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# REMARKS ON ONE-DIMENSIONAL SEMINORMAL RINGS

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Various characterizations of reduced seminormal rings of dimension one are given in Salmon [4], Bombieri [1] and Davis [2]. Among others it is shown that if $(A, \mathfrak{m})$ is a local ring of a closed point on an algebraic curve defined over an algebraically closed field $k$, then $A$ is seminormal if and only if the completion $\hat{A}$ is $k$-isomorphic to $k[[X_1, \cdots, X_n]]/(\cdots, X_iX_j, \cdots)$ where $i \neq j$ ([1]) or the associated graded ring $Gr^{\cdot}(A)$ is $k$-isomorphic to $k[X_1, \cdots, X_n]/(\cdots, X_iX_j, \cdots)$ where $i \neq j$ ([2]). Generalizing these results we prove the following in the first section. Under certain moderate assumptions on $A$ there exist an integer $n$ and an ideal $I$ in $k[X_1, \cdots, X_n]$ such that $A$ is seminormal if and only if $\hat{A} \simeq k[[X_1, \cdots, X_n]]/Ik[[X_1, \cdots, X_n]]$ or $Gr^{\cdot}(A) \simeq k[X_1, \cdots, X_n]/I$. Moreover the ideal $I$ is generated by quadratic forms and these forms and integer $n$ are determined solely by the $k$-algebra structure of $\bar{A}/J(\bar{A})$, where $\bar{A}$ is the integral closure of $A$ in the total quotient ring of $A$ and $J(\bar{A})$ is the Jacobson radical of $\bar{A}$. Let $C$ be a plane algebraic curve and let $P$ be a closed point on $C$. Then it is known that the local ring $O_{P,C}$ is seminormal if and only if $P$ is a simple point or a node (cf. [1], [2], [4]). It is then natural to ask what the seminormalization of $O_{P,C}$ is when $P$ is not a seminormal point. The answer to this question is given in the second section in the case where $P$ is an ordinary multiple point.

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## 0. Notations and conventions

The following notations and conventions are fixed throughout this article. When $R$ is a ring, $J(R)$ stands for the Jacobson radical of $R$, $Q(R)$ for the total quotient ring of $R$, $\bar{R}$ for the integral closure of $R$ in $Q(R)$ and ${}^+R$ for the seminormalization of $R$. We denote by $\simeq_R$ an $R$-algebra isomorphism. An $R$-algebra is always assumed to be commutative, associative and containing 1. The symbols $X, Y, Z, T, X_i$, etc. are used to denote indeterminates or variables. When we say that $(R, \mathfrak{M})$ is a quasi-local ring, we mean that $R$ is a ring which has the unique maximal ideal $\mathfrak{M}$. A noetherian quasi-local ring is called a local ring.

## 1. Characterizations of one-dimensional seminormal rings

**Lemma 1.1.** *Let $(R, \mathfrak{M})$ be a reduced one-dimensional quasi-local ring. Then $R$ is seminormal if and only if $\mathfrak{M}=J(\bar{R})$.*

Proof. By definition, we have ${}^{+}R=R+J(\bar{R})$. Therefore, $R={}^{+}R$ if and only if $\mathfrak{M}=J(\bar{R})$. Q.E.D.

**Lemma 1.2.** *Let $k$ be a field and let $L$ be a reduced noetherian $k$-algebra of dimension 0. Then, both $k+TL[T]$ and $k+TL[[T]]$ are seminormal, where $k+TL[T]$ is identified with a subring of $L[T]$ and $k+TL[[T]]$ is identified with a quasi-local subring of $L[[T]]$ with residue field $k$.*

Proof. Set $R=k+TL[T]$ and $S=k'+TL[T]$, where $k'$ is the integral closure of $k$ in $L$. First we prove that $\bar{R}=S$. By assumption $L$ is a direct product of fields, hence $L[T]$ is normal. Thus we have $\bar{R}\subset L[T]$. Let $f(T)$ be an element in $L[T]$. Then, it is easy to see that $f(T)\in\bar{R}$ if and only if $f(0)\in k'$, which shows $\bar{R}=S$. Let $\mathfrak{P}$ be the prime ideal $TL[T]$ of $R$. Then, as is readly seen, we have $\mathfrak{P}R_{\mathfrak{P}}=J(S_{\mathfrak{P}})$, and hence $R_{\mathfrak{P}}$ is seminormal by Lemma 1.1. Let $g(T)$ be an arbitrary element in ${}^{+}R$. Then we have $h(T)g(T)\in R$ for some element $h(T)$ in $R$ but not in $\mathfrak{P}$, because $({}^{+}R)_{\mathfrak{P}}\subset{}^{+}(R_{\mathfrak{P}})=R_{\mathfrak{P}}$. Then we have $h(0)g(0)\in k$, which implies that $g(0)\in k$ because $h(0)\neq 0$. This shows that $g(T)\in R$, and $R$ is seminormal. We can verify that $k+TL[[T]]$ is seminormal by the similar way. Q.E.D.

1.3. In the rest of this section we fix the following notations: Let $(A, \mathfrak{m})$ be a reduced one-dimensional local ring containing the field $k$ isomorphic to $A/\mathfrak{m}$. We assume that $\bar{A}$ is a finite $A$-module. Then $\bar{A}$ is a semi-local ring. Let $J=J(\bar{A})$ and let $K=\bar{A}/J$. When $M$ is a finite $A$-module we denote by $\hat{M}$ the $\mathfrak{m}$-adic completion of $M$.

Then the following lemma is proved in Davis [2].

**Lemma.** *The following conditions are equivalent to each other.*

(1) *$A$ is seminormal.*

(2) *$Gr^{\cdot}(A)$ is $k$-isomorphic to $k+TK[T]$.*

(3) *$Gr^{\cdot}(A)$ is reduced and seminormal.*

1.4. Since the $\mathfrak{m}$-adic completion of $\bar{A}$ coincides with the $J$-adic completion of $\bar{A}$, we have $\hat{\bar{A}}\cong_k K[[T]]$ and $\hat{J}=J(\hat{\bar{A}})\cong_k TK[[T]]$. Notice that $\bar{\hat{A}}=\hat{\bar{A}}$, because $\hat{\bar{A}}$ is a finite $\hat{A}$-module with $\hat{\bar{A}}\subset Q(\hat{A})$ and $\hat{\bar{A}}\cong_k K[[T]]$ is normal.

**Lemma 1.5.** *The following conditions are equivalent to each other.*

(1) *$A$ is seminormal.*

(2) *$\hat{A}$ is $k$-isomorphic to $k+TK[[T]]$.*

(3) *$\hat{A}$ is seminormal.*

Proof. (1)$\Rightarrow$(2): $\hat{A}$ is a local ring with maximal ideal $\hat{\mathfrak{m}}$ and $k\subset\hat{A}$. Hence we have $\hat{A}\cong_k k+\hat{\mathfrak{m}}=k+\hat{J}\cong_k k+TK[[T]]$, because seminormality of $A$ implies that $\mathfrak{m}=J$.
(2)$\Rightarrow$(3): By Lemma 1.2, if $\hat{A}\cong_k k+TK[[T]]$ then $\hat{A}$ is seminormal.
(3)$\Rightarrow$(1): Since $\hat{A}$ is seminormal we have $\hat{\mathfrak{m}}=\hat{J}$, where $\hat{\mathfrak{m}}=\mathfrak{m}\otimes_A\hat{A}$ and $\hat{J}=J\otimes_A\hat{A}$. Therefore we have $\mathfrak{m}=J$ because $\hat{A}$ is faithfully flat over $A$. By Lemma 1.1 we see that $A$ is seminormal. Q.E.D.

**Lemma 1.6.** *Let $L$ be a $k$-algebra and let $v_1, \cdots, v_n$ be a $k$-basis of $L$. Let $\rho_{ijk}$ $(1\leqq i, j, k\leqq n)$ be the structure constants of $L$, i.e., $\rho_{ijk}$'s are elements of $k$ such that*

$$v_iv_j=\sum_{k=1}^n \rho_{ijk}v_k\,.$$

*Let $\sigma$: $k[X_1, \cdots, X_n]\to L[T]$ (or $k[[X_1, \cdots, X_n]]\to L[[T]]$) be a $k$-algebra homomorphism defined by $\sigma(X_i)=v_iT$ $(i=1, \cdots, n)$. Then the kernel $I$ of $\sigma$ is generated by the quadratic polynomials*

$$\psi_{ijk}=\Big(\sum_{m=1}^n \rho_{ikm}X_m\Big)X_j-\Big(\sum_{m=1}^n \rho_{jkm}X_m\Big)X_i \qquad (1\leqq i, j, k\leqq n)\,.$$

Proof. First notice that $\rho_{ijk}$'s satisfy the relations

$$(1)\qquad \rho_{ijk}=\rho_{jik}\,,$$

$$(2)\qquad \sum_{m=1}^n \rho_{ikm}\rho_{mjs}=\sum_{m=1}^n \rho_{jkm}\rho_{mis}\,.$$

Let

$$1=\sum_{m=1}^n c_mv_m$$

where $c_m\in k$ $(m=1, \cdots, n)$. From $\big(\sum_{m=1}^n c_mv_m\big)v_i=v_i$ it follows that

$$(3)\qquad \sum_{m=1}^n c_m\rho_{mis}=\delta_{is}\,.$$

Let $I'$ be the ideal generated by $\{\psi_{ijk}\}$. As is readily seen $I$ is a homogeneous ideal, and $I'\subset I$ from (2). We set $Y=\sum_{m=1}^n c_mX_m$. Then, using (3) we get

$$\sum_{i=1}^n c_i\psi_{ijk}=X_jX_k-Y\sum_{s=1}^n \rho_{jks}X_s$$

i.e.,

$$(4)\qquad X_jX_k\equiv Y\sum_{s=1}^n \rho_{jks}X_s \qquad (\mathrm{mod}\ I')\,.$$

Let $F(X_1, \cdots, X_n)$ be a homogeneous polynomial of degree $N$ in $I$. Clearly $N>1$, and from (4) we easily see that

$$F(X_1, \cdots, X_n) \equiv Y^{N-1}\sum_{s=1}^{n} a_s X_s \qquad (\mathrm{mod}\ I')$$

for some $a_s \in k$. Since $\sigma(F)=0$, $\sigma(Y)=1$ and $\sigma(X_s)=v_s T$ we have $a_s=0$ for $s=1, \cdots, n$ because $v_1, \cdots, v_n$ are linearly independent over $k$. This proves that $F \in I'$. Q.E.D.

For practical purpose the generators $\{\psi_{ijk}\}$ of $I$ are not easy to handle. For later use we prove the following lemma.

**Lemma 1.7.** *Let $L=k[X]/(f(X))$, where $f(X)=X^n+a_{n-1}X^{n-1}+\cdots+a_0$ and let $\alpha$=the residue class of $X$ in $L$. Define a $k$-algebra homomorphism $\sigma$: $k[X_0, \cdots, X_{n-1}] \to L[T]$ (or $k[[X_0, \cdots, X_{n-1}]] \to L[[T]]$) by $\sigma(X_i)=\alpha^i T$. We set $\xi_m = X_{[m/2]}X_{[(m+1)/2]}$ for $m=0, \cdots, 2n-2$, where [ ] is the Gauss symbol. Let $I_f$ be the ideal generated by quadratic polynomilas*

$$g_{ij} = X_i X_j - \xi_{i+j} \qquad (0 \leqq i,\ j \leqq n-1)\,,$$
$$h_s = \sum_{m=0}^{n} a_m \xi_{m+s} \qquad (0 \leqq s \leqq n-2),\ \textit{where}\ a_n = 1\,.$$

*Then the kernel of $\sigma$ is equal to $I_f$.*

Proof. Let $\alpha^i\alpha^j=\sum_{k=0}^{n-1} \rho_{ijk}\alpha^k$ and set $\psi_{ijk}=(\sum_{m=0}^{n-1} \rho_{ikm}X_m)X_j-(\sum_{m=0}^{n-1} \rho_{jkm}X_m)X_i$. Then, by virtue of Lemma 1.6, Ker $\sigma$ is generated by $\psi_{ijk}$'s. It is easy to check that $I_f$ is contained in Ker $\sigma$. We shall prove the inverse containment $I_f \supset$ Ker $\sigma$. Notice that $\rho_{ijk}=\rho_{stk}$ if $i+j=s+t$. We set $\rho_{i+j,k}=\rho_{ijk}$. Then we have

(1) $\quad \rho_{N,m} = \delta_{Nm}$ if $N<n$ and $\rho_{n,m} = -a_m$ for $0 \leqq m \leqq n-1$,

and

(2) $\quad \rho_{N,0} = -a_0\rho_{N-1,n-1}$ and $\rho_{N,k} = \rho_{N-1,k-1} - a_k\rho_{N-1,n-1}$ for $1 \leqq k \leqq n-1$.

From (2), if $s<n-1$ and $N \geqq 1$ we easily get the relation

$$\sum_{m=0}^{n-1} \rho_{N,m}\xi_{m+s} \equiv \sum_{m=0}^{n-1} \rho_{N-1,m}\xi_{m+s+1} \qquad (\mathrm{mod}\ I_f)\,. \tag{3}$$

To prove $I_f \supset$ Ker $\sigma$ it suffices to prove $\psi_{ijk} \in I_f$. Notice that

$$\psi_{ijk} \equiv \sum_{m=0}^{n-1} \rho_{i+k,m}\xi_{m+j} - \sum_{m=0}^{n-1} \rho_{j+k,m}\xi_{m+i} \qquad (\mathrm{mod}\ I_f) \tag{4}$$

because $X_m X_j \equiv \xi_{m+j}$ and $X_m X_i \equiv \xi_{m+i}$ (mod $I_f$). We may assume $i \geqq j$ because $\psi_{ijk} = -\psi_{jik}$. If $i+k<n$, from (1) and (4) we have

$$\psi_{ijk} \equiv \xi_{i+j+k} - \xi_{i+j+k} = 0 \qquad (\mathrm{mod}\ I_f)\,.$$

If $i+k \geqq n$ and $j+k<n$, from (1), (3) and (4) we have

$$\psi_{ijk} \equiv \sum_{m=0}^{n-1} \rho_{n,m}\xi_{m+i+j+k-n}-\xi_{i+j+k} \qquad (\mathrm{mod}\ I_f)$$
$$= -\sum_{m=0}^{n-1} a_m \xi_{m+i+j+k-n}-\xi_{i+j+k} = -h_{i+j+k-n}\,.$$

If $j+k\geqq n$, from (3) and (4) we have

$$\psi_{ijk} \equiv \sum_{m=0}^{n-1} \rho_{i+j+k+1-n,m}\xi_{m+n-1}-\sum_{m=0}^{n-1} \rho_{i+j+k+1-n,m}\xi_{m+n-1} \qquad (\mathrm{mod}\ I_f)$$
$$=0\,.$$

Thus we have $\psi_{ijk}\equiv 0$ (mod $I_f$) and we proved the assertion. Q.E.D.

By virtue of Lemmas 1.3, 1.5 and 1.6 we have the following

**Theorem 1.8.** *Let $v_1, \cdots, v_n$ be a k-basis of a k-algebra $K=\bar{A}/J(\bar{A})$ and let $\rho_{ijk}$ be the structure constants of k-algebra K. Set*

$$\psi_{ijk} = (\sum_{m=1}^{n} \rho_{ikm}X_m)X_j-(\sum_{m=1}^{n} \rho_{jkm}X_m)X_i \qquad (1\leqq i,\, j,\, k\leqq n)\,.$$

*Then the following conditions are equivalent to each other.*
(1) *$A$ is seminormal.*
(2) *$\hat{A}$ is k-isomorphic to $k[[X_1, \cdots, X_n]]/(\cdots, \psi_{ijk}, \cdots)$.*
(3) *$Gr^{\cdot}(A)$ is k-isomorphic to $k[X_1, \cdots, X_n]/(\cdots, \psi_{ijk}, \cdots)$.*

**Lemma 1.9.** *We say that a polynomial $f(X)$ in $k[X]$ is reduced in $k[X]$ if $f(X)$ has no multiple factors in $k[X]$, i.e., if the residue ring $k[X]/(f(X))$ is reduced. Let the ideal $I_f$ have the same meaning as in* 1.7. *If $\hat{A}$ is k-isomorphic to $k[[X_0, \cdots, X_{n-1}]]/I_f$ or $Gr^{\cdot}(A)$ is k-isomorphic to $k[X_0, \cdots, X_{n-1}]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$ then $A$ is seminormal.*

Proof. This lemma follows from Lemmas 1.2 and 1.7. Q.E.D.

**Theorem 1.10.** *Let the ideal $I_f$ have the same meaning as in* 1.7. *Assume that k is a perfect infinite field. Then the following conditions are equivalent to each other.*
(1) *$A$ is seminormal.*
(2) *$\hat{A}$ is k-isomorphic to $k[[X_0, \cdots, X_{n-1}]]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$.*
(3) *$Gr^{\cdot}(A)$ is k-isomorphic to $k[X_0, \cdots, X_{n-1}]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$.*

Proof. If we see that $K$ is $k$-isomorphic to $k[X]/(f(X))$ for some reduced monic polynomial $f(X)$ in $k[X]$ then the assertion follows from Theorem 1.8, Lemma 1.7 and Lemma 1.9. Let $\mathfrak{M}_1, \cdots, \mathfrak{M}_r$ be all the maximal ideals of $\bar{A}$ and set $K_i=\bar{A}/\mathfrak{M}_i$ for $i=1, \cdots, r$. Then we have $K\cong_k K_1\times\cdots\times K_r$. Since $k$ is a perfect field we have $K_i \cong_k k[X]/(f_i(X))$ for some irreducible monic polynomial $f_i(X)$ in $k[X]$. We may assume that $f_1(X), \cdots, f_r(X)$ are relatively prime because $k$ is an infinite field. Set $f(X)=f_1(X)\cdots f_r(X)$. Then $f(X)$ is a rdeuced monic polynomial and we have $K \cong_k k[X]/(f(X))$. Q.E.D.

REMARK 1.11. In Theorem 1.10 we assumed that the local ring $A$ is reduced. We may replace this assumption by the condition that depth $A \geqq 1$. In fact the following lemma holds in general.

**Lemma.** *Let $(R, \mathfrak{M})$ be a seminormal local ring with depth $R \geqq 1$. Then $R$ is reduced.*

Proof. Let $\mathfrak{n}=\mathrm{nil}(R)$, where $\mathrm{nil}(R)$ denotes the nilradical of $R$, and let $S$ be the set of regular elements of $R$. Then we have $\mathrm{nil}(\bar{R})=S^{-1}\mathfrak{n}$. From definition, it is easy to check that $R+\mathrm{nil}(\bar{R}) \subset {}^+R$, hence we have $S^{-1}\mathfrak{n} \subset R$. Let $x$ be an arbitrary element in $\mathfrak{n}$. Notice that $S \cap \mathfrak{M} \neq \phi$ because depth $R \geqq 1$. Let $a$ be an element in $S \cap \mathfrak{M}$. Then we have $x/a^n \in R$ and $x \in a^n R \subset \mathfrak{M}^n$ for any integer $n$. Thus we have $x \in \bigcap_{n=0}^{\infty} \mathfrak{M}^n$, and hence we have $x=0$ by Krull's intersection theorem. Q.E.D.

## 2. Seminormalization of local rings of plane algebraic curves

2.1. Let $V$ be an algebraic variety defined over a field $k$ and let $P$ be a closed point on $V$. Then $P$ is said to be seminormal if the local ring $O_{P,V}$ is seminormal. Let $V$ be a reduced plane algebraic curve. Then it is known that $P$ is a seminormal point if and only if $P$ is a smooth point or $P$ is a node (cf. [1], [2], [4]). Consequently, if $P$ is a singular point and $P$ is not a node, $O_{P,V}$ is not seminormal. We are interested in how the seminormalization of $O_{P,V}$ can be obtained. For this purpose, we may assume that $V$ is a plane curve defined by a polynomial $F(X, Y)=\sum_{i+j \geqq n} a_{ij}X^iY^j$ in $k[X, Y]$ with $a_{0n}=1$ and $P$ is the origin $(0, 0)$. In this section we shall determine the seminormalization of $O_{P,V}=(k[X, Y]/(F(X, Y)))_{(X,Y)}$ when $f_F(X):=\sum_{i+j=n} a_{ij}X^j$ is a reduced monic polynomial in $k[X]$.

**Lemma 2.2.** *Let $I_0$ be the ideal of $k[X_0, \cdots, X_{n-1}]$ $(n \geqq 2)$ generated by $g_{ij}=X_iX_j-\xi_{i+j}$ $(0 \leqq i<j \leqq n-1)$, where $\xi_m$ is the same as in 1.7. Then we have $X_{i_1}\cdots X_{i_s} \equiv X_{j_1}\cdots X_{j_s}$ (mod $I_0$) if $i_1+\cdots+i_s=j_1+\cdots+j_s$. In paticular we have $X_0^{t-1}X_t \equiv X_1^t$ (mod $I_0$).*

Proof. This is easily seen by simple calculations and we omit the proof.

2.3. Let $F(X, Y)=\sum_{i+j \geqq n} a_{ij}X^iY^j$ be an element in $k[X, Y]$ with $n \geqq 2$ and $a_{0n}=1$. For each ordered pair $(i, j)$ with $i+j \geqq n$, we choose a fixed integer $t=t(i, j)$ satisfying

$$\mathrm{Max}(0, n-i) \leqq t \leqq \mathrm{Min}(n, j)$$

and set

$$\phi_s = \sum_{i+j \geqq n} a_{ij} X_0^{i+t-n} X_1^{j-t} \xi_{t+s} \qquad (0 \leqq s \leqq n-2) .$$

Let $I$ be the ideal of $k[X_0, \cdots, X_{n-1}]$ generated by $I_0$ and $\phi_s$'s. It should be noticed that $I$ is independent of the choice of $t$ by Lemma 2.2, and $t(n-m,m)=m$ for $m=0, \cdots, n$.

**Lemma 2.4.** *Let $F(X, Y)$, $I_0$ and $I$ be the same as in* 2.2 *and* 2.3. *Then the ring homomorphism $\tau$: $(k[X, Y]/(F(X, Y)))_{(X,Y)} \to (k[X_0, \cdots, X_{n-1}]/I)_{(X_0, \cdots, X_{n-1})}$ defined by $\tau(X)=X_0$ and $\tau(Y)=X_1$ is injective, birational and integral.*

Proof. The proof is divided into three steps.

Step 1. *The homomorphism $\tau$: $k[X, Y]/(F(X, Y)) \to k[X_0, \cdots, X_{n-1}]/I$ defined by $\tau(X)=X_0$ and $\tau(Y)=X_1$ is well-defined and injective.*

In fact by Lemma 2.2 we have

$$(1) \qquad \begin{cases} X_0^{n-1}\phi_s \equiv \sum\limits_{i+j \geqq n} a_{ij} X_0^i X_1^j X_s = X_s F(X_0, X_1) & (\mathrm{mod}\ I_0) \\ X_0^{n-2}\phi_0 \equiv \sum\limits_{i+j \geqq n} a_{ij} X_0^i X_1^j = F(X_0, X_1) & (\mathrm{mod}\ I_0) \end{cases}$$

because $X_0^{i+t-1}\xi_{t+s} \equiv X_0^i X_1^t X_s$ and $X_0^{i+t-2}\xi_t \equiv X_0^i X_1^t$ (mod $I_0$) (notice that $i+t \geqq n \geqq 2$). In paticular $F(X_0, X_1) \in I$ and the homomorphism $\tau$ is well-defined. To prove that $\tau$ is injective let

$$G(X_0, X_1) = \sum_{0 \leqq i < j \leqq n-1} h_{ij} g_{ij} + \sum_{s=0}^{n-2} \lambda_s \phi_s$$

be an element of $I \cap k[X_0, X_1]$ where $h_{ij}$'s and $\lambda_s$'s are elements of $k[X_0, \cdots, X_{n-1}]$. Then from (1) we get

$$X_0^{n-1} G(X_0, X_1) \equiv X_0^{n-1} \sum_{0 \leqq i < j \leqq n-1} h_{ij} g_{ij} + F(X_0, X_1) \sum_{s=0}^{n-2} X_s \lambda_s \qquad (\mathrm{mod}\ I_0) .$$

Therefore we can write

$$X_0^{n-1} G(X_0, X_1) = \sum l_{ij} g_{ij} + \lambda(X_0, \cdots, X_{n-1}) F(X_0, X_1)$$

for some elements $l_{ij}$ in $k[X_0, \cdots, X_{n-1}]$, where $\lambda(X_0, \cdots, X_{n-1}) = \sum_{s=0}^{n-2} X_s \lambda_s$. As is readily seen, there exist an integer $N$ and a polynomial $\tilde{\lambda}(X, Y)$ in two variables such that

$$\tilde{\lambda}(X, ZX) = X^N \lambda(X, ZX, \cdots, Z^{n-1}X) .$$

Since $g_{ij}(X, ZX, \cdots, Z^{n-1}X)=0$, we get, by specializing $X_i \mapsto Z^i X$, the relation

$$X^{N+n-1}G(X, ZX) = \tilde{\lambda}(X, ZX)F(X, ZX).$$

From this we get the identity

$$X_0^{N+n-1}G(X_0, X_1) = \tilde{\lambda}(X_0, X_1)F(X_0, X_1)$$

because $X$ and $ZX$ are independent variables over $k$. From our assumption $F(X_0, X_1)$ is not divisible by $X_0$. Hence $\tilde{\lambda}(X_0, X_1)$ is divisible by $X_0^{N+n-1}$, and we have $G(X_0, X_1) \in F(X_0, X_1)k[X_0, X_1]$.

Step 2. *Let $\mathfrak{Q}$ be a proper prime ideal of $k[X_0, \cdots, X_{n-1}]$ containing $I$ and $X_0$. Then $\mathfrak{Q}$ is necessarily equal to the maximal ideal $(X_0, \cdots, X_{n-1})$.*

If $X_i \in \mathfrak{Q}$ for some $i \leqq n-3$ we have $X_{i+1}^2 \equiv X_i X_{i+2} \equiv 0 \pmod{\mathfrak{Q}}$ whence $X_{i+1} \in \mathfrak{Q}$. Thus we have $X_0, \cdots, X_{n-2} \in \mathfrak{Q}$. On the other hand, we have $\phi_{n-2} \in I \subset \mathfrak{Q}$ and $\phi_{n-2} = X_{n-1}^2 + \eta$ with $\eta \in \mathfrak{Q}$ because $t(0, n) = n$, $a_{0n} = 1$ and $\xi_s \in \mathfrak{Q}$ for $s < 2n-2$. Thus we have $X_{n-1}^2 \in \mathfrak{Q}$, whence $X_{n-1} \in \mathfrak{Q}$. Obviously, $(X_0, \cdots, X_{n-1})$ is a maximal ideal of $k[X_0, \cdots, X_{n-1}]$, and thence we have $\mathfrak{Q} = (X_0, \cdots, X_{n-1})$.

Step 3. *The ring homomorphism $\tau$: $(k[X, Y]/(F(X, Y)))_{(X,Y)} \to (k[X_0, \cdots, X_{n-1}]/I)_{(X_0,\cdots,X_{n-1})}$ induced naturally by $\tau$ is birational and integral.*

Set $B = k[X_0, X_1]/(F(X_0, X_1))$, $C = k[X_0, \cdots, X_{n-1}]/I$, $\mathfrak{p} = (X_0, X_1)B$ and $\mathfrak{M} = (X_0, \cdots, X_{n-1})C$. We shall denote by $x_i$ the residue class of $X_i$ modulo $I$. Notice that $x_i x_0^{i-1} = x_1^i$ for $1 \leqq i \leqq n-1$ and $x_0$ is a regular element of $B$. This implies that $C_\mathfrak{p} \subset Q(B_\mathfrak{p})$. Next we show that $C_\mathfrak{p}$ is integral over $B_\mathfrak{p}$. In fact, we have

$$F(x_0, x_1) = \sum_{i+j \geqq n} a_{ij} x_0^i x_1^j = 0$$

in $B$. From this it is easy to check that $x_1/x_0$ is integral over $B_\mathfrak{p}$. Hence $x_i = x_1(x_1/x_0)^{i-1}$ is also integral over $B_\mathfrak{p}$ for $1 \leqq i \leqq n-1$, which shows that $C_\mathfrak{p}$ is integral over $B_\mathfrak{p}$. It remains to prove that $C_\mathfrak{p} = C_\mathfrak{M}$. Let $\mathfrak{Q}C_\mathfrak{p}$ be a maximal ideal of $C_\mathfrak{p}$, where $\mathfrak{Q}$ is a maximal ideal of $C$. Then we have $\mathfrak{Q}C_\mathfrak{p} \cap B_\mathfrak{p} = \mathfrak{p}B_\mathfrak{p}$ because $C_\mathfrak{p}$ is integral over $B_\mathfrak{p}$, whence $\mathfrak{Q} \cap B = \mathfrak{p}$. Thus we have $\mathfrak{Q} = \mathfrak{M}$ by the step 2, from which we see that $C_\mathfrak{p}$ is a local ring. Therefore, we have $C_\mathfrak{p} = C_\mathfrak{M}$ and the assertion is verified.

**Lemma 2.5.** *Assume that $f_F(X) = \sum_{i+j=n} a_{ij} X^j$ is a reduced monic polynomial in $k[X]$. Then the local ring $(k[X_0, \cdots, X_{n-1}]/I)_{(X_0,\cdots,X_{n-1})}$ is seminormal.*

Proof. Set $R = (k[X_0, \cdots, X_{n-1}]/I)_{(X_0,\cdots,X_{n-1})}$. Notice that the leading form of $\phi_s$ is $\sum_{i+j=n} a_{ij} \xi_{j+s}$. Hence, if we set $f = f_F(X)$, we have $Gr^{\cdot}(R) \cong_k k[X_0, \cdots, X_{n-1}]/I_f$, where $I_f$ is the ideal defined in 1.7 (cf. [3; p. 118]). Therefore, by virtue of Lemma 1.9, we see that $R$ is seminormal. Q.E.D.

Summarizing the results stated in Lemmas 2.4 and 2.5, we have the following theorem.

**Theorem 2.6.** *Let $F(X, Y)=\sum_{i+j\geqq n} a_{ij}X^iY^j$ be an element in $k[X, Y]$ with $n\geqq 2$ and $a_{0n}=1$. Assume that $f_F(X)=\sum_{i+j=n} a_{ij}X^j$ is a reduced monic polynomial in $k[X]$. Then the seminormalization of the local ring $(k[X, Y]/(F(X, Y)))_{(X,Y)}$ is isomorphic to $(k[X_0, \cdots, X_{n-1}]/I)_{(X_0,\cdots,X_{n-1})}$ where the ideal $I$ is generated by $g_{ij}$ $(0\leqq i<j\leqq n-1)$ and $\phi_s$ $(0\leqq s\leqq n-2)$ given in 2.2 and 2.3, respectively.*

REMARK 2.7. When $f_F(X)$ is a reduced monic polynomial in $k[X]$ the seminormalization of $O_{P,V}$ is determined by the leading form of the defining equation even if $f_F(X)$ is not reduced in $\bar{k}[X]$, where $\bar{k}$ denotes the algebraic closure of $k$. But when $f_F(X)$ is not reduced in $k[X]$ the seminormalization of $O_{P,V}$ is more complicated as is shown by the following examples.
(1) Let $R=(k[X, Y]/(Y^3-XY^2-X^4))_{(X,Y)}$. Then $^+R$ is $(k[X_0, X_1]/(X_1^2-X_0X_1-X_0^3)_{(X_0,X_1)}$, and the embedding $R\rightarrow {}^+R$ is given by $X\mapsto X_1$ and $Y\mapsto X_0^2+X_1$.
(2) Let $S=(k[X, Y]/(Y^3-XY^2-X^5))_{(X,Y)}$. Then $^+S$ is $(k[X_0, X_1, X_2]/(X_1^2-X_0X_2, X_1X_2+X_0X_1-X_0^3, X_2^2+X_1^2-X_0^2X_1))_{(X_0,X_1,X_2)}$ and the embedding $S\rightarrow {}^+S$ is given by $X\mapsto X_0$ and $Y\mapsto X_2+X_0$.

REMARK 2.8. Let $F(X, Y)$ and $I$ have the same meaning as in 2.6. Assume that the origin $P$ is a unique singular point and $k[X_0, \cdots, X_{n-1}]/I$ is integral over $k[X, Y]/(F(X, Y))$. Then the seminormalization of $k[X, Y]/(F(X, Y))$ is $k[X_0, \cdots, X_{n-1}]/I$.

EXAMPLE. Let $R=k[X, Y]/(Y^3-aX^3-X^6)$. Assume that $\mathrm{char}(k)\neq 3$ or $a^{1/3}\notin k$. Then $^+R=k[X, Y, Z]/(Y^2-XZ, YZ-aX^2-X^5, Z^2-aXY-X^4Y)$.

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