J', X \cup J)} = 2^{-s} \sum_{J \in \mathcal{P}_s} B(I, J)\, \lambda_{\rho(X \cup J', X \cup J)} \dim \rho_{(X \cup J', X \cup J)}$ by Lemma 2.5.2. This relation shows our statement.

Remark 2.5.4. (i) The statement of Theorem 2.5.3 is a counterpart of the statements for some families of the unipotent representations of the exceptional groups given in Lusztig [12], p. 45 and [13], p. 335.

(ii) Assume char $\boldsymbol{F}_q \neq 2$. Lemma 2.4.12 and the proof of Theorem 2.5.3 show that Δ and $\tilde{\Delta}$ coincide on the subspace $\mathfrak{S}$ of $\mathfrak{R}_{X,Y}$ which is spanned by $\{\rho_{(X \cup I', X \cup I)}, \rho^{\Delta}_{(X \cup I', X \cup I)}; I \in \mathcal{P}_s, |I| = s\}$ and $\{R_{(X \cup I', X \cup I)}; I \in \mathcal{P}_s, |I| = s\}$ under the assumption of Theorem 2.5.3. If $|Y| = 1$, 3 or 4, $\mathfrak{S} = \mathfrak{R}_{X\ Y}$. If $|Y| = 5$ or 6, $\dim \mathfrak{S} = \dim \mathfrak{R}_{X,Y} - 1$. We may ask if the following is true (cf. Remark 2.4.11).

Conjecture 2.5.5. $\tilde{\Delta} = \Delta$.

3. Unitary case

The method which we applied in the case of split classical groups is also effective for the unitary groups. Let G be the unitary group U_n over $\boldsymbol{F}_q$ and we assume m is an even integer. The Weyl group W is canonically identified with the symmetric group S_n and we assume the generic algebra $\mathfrak{A}(W)$ (cf. 2.1) is over the extension field of $\mathfrak{A}(X)$ which contains $X^{1/2}$ ($X^{1/2}$ being fixed). Let $\mathcal{P}(n)$ be the set of all partitions of n. For $\alpha \in \mathcal{P}(n)$, let χ_α (resp. ν_{χ_α}) be the corresponding irreducible representation (or its character) of W (resp. $\mathfrak{A}(W)$). The following lemma is easily checked by the induction on n.

Lemma 3.1. *Let w_0 be the longest element of W and $\alpha=(\alpha_1\geqslant\cdots\geqslant\alpha_s>0)$ $\in\mathcal{P}(n)$. Define $C_\alpha=\binom{n}{2}+\sum_{\substack{1\leqslant i\leqslant s\\ 1\leqslant j\leqslant\alpha_i}}(j-i)$. Then $a_{w_0}^2$ acts as a scalar X^{C_α} on the representation ν_{χ_α} of $\mathfrak{A}(W)$.*

Let the notation be as in 2.1. We write ρ_α^K instead $\rho_{\chi_\alpha}^K$ of for $\alpha\in\mathcal{P}(n)$ to simplify the notation. Since $a_{w_0}^K I_\sigma$ commutes with $\mathfrak{A}^K(W)$, each irreducible component ρ_α^K of Z^K is regarded as a G^{F^m} A-module ρ_α^K by the mapping $\sigma\mapsto(q^m)^{-C_\alpha/m}$ $a_{w_0}^K I_\rho$.

For $\alpha\in\mathcal{P}(n)$, let $\rho_\alpha^k=|W|^{-1}\sum_{w\in W}\chi_\alpha(ww_0)R_{T_w}^1$. If we put η_α=the signature of dim ρ_α^k, then by [14], $\eta_\alpha\rho_\alpha^k$ is the irreducible representation of G^F and all the unipotent representations of G^F are of this form. For the simplification of the notation we let $f_{\alpha,w}(X)=\eta_\alpha f_{\eta_\alpha\rho_\alpha^k,w}(X)$ $(\alpha\in\mathcal{P}(n),\ w\in W)$, $\lambda_\alpha=\lambda_{\eta_\alpha\rho_\alpha^k}$. Then

Theorem 3.2. *Assume* char $\boldsymbol{F}_q\neq 2$. *Let $\alpha\in\mathcal{P}(n)$ and $w\in W$.* Then

(i) $\rho_\alpha^k(n_{K/k}y)=(-1)^{mC_\alpha/2}\tilde{\rho}_\alpha^K(y\sigma)$ *for any even integer m and $y\in G^{F^m}$,*

(ii) $f_{\alpha,w}(q^m)\lambda_\alpha^{m/2}=\nu_{\chi_\alpha}^K(a_w^K a_{w_0}^K)(-q)$ *for any even integer m,*

(iii) $|X(w)^{F^m}|=\sum_{\alpha\in\mathcal{P}(n)}\nu_{\chi_\alpha}^K(a_w^K a_{w_0}^K)(-q)^{-mC_\alpha/2}$ dim ρ_α^k *for any even integer m.*

Our proof is based on Kawanaka [7] as is stated in the introduction. In this respect, (i) of the theorem for cuspidal or subcuspidal ρ_α^k's is essential. The detailed arguments, which is slightly tedious, are omitted.

References

[1] C.T. Benson and C.W. Curtis: *On the degrees and rationality of certain characters of finite Chevalley groups,* Trans. Amer. Math. Soc. **165** (1972), 251–273 and **202** (1975), 405–406.

[2] F. Digne et J. Michel: *Nombre de points rationnels des varietes de Deligne-Lusztig et charactères de l'algebre de Hecke,* C. R. Acad. Sci. Paris **287**, Sér. A (1978), 811–814.

[3] P. Deligne and G. Lusztig: *Representations of reductive groups over finite fields,* Ann. of Math. (2) **103** (1976), 103–161.

[4] T.V. Fossum: *Characters and centers of symmetric algebras,* J. Algebra **16** (1970), 4–13.

[5] A. Gyoja: *Liftings of irreducible characters of finite reductive groups,* Osaka J. Math. **16** (1979), 1–30.

[6] P.N. Hoeffsmit: *Representations of Hecke algebras of finite groups with BN-pairs of classical type,* Ph. D. Dissertation, Univ. British Columbia, Vancouver, 1974.

[7] N. Kawanaka: *On the irreducible characters of finite unitary groups,* J. Math. Soc. Japan **29** (1977), 425–450.

[8] N. Kawanaka: *Lifting of irreducible characters of finite classical groups,* J. Fac. Sci. Univ. Tokyo Sect. IA Math.

[9] G. Lusztig: *On the Green polynomials of classical groups,* Proc. London Math. Soc. (3) **33** (1976), 443–475.

[10] G. Lusztig: *Coxeter orbits and eigenspaces of Frobenius,* Invent. Math. **38** (1976), 101–159.

[11] G. Lusztig: *Irreducible representations of finite classical groups,* Invent. Math. **43** (1977), 125–175.

[12] G. Lusztig: Representations of finite Chevalley groups, C.B.M.S. Regional conference series in math. 39, AMS, 1978.

[13] G. Lusztig: *Unipotent representations of a finite Chevalley group of type E_8*, Quart. J. Math. Oxford Ser. (2) **30** (1979), 315–338.

[14] G. Lusztig and B. Srinivasan: *The characters of the finite unitary groups,* J. Algebra **49** (1977), 167–171.

[15] T. Shintani: *Two remarks on irreducible characters of finite general linear groups,* J. Math. Soc. Japan **28** (1976), 396–414.

[16] T.A. Springer and R. Steinberg: *Conjugacy classes,* in Lecture Notes in Math. 131, Springer-Verlag, 1970.

Department of Mathematics
Nara University of Educatinn
Nara 630, Japan