On the Concept and Mechanism of the (MN – r_p – σ_θ) Fracture Theory†

EL-SHEIKHY Refat*and NAKA Masaaki**

Abstract

The current theory of "Circumferential Strain Energy (MN- r_p - σ_θ)" is new fracture theory recently developed. This theory depends on the calculation of the components of critical strain energy density factors at the position of maximum circumferential tensile stress (σ_θ) $_{max}$ regarding the minimum critical radius (r_{pc}) $_{min}$ of the plastic zone at the crack vicinity. This theory can solve the problems of cracks under any load condition with a higher accuracy than other theories. It can be applied in linear elastic fracture mechanics LEFM for brittle and quasi-brittle homogeneous materials and composites. It can be extended to elastic plastic fracture mechanics EPFM. It can be extended and can be applied to ductile materials. This theory, by using its fracture mechanism and hypotheses can investigate each of the cracking load and crack propagation directions. It can predict also the fracture toughness, crack growth path and the crack history. Itcan be used for estimating the safe life time of any structure. It is developed for mixed modecracks for isotropic brittle materials and can be applied for group of cracks under static, cyclic anddynamic loading. Also, it can be applied for cracks under thermal stresses.

KEY WORDS: (Fracture),(Strain energy components)(Circumferential stress)(Plastic zone),(Crack tip)

1. Introduction

This theory is newly developed and represents efforts to develop and investigate an accurate method to deal with the complicated problems of cracks [1-7]. The new fracture theory is called "Circumferential Strain Energy Theory (MN- r_p - σ_θ)". This theory depends on the calculation of the components of strain energy density factors at the position of maximum circumferential tensile stress, which will be at the minimum critical radius of the plastic zone (r_{pc}) at the crack vicinity. This theory can solve the problems of cracks under any load condition with higher accuracy than other theories [1-7]. This theory can be applied in linear elastic fracture mechanics LEFM and in elastic plastic fracture mechanics EPFM for quasi-brittle homogeneous materials and brittle composites. It can be extended and used for ductile materials. This theory can investigate cracking load, crack propagation mechanism, crack propagation direction and crack propagation length. It can predict also the fracture toughness, crack growth path and the crack history. It can be used for estimating the safe life time of any structure. It is developed in mixed mode cracks for isotropic brittle materials and can be applied for group of cracks under static, cyclic or dynamic loading. Also, it can be applied for cracks under thermal stresses.

Because old fracture theories [1-7] could not solve many of the crack problems especially for complicated fracture and cracks under complicated cases of loading, the fracture theory called NN-theory [8, 9] was developed. This theory considered the energy components (S_v, S_d) without considering the plastic zone due to stress concentration at the crack tip in the analysis. It neglected the effect of the plastic zone in the calculations for facilitating the analysis without more complicated equations. But, actually the plastic zone has an important role in the mechanism of crack propagation. The radius of the plastic zone has a minimum length at the point of propagation; r_{pc}.

For the case of the σ_θ criterion [1- 4], it was recognized that it cannot alone represent the physical meaning of the actual mechanism. The $\sigma_{\theta c}$ criterion can not consider the mechanical properties such as Poisson s ratio (ν) or Young s modulus of elasticity (E). Also, it cannot accommodate differences between plane stress and plane strain loading conditions. Therefore, it cannot consider the material types or mechanical properties. It cannot take the material size into consideration in the mechanism or analysis. It just can be used as a preliminary investigation for the analysis. This is very dangerous in cracking analysis since it neglects the most important factors which may give errors in the results. Therefore, it was very important to develop the current theory (MN-r_p-σ_θ). It can consider all fracture parameters at once.

† Received on November 7, 2005
* Foreign Guest Researcher
** Professor

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2. Aim and Scope of Current Research

This research is aimed to study the crack propagation process by the consideration of all of the fracture factors S_v, S_d, r_p, σ_θ simultaneously at once in order to provide more accurate results. This research is intended to use these fracture factors together to determine the fracture direction, fracture load and fracture toughness for cracks under tension stresses, shear stresses, tension and shear stresses and cracks under compression and shear stresses. In the propagation period, the circumferential stresses at the plastic zone boundary at which the circumferential stresses will affect the crack propagation should be considered. It has been recognized by the researcher and others [8,9] that crack propagation usually occurs at the minimum length of the plastic zone radius r_{pmin}., at the same time the circumferential stresses becomes critical ($\sigma_{\theta\, c}$.) Therefore, the maximum circumferential stresses $\sigma_{\theta\, c}$ will be at the minimum critical radius of the plastic zone ($r_{p\, c\, min}$). At the same time, the strain energy should be critical at the crack propagation point in the same propagation direction θ_c. The mechanism of the cracking will consider the strain energy at the point of minimum plastic zone radius which passes by the point of the maximum circumferential tensile stresses at the same time.

3. Theoretical Analysis

The theoretical analysis of the proposed theory (MN-r_p- σ_θ) is presented in the following section for all cracks under single mode which are both of the opening mode (mode I) under static tension stress or sliding shape change mode (mode II) under static shear stress. The analysis is also presented for both mixed mode cracks which are mixed mode crack of opening and sliding modes together under tension and shear stresses and mixed mode under compression and shear stresses. This study employs in linear elastic fracture mechanics (LEFM) near the crack tip for isotropic brittle materials.

3.1 Opening volume change crack mode (mode I crack) under static normal stresses

3.1.1 Fracture condition of propagation direction

A crack will propagate in the direction of maximum/minimum distortional strain energy density factor ($S_{dI\, min}$)$_{max}$ when it will be in the same direction as minimum length of the plastic zone radius ($r_{PI\, min}$) and at the same time pass on the plastic zone boundaries through the point of maximum value of the circumferential tensile stress ($\sigma_{\theta\, I\ max}$). **Equations 1** to **4** represent the fracture conditions of mode I cracks for determination of the propagation direction. These four conditions should be realized at the same time to predict the same fracture angle.

$$\frac{\partial \left(S_{dI\,min}\right)^{max}}{\partial \theta} = 0 \qquad (1)$$

$$\frac{\partial^2 \left(S_{dI\,min}\right)^{max}}{\partial \theta^2} \geq 0 \qquad (2)$$

$$\theta_I = 0\ ,\ \beta_I = 90$$

$$\frac{\partial \sigma_{\theta I}}{\partial \theta} = 0 \quad \& \quad \frac{\partial^2 \sigma_{\theta I}}{\partial \theta} \leq 0 \qquad (3)$$

$$\frac{\partial r_{PI}}{\partial \theta} = 0 \quad \& \quad \frac{\partial^2 r_{PI}}{\partial \theta^2} \geq 0 \qquad (4)$$

3.1.2 Fracture condition of starting propagation and propagation load

Crack will start propagation when all of the three fracture factors the maximum volumetric strain energy density factor ($S_{VI\, max}$) , minimum radius of the plastic zone ($r_{PI\, min}$) and maximum circumferential tensile stress factor ($\sigma_{\theta\, I\ max}$) reaches their critical quantity in the same time as indicated in the **Eqs. 5 -7** and **Figs. 1** to **8** respectively. **Equations 5 -7** represent the fracture conditions for starting propagation and predicting the propagating load. All of the conditions should be realized simultaneously.

The crack will start the extension when

$$\left(\sigma_{\theta II}\right) \geq \left(\sigma_{\theta II c}\right) \qquad (5)$$

$$\left(r_{PII}\right) \geq \left(r_{PIIc}\right) \qquad (6)$$

$S_{vI} > S_{vIc}$ (7)

$S_{vI} \rightarrow S_{vIc}$, (7a)

$r_{PI} \rightarrow r_{PIc}$, (7b)

$\sigma_{\theta I} \rightarrow \sigma_{\theta I c}$ (7c)

3.2 Sliding shape change crack mode (mode II crack) under static normal stresses

3.2.1 Fracture condition of propagation direction

A crack will propagate in the direction of minimum distortional strain energy density factor ($S_{dII\, min}$) when it is in the same direction as the minimum length of the plastic zone radius $r_{PII\, min}$ when at the same time pass the plastic zone boundaries through the point of maximum value of the circumferential tensile stress $\sigma_{\theta II}$ max . **Equations 8 -11** represent the fracture conditions of mode I crack for determination the propagation direction. These four conditions should be realized at the same time to predict the same fracture angle.

Then, mode II crack will propagate in the direction of

$$\frac{\partial S_{\theta I}}{\partial \theta}=0 \quad \& \quad \frac{\partial^2 S_{\theta I}}{\partial \theta} \le 0 \tag{8}$$

$$\frac{\partial r_{PI}}{\partial \theta}=0 \quad \& \quad \frac{\partial^2 r_{PI}}{\partial \theta^2} \ge 0 \tag{9}$$

$$\frac{\partial (S_{dII})_{min}}{\partial \theta}=0 \tag{10}$$

$$\frac{\partial^2 (S_{dII})_{min}}{\partial \theta^2} \ge 0 \tag{11}$$

3.2.2 Fracture condition of starting propagation and propagation load

Crack will start propagation when all of the three fracture factors, the maximum volumetric strain energy density factor ($S_{VII\,max}$) , minimum radius of the plastic zone ($r_{PII\,min}$) and maximum circumferential tensile stress factor ($\sigma_{\theta II\,max}$) reach their critical quantity at the same time as indicated in the **Eqs. 12-14** and **Figs. 1** to 8. **Equations 12–14** represent the fracture conditions for starting propagation and predicting the propagating load. All of the conditions should be realized simultaneously.

Then, it will start the extension when:

$$(S_{\theta II}) \ \ge \ (S_{\theta II c}) \tag{12}$$

$$(r_{PII}) \ \ge \ (r_{PII c}) \tag{13}$$

$S_{vII} > S_{vIIc}$ (14)

$S_{vII} \rightarrow S_{vIIc}$, (14a)

$r_{PII} \rightarrow r_{PIIc}$, (14b)

$\sigma_{\theta II} \rightarrow \sigma_{\theta II c}$ (14c)

3.3 Mixed mode crack under (tension and shear) or (compression and shear) stresses

3.3.1 Fracture condition of propagation direction

A crack will propagate in the direction of maximum minimum distortional strain energy density factor ($S_{dI\,max\,min\,-dII\,min}$) when it will be in the same direction of minimum length of the plastic zone radius ($r_{PI\,min-PII\,min}$) and at the same time pass the plastic zone boundaries through the point of maximum value of the circumferential tensile stress ($\sigma_{\theta I\,max-\,\theta II\,max}$).

Then, mixed mode cracks will propagate in the direction of the following conditions of **Equations 15-19** considering that **Eq. 18** is for mixed mode under tension and shear while **Eq. 19** is for mixed mode of compression and shear stresses.

$$(S_{\theta}) = (S_{\theta I}) + (S_{\theta II}) \tag{15}$$

$$(r_{PI-II}) = (r_{PI}) + (r_{PII}) \tag{16}$$

$$S^{tension}_{dI-II} = S_{dI} + S_{dII} \tag{17}$$

$$S^{tension}_{dI-II} = \left[\left(s_{dI} + s_{dII}\cos\theta_{II} \right)^2 + \left(s_{dII}\sin\theta_{II} \right)^2 \right]^{1/2} \tag{18}$$

$$S^{compression}_{dI-II} = \left[\left(-s_{dI} + s_{dII}\cos\theta_{II} \right)^2 + \left(s_{dII}\sin\theta_{II} \right)^2 \right]^{1/2} \tag{19}$$

3.3.2 Fracture condition of starting propagation and propagation load.

A crack will start propagation when all of the three fracture factors the maximum volumetric strain energy density factor ($S_{VI\,max\,-\,VII\,max}$) , minimum radius of the plastic zone ($r_{PI\,min\,-\,PII\,min}$) and maximum circumferential tensile stress factor ($\sigma_{\theta I\,max\,-\theta II\,max}$) reach their critical quantity in the same time as indicated in the **Eq. 20 -22** and **Figs. 1** to **8** respectively.

Then, it will start the extension under the conditions of **Eqs. 20 – 22** as the following:

$$(S_{\theta}) \ \ge \ (S_{\theta c}) \tag{20}$$

$$(r_{PI-II}) \ \ge \ (r_{PI-II})_c \tag{21}$$

$S_{vI-II} \rightarrow S_{vI-IIc}$, (22a)

$r_{PI-II} \rightarrow r_{PI-IIc}$, (22b)

$\sigma_{\theta I-II} \rightarrow \sigma_{\theta I-II c}$ (22c)

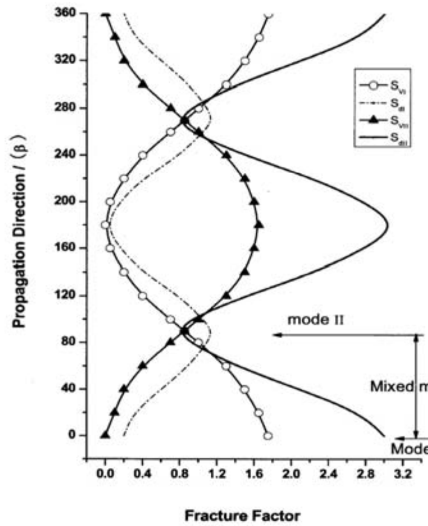

Fig. 1 Variation of energy components around crack vicinity.

Where

$\sigma_{\theta I} = K_I\,(3\cos(\theta_I/2)+\cos(3\theta_I/2))/(2\pi r)^{0.5}$

$\sigma_{\theta I} = K_I\,(\cos^3(\theta_I/2)/(2\pi r)^{0.5}$

$\sigma_{\theta II} = K_{II}\,[-0.75(3\sin(\theta_{II}/2)+\sin(3\theta_{II}/2))]/(2\pi r)^{0.5}$

$\sigma_{\theta II} = K_{II}\,[-1.5(\cos(\theta_{II}/2)\sin(3\theta_{II}/2))]/(2\pi r)^{0.5}$

$S_{vI} = K_I^2(1+\nu)^2\cos^2(\theta/2)/(\triangle\pi r)$ (Plain strain)

$S_{vI} = K_I^2\cos^2(\theta/2)/(\triangle\pi r)$ (Plain stress)

$S_{dI} = K_I^2(1+\cos\theta)[0.66((1-2\nu)^2+(1-\cos^2\theta)/(16G\pi r)$ (Plain strain)

$S_{vII} = K_{II}^2(1+\nu)^2\cos(\theta/2)/(9\triangle\pi r)$ (Plain strain)

$S_{dII} = [K_{II}^2/12G\pi r][4(\nu^2-\nu+1)\sin^2\theta/2+3\sin\theta/2\sin\theta\cos3\theta/2+0.75\sin^2\theta+3\cos2\theta/2-3\cos\theta/2\sin\theta\sin3\theta/2]$ (Plain strain)

$\triangle = E/[3(1-2\nu)$

$G = E/[2(1+\nu)]$

$(\overset{\omega}{\sigma}_\theta)$ = directional circumferential tensile stress for mixed modes,

$K_i = \sigma(\pi a)^{0.5}\sin^2\beta$ stress intensity factor for Mode I,

r = radius of plastic zone, ν = Poisson's ratio, θ = fracture angle, β = crack angle

σ = far field stress, a = half crack length, E = modulus of elasticity,

$(\overset{\omega}{S}_{vI})$ = directional volumetric strain energy density factor for Mode I,

$K_{II} = \sigma(\pi a)^{0.5}\sin^2\beta\cos\beta$,= stress intensity factor for Mode II,

$(\overset{\omega}{S}_{dII})$ = directional distortional strain energy density factor for Mode II.

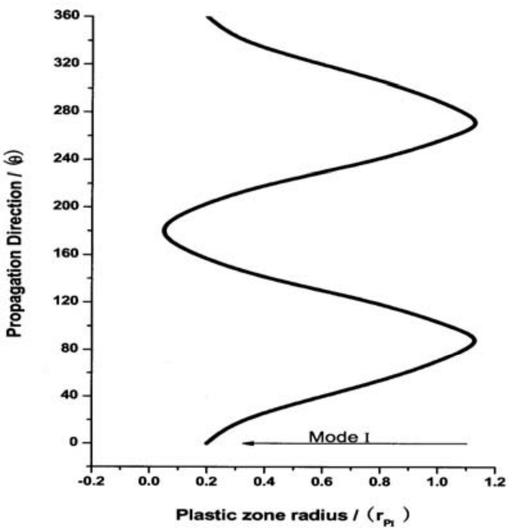

Fig. 2 Variation of plastic zone radius for mode I around crack vicinity.

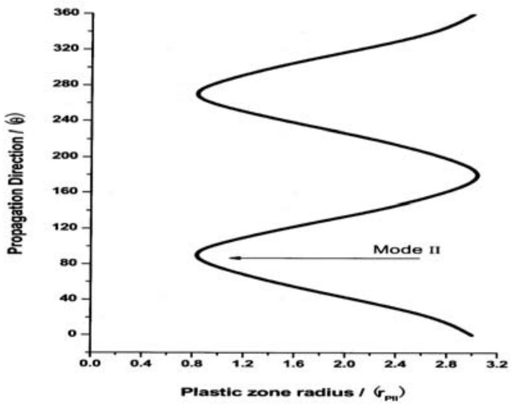

Fig. 3 Variation of plastic zone radius for mode II around crack vicinity.

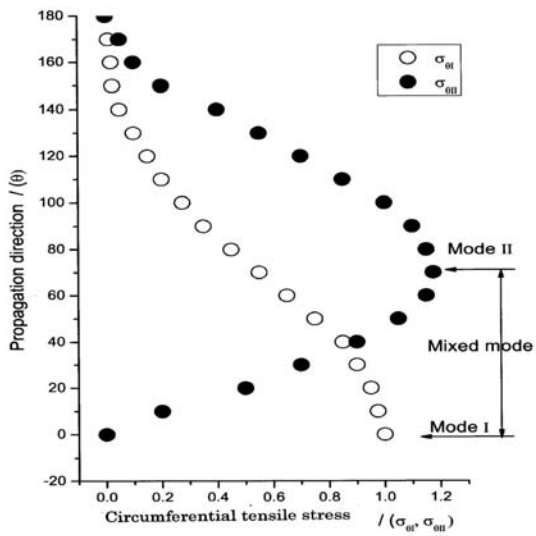

Fig. 4 Variation of $\sigma\theta$ for both mode I and mode II around crack vicinity.

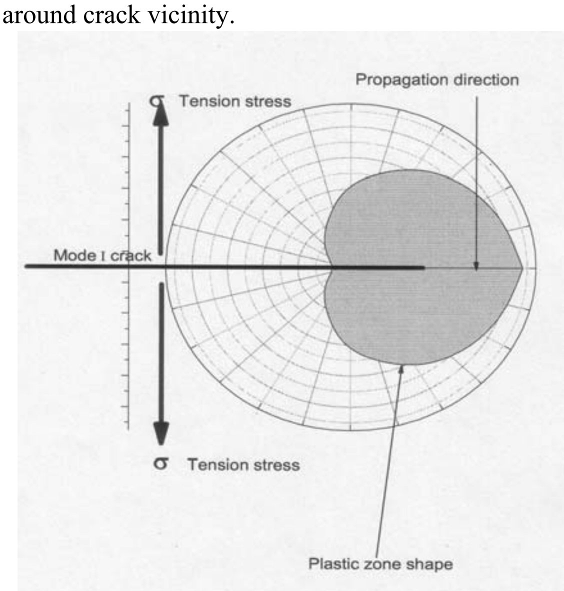

Fig. 5 Plastic zone for tension mode I at crack tip.

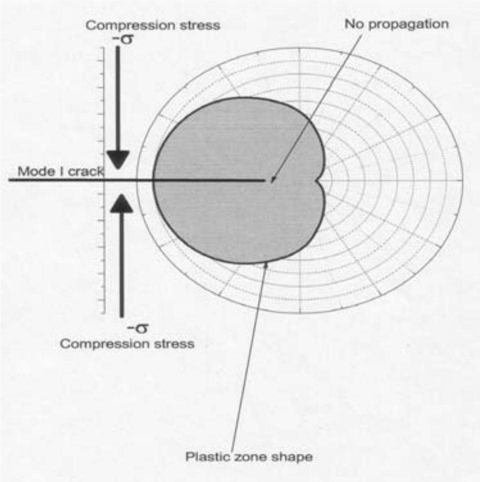

Fig. 6 Plastic zone for compression mode I at crack tip.

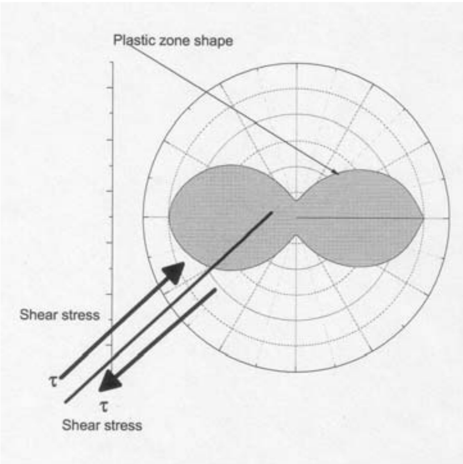

Fig. 7 Plastic zone for mode II at crack vicinity.

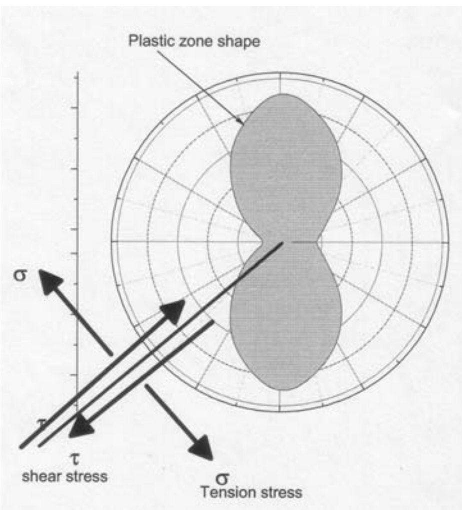

Fig. 8 Plastic zone for mixed mode I+II at crack tip.

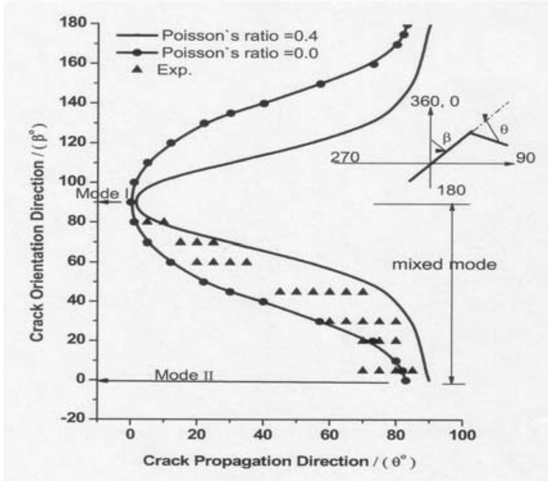

Fig. 9 Crack propagation direction for mode I, mode II and tension mixed mode.

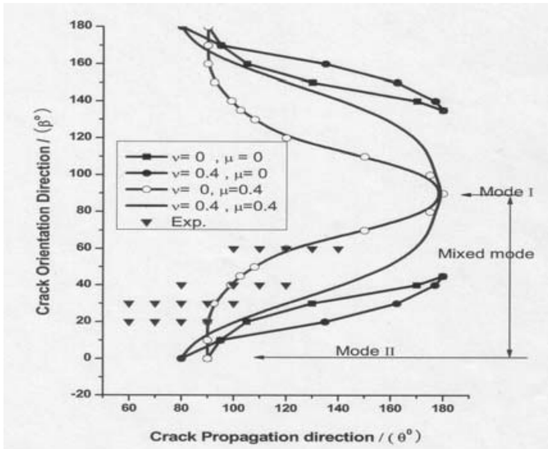

Fig. 10 Crack propagation direction for mode I, mode II and compression mixed mode.

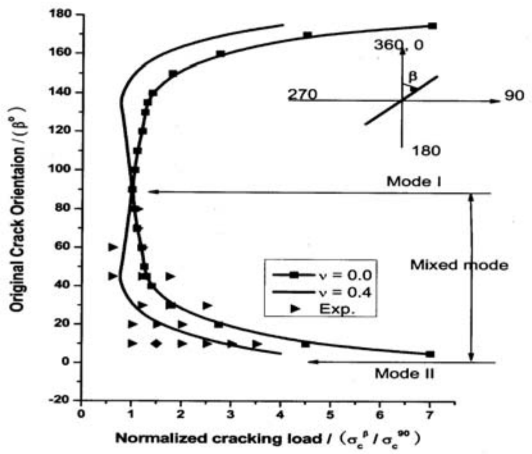

Fig. 11 Crack propagation load for mode I, mode II and tension mixed mode.

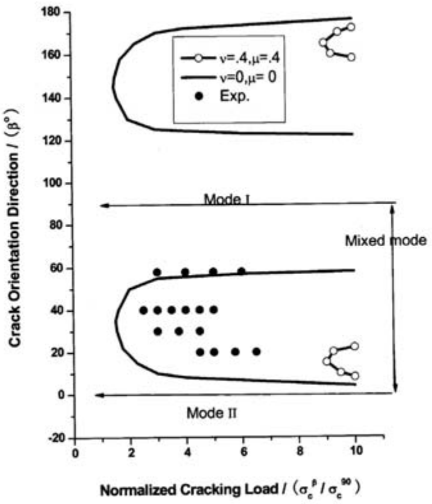

Fig. 12 Crack propagation load for mode I, mode II and compression mixed mode.

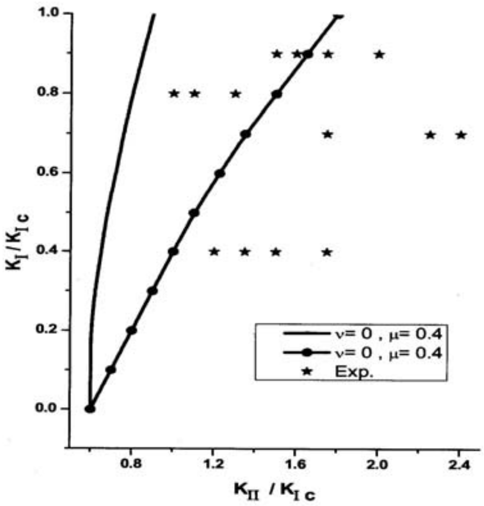

Fig. 13 Cracking toughness for mode I, mode II and tension mixed mode.

4. Results and Discussions

The results of propagation direction, propagation load and the fracture toughness envelope (K_I-K_{II}) for both mixed modes under tension and shear stresses and mixed mode under shear and compression stresses are presented as follows:

4.1 Propagation direction

The propagation direction of mixed mode cracks is obtained based on the assumptions and analysis of the theory for each of mode I, mode II and mixed modes. They are plotted as the following:

(1) Mixed mode under tension results are shown in **Fig. 9** with comparison to experimental work [10)].

Figure 9 indicates the relation between the

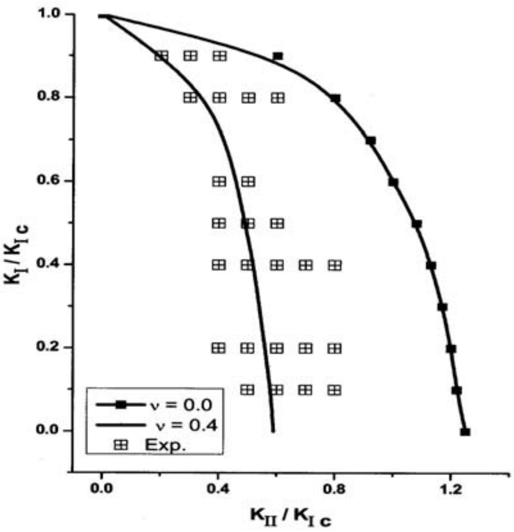

Fig. 14 Cracking toughness for mode I, mode II and compression mixed mode.

inclination angles (β) of the original cracks and the propagation direction (θ). Then, depending on the measurements of the original crack inclination and the material Poisson's ratio (ν), the propagation direction (θ) can be easily predicted without making any complicated analysis for the cracks under static tension and shear stresses.

(2) Mixed mode under compression results are shown in **Fig. 10** showing the comparison to experimental work [10)].

Figure 10 represents the relation between the original crack angle (β) and the propagation angle (θ). By measuring crack angle for inclined cracks under compression loading it is easy to predict the expected propagation direction.

From both **Fig. 9** and **Fig. 10** it can be recognized that the crack under tension and shear stresses propagate in a direction tending to be parallel to the loading direction while the cracks under compression and shear stresses propagate in a direction tending to be normal to the loading direction.

4.2 Propagation load

The fracture load of mixed modes can be predicted in normalized relation to the fracture load of mode I. They can be determined as functions of the critical fracture factors of both mixed mode and mode I by developing the following:

(1) Results of mixed mode crack under tension and shear stresses compared to experimental work are shown in **Fig. 11**

(2) Results of mixed mode crack under compression and shear stresses compared to Experimental work are shown in **Fig.** 12.

Fig. 11 and **Fig. 12** show the relations between the propagation load and the original crack angle. These relations make it easy for users and engineers to predict the cracking load by just measuring the original crack angle and the material's Poisson ratio (ν). The cracking load relations are plotted in normalized values to the cracking load of mode I crack which is material intrinsic parameter. This material intrinsic parameter can be predicted experimentally or can be obtained from standards.

4.3 Fracture toughness

The fracture toughness envelope (K_I-K_{II}) for mixed mode crack under both (tension and shear stresses) and (compression and shear stresses) in comparison to experimental work is shown in **Fig. 13** and **Fig. 14** respectively.

Figure 13 and **Fig. 14** represent the relations between fracture toughness of mode I cracks (K_I) and fracture toughness of mode II cracks (K_{II}). The relations are plotted in normalized relation to critical fracture toughness for mode I crack which is a material intrinsic parameter. Critical fracture toughness of (K_{Ic}) for mode I crack can be predicted experimentally or from standards. The fracture toughness for mixed mode cracking can be predicted from **Fig. 13** and **Fig. 14** by measuring the values of (K_I/ K_{Ic}) and (K_{II}/K_{Ic}) for the crack by using the inclination angle (β). Then by using K_I and K_{II} can be calculated because K_{Ic} is material intrinsic parameter. Then from the following relation **Eq. 23** , the fracture toughness for mixed mode can be predicted.

$K = K_I \sin^2 \beta + K_{II} \cos^2 \beta$

Using $\beta = 90^\circ$ for mode I crack, $\beta = 0^\circ$ for mode II crack and for mixed mode $0^\circ < \beta < 90^\circ$

4.4 Fracture components of strain energy

The strain energy for any material has two components. First component is the dilatational or volumetric component (S_v) while the second component is the shape change or distortional component (S_d). The basic crack modes are two modes. The first mode is the opening mode or volume change mode which is a mode I crack. The second mode is the shape change mode or distortional sliding mode which is a mode II crack. A mode I crack is made by tension stresses while a mode II crack is made by plane shear stresses. The strain energy for mode I has two components which are (S_{vI}, S_{dI}) as shown in **Fig.** 1. The main strain energy component for mode I is (S_{vI}) which is responsible on the crack propagation as shown in **Fig.** 1.

Mode II crack has two strain energy components which are (S_{vII} , S_{dII}) as shown in **Fig.** 1. The shape change component of strain energy (S_{dII}) is responsible of predicting the fracture of mode II crack at $\beta = 90$, $90 > \theta > 0$.

For mixed modes which include both mode I and mode II cracks, both of (S_{vI}) and (S_{dII}) are responsible of predicting the fracture of mixed mode crack.

4.5 Circumferential tensile stresses

The circumferential tensile stresses factors for both mode I ($\sigma_{\theta I}$) and mode II ($\sigma_{\theta II}$) cracks are represented in **Fig. 4.** As shown in **Fig.** 4, the maximum values of ($\sigma_{\theta I}$) and ($\sigma_{\theta II}$) represent the critical values around the crack tip for both mode I and mode II cracks. For mixed mode the both
values will be considered.

4.6 Plastic zone radii

The radius of the plastic zone around the crack tip for mode I (r_{pI}) is represented in **Fig. 3** while (r_{pII}) for mode II is represented in **Fig. 4** . They are plotted in comparison to the fracture direction (θ) at the crack tip. It can be recognized that the values or (r_{pIc}) and (r_{pIIc}) at the propagation direction of both of mode I and mode II respectively are the minimum values. **Fig. 5** and **Fig. 6** show the plastic zone shape of mode I under tension and compression respectively. **Fig. 7** shows the shape of plastic zone of a mode II crack while **Fig. 8** shows the shape of plastic zone of mixed mode under tension and shear stresses.

5. Conclusions

Based on the study and analysis of the current theory, it is clear that the concept and formulation of the theory are new. It is the only theory which considers the strain energy approach, plastic zone and the circumferential stresses at the crack tip simultaneously. The available experimental results are compared to the theoretical results. Both of the theoretical results and the experimental work are shown to be in good agreement. The new developed theory; MN-r_p-σ_θ Theory regards all fracture factors with a logical concept. It considered all energy fracture factors (S_v, S_d, r_p, σ_θ) in a single point of view considering that each factor has a certain effect on the fracture process. The assumption of using all fracture factors together is presented for the first time by MN-theory with this new concept. Depending on the results of this theory the critical loads, critical stress intensity factors and fracture direction can be predicted easily by users and engineers using the charts of the results. The fracture trajectory and crack history can be predicted.

The mechanism and hypotheses of the current theory are new and completely reflect to the physical meaning. Engineers and users can depend safely on the MN- theory for studying cracking with a full understanding of the cracking process. The theory is applied in this research for brittle isotropic materials under static loads. It can be extended to all fracture fields with the same basics and the proper modifications for each fracture factor, case of loading and material properties.

This theory by using its fracture mechanism and hypotheses can investigate each of cracking load and crack propagation direction. It can predict also the fracture toughness, crack growth path and the crack history. It can be used for estimating of the safe life time of any structure. It is developed for mixed mode cracks in isotropic brittle materials and can be applied for group of cracks under static, cyclic and dynamic loading. Also, it can be applied for cracks under thermal stresses.

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