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REMARKS ON ONE-DIMENSIONAL SEMINORMAL RINGS

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Various characterizations of reduced seminormal rings of dimension one are given in Salmon [4], Bombieri [1] and Davis [2]. Among others it is shown that if (A, \mathfrak{m}) is a local ring of a closed point on an algebraic curve defined over an algebraically closed field k , then A is seminormal if and only if the completion \hat{A} is k -isomorphic to $k[[X_1, \dots, X_n]]/(\dots, X_i X_j, \dots)$ where $i \neq j$ ([1]) or the associated graded ring $Gr^*(A)$ is k -isomorphic to $k[X_1, \dots, X_n]/(\dots, X_i X_j, \dots)$ where $i \neq j$ ([2]). Generalizing these results we prove the following in the first section. Under certain moderate assumptions on A there exist an integer n and an ideal I in $k[X_1, \dots, X_n]$ such that A is seminormal if and only if $\hat{A} \cong k[[X_1, \dots, X_n]]/Ik[[X_1, \dots, X_n]]$ or $Gr^*(A) \cong k[X_1, \dots, X_n]/I$. Moreover the ideal I is generated by quadratic forms and these forms and integer n are determined solely by the k -algebra structure of $\bar{A}/J(\bar{A})$, where \bar{A} is the integral closure of A in the total quotient ring of A and $J(\bar{A})$ is the Jacobson radical of \bar{A} . Let C be a plane algebraic curve and let P be a closed point on C . Then it is known that the local ring $O_{P,C}$ is seminormal if and only if P is a simple point or a node (cf. [1], [2], [4]). It is then natural to ask what the seminormalization of $O_{P,C}$ is when P is not a seminormal point. The answer to this question is given in the second section in the case where P is an ordinary multiple point.

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0. Notations and conventions

The following notations and conventions are fixed throughout this article. When R is a ring, $J(R)$ stands for the Jacobson radical of R , $Q(R)$ for the total quotient ring of R , \bar{R} for the integral closure of R in $Q(R)$ and ${}^+R$ for the seminormalization of R . We denote by \cong_R an R -algebra isomorphism. An R -algebra is always assumed to be commutative, associative and containing 1. The symbols X, Y, Z, T, X_i , etc. are used to denote indeterminates or variables. When we say that (R, \mathfrak{M}) is a quasi-local ring, we mean that R is a ring which has the unique maximal ideal \mathfrak{M} . A noetherian quasi-local ring is called a local ring.

1. Characterizations of one-dimensional seminormal rings

Lemma 1.1. *Let (R, \mathfrak{M}) be a reduced one-dimensional quasi-local ring. Then R is seminormal if and only if $\mathfrak{M} = J(\bar{R})$.*

Proof. By definition, we have ${}^+R = R + J(\bar{R})$. Therefore, $R = {}^+R$ if and only if $\mathfrak{M} = J(\bar{R})$. Q.E.D.

Lemma 1.2. *Let k be a field and let L be a reduced noetherian k -algebra of dimension 0. Then, both $k + TL[T]$ and $k + TL[[T]]$ are seminormal, where $k + TL[T]$ is identified with a subring of $L[T]$ and $k + TL[[T]]$ is identified with a quasi-local subring of $L[[T]]$ with residue field k .*

Proof. Set $R = k + TL[T]$ and $S = k' + TL[T]$, where k' is the integral closure of k in L . First we prove that $\bar{R} = S$. By assumption L is a direct product of fields, hence $L[T]$ is normal. Thus we have $\bar{R} \subset L[T]$. Let $f(T)$ be an element in $L[T]$. Then, it is easy to see that $f(T) \in \bar{R}$ if and only if $f(0) \in k'$, which shows $\bar{R} = S$. Let \mathfrak{P} be the prime ideal $TL[T]$ of R . Then, as is readily seen, we have $\mathfrak{P}R_{\mathfrak{P}} = J(S_{\mathfrak{P}})$, and hence $R_{\mathfrak{P}}$ is seminormal by Lemma 1.1. Let $g(T)$ be an arbitrary element in ${}^+R$. Then we have $h(T)g(T) \in R$ for some element $h(T)$ in R but not in \mathfrak{P} , because $({}^+R)_{\mathfrak{P}} \subset {}^+(R_{\mathfrak{P}}) = R_{\mathfrak{P}}$. Then we have $h(0)g(0) \in k$, which implies that $g(0) \in k$ because $h(0) \neq 0$. This shows that $g(T) \in R$, and R is seminormal. We can verify that $k + TL[[T]]$ is seminormal by the similar way. Q.E.D.

1.3. In the rest of this section we fix the following notations: Let (A, \mathfrak{m}) be a reduced one-dimensional local ring containing the field k isomorphic to A/\mathfrak{m} . We assume that \bar{A} is a finite A -module. Then \bar{A} is a semi-local ring. Let $J = J(\bar{A})$ and let $K = \bar{A}/J$. When M is a finite A -module we denote by \hat{M} the \mathfrak{m} -adic completion of M .

Then the following lemma is proved in Davis [2].

Lemma. *The following conditions are equivalent to each other.*

- (1) A is seminormal.
- (2) $Gr^*(A)$ is k -isomorphic to $k + TK[T]$.
- (3) $Gr^*(A)$ is reduced and seminormal.

1.4. Since the \mathfrak{m} -adic completion of \bar{A} coincides with the J -adic completion of \bar{A} , we have $\hat{\bar{A}} \cong_k K[[T]]$ and $\hat{J} = J(\hat{\bar{A}}) \cong_k TK[[T]]$. Notice that $\hat{\bar{A}} = \hat{\hat{A}}$, because $\hat{\bar{A}}$ is a finite \hat{A} -module with $\hat{\bar{A}} \subset Q(\hat{A})$ and $\hat{\bar{A}} \cong_k K[[T]]$ is normal.

Lemma 1.5. *The following conditions are equivalent to each other.*

- (1) A is seminormal.
- (2) \hat{A} is k -isomorphic to $k + TK[[T]]$.
- (3) \hat{A} is seminormal.

Proof. (1) \Rightarrow (2): \hat{A} is a local ring with maximal ideal $\hat{\mathfrak{m}}$ and $k \subset \hat{A}$. Hence we have $\hat{A} \cong_k k + \hat{\mathfrak{m}} = k + \hat{J} \cong_k k + TK[[T]]$, because seminormality of A implies that $\mathfrak{m} = J$.

(2) \Rightarrow (3): By Lemma 1.2, if $\hat{A} \cong_k k + TK[[T]]$ then \hat{A} is seminormal.

(3) \Rightarrow (1): Since \hat{A} is seminormal we have $\hat{\mathfrak{m}} = \hat{J}$, where $\hat{\mathfrak{m}} = \mathfrak{m} \otimes_A \hat{A}$ and $\hat{J} = J \otimes_A \hat{A}$. Therefore we have $\mathfrak{m} = J$ because \hat{A} is faithfully flat over A . By Lemma 1.1 we see that A is seminormal. Q.E.D.

Lemma 1.6. *Let L be a k -algebra and let v_1, \dots, v_n be a k -basis of L . Let ρ_{ijk} ($1 \leq i, j, k \leq n$) be the structure constants of L , i.e., ρ_{ijk} 's are elements of k such that*

$$v_i v_j = \sum_{k=1}^n \rho_{ijk} v_k.$$

Let $\sigma: k[X_1, \dots, X_n] \rightarrow L[[T]]$ (or $k[[X_1, \dots, X_n]] \rightarrow L[[T]]$) be a k -algebra homomorphism defined by $\sigma(X_i) = v_i T$ ($i=1, \dots, n$). Then the kernel I of σ is generated by the quadratic polynomials

$$\psi_{ijk} = \left(\sum_{m=1}^n \rho_{ikm} X_m \right) X_j - \left(\sum_{m=1}^n \rho_{jkm} X_m \right) X_i \quad (1 \leq i, j, k \leq n).$$

Proof. First notice that ρ_{ijk} 's satisfy the relations

$$\begin{aligned} (1) \quad & \rho_{ijk} = \rho_{jik}, \\ (2) \quad & \sum_{m=1}^n \rho_{ikm} \rho_{mjs} = \sum_{m=1}^n \rho_{jkm} \rho_{mis}. \end{aligned}$$

Let

$$1 = \sum_{m=1}^n c_m v_m$$

where $c_m \in k$ ($m=1, \dots, n$). From $(\sum_{m=1}^n c_m v_m) v_i = v_i$ it follows that

$$(3) \quad \sum_{m=1}^n c_m \rho_{mis} = \delta_{is}.$$

Let I' be the ideal generated by $\{\psi_{ijk}\}$. As is readily seen I is a homogeneous ideal, and $I' \subset I$ from (2). We set $Y = \sum_{m=1}^n c_m X_m$. Then, using (3) we get

$$\sum_{i=1}^n c_i \psi_{ijk} = X_j X_k - Y \sum_{s=1}^n \rho_{jks} X_s$$

i.e.,

$$(4) \quad X_j X_k \equiv Y \sum_{s=1}^n \rho_{jks} X_s \pmod{I'}.$$

Let $F(X_1, \dots, X_n)$ be a homogeneous polynomial of degree N in I . Clearly $N > 1$, and from (4) we easily see that

$$F(X_1, \dots, X_n) \equiv Y^{N-1} \sum_{s=1}^n a_s X_s \pmod{I'}$$

for some $a_s \in k$. Since $\sigma(F)=0$, $\sigma(Y)=1$ and $\sigma(X_s)=v_s T$ we have $a_s=0$ for $s=1, \dots, n$ because v_1, \dots, v_n are linearly independent over k . This proves that $F \in I'$. Q.E.D.

For practical purpose the generators $\{\psi_{ijk}\}$ of I are not easy to handle. For later use we prove the following lemma.

Lemma 1.7. *Let $L=k[X]/(f(X))$, where $f(X)=X^n+a_{n-1}X^{n-1}+\dots+a_0$ and let α =the residue class of X in L . Define a k -algebra homomorphism $\sigma: k[X_0, \dots, X_{n-1}] \rightarrow L[T]$ (or $k[[X_0, \dots, X_{n-1}]] \rightarrow L[[T]]$) by $\sigma(X_i)=\alpha^i T$. We set $\xi_m = X_{[m/2]} X_{[(m+1)/2]}$ for $m=0, \dots, 2n-2$, where $[]$ is the Gauss symbol. Let I_f be the ideal generated by quadratic polynomials*

$$\begin{aligned} g_{ij} &= X_i X_j - \xi_{i+j} \quad (0 \leq i, j \leq n-1), \\ h_s &= \sum_{m=0}^n a_m \xi_{m+s} \quad (0 \leq s \leq n-2), \text{ where } a_n = 1. \end{aligned}$$

Then the kernel of σ is equal to I_f .

Proof. Let $\alpha^i \alpha^j = \sum_{k=0}^{n-1} \rho_{ijk} \alpha^k$ and set $\psi_{ijk} = (\sum_{m=0}^{n-1} \rho_{ikm} X_m) X_j - (\sum_{m=0}^{n-1} \rho_{jkm} X_m) X_i$.

Then, by virtue of Lemma 1.6, $\text{Ker } \sigma$ is generated by ψ_{ijk} 's. It is easy to check that I_f is contained in $\text{Ker } \sigma$. We shall prove the inverse containment $I_f \supset \text{Ker } \sigma$. Notice that $\rho_{ijk} = \rho_{stk}$ if $i+j=s+t$. We set $\rho_{i+j,k} = \rho_{ijk}$. Then we have

$$(1) \quad \rho_{N,m} = \delta_{Nm} \text{ if } N < n \text{ and } \rho_{n,m} = -a_m \quad \text{for } 0 \leq m \leq n-1,$$

and

$$(2) \quad \rho_{N,0} = -a_0 \rho_{N-1,n-1} \text{ and } \rho_{N,k} = \rho_{N-1,k-1} - a_k \rho_{N-1,n-1} \quad \text{for } 1 \leq k \leq n-1.$$

From (2), if $s < n-1$ and $N \geq 1$ we easily get the relation

$$(3) \quad \sum_{m=0}^{n-1} \rho_{N,m} \xi_{m+s} \equiv \sum_{m=0}^{n-1} \rho_{N-1,m} \xi_{m+s+1} \pmod{I_f}.$$

To prove $I_f \supset \text{Ker } \sigma$ it suffices to prove $\psi_{ijk} \in I_f$. Notice that

$$(4) \quad \psi_{ijk} \equiv \sum_{m=0}^{n-1} \rho_{i+k,m} \xi_{m+j} - \sum_{m=0}^{n-1} \rho_{j+k,m} \xi_{m+i} \pmod{I_f}$$

because $X_m X_j \equiv \xi_{m+j}$ and $X_m X_i \equiv \xi_{m+i} \pmod{I_f}$. We may assume $i \geq j$ because $\psi_{ijk} = -\psi_{jik}$. If $i+k < n$, from (1) and (4) we have

$$\psi_{ijk} \equiv \xi_{i+j+k} - \xi_{i+j+k} = 0 \pmod{I_f}.$$

If $i+k \geq n$ and $j+k < n$, from (1), (3) and (4) we have

$$\begin{aligned}\psi_{ijk} &\equiv \sum_{m=0}^{n-1} \rho_{n,m} \xi_{m+i+j+k-n} - \xi_{i+j+k} \pmod{I_f} \\ &= - \sum_{m=0}^{n-1} a_m \xi_{m+i+j+k-n} - \xi_{i+j+k} = -h_{i+j+k-n}.\end{aligned}$$

If $j+k \geq n$, from (3) and (4) we have

$$\begin{aligned}\psi_{ijk} &\equiv \sum_{m=0}^{n-1} \rho_{i+j+k+1-n,m} \xi_{m+n-1} - \sum_{m=0}^{n-1} \rho_{i+j+k+1-n,m} \xi_{m+n-1} \pmod{I_f} \\ &= 0.\end{aligned}$$

Thus we have $\psi_{ijk} \equiv 0 \pmod{I_f}$ and we proved the assertion.

Q.E.D.

By virtue of Lemmas 1.3, 1.5 and 1.6 we have the following

Theorem 1.8. *Let v_1, \dots, v_n be a k -basis of a k -algebra $K = \bar{A}/J(\bar{A})$ and let ρ_{ijk} be the structure constants of k -algebra K . Set*

$$\psi_{ijk} = \left(\sum_{m=1}^n \rho_{ikm} X_m \right) X_j - \left(\sum_{m=1}^n \rho_{jkm} X_m \right) X_i \quad (1 \leq i, j, k \leq n).$$

Then the following conditions are equivalent to each other.

- (1) A is seminormal.
- (2) \hat{A} is k -isomorphic to $k[[X_1, \dots, X_n]]/(\dots, \psi_{ijk}, \dots)$.
- (3) $Gr^*(A)$ is k -isomorphic to $k[X_1, \dots, X_n]/(\dots, \psi_{ijk}, \dots)$.

Lemma 1.9. *We say that a polynomial $f(X)$ in $k[X]$ is reduced in $k[X]$ if $f(X)$ has no multiple factors in $k[X]$, i.e., if the residue ring $k[X]/(f(X))$ is reduced. Let the ideal I_f have the same meaning as in 1.7. If \hat{A} is k -isomorphic to $k[[X_0, \dots, X_{n-1}]]/I_f$ or $Gr^*(A)$ is k -isomorphic to $k[X_0, \dots, X_{n-1}]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$ then A is seminormal.*

Proof. This lemma follows from Lemmas 1.2 and 1.7.

Q.E.D.

Theorem 1.10. *Let the ideal I_f have the same meaning as in 1.7. Assume that k is a perfect infinite field. Then the following conditions are equivalent to each other.*

- (1) A is seminormal.
- (2) \hat{A} is k -isomorphic to $k[[X_0, \dots, X_{n-1}]]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$.
- (3) $Gr^*(A)$ is k -isomorphic to $k[X_0, \dots, X_{n-1}]/I_f$ for some reduced monic polynomial $f(X)$ of degree n in $k[X]$.

Proof. If we see that K is k -isomorphic to $k[X]/(f(X))$ for some reduced monic polynomial $f(X)$ in $k[X]$ then the assertion follows from Theorem 1.8, Lemma 1.7 and Lemma 1.9. Let $\mathfrak{M}_1, \dots, \mathfrak{M}_r$ be all the maximal ideals of \bar{A} and set $K_i = \bar{A}/\mathfrak{M}_i$ for $i=1, \dots, r$. Then we have $K \cong_k K_1 \times \dots \times K_r$. Since k is a

perfect field we have $K_i \cong_k k[X]/(f_i(X))$ for some irreducible monic polynomial $f_i(X)$ in $k[X]$. We may assume that $f_1(X), \dots, f_r(X)$ are relatively prime because k is an infinite field. Set $f(X) = f_1(X) \cdots f_r(X)$. Then $f(X)$ is a reduced monic polynomial and we have $K \cong_k k[X]/(f(X))$. Q.E.D.

REMARK 1.11. In Theorem 1.10 we assumed that the local ring A is reduced. We may replace this assumption by the condition that $\text{depth } A \geq 1$. In fact the following lemma holds in general.

Lemma. *Let (R, \mathfrak{M}) be a seminormal local ring with $\text{depth } R \geq 1$. Then R is reduced.*

Proof. Let $\mathfrak{n} = \text{nil}(R)$, where $\text{nil}(R)$ denotes the nilradical of R , and let S be the set of regular elements of R . Then we have $\text{nil}(\bar{R}) = S^{-1}\mathfrak{n}$. From definition, it is easy to check that $R + \text{nil}(\bar{R}) \subset {}^+R$, hence we have $S^{-1}\mathfrak{n} \subset R$. Let x be an arbitrary element in \mathfrak{n} . Notice that $S \cap \mathfrak{M} \neq \emptyset$ because $\text{depth } R \geq 1$. Let a be an element in $S \cap \mathfrak{M}$. Then we have $x/a^n \in R$ and $x \in a^n R \subset \mathfrak{M}^n$ for any integer n . Thus we have $x \in \bigcap_{n=0}^{\infty} \mathfrak{M}^n$, and hence we have $x=0$ by Krull's intersection theorem. Q.E.D.

2. Seminormalization of local rings of plane algebraic curves

2.1. Let V be an algebraic variety defined over a field k and let P be a closed point on V . Then P is said to be seminormal if the local ring $O_{P,V}$ is seminormal. Let V be a reduced plane algebraic curve. Then it is known that P is a seminormal point if and only if P is a smooth point or P is a node (cf. [1], [2], [4]). Consequently, if P is a singular point and P is not a node, $O_{P,V}$ is not seminormal. We are interested in how the seminormalization of $O_{P,V}$ can be obtained. For this purpose, we may assume that V is a plane curve defined by a polynomial $F(X, Y) = \sum_{i+j \geq n} a_{ij} X^i Y^j$ in $k[X, Y]$ with $a_{0n} = 1$ and P is the origin $(0, 0)$. In this section we shall determine the seminormalization of $O_{P,V} = (k[X, Y]/(F(X, Y)))_{(X,Y)}$ when $f_F(X) := \sum_{i+j=n} a_{ij} X^j$ is a reduced monic polynomial in $k[X]$.

Lemma 2.2. *Let I_0 be the ideal of $k[X_0, \dots, X_{n-1}]$ ($n \geq 2$) generated by $g_{ij} = X_i X_j - \xi_{i+j}$ ($0 \leq i < j \leq n-1$), where ξ_m is the same as in 1.7. Then we have $X_{i_1} \cdots X_{i_s} \equiv X_{j_1} \cdots X_{j_s} \pmod{I_0}$ if $i_1 + \cdots + i_s = j_1 + \cdots + j_s$. In particular we have $X_0^{t-1} X_t \equiv X_1^t \pmod{I_0}$.*

Proof. This is easily seen by simple calculations and we omit the proof.

2.3. Let $F(X, Y) = \sum_{i+j \geq n} a_{ij} X^i Y^j$ be an element in $k[X, Y]$ with $n \geq 2$ and $a_{0n} = 1$. For each ordered pair (i, j) with $i+j \geq n$, we choose a fixed integer $t = t(i, j)$ satisfying

$$\text{Max}(0, n-i) \leq t \leq \text{Min}(n, j)$$

and set

$$\phi_s = \sum_{i+j \geq n} a_{ij} X_0^{i+t-n} X_1^{j-t} \xi_{t+s} \quad (0 \leq s \leq n-2).$$

Let I be the ideal of $k[X_0, \dots, X_{n-1}]$ generated by I_0 and ϕ_s 's. It should be noticed that I is independent of the choice of t by Lemma 2.2, and $t(n-m, m) = m$ for $m=0, \dots, n$.

Lemma 2.4. *Let $F(X, Y)$, I_0 and I be the same as in 2.2 and 2.3. Then the ring homomorphism $\tau: (k[X, Y]/(F(X, Y)))_{(X, Y)} \rightarrow (k[X_0, \dots, X_{n-1}]/I)_{(X_0, \dots, X_{n-1})}$ defined by $\tau(X) = X_0$ and $\tau(Y) = X_1$ is injective, birational and integral.*

Proof. The proof is divided into three steps.

Step 1. *The homomorphism $\tau: k[X, Y]/(F(X, Y)) \rightarrow k[X_0, \dots, X_{n-1}]/I$ defined by $\tau(X) = X_0$ and $\tau(Y) = X_1$ is well-defined and injective.*

In fact by Lemma 2.2 we have

$$(1) \quad \begin{cases} X_0^{n-1} \phi_s \equiv \sum_{i+j \geq n} a_{ij} X_0^i X_1^j X_s = X_s F(X_0, X_1) \pmod{I_0} \\ X_0^{n-2} \phi_0 \equiv \sum_{i+j \geq n} a_{ij} X_0^i X_1^j = F(X_0, X_1) \pmod{I_0} \end{cases}$$

because $X_0^{i+t-1} \xi_{t+s} \equiv X_0^i X_1^t X_s$ and $X_0^{i+t-2} \xi_t \equiv X_0^i X_1^t \pmod{I_0}$ (notice that $i+t \geq n \geq 2$). In particular $F(X_0, X_1) \in I$ and the homomorphism τ is well-defined. To prove that τ is injective let

$$G(X_0, X_1) = \sum_{0 \leq i < j \leq n-1} h_{ij} g_{ij} + \sum_{s=0}^{n-2} \lambda_s \phi_s$$

be an element of $I \cap k[X_0, X_1]$ where h_{ij} 's and λ_s 's are elements of $k[X_0, \dots, X_{n-1}]$. Then from (1) we get

$$X_0^{n-1} G(X_0, X_1) \equiv X_0^{n-1} \sum_{0 \leq i < j \leq n-1} h_{ij} g_{ij} + F(X_0, X_1) \sum_{s=0}^{n-2} X_s \lambda_s \pmod{I_0}.$$

Therefore we can write

$$X_0^{n-1} G(X_0, X_1) = \sum l_{ij} g_{ij} + \lambda(X_0, \dots, X_{n-1}) F(X_0, X_1)$$

for some elements l_{ij} in $k[X_0, \dots, X_{n-1}]$, where $\lambda(X_0, \dots, X_{n-1}) = \sum_{s=0}^{n-2} X_s \lambda_s$. As is readily seen, there exist an integer N and a polynomial $\tilde{\lambda}(X, Y)$ in two variables such that

$$\tilde{\lambda}(X, ZX) = X^N \lambda(X, ZX, \dots, Z^{n-1} X).$$

Since $g_{ij}(X, ZX, \dots, Z^{n-1} X) = 0$, we get, by specializing $X_i \mapsto Z^i X$, the relation

$$X^{N+n-1}G(X, ZX) = \tilde{\lambda}(X, ZX)F(X, ZX).$$

From this we get the identity

$$X_0^{N+n-1}G(X_0, X_1) = \tilde{\lambda}(X_0, X_1)F(X_0, X_1)$$

because X and ZX are independent variables over k . From our assumption $F(X_0, X_1)$ is not divisible by X_0 . Hence $\tilde{\lambda}(X_0, X_1)$ is divisible by X_0^{N+n-1} , and we have $G(X_0, X_1) \in F(X_0, X_1)k[X_0, X_1]$.

Step 2. Let \mathfrak{Q} be a proper prime ideal of $k[X_0, \dots, X_{n-1}]$ containing I and X_0 . Then \mathfrak{Q} is necessarily equal to the maximal ideal (X_0, \dots, X_{n-1}) .

If $X_i \in \mathfrak{Q}$ for some $i \leq n-3$ we have $X_{i+1}^2 \equiv X_i X_{i+2} \equiv 0 \pmod{\mathfrak{Q}}$ whence $X_{i+1} \in \mathfrak{Q}$. Thus we have $X_0, \dots, X_{n-2} \in \mathfrak{Q}$. On the other hand, we have $\phi_{n-2} \in I \subset \mathfrak{Q}$ and $\phi_{n-2} = X_{n-1}^2 + \eta$ with $\eta \in \mathfrak{Q}$ because $t(0, n) = n$, $a_{0n} = 1$ and $\xi_s \in \mathfrak{Q}$ for $s < 2n-2$. Thus we have $X_{n-1}^2 \in \mathfrak{Q}$, whence $X_{n-1} \in \mathfrak{Q}$. Obviously, (X_0, \dots, X_{n-1}) is a maximal ideal of $k[X_0, \dots, X_{n-1}]$, and thence we have $\mathfrak{Q} = (X_0, \dots, X_{n-1})$.

Step 3. The ring homomorphism $\tau: (k[X, Y]/(F(X, Y)))_{(X, Y)} \rightarrow (k[X_0, \dots, X_{n-1}]/I)_{(X_0, \dots, X_{n-1})}$ induced naturally by τ is birational and integral.

Set $B = k[X_0, X_1]/(F(X_0, X_1))$, $C = k[X_0, \dots, X_{n-1}]/I$, $\mathfrak{p} = (X_0, X_1)B$ and $\mathfrak{M} = (X_0, \dots, X_{n-1})C$. We shall denote by x_i the residue class of X_i modulo I . Notice that $x_i x_0^{i-1} = x_1^i$ for $1 \leq i \leq n-1$ and x_0 is a regular element of B . This implies that $C_{\mathfrak{p}} \subset Q(B_{\mathfrak{p}})$. Next we show that $C_{\mathfrak{p}}$ is integral over $B_{\mathfrak{p}}$. In fact, we have

$$F(x_0, x_1) = \sum_{i+j \geq n} a_{ij} x_0^i x_1^j = 0$$

in B . From this it is easy to check that x_1/x_0 is integral over $B_{\mathfrak{p}}$. Hence $x_i = x_1(x_1/x_0)^{i-1}$ is also integral over $B_{\mathfrak{p}}$ for $1 \leq i \leq n-1$, which shows that $C_{\mathfrak{p}}$ is integral over $B_{\mathfrak{p}}$. It remains to prove that $C_{\mathfrak{p}} = C_{\mathfrak{M}}$. Let $\mathfrak{Q}C_{\mathfrak{p}}$ be a maximal ideal of $C_{\mathfrak{p}}$, where \mathfrak{Q} is a maximal ideal of C . Then we have $\mathfrak{Q}C_{\mathfrak{p}} \cap B_{\mathfrak{p}} = \mathfrak{p}B_{\mathfrak{p}}$ because $C_{\mathfrak{p}}$ is integral over $B_{\mathfrak{p}}$, whence $\mathfrak{Q} \cap B = \mathfrak{p}$. Thus we have $\mathfrak{Q} = \mathfrak{M}$ by the step 2, from which we see that $C_{\mathfrak{p}}$ is a local ring. Therefore, we have $C_{\mathfrak{p}} = C_{\mathfrak{M}}$ and the assertion is verified.

Lemma 2.5. Assume that $f_F(X) = \sum_{i+j=n} a_{ij} X^j$ is a reduced monic polynomial in $k[X]$. Then the local ring $(k[X_0, \dots, X_{n-1}]/I)_{(X_0, \dots, X_{n-1})}$ is seminormal.

Proof. Set $R = (k[X_0, \dots, X_{n-1}]/I)_{(X_0, \dots, X_{n-1})}$. Notice that the leading form of ϕ_s is $\sum_{i+j=n} a_{ij} \xi_{j+s}$. Hence, if we set $f = f_F(X)$, we have $Gr^*(R) \cong_k k[X_0, \dots, X_{n-1}]/I_f$, where I_f is the ideal defined in 1.7 (cf. [3; p. 118]). Therefore, by virtue of Lemma 1.9, we see that R is seminormal. Q.E.D.

Summarizing the results stated in Lemmas 2.4 and 2.5, we have the following theorem.

Theorem 2.6. Let $F(X, Y) = \sum_{i+j \geq n} a_{ij} X^i Y^j$ be an element in $k[X, Y]$ with $n \geq 2$ and $a_{0n} = 1$. Assume that $f_F(X) = \sum_{i+j=n} a_{ij} X^i$ is a reduced monic polynomial in $k[X]$. Then the seminormalization of the local ring $(k[X, Y]/(F(X, Y)))_{(X, Y)}$ is isomorphic to $(k[X_0, \dots, X_{n-1}]/I)_{(X_0, \dots, X_{n-1})}$ where the ideal I is generated by g_{ij} ($0 \leq i < j \leq n-1$) and ϕ_s ($0 \leq s \leq n-2$) given in 2.2 and 2.3, respectively.

REMARK 2.7. When $f_F(X)$ is a reduced monic polynomial in $k[X]$ the seminormalization of $O_{P, V}$ is determined by the leading form of the defining equation even if $f_F(X)$ is not reduced in $\bar{k}[X]$, where \bar{k} denotes the algebraic closure of k . But when $f_F(X)$ is not reduced in $k[X]$ the seminormalization of $O_{P, V}$ is more complicated as is shown by the following examples.

- (1) Let $R = (k[X, Y]/(Y^3 - XY^2 - X^4))_{(X, Y)}$. Then ${}^+R$ is $(k[X_0, X_1]/(X_1^2 - X_0X_1 - X_0^3))_{(X_0, X_1)}$, and the embedding $R \rightarrow {}^+R$ is given by $X \mapsto X_1$ and $Y \mapsto X_0^2 + X_1$.
- (2) Let $S = (k[X, Y]/(Y^3 - XY^2 - X^5))_{(X, Y)}$. Then ${}^+S$ is $(k[X_0, X_1, X_2]/(X_1^2 - X_0X_2, X_1X_2 + X_0X_1 - X_0^3, X_2^2 + X_1^2 - X_0^2X_1))_{(X_0, X_1, X_2)}$ and the embedding $S \rightarrow {}^+S$ is given by $X \mapsto X_0$ and $Y \mapsto X_2 + X_0$.

REMARK 2.8. Let $F(X, Y)$ and I have the same meaning as in 2.6. Assume that the origin P is a unique singular point and $k[X_0, \dots, X_{n-1}]/I$ is integral over $k[X, Y]/(F(X, Y))$. Then the seminormalization of $k[X, Y]/(F(X, Y))$ is $k[X_0, \dots, X_{n-1}]/I$.

EXAMPLE. Let $R = k[X, Y]/(Y^3 - aX^3 - X^6)$. Assume that $\text{char}(k) \neq 3$ or $a^{1/3} \notin k$. Then ${}^+R = k[X, Y, Z]/(Y^2 - XZ, YZ - aX^2 - X^5, Z^2 - aXY - X^4Y)$.

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