



Title	Integral formulas for harmonic functions associated with boundaries of a bounded symmetric domain
Author(s)	Inoue, Toru
Citation	Osaka Journal of Mathematics. 1989, 26(3), p. 527-534
Version Type	VoR
URL	<a href="https://doi.org/10.18910/6395">https://doi.org/10.18910/6395</a>
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Osaka J. Math.  
26 (1989), 527-534

## INTEGRAL FORMULAS FOR HARMONIC FUNCTIONS ASSOCIATED WITH BOUNDARIES OF A BOUNDED SYMMETRIC DOMAIN

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(Received October 20, 1988)

### 1. Introduction

In the case of the unit disc, or the upper half-plane in the theory of one complex variable, the Poisson kernel can be expressed in terms of the Cauchy kernel in the following simple way; in either case, denoting the Cauchy and the Poisson kernels by  $\mathcal{G}(z, w)$ ,  $\mathcal{P}(z, u)$  respectively

$$(1) \quad \mathcal{P}(z, u) = \frac{|\mathcal{G}(z, u)|^2}{\mathcal{G}(z, z)}.$$

It is natural, therefore, to extend this definition whenever the Cauchy kernel is defined. Hua [3] did this for four classical types of bounded symmetric domains and established some of its basic properties. For generalized half-planes this was done by Korányi [6] who then used the theory of Cayley transform to determine the Cauchy and Poisson kernels for all the bounded symmetric domains (See also [8], [5]).

It is known that the Poisson kernel has another interpretation; it can be regarded as the Jacobian of an automorphism restricted to the boundary. This way of viewing the Poisson kernel was shown to work on arbitrary non-compact Riemannian symmetric spaces by Furstenberg [1]. For any symmetric domain it turns out that these two possible definitions of the Poisson kernel coincide (See [6], though it is not explicitly stated).

Now let  $D$  be an irreducible bounded symmetric domain in the canonical Harish-Chandra realization. If  $r$  is the rank of  $D$ , then the topological boundary  $\partial D$  breaks into  $r$  boundaries  $B_1, \dots, B_r$ , such that  $\bar{B}_i \supset B_{i+1}$  ( $1 \leq i \leq r-1$ ), and  $B_r$  is the Silov boundary. As is shown in [4], for each boundary  $B_i$  ( $1 \leq i \leq r$ ), there is a natural measure  $\sigma_i$  on  $B_i$  and a Cauchy type kernel function  $\mathcal{G}_i(z, w)$  such that

$$f(z) = \int_{B_i} \mathcal{G}_i(z, u) f(u) d\sigma_i(u)$$

whenever  $f$  is holomorphic in a neighborhood of  $\bar{D}$ , the closure of  $D$ . For the

Silov boundary  $B_r$ , the function  $\mathcal{G}_r(z, w)$  is the usual Cauchy(-Szegő) kernel of  $D$ , from which Hua et al. defined the Poisson kernel by (1). Therefore it is natural to define, for each boundary  $B_i$ , the Poisson type kernel  $\mathcal{P}_i(z, u)$  by putting

$$\mathcal{P}_i(z, u) = \frac{|\mathcal{G}_i(z, u)|^2}{\mathcal{G}_i(z, z)}, \quad z \in D, u \in B_i.$$

In this note we show that the kernel  $\mathcal{P}_i(z, u)$  represents harmonic functions  $f$  in  $D$  in terms of the boundary values on  $B_i$ , i.e.,

$$f(z) = \int_{B_i} \mathcal{P}_i(z, u) f(u) d\sigma_i(u)$$

whenever  $f$  is harmonic in  $D$  and continuous on its closure  $\bar{D}$ . We also show that the kernel  $\mathcal{P}_i(z, u)$  can be regarded as the Jacobian of an automorphism restricted to the boundary  $B_i$ , i.e., if  $g$  is an automorphism of  $D$ ,

$$\mathcal{P}_i(g \cdot o, u) = \frac{d\sigma_i(g^{-1} \cdot u)}{d\sigma_i(u)},$$

where  $o$  is the origin of  $D$ .

## 2. Preliminaries

We begin by reviewing the general background on bounded symmetric domains (cf. [2], [9]). Every bounded symmetric domain  $D$  can be written as  $D=G/K$ , where  $G$  is a connected semisimple linear Lie group and  $K$  is a maximal compact subgroup of  $G$ , such that  $G$  operates holomorphically on  $D$ . In this note we assume that  $G$  is simple, i.e., that  $D$  is irreducible. We further assume that the complexification  $G_c$  of  $G$  is simply connected. Let  $\mathfrak{g}$ ,  $\mathfrak{k}$  be the Lie algebras of  $G$ ,  $K$  and  $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$  be the corresponding Cartan decomposition. We denote the complexifications of  $\mathfrak{g}$ ,  $\mathfrak{k}$ ,  $\mathfrak{p}$  by  $\mathfrak{g}_c$ ,  $\mathfrak{k}_c$ ,  $\mathfrak{p}_c$ , respectively. Then  $\mathfrak{p}_c$  is decomposed into the direct sum of two complex subalgebras  $\mathfrak{p}^+$ ,  $\mathfrak{p}^-$ , which are  $(\pm\sqrt{-1})$ -eigenspaces of the complex structure of  $\mathfrak{p}$ , respectively, and are abelian subalgebras of  $\mathfrak{g}_c$  normalized by  $\mathfrak{k}_c$ . Let  $P^\pm$ ,  $K_c$ , be the connected subgroups of  $G_c$  corresponding to  $\mathfrak{p}^\pm$ ,  $\mathfrak{k}_c$ , respectively. Then the map  $\mathfrak{p}^+ \times K_c \times \mathfrak{p}^- \rightarrow G_c$ , given by  $(X^+, k, X^-) \rightarrow \exp X^+ \cdot k \cdot \exp X^-$ , is a holomorphic diffeomorphism onto a dense open subset  $P^+ K_c P^-$  of  $G_c$ , which contains  $G$ . Therefore every element  $g \in P^+ K_c P^-$  can be written in a unique way as

$$(2) \quad g = \pi_+(g) \cdot \pi_0(g) \cdot \pi_-(g), \quad \pi_0(g) \in K_c, \quad \pi_\pm(g) \in P^\pm.$$

Furthermore, the map  $\zeta: P^+ K_c P^- \rightarrow \mathfrak{p}^+$ , given by  $\zeta(g) = \log(\pi_+(g))$  induces a holomorphic diffeomorphism of  $D=G/K$  onto  $\zeta(G)$ , and  $\zeta(G)$  is a bounded domain in  $\mathfrak{p}^+$ . Henceforce we assume that  $D$  is a bounded symmetric domain

in  $\mathfrak{p}^+$  realized in this manner. In this realization the action of  $G$  on  $D$  is given by

$$g \cdot z = \zeta(g \exp z), \quad g \in G, z \in D,$$

and extends smoothly to  $\bar{D}$ .

Let  $\mathfrak{t}$  be a maximal abelian subalgebra of  $\mathfrak{k}$ . Then  $\mathfrak{t}_c$ , the complexification of  $\mathfrak{t}$ , is a Cartan subalgebra of  $\mathfrak{g}_c$ . Let  $\Phi$  be the root system of  $\mathfrak{g}_c$  relative to  $\mathfrak{t}_c$ . For each  $\alpha \in \Phi$ , let  $H_\alpha, E_\alpha$  denote the usual basis elements of  $\mathfrak{g}_c$ . We can choose a linear order in the dual of the real vector space  $\sqrt{-1}\mathfrak{t}$  such that  $\mathfrak{p}^+$  is spanned by the root spaces for noncompact positive roots. We let  $\Phi^+$  be the resulting set of positive roots.

We choose a maximal set  $\{\gamma_1, \dots, \gamma_r\}$  of strongly orthogonal noncompact positive roots as follows. Let  $\gamma_1$  be the highest root of  $\Phi$  and for each  $j$ ,  $\gamma_{j+1}$  be the highest positive noncompact root that is strongly orthogonal to each of  $\{\gamma_1, \dots, \gamma_j\}$ . We write  $H_j, E_j$  for  $H_{\gamma_j}, E_{\gamma_j}$ . For each  $1 \leq i \leq r$ , we define the partial Cayley transform  $c_i \in G_c$  by

$$c_i = \prod_{j=1}^i \exp \frac{\pi}{4} (E_{-j} - E_j).$$

Since  $c_i \in P^+ K_c P^-$ , we can define  $o_i = \zeta(c_i)$ . Let  $B_i$  denote the  $G$ -orbit of  $o_i$ . Then

$$\bar{D} - D = \bigcup_{1 \leq i \leq r} B_i \quad (\text{disjoint union}).$$

Moreover  $\bar{B}_i \supset B_{i+1}$  ( $1 \leq i \leq r-1$ ), and  $B_r$  is the Silov boundary.

Let  $C_i (\subset B_i)$  be the boundary component of  $D$  containing  $o_i$ , and let  $P_i = \{g \in G; g \cdot C_i = C_i\}$  and  $S_i = \{g \in G; g \cdot o_i = o_i\}$ . Then  $P_i$  is a maximal parabolic subgroup of  $G$ , and we have a Langlands decomposition  $P_i = M_i A_i N_i$  such that if we put  $L_i = M_i \cap S_i$  then  $S_i = L_i A_i N_i$  (cf. [4]). Further there exists a semisimple subgroup  $G_i$  of  $G$  such that  $C_i = G_i \cdot o_i$ .

For each  $1 \leq i \leq r$ , we define a  $C^\infty$  function  $\rho_i$  on  $G$  as follows. Since  $P_i = M_i A_i N_i$  is a parabolic subgroup, each  $g \in G$  can be uniquely written in the form  $g = k m a n$  ( $k \in K$ ,  $m \in M_i \cap \exp \mathfrak{p}$ ,  $a \in A_i$ ,  $n \in N_i$ ); so put  $\rho_i(g) = (\det(\text{Ad}(a)|_{\mathfrak{n}_i}))^{-1}$ , where  $\mathfrak{n}_i$  is the Lie algebra of  $N_i$ . Let  $dk$  denote the Haar measure on  $K$  such that  $\int_K dk = 1$ . Then we can normalize various left Haar measures in such a way that

$$(3) \quad \int_G f(g) dg = \int_{K \times G_i \times S_i} f(k g_i s_i) \rho_i(s_i)^{-1} dk dg_i ds_i$$

for any integrable  $f$  on  $G$  (cf. [4], p. 89).

Let  $\{\alpha_1, \dots, \alpha_r\}$  be an enumeration of the set of simple roots for  $\Phi^+$  such

that  $\alpha_1$  is the unique noncompact simple root for  $\Phi^+$ , and let  $\lambda$  be the linear form on  $t_c$  such that

$$2(\lambda, \alpha_1)/(\alpha_1, \alpha_1) = 1 \quad \text{and} \quad (\lambda, \alpha_j) = 0 \quad \text{for } j = 2, \dots, l$$

where  $(\ , \ )$  is the inner product induced by Killing form of  $\mathfrak{g}_c$ . Then  $\lambda$  is the differential of a holomorphic character of  $K_c$ .

Let  $t_{\bar{c}} = \sum_{i=1}^r \mathbf{R} H_i$ . Then the restrictions of  $t_c$ -roots to  $t_{\bar{c}}$  are of the form  $\pm \gamma_j$  (each with multiplicity one),  $\pm \frac{1}{2}(\gamma_j \pm \gamma_k)$  ( $j < k$ , each with the same multiplicity  $u > 0$ ),  $\pm \frac{1}{2} \gamma_j$  (each with the same multiplicity  $2v \geq 0$ ). For each  $1 \leq i \leq r$ , let (as in [4], p. 91)

$$(4) \quad p_i = \frac{1}{2} u(i-1) + u(r-i) + v + 1,$$

and set  $\omega_i = -p_i \lambda$ . Note that each  $p_i$  is an integer or a half-integer. If  $p_i$  is an integer,  $\omega_i$  is also the differential of a holomorphic character of  $K_c$ . For the moment we assume that this is the case and let  $\tau_i$  be the corresponding character of  $K_c$ , i.e.,  $\tau_i = e^{\omega_i}$ . We define  $J_i: G \times \bar{D} \rightarrow \mathbf{C}^\times$  ( $\mathbf{C}^\times$  = the multiplicative group of non-zero complex numbers) by

$$J_i(g, z) = \tau_i(\pi_0(g \exp z))$$

where  $\pi_0$  is as in (2). Then we have

$$J_i(g_1 g_2, z) = J_i(g_1, g_2 \cdot z) J_i(g_2, z), \quad g_1, g_2 \in G, z \in \bar{D}.$$

Let  $\chi: K_c \rightarrow \mathbf{C}^\times$  be a holomorphic character of  $K_c$  defined by

$$\chi(k) = \det(\text{Ad}(k)|_{\mathfrak{p}^+}),$$

and let  $\mathcal{K}: D \times D \rightarrow \mathbf{C}^\times$  be a function defined by

$$\mathcal{K}(z, w) = \chi(\pi_0(\exp(-\bar{w}) \exp z))$$

where  $w \rightarrow \bar{w}$  denotes the complex conjugation of  $\mathfrak{g}_c$  with respect to  $\mathfrak{g}$ . Then, up to a constant factor,  $\mathcal{K}(z, w)$  is the Bergman kernel function of  $D$  (cf. [7], [4]). Let  $n = \dim_{\mathbf{C}} D$ ,  $n_i = \dim_{\mathbf{C}} C_i$ , and  $d_i = \dim_{\mathbf{R}} B_i$ , and set

$$q_i = \frac{n - n_i}{3n - n_i - d_i}.$$

Since  $D \times D$  is simply connected, we can define powers  $\mathcal{K}(z, w)^{q_i}$  of  $\mathcal{K}(z, w)$  with  $\mathcal{K}(o, o)^{q_i} = 1$ . We let

$$\mathcal{G}_i(z, w) = \mathcal{K}(z, w)^{q_i}.$$

For a fixed  $z \in D$ ,  $\mathcal{G}_i(z, \cdot)$  extends smoothly to  $\bar{D}$ . If  $p_i$  (in (4)) is an integer, then it follows from Lemma 6.24 of [4] that

$$\mathcal{G}_i(z, w) = \tau_i(\pi_0(\exp(-\bar{w}) \exp z)),$$

and we have

$$\mathcal{G}_i(g \cdot z, g \cdot w) = J_i(g, z) \mathcal{G}_i(z, w) \overline{J_i(g, w)}.$$

Up to now we have assumed that the  $p_i$  is an integer. We note that, even if  $p_i$  is a half-integer,  $J_i(g, z)^2$  is a well defined function on  $G \times \bar{D}$ , and satisfies the following properties

$$(5) \quad J_i(g_1 g_2, z)^2 = J_i(g_1, g_2 \cdot z)^2 J_i(g_2, z)^2,$$

$$(6) \quad \mathcal{G}_i(g \cdot z, g \cdot w)^2 = J_i(g, z)^2 \mathcal{G}_i(z, w)^2 \overline{J_i(g, w)^2}.$$

Let  $d\sigma_i$  be the quasi-invariant measure on  $B_i = G/S_i$  defined by

$$\int_{B_i} f(u) d\sigma_i(u) = \int_{K \times G_i} |J_i(kg_i, o_i)^{-2}| f(kg_i \cdot o_i) dk dg_i$$

for all  $f \in C_c(B_i)$  (continuous functions with compact support). Then Proposition 4.38 of [4] implies that

$$\int_{B_i} d\sigma_i(u) = \int_{K \times G_i} |J_i(kg_i, o_i)^{-2}| dk dg_i < \infty.$$

Therefore we can (and do) normalize the Haar measure  $dg_i$  on  $G_i$  so that

$$(7) \quad \int_{B_i} d\sigma_i(u) = 1.$$

It then follows from formula (6.15) of [4] that

$$f(z) = \int_{B_i} \mathcal{G}_i(z, u) f(u) d\sigma_i(u)$$

whenever  $f$  is holomorphic in the neighborhood of  $\bar{D}$ .

### 3. Integral formulas

As in the introduction we define, for each boundary  $B_i$  ( $1 \leq i \leq r$ ), the Poisson type kernel function  $\mathcal{P}_i(z, u)$  by putting

$$\mathcal{P}_i(z, u) = \frac{|\mathcal{G}_i(z, u)|^2}{\mathcal{G}_i(z, z)}, \quad z \in D, u \in B_i.$$

**Proposition.** *For  $g \in G$ ,  $u \in B_i$ , we have*

$$\mathcal{P}_i(g \cdot o, u) = |J_i(g^{-1}, u)^{-2}| = \frac{d\sigma_i(g^{-1} \cdot u)}{d\sigma_i(u)}.$$

Proof. Since  $\mathcal{G}_i(o, w) = 1$ , (5) and (6) imply

$$\mathcal{G}_i(g \cdot o, u)^2 = J_i(g, o)^2 \overline{J_i(g^{-1}, u)^{-2}}.$$

and

$$\mathcal{G}_i(g \cdot o, g \cdot o) = |J_i(g, o)^2|.$$

So the first equality follows from the definition of  $\mathcal{P}_i(g \cdot o, u)$ .

For the second equality, it suffices to show that

$$\int_{B_i} f(g \cdot u) d\sigma_i(u) = \int_{B_i} |J_i(g^{-1}, u)^{-2}| f(u) d\sigma_i(u)$$

for  $f \in C_c(B_i)$ . We first note that

$$(8) \quad |J_i(s_i, o_i)^{-2}| = \rho_i(s_i) \quad \text{for } s_i \in S_i;$$

this follows from the argument in the proof of Lemma 6.30 of [4]. Now for each  $f \in C_c(B_i)$ , we can take  $\tilde{f} \in C_c(G)$  such that

$$f(h \cdot o_i) = \int_{S_i} \tilde{f}(hs_i) ds_i, \quad h \in G.$$

Hence

$$\begin{aligned} & \int_{B_i} f(g \cdot u) d\sigma_i(u) \\ &= \int_{K \times G_i} |J_i(kg_i, o_i)^{-2}| f(gkg_i \cdot o_i) dk dg_i \\ &= \int_{K \times G_i \times S_i} |J_i(kg_i s_i, o_i)^{-2}| \tilde{f}(gkg_i s_i) \rho_i(s_i)^{-1} dk dg_i ds_i \quad (\text{by (5) and (8)}) \\ &= \int_G |J_i(h, o_i)^{-2}| \tilde{f}(gh) dh \quad (\text{by (3)}) \\ &= \int_G |J_i(g^{-1}h, o_i)^{-2}| \tilde{f}(h) dh \\ &= \int_{K \times G_i \times S_i} |J_i(g^{-1}kg_i s_i, o_i)^{-2}| \tilde{f}(kg_i s_i) \rho_i(s_i)^{-1} dk dg_i ds_i \\ &= \int_{K \times G_i} |J_i(g^{-1}kg_i, o_i)^{-2}| f(kg_i \cdot o_i) dk dg_i \\ &= \int_{K \times G_i} |J_i(g^{-1}, kg_i \cdot o_i)^{-2}| |J_i(kg_i, o_i)^{-2}| f(kg_i \cdot o_i) dk dg_i \quad (\text{by (5)}) \\ &= \int_{B_i} |J_i(g^{-1}, u)^{-2}| f(u) d\sigma_i(u). \end{aligned}$$

This proves the Proposition.

**Theorem.** *If  $f$  is harmonic on  $D$  and continuous on  $\bar{D}$ , then for all  $z \in D$ ,*

$$f(z) = \int_{B_i} \mathcal{P}_i(z, u) f(u) d\sigma_i(u).$$

Proof. For each  $z \in D$ , choose  $g \in G$  such that  $z = g \cdot o$ . Then, by the mean value theorem for harmonic functions (cf. [2]), we have

$$f(z) = \int_K f(gk \cdot w) dk$$

for  $w \in D$ . The continuity of  $f$  on  $\bar{D}$  implies that this formula is valid for all  $w \in \bar{D}$ . Therefore

$$\begin{aligned} f(z) &= \int_{B_i} f(z) d\sigma_i(u) \quad (\text{by (7)}) \\ &= \int_{B_i} \left( \int_K f(gk \cdot u) dk \right) d\sigma_i(u) \\ &= \int_K \left( \int_{B_i} f(gk \cdot u) d\sigma_i(u) \right) dk \\ &= \int_K \left( \int_{B_i} f(g \cdot u) d\sigma_i(u) \right) dk \quad (\text{by } K\text{-invariance of } d\sigma_i) \\ &= \int_{B_i} f(g \cdot u) d\sigma_i(u) \\ &= \int_{B_i} \mathcal{P}_i(z, u) f(u) d\sigma_i(u) \quad (\text{by Proposition}) . \end{aligned}$$

This finished the proof.

REMARK. If  $i \neq r$ , the maximal compact subgroup  $K$  of  $G$  does not act transitively on the boundary  $B_i$ . Therefore Proposition implies that the Poisson type kernel  $\mathcal{P}_i(z, u)$  is not necessarily harmonic in the variable  $z$ .

EXAMPLE. Let  $p \geq q$  and

$$D = \{z \in M_{p,q}(\mathbf{C}); 1_q - z^* z > 0\} .$$

Here  $M_{p,q}(\mathbf{C})$  refers to all  $p$  by  $q$  complex matrices,  $1_q$  is the identity matrix of size  $q$ ,  $z^*$  is the conjugate transpose of  $z$  and “ $> 0$ ” means “is positive definite”. Then (cf. [9])  $D$  is the bounded symmetric domain of rank  $q$ , and for each  $1 \leq i \leq q$ , the  $i$ -th boundary  $B_i$  is given by

$$B_i = \{z \in M_{p,q}(\mathbf{C}); 1_q - z^* z \geq 0 \text{ and } \text{rank}(1_q - z^* z) = q - i\} .$$

On the other hand the Cauchy type kernel function  $\mathcal{G}_i(z, w)$  associated with the boundary  $B_i$  is given by (cf. [4], p. 129)

$$\mathcal{G}_i(z, w) = \det(1_q - w^* z)^{-(p+q-i)} .$$

Therefore the Poisson type kernel function  $\mathcal{P}_i(z, u)$  is given by

$$\mathcal{P}_i(z, u) = \frac{\det(1_q - z^* z)^{p+q-i}}{|\det(1_q - u^* z)|^{2(p+q-i)}}, \quad (1 \leq i \leq q) .$$

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