

Title	Integral formulas for harmonic functions associated with boundaries of a bounded symmetric domain
Author(s)	Inoue, Toru
Citation	Osaka Journal of Mathematics. 1989, 26(3), p. 527-534
Version Type	VoR
URL	https://doi.org/10.18910/6395
rights	
Note	

Osaka University Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

Osaka University

INTEGRAL FORMULAS FOR HARMONIC FUNCTIONS ASSOCIATED WITH BOUNDARIES OF A BOUNDED SYMMETRIC DOMAIN

TORU INOUE

(Received October 20, 1988)

1. Introduction

In the case of the unit disc, or the upper half-plane in the theory of one complex variable, the Poisson kernel can be expressed in terms of the Cauchy kernel in the following simple way; in either case, denoting the Cauchy and the Poisson kernels by $\mathcal{G}(z, w)$, $\mathcal{P}(z, u)$ respectively

$$(1) \quad \mathcal{P}(z, u) = \frac{|\mathcal{G}(z, u)|^2}{\mathcal{G}(z, z)}.$$

It is natural, therefore, to extend this definition whenever the Cauchy kernel is defined. Hua [3] did this for four classical types of bounded symmetric domains and established some of its basic properties. For generalized half-planes this was done by Korányi [6] who then used the theory of Cayley transform to determine the Cauchy and Poisson kernels for all the bounded symmetric domains (See also [8], [5]).

It is known that the Poisson kernel has another interpretation; it can be regarded as the Jacobian of an automorphism restricted to the boundary. This way of viewing the Poisson kernel was shown to work on arbitrary non-compact Riemannian symmetric spaces by Furstenberg [1]. For any symmetric domain it turns out that these two possible definitions of the Poisson kernel coincide (See [6], though it is not explicitly stated).

Now let D be an irreducible bounded symmetric domain in the canonical Harish-Chandra realization. If r is the rank of D , then the topological boundary ∂D breaks into r boundaries B_1, \dots, B_r , such that $\bar{B}_i \supset B_{i+1} (1 \leq i \leq r-1)$, and B_r is the Silov boundary. As is shown in [4], for each boundary $B_i (1 \leq i \leq r)$, there is a natural measure σ_i on B_i and a Cauchy type kernel function $\mathcal{G}_i(z, w)$ such that

$$f(z) = \int_{B_i} \mathcal{G}_i(z, u) f(u) d\sigma_i(u)$$

whenever f is holomorphic in a neighborhood of \bar{D} , the closure of D . For the

Silov boundary B_i , the function $\mathcal{G}_i(z, w)$ is the usual Cauchy(-Szegő) kernel of D , from which Hua et al. defined the Poisson kernel by (1). Therefore it is natural to define, for each boundary B_i , the Poisson type kernel $\mathcal{P}_i(z, u)$ by putting

$$\mathcal{P}_i(z, u) = \frac{|\mathcal{G}_i(z, u)|^2}{\mathcal{G}_i(z, z)}, \quad z \in D, u \in B_i.$$

In this note we show that the kernel $\mathcal{P}_i(z, u)$ represents harmonic functions f in D in terms of the boundary values on B_i , i.e.,

$$f(z) = \int_{B_i} \mathcal{P}_i(z, u) f(u) d\sigma_i(u)$$

whenever f is harmonic in D and continuous on its closure \bar{D} . We also show that the kernel $\mathcal{P}_i(z, u)$ can be regarded as the Jacobian of an automorphism restricted to the boundary B_i , i.e., if g is an automorphism of D ,

$$\mathcal{P}_i(g \cdot o, u) = \frac{d\sigma_i(g^{-1} \cdot u)}{d\sigma_i(u)},$$

where o is the origin of D .

2. Preliminaries

We begin by reviewing the general background on bounded symmetric domains (cf. [2], [9]). Every bounded symmetric domain D can be written as $D=G/K$, where G is a connected semisimple linear Lie group and K is a maximal compact subgroup of G , such that G operates holomorphically on D . In this note we assume that G is simple, i.e., that D is irreducible. We further assume that the complexification G_c of G is simply connected. Let $\mathfrak{g}, \mathfrak{k}$ be the Lie algebras of G, K and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition. We denote the complexifications of $\mathfrak{g}, \mathfrak{k}, \mathfrak{p}$ by $\mathfrak{g}_c, \mathfrak{k}_c, \mathfrak{p}_c$, respectively. Then \mathfrak{p}_c is decomposed into the direct sum of two complex subalgebras $\mathfrak{p}^+, \mathfrak{p}^-$, which are $(\pm\sqrt{-1})$ -eigenspaces of the complex structure of \mathfrak{p} , respectively, and are abelian subalgebras of \mathfrak{g}_c normalized by \mathfrak{k}_c . Let P^\pm, K_c , be the connected subgroups of G_c corresponding to $\mathfrak{p}^\pm, \mathfrak{k}_c$, respectively. Then the map $\mathfrak{p}^+ \times K_c \times \mathfrak{p}^- \rightarrow G_c$, given by $(X^+, k, X^-) \rightarrow \exp X^+ \cdot k \cdot \exp X^-$, is a holomorphic diffeomorphism onto a dense open subset $P^+K_cP^-$ of G_c , which contains G . Therefore every element $g \in P^+K_cP^-$ can be written in a unique way as

$$(2) \quad g = \pi_+(g) \cdot \pi_0(g) \cdot \pi_-(g), \quad \pi_0(g) \in K_c, \pi_\pm(g) \in P^\pm.$$

Furthermore, the map $\zeta: P^+K_cP^- \rightarrow \mathfrak{p}^+$, given by $\zeta(g)=\log(\pi_+(g))$ induces a holomorphic diffeomorphism of $D=G/K$ onto $\zeta(G)$, and $\zeta(G)$ is a bounded domain in \mathfrak{p}^+ . Henceforce we assume that D is a bounded symmetric domain

in \mathfrak{p}^+ realized in this manner. In this realization the action of G on D is given by

$$g \cdot z = \zeta(g \exp z), \quad g \in G, z \in D,$$

and extends smoothly to \bar{D} .

Let \mathfrak{t} be a maximal abelian subalgebra of \mathfrak{k} . Then $\mathfrak{t}_\mathbb{C}$, the complexification of \mathfrak{t} , is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. Let Φ be the root system of $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{t}_\mathbb{C}$. For each $\alpha \in \Phi$, let H_α, E_α denote the usual basis elements of $\mathfrak{g}_\mathbb{C}$. We can choose a linear order in the dual of the real vector space $\sqrt{-1} \mathfrak{t}$ such that \mathfrak{p}^+ is spanned by the root spaces for noncompact positive roots. We let Φ^+ be the resulting set of positive roots.

We choose a maximal set $\{\gamma_1, \dots, \gamma_r\}$ of strongly orthogonal noncompact positive roots as follows. Let γ_1 be the highest root of Φ and for each j, γ_{j+1} be the highest positive noncompact root that is strongly orthogonal to each of $\{\gamma_1, \dots, \gamma_j\}$. We write H_j, E_j for $H_{\gamma_j}, E_{\gamma_j}$. For each $1 \leq i \leq r$, we define the partial Cayley transform $c_i \in G_\mathbb{C}$ by

$$c_i = \prod_{j=1}^i \exp \frac{\pi}{4} (E_{-\gamma_j} - E_{\gamma_j}).$$

Since $c_i \in P^+ K_\mathbb{C} P^-$, we can define $o_i = \zeta(c_i)$. Let B_i denote the G -orbit of o_i . Then

$$\bar{D} - D = \bigcup_{1 \leq i \leq r} B_i \quad (\text{disjoint union}).$$

Moreover $\bar{B}_i \supset B_{i+1}$ ($1 \leq i \leq r-1$), and B_r is the Silov boundary.

Let $C_i (\subset B_i)$ be the boundary component of D containing o_i , and let $P_i = \{g \in G; g \cdot C_i = C_i\}$ and $S_i = \{g \in G; g \cdot o_i = o_i\}$. Then P_i is a maximal parabolic subgroup of G , and we have a Langlands decomposition $P_i = M_i A_i N_i$ such that if we put $L_i = M_i \cap S_i$ then $S_i = L_i A_i N_i$ (cf. [4]). Further there exists a semisimple subgroup G_i of G such that $C_i = G_i \cdot o_i$.

For each $1 \leq i \leq r$, we define a C^∞ function ρ_i on G as follows. Since $P_i = M_i A_i N_i$ is a parabolic subgroup, each $g \in G$ can be uniquely written in the form $g = kman$ ($k \in K, m \in M_i \cap \exp \mathfrak{p}, a \in A_i, n \in N_i$); so put $\rho_i(g) = (\det(\text{Ad}(a)|_{\mathfrak{n}_i}))^{-1}$, where \mathfrak{n}_i is the Lie algebra of N_i . Let dk denote the Haar measure on K such that $\int_K dk = 1$. Then we can normalize various left Haar measures in such a way that

$$(3) \quad \int_G f(g) dg = \int_{K \times G_i \times S_i} f(kg_i s_i) \rho_i(s_i)^{-1} dk dg_i ds_i$$

for any integrable f on G (cf. [4], p. 89).

Let $\{\alpha_1, \dots, \alpha_j\}$ be an enumeration of the set of simple roots for Φ^+ such

that α_1 is the unique noncompact simple root for Φ^+ , and let λ be the linear form on \mathfrak{t}_c such that

$$2(\lambda, \alpha_1)/(\alpha_1, \alpha_1) = 1 \quad \text{and} \quad (\lambda, \alpha_j) = 0 \quad \text{for} \quad j = 2, \dots, l$$

where $(\ , \)$ is the inner product induced by Killing form of \mathfrak{g}_c . Then λ is the differential of a holomorphic character of K_C .

Let $\mathfrak{t}_{\bar{c}} = \sum_{i=1}^r \mathbf{R} H_j$. Then the restrictions of \mathfrak{t}_c -roots to $\mathfrak{t}_{\bar{c}}$ are of the form $\pm \gamma_j$ (each with multiplicity one), $\pm \frac{1}{2}(\gamma_j \pm \gamma_k)$ ($j < k$, each with the same multiplicity $u > 0$), $\pm \frac{1}{2} \gamma_j$ (each with the same multiplicity $2v \geq 0$). For each $1 \leq i \leq r$, let (as in [4], p. 91)

$$(4) \quad p_i = \frac{1}{2} u(i-1) + u(r-i) + v + 1,$$

and set $\omega_i = -p_i \lambda$. Note that each p_i is an integer or a half-integer. If p_i is an integer, ω_i is also the differential of a holomorphic character of K_C . For the moment we assume that this is the case and let τ_i be the corresponding character of K_C , i.e., $\tau_i = e^{\omega_i}$. We define $J_i: G \times \bar{D} \rightarrow \mathbf{C}^\times$ (\mathbf{C}^\times = the multiplicative group of non-zero complex numbers) by

$$J_i(g, z) = \tau_i(\pi_0(g \exp z))$$

where π_0 is as in (2). Then we have

$$J_i(g_1 g_2, z) = J_i(g_1, g_2 \cdot z) J_i(g_2, z), \quad g_1, g_2 \in G, z \in \bar{D}.$$

Let $\chi: K_C \rightarrow \mathbf{C}^\times$ be a holomorphic character of K_C defined by

$$\chi(k) = \det(\text{Ad}(k)|_{\mathfrak{p}^+}),$$

and let $\mathcal{K}: D \times D \rightarrow \mathbf{C}^\times$ be a function defined by

$$\mathcal{K}(z, w) = \chi(\pi_0(\exp(-\bar{w}) \exp z))$$

where $w \rightarrow \bar{w}$ denotes the complex conjugation of \mathfrak{g}_c with respect to \mathfrak{g} . Then, up to a constant factor, $\mathcal{K}(z, w)$ is the Bergman kernel function of D (cf. [7], [4]). Let $n = \dim_{\mathbf{C}} D$, $n_i = \dim_{\mathbf{C}} C_i$, and $d_i = \dim_{\mathbf{R}} B_i$, and set

$$q_i = \frac{n - n_i}{3n - n_i - d_i}.$$

Since $D \times D$ is simply connected, we can define powers $\mathcal{K}(z, w)^{q_i}$ of $\mathcal{K}(z, w)$ with $\mathcal{K}(o, o)^{q_i} = 1$. We let

$$\mathcal{G}_i(z, w) = \mathcal{K}(z, w)^{q_i}.$$

For a fixed $z \in D$, $\mathcal{G}_i(z, \cdot)$ extends smoothly to \bar{D} . If p_i (in (4)) is an integer, then it follows from Lemma 6.24 of [4] that

$$\mathcal{G}_i(z, w) = \tau_i(\pi_0(\exp(-\bar{w}) \exp z)),$$

and we have

$$\mathcal{G}_i(g \cdot z, g \cdot w) = J_i(g, z) \mathcal{G}_i(z, w) \overline{J_i(g, w)}.$$

Up to now we have assumed that the p_i is an integer. We note that, even if p_i is a half-integer, $J_i(g, z)^2$ is a well defined function on $G \times \bar{D}$, and satisfies the following properties

$$(5) \quad J_i(g_1 g_2, z)^2 = J_i(g_1, g_2 \cdot z)^2 J_i(g_2, z)^2,$$

$$(6) \quad \mathcal{G}_i(g \cdot z, g \cdot w)^2 = J_i(g, z)^2 \mathcal{G}_i(z, w)^2 \overline{J_i(g, w)^2}.$$

Let $d\sigma_i$ be the quasi-invariant measure on $B_i = G/S_i$ defined by

$$\int_{B_i} f(u) d\sigma_i(u) = \int_{K \times G_i} |J_i(kg_i, o_i)|^{-2} |f(kg_i \cdot o_i)| dk dg_i$$

for all $f \in C_c(B_i)$ (continuous functions with compact support). Then Proposition 4.38 of [4] implies that

$$\int_{B_i} d\sigma_i(u) = \int_{K \times G_i} |J_i(kg_i, o_i)|^{-2} dk dg_i < \infty.$$

Therefore we can (and do) normalize the Haar measure dg_i on G_i so that

$$(7) \quad \int_{B_i} d\sigma_i(u) = 1.$$

It then follows from formula (6.15) of [4] that

$$f(z) = \int_{B_i} \mathcal{G}_i(z, u) f(u) d\sigma_i(u)$$

whenever f is holomorphic in the neighborhood of \bar{D} .

3. Integral formulas

As in the introduction we define, for each boundary $B_i (1 \leq i \leq r)$, the Poisson type kernel function $\mathcal{P}_i(z, u)$ by putting

$$\mathcal{P}_i(z, u) = \frac{|\mathcal{G}_i(z, u)|^2}{\mathcal{G}_i(z, z)}, \quad z \in D, u \in B_i.$$

Proposition. For $g \in G, u \in B_i$, we have

$$\mathcal{P}_i(g \cdot o, u) = |J_i(g^{-1}, u)|^{-2} = \frac{d\sigma_i(g^{-1} \cdot u)}{d\sigma_i(u)}.$$

Proof. Since $\mathcal{G}_i(o, w)=1$, (5) and (6) imply

$$\mathcal{G}_i(g \cdot o, u)^2 = J_i(g, o)^2 \overline{J_i(g^{-1}, u)^{-2}} .$$

and

$$\mathcal{G}_i(g \cdot o, g \cdot o) = |J_i(g, o)|^2 .$$

So the first equality follows from the definition of $\mathcal{P}_i(g \cdot o, u)$.

For the second equality, it suffices to show that

$$\int_{B_i} f(g \cdot u) d\sigma_i(u) = \int_{B_i} |J_i(g^{-1}, u)^{-2}| f(u) d\sigma_i(u)$$

for $f \in C_c(B_i)$. We first note that

$$(8) \quad |J_i(s_i, o_i)^{-2}| = \rho_i(s_i) \quad \text{for } s_i \in S_i ;$$

this follows from the argument in the proof of Lemma 6.30 of [4]. Now for each $f \in C_c(B_i)$, we can take $\tilde{f} \in C_c(G)$ such that

$$f(h \cdot o_i) = \int_{S_i} \tilde{f}(hs_i) ds_i, \quad h \in G .$$

Hence

$$\begin{aligned} & \int_{B_i} f(g \cdot u) d\sigma_i(u) \\ &= \int_{K \times G_i} |J_i(kg_i, o_i)^{-2}| f(gkg_i \cdot o_i) dk dg_i \\ &= \int_{K \times G_i \times S_i} |J_i(kg_i s_i, o_i)^{-2}| \tilde{f}(gkg_i s_i) \rho_i(s_i)^{-1} dk dg_i ds_i \quad (\text{by (5) and (8)}) \\ &= \int_G |J_i(h, o_i)^{-2}| \tilde{f}(gh) dh \quad (\text{by (3)}) \\ &= \int_G |J_i(g^{-1} h, o_i)^{-2}| \tilde{f}(h) dh \\ &= \int_{K \times G_i \times S_i} |J_i(g^{-1} kg_i s_i, o_i)^{-2}| \tilde{f}(kg_i s_i) \rho_i(s_i)^{-1} dk dg_i ds_i \\ &= \int_{K \times G_i} |J_i(g^{-1} kg_i, o_i)^{-2}| f(kg_i \cdot o_i) dk dg_i \\ &= \int_{K \times G_i} |J_i(g^{-1}, kg_i \cdot o_i)^{-2}| |J_i(kg_i, o_i)^{-2}| f(kg_i \cdot o_i) dk dg_i \quad (\text{by (5)}) \\ &= \int_{B_i} |J_i(g^{-1}, u)^{-2}| f(u) d\sigma_i(u) . \end{aligned}$$

This proves the Proposition.

Theorem. *If f is harmonic on D and continuous on \bar{D} , then for all $z \in D$,*

$$f(z) = \int_{B_i} \mathcal{P}_i(z, u) f(u) d\sigma_i(u) .$$

Proof. For each $z \in D$, choose $g \in G$ such that $z = g \cdot o$. Then, by the mean value theorem for harmonic functions (cf. [2]), we have

$$f(z) = \int_K f(gk \cdot w) dk$$

for $w \in D$. The continuity of f on \bar{D} implies that this formula is valid for all $w \in \bar{D}$. Therefore

$$\begin{aligned} f(z) &= \int_{B_i} f(z) d\sigma_i(u) \quad (\text{by (7)}) \\ &= \int_{B_i} \left(\int_K f(gk \cdot u) dk \right) d\sigma_i(u) \\ &= \int_K \left(\int_{B_i} f(gk \cdot u) d\sigma_i(u) \right) dk \\ &= \int_K \left(\int_{B_i} f(g \cdot u) d\sigma_i(u) \right) dk \quad (\text{by } K\text{-invariance of } d\sigma_i) \\ &= \int_{B_i} f(g \cdot u) d\sigma_i(u) \\ &= \int_{B_i} \mathcal{P}_i(z, u) f(u) d\sigma_i(u) \quad (\text{by Proposition}). \end{aligned}$$

This finished the proof.

REMARK. If $i \neq r$, the maximal compact subgroup K of G does not act transitively on the boundary B_i . Therefore Proposition implies that the Poisson type kernel $\mathcal{P}_i(z, u)$ is not necessarily harmonic in the variable z .

EXAMPLE. Let $p \geq q$ and

$$D = \{z \in M_{p,q}(\mathbf{C}); 1_q - z^*z > 0\}.$$

Here $M_{p,q}(\mathbf{C})$ refers to all p by q complex matrices, 1_q is the identity matrix of size q , z^* is the conjugate transpose of z and “ >0 ” means “is positive definite”. Then (cf. [9]) D is the bounded symmetric domain of rank q , and for each $1 \leq i \leq q$, the i -th boundary B_i is given by

$$B_i = \{z \in M_{p,q}(\mathbf{C}); 1_q - z^*z \geq 0 \text{ and } \text{rank}(1_q - z^*z) = q - i\}.$$

On the other hand the Cauchy type kernel function $\mathcal{G}_i(z, w)$ associated with the boundary B_i is given by (cf. [4], p. 129)

$$\mathcal{G}_i(z, w) = \det(1_q - w^*z)^{-(p+q-i)}.$$

Therefore the Poisson type kernel function $\mathcal{P}_i(z, u)$ is given by

$$\mathcal{P}_i(z, u) = \frac{\det(1_q - z^*z)^{p+q-i}}{|\det(1_q - u^*z)|^{2(p+q-i)}}, \quad (1 \leq i \leq q).$$

References

- [1] H. Furstenberg: *A Poisson formula for semi-simple Lie groups*, Ann. of Math. **77** (1963), 335–386.
- [2] S. Helgason: *Differential Geometry and Symmetric Spaces*, Academic press, 1962.
- [3] L.K. Hua: *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains*, Trans. Math. Monographs, vol. 6, Amer. Math. Soc. 1963.
- [4] T. Inoue: *Unitary representations and kernel functions associated with boundaries of a bounded symmetric domain*, Hiroshima Math. J. **10** (1980), 75–140.
- [5] A. Koranyi: *A Poisson formula for homogeneous wedge domains*, J. Analyse Math. **14** (1965), 275–284.
- [6] ———: *The Poisson integral for generalized half-planes and bounded symmetric domains*, Ann. of math. **82** (1965), 332–350.
- [7] I. Satake: *Algebraic Structures of Symmetric Domains*, Pub. Math. Soc. Japan **14**, Iwanami, Tokyo and Princeton Univ. Press, 1980.
- [8] E.M. Stein, G. Weiss and M. Weiss: *H^p -classes of holomorphic functions in tube domains*, Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 1035–1039.
- [9] J.A. Wolf: *Fine structure of hermitian symmetric spaces*, in “Symmetric spaces”, (ed. W. Boothby and G. Weiss), Marcel Dekker, 1972, 271–357.

Department of Mathematics
Faculty of Science
Yamaguchi University
Yamaguchi 753, Japan