Podestà, F. and Spiro, A.
Osaka J. Math.
36 (1999), 805 – 833

KÄHLER MANIFOLDS WITH LARGE ISOMETRY GROUP

FABIO PODESTÀ AND ANDREA SPIRO

(Received January 5, 1998)

1. Introduction

A Riemannian manifold is called of cohomogeneity one if it is acted on by a closed Lie group G of isometries with principal orbits of codimension one. This class of manifolds have at least two good reasons for being considered particularly appealing: their degree of symmetry is so high that classification theorems of purely algebraic nature are still possible in several situations (see e.g. [5], [1], [2], [10]); at the same time, they allow to construct non homogeneous examples of Riemannian manifolds with special geometric properties, like Einstein metrics, exceptional holonomy (see e.g. [4]).

We are interested in compact non homogeneous Kähler-Einstein manifolds and several of them have been already constructed by Koiso and Sakane in the cohomogeneity one category (see e.g. [15], [13]). The aims of the present paper consist in giving an explicit classification of compact cohomogeneity one Kähler manifolds with vanishing first Betti number and in using it for obtaining a complete list of the Kähler-Einstein manifolds in that family.

It is well known (see e.g. [9]) that the vanishing of $b_1(M)$ implies the existence of a G-equivariant moment mapping $\mu\colon M \to \mathfrak{g}^*$ and this fact has an important consequence on the algebraic structure of G. In fact, we prove that (see Lemma 2.2) *if G is semisimple and $G/L = G(p)$ is a regular orbit on M, with $\mathfrak{g}$ and $\mathfrak{l}$ Lie algebras of G and L respectively, then the centralizer $C_{\mathfrak{g}}(\mathfrak{l})$ is of the form*

$$C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{a} \tag{1.1}$$

where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$ and $\mathfrak{a}$ is either one dimensional or it is a rank one simple Lie algebra. In the following we will distinguish these two cases by saying that the action of G on M is *ordinary* if $\mathfrak{a}$ is one dimensional and *extra-ordinary* if it is 3-dimensional.

For the ordinary actions, the moment mapping μ determines a fibration of any regular orbit of M onto a flag manifold

$$\mu|_{G/L}\colon G/L \to G/K \tag{1.2}$$

where $K = N^o_G(L)$. We say that the complex structure J on M is *projectable* if the G-invariant CR structure $J_{G/L}$ induced on any regular orbit is also $Ad(K)$-invariant and hence it descends to a G-invariant complex structure J_K on G/K (see §3). Note that a generic complex structure is projectable and that the non-projectable complex structures can occur only if the pair (G, L) verifies some special algebraic conditions (see e.g. Lemma 3.2).

In this paper we obtain a classification of the cohomogeneity one, compact Kähler manifolds with ordinary action and projectable complex structures. We will call them briefly *compact K.O.P. manifolds* (from Kähler with Ordinary, Projectable, cohomogeneity one action) and we leave the analysis of the remaining cases to a forthcoming paper.

Let γ be a normal geodesic of the cohomogeneity one Kähler manifold (M, g, J) and for any $t \in \mathbb{R}$ consider the two form $\omega_t \in \Lambda^2\mathfrak{g}$

$$\omega_t(X,Y) \stackrel{\text{def}}{=} \omega_{\gamma_t}(\hat{X}, \hat{Y})$$

where ω is the Kähler form of M and $\hat{X}$, $\hat{Y}$ are the infinitesimal transformations on M corresponding to X, $Y \in \mathfrak{g}$. From the semisimplicity of G, it can be shown that there exists a unique map $Z_\omega\colon \mathbb{R} \to C_\mathfrak{g}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{a}$ such that $\omega_t(X, Y) = \mathcal{B}([Z_\omega(t), X], Y)$, where $\mathcal{B}$ is the Cartan-Killing form of $\mathfrak{g}$. The basic fact of our study is that *the complex structure J and the Kahler metric g on M can be completely recovered from the complex structure J_K induced on the flag manifold G/K and from the function $Z_\omega(t)$.*

In order to state this result in a more precise way, let us point out some crucial properties of $Z_\omega(t)$ (see §4). M is known to admit exactly two singular orbit G/H_1 and G/H_2 and we may always assume that γ is parametrized so that $\gamma(0)$ and $\gamma(\mathrm{d})$ are the intersections with those singular orbits. Consider also an element $Z \in C_\mathfrak{g}(\mathfrak{l})$ such that $\mathfrak{a} = \mathbb{R}Z$ and $\mathcal{B}(Z, Z) = -1$. Then, we prove that:

a) the image $\ell_Z \stackrel{\text{def}}{=} Z_\omega(\mathbb{R})$ is a segment parallel to Z with endpoints given by $Z_1 = Z_\omega(0)$ and $Z_2 = Z_\omega(\mathrm{d})$ and each inner point $P \in \ell_Z$ is a regular element for the flag manifold G/K, i.e. $C_G(P) = K$; furthermore, if α is any root so that E_α is a $+i$-eigenvector of J, then $\alpha(P) \in i\mathbb{R}^+$;
b) for each endpoint Z_i, $i = 1, 2$, $C_G(Z_i)/K \simeq \mathbb{C}P^{n_i}$ for some $n_i \geq 0$ and the complex structure J_K projects naturally onto an invariant complex structure on $G/C_G(Z_i)$.

For a pair (G, L) with $C_\mathfrak{g}(\mathfrak{l}) = \mathfrak{l} + \mathbb{R}Z$, we call *admissible* any segment which verifies a) and b) (see Definition 5.1). Moreover, if Z_i is one endpoint of an admissible segment, we call *degree of the endpoint* the integer $m_i = n + 1$ if $C_G(Z_i)/K = \mathbb{C}P^{n_i}$. Finally, if we denote by $H_i = C_G(Z_i)$, we say that two admissible segments ℓ_Z and ℓ'_Z are *equivalent* if the corresponding triples (H_1, L, H_2) and (H'_1, L, H'_2) determine two equivalent abstract models of compact cohomogenity one G-manifolds with singular orbits (for the precise defintion, see [1]).

We call *abstract model of a K.O.P. manifold* a pentuple $\mathcal{K} = (G, L, J, \ell_Z, f)$ of the following form:

i) G is a compact semisimple Lie group; $L \subset G$ is a closed subgroup with $C_\mathfrak{g}(\mathfrak{l}) =$

$\mathfrak{z}(\mathfrak{l}) + \mathbb{R}Z$;

ii) J is a G-invariant complex structure on the flag manifold G/K, $K = N^o_G(L)$;

iii) ℓ_Z is an admissible segment;

iv) f is a smooth real function on $\mathbb{R}$ with values in $[0, \mathrm{length}(\ell_Z)]$, which is monotone on some interval $]0, \mathrm{d}[$, it is invariant by the symmetries $t \to -t$, $t \to 2\mathrm{d} - t$, and $f(0) = 0$, $f(\mathrm{d}) = \mathrm{length}(\ell_Z)$, $f''(0) = -f''(\mathrm{d}) = 1$.

(see Definition 5.1). Then our first main theorem is the following.

Theorem 1.2. *Let $\mathcal{M}$ be the moduli space (w.r.t. G-equivariant biholomorphic isometries) of compact K.O.P. manifolds (M, g, J), with vanishing first Betti number and let $\mathcal{A}$ be the collection of equivalence classes of abstract models. Then:*

(1) *there exists a bijective correspondence between $\mathcal{M}$ and $\mathcal{A}$;*

(2) *if (M, J, g) and (M', J', g') are realizations of the abstract models $\mathcal{K}, \mathcal{K}'$ given by $\mathcal{K} = (G, L, J, \ell_Z, f)$ and $\mathcal{K}' = (G', L', J', \ell'_Z, f')$ respectively, then M and M' are equivalent as G-manifolds if and only if $G/L = G'/L'$ and $\ell_Z \simeq \ell'_Z$.*

For the exact relation between an abstract model and the geometric data of its realizations, see Cor. 4.5 and Thm. 5.2.

Let us come now to the Kähler-Einstein manifolds. From Theorem 1.2, the classification of compact Kähler-Einstein manifolds reduces to characterizing the abstract models associated to Einstein metrics. The result is easily obtained and it is the following second main theorem.

In the statement, we denote by $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ the orthogonal decomposition w.r.t. $\mathcal{B}$ of the Lie algebra $\mathfrak{g}$ of G with $\mathfrak{l} + \mathbb{R}Z = C_{\mathfrak{g}}(\mathfrak{l})$. By R we denote the roots of $\mathfrak{g}^{\mathbb{C}}$ corresponding to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}^{\mathbb{C}}$ and by $R_{\mathfrak{m}}$ the roots corresponding to root vectors in $\mathfrak{m}^{\mathbb{C}}$. We also assume that $R^+_{\mathfrak{m}}$ stands for the roots so that $J_K(E_\alpha) = +iE_\alpha$ and Z^κ denotes the element $Z^\kappa = \sum_{\alpha \in R^+_{\mathfrak{m}}} H_\alpha$, where H_α denotes the element of $\mathfrak{h}$ corresponding to the root α under the isomorphism $\mathcal{B}^* : \mathfrak{h}^* \to \mathfrak{h}$ induced by the Cartan-Killing form $\mathcal{B}$.

Theorem 1.3. *An abstract model $\mathcal{K} = (G, L, J, \ell_Z, f)$ corresponds to an Einstein-Kähler manifold with Einstein constant $c = 1$ if and only if the following conditions holds:*

(1) *if Z_1 and Z_2 are the two endpoints of ℓ_Z and m_1 and m_2 are their degrees, then*

$$|Z_1 - Z_2| = m_1 + m_2 \quad , \quad \frac{m_2}{m_1 + m_2} Z_1 + \frac{m_1}{m_1 + m_2} Z_2 = Z^\kappa ; \tag{1.3}$$

(2) *the following Futaki integral vanishes*

$$\int_{-m_1}^{m_2} y \prod_{\alpha \in R^+_{\mathfrak{m}}} \alpha(Z^\kappa - yZ) dy = 0 ; \tag{1.4}$$

(3) $f(d) = m_1 + m_2$ *and the inverse function* $t(f)$ *of* $f|_{[0,d]}$ *is*

$$t(f) = \int_0^f \frac{\sqrt{\prod_{\alpha\in R^+_{\mathfrak{m}}} \alpha(Z^\kappa - s)Z}}{-2\int_{-m_1}^s v\prod_{\alpha\in R^+_{\mathfrak{m}}} \alpha(Z^\kappa - vZ)dv} ds\ . \tag{1.5}$$

A realization (M, J) *of an abstract model* $\mathcal{K}$ *has two singular orbits of codimension* $2m_1$ *and* $2m_2$*, respectively, and it admits a Kähler-Einstein metric with Einstein constant* $c = 1$ *if and only if the admissible segment* ℓ_Z *verifies* (1) *and* (2); *on such complex manifold the Kähler-Einstein metric is unique.*

REMARK 1.4. All the examples of compact Kähler-Einstein metrics given by Koiso and Sakane in [13] have ordinary action and projectable complex structure and they exhaust the realizations of the abstract models of Theorem 1.3. Therefore, our result can be restated saying that *the compact, Kähler-Einstein K.O.P. manifolds are exactly all those given by Koiso and Sakane in* [13].[1]

We note that if a quadruple (G, L, J, ℓ_Z) verifies part 1) and 2) of Thm. 1.3, then there exists an f which verifies condition 3); from this, it can be inferred that our result gives an independent proof of Theorem 5.2 in [12].

REMARK 1.5. The admissible segments with $m_1 = m_2 = 1$, which verify the condition (1) of Theorem 1.3 are diameters of the sphere $\mathcal{S} = \{Z\colon |Z - Z^\kappa| = 1\}$. If $\mathcal{S}$ is included in the T-chamber of Z^κ, it is not hard to select at least one admissible segment which verifies also condition (2). This gives a way to build compact K.O.P. manifolds, with singular orbits of codimension 2 and which admit a Kähler-Einstein metric.

The admissible segments with $m_1 \neq 1$ or $m_2 \neq 1$, which verify (1) of Theorem 1.3 are much more rare, so that condition (2) of the same theorem creates an extremely strong obstraction to the existence of K.O.P. manifolds with Einstein metric and singular orbits with codimension higher than 2. Examples of this kind are given by the complex projective space $\mathbb{CP}^n$: the group $G \subset U(n+1)$ can be taken either to have one fixed point or to be a product group $G = G_1 \times G_2$, where the semisimple parts $G_i^{ss} = SU(n_i)$ or $Sp(\frac{n_i}{2})$ for $i = 1, 2$ and $n_1 + n_2 = n + 1$. It would be interesting to know other non trivial examples of such K.O.P. manifolds and if they can be explicitly listed.

We conclude observing that any compact K.O.P. manifold is also almost homogeneous w.r.t. to the action of the complexified group $G^{\mathbb{C}}$; we refer to [11] for a general theory.

2. First properties of compact cohomogeneity one Kähler manifolds

[1] We like to mention that, after a preliminary version of this paper ([14]) had been completed, we received an interesting preprint by A. Dancer and M. Wang ([8]) concerning non compact Kähler-Einstein metrics of cohomogeneity one.

2.1 Cohomogeneity one compact Kähler manifolds Throughout the following (M, g, J) will denote a compact Kähler manifold with vanishing first Betti number, where g is the metric tensor, J the complex structure and $\omega = g(\cdot, J\cdot)$ is the associated Kähler fundamental form. The Lie algebra of a Lie group acting on M will be always denoted by the corresponding gothic letter. Furthermore, if X is any element of $\mathfrak{g}$, the symbol $\hat{X}$ will be used for the corresponding Killing vector field on M. Finally, $\mathcal{B}$ denotes the Cartan-Killing form of any semisimple Lie algebra.

We assume that G is a compact connected Lie group of isometries of M (and hence also of holomorphic transformations, since M is compact) with at least one hypersurface orbit. In this situation, it is well known (see [5], for instance) that either all orbits are regular and of codimension one or they are all regular except exactly two singular orbits. The orbit space $\Omega = M/G$ is diffeomorphic to S^1 in the first case, while it is equivalent to $[0, 2\pi]$ in the second case. Indeed, assuming that $b_1(M) = 0$, there is no fibration $M \to S^1$ and so, *M admits exactly two singular orbits S_1 and S_2*. The subset M_{reg} of all regular points is dense in M and the stability subgroups are all conjugate to the same compact subgroup, say L.

Since the manifold M is orientable, every regular orbit is orientable and we may define a unit normal vector field ξ on the whole subset M_{reg}. It is known (see e.g. [2]) that any integral curve of ξ in M_{reg} is a geodesic. Actually, for any point $p \in M_{\text{reg}}$, the geodesic $\gamma : \mathbb{R} \to M$ such that $\gamma'_0 = \xi(p)$ intersects orthogonally all regular orbits and it is called *normal geodesic*. It is clear tha for any t, γ'_t is equal or opposite to ξ_{γ_t}. More precisely, assume that γ is parametrized so that $A = \gamma^{-1}(M_{\text{reg}}) \subset \mathbb{R}$ is of the form $A = \bigcup_{k\in\mathbb{Z}}]k\mathrm{d}, (k+1)\mathrm{d}[$, where $\mathrm{d} = \mathrm{dist}(S_1, S_2)$. Then, up to reversing the orientation, it is easily seen that

$$\gamma'_t = (-1)^k \xi_{\gamma_t} \quad t \in]k\mathrm{d}, (k+1)\mathrm{d}[. \tag{2.1}$$

2.2 The moment mapping and the flag manifolds associated to the orbits. We briefly recall the definition of the *moment mapping*. For a complete discussion of its main properties see e.g. [9].

For any $X \in \mathfrak{g} = Lie(G)$ the facts that $\mathcal{L}_{\hat{X}}\omega = d(\imath_{\hat{X}}\omega) = 0$ and that $b_1(M) = 0$ imply the existence a potential $f_X \in C^\infty(M)$ such that $df_X = \imath_{\hat{X}}\omega$. By the compactness of M, it can be proved the potential f_X can be chosen for all X in a consistent way so that the mapping $\mu : M \to \mathfrak{g}^*$

$$\mu(m)(X) = f_X(m), \quad m \in M$$

is well defined and G-equivariant (we consider the coadjoint action of G on $\mathfrak{g}^*$). Any such map is usually called *moment mapping* and it is unique in case G is semisimple.

The existence of a moment mapping μ brings to the following basic result.

Proposition 2.1. *Suppose that G is a compact connected Lie group acting on M with cohomogenity one. Let $\gamma\colon \mathbb{R} \to M$ be a normal geodesic, $L \subset G$ be the common stability subgroup at all the points $\gamma(\mathbb{R}) \cap M_{\text{reg}}$ and, for any t, let*

$$K_t = \{g \in G;\ Ad(g)^*(\mu(\gamma_t)) = \mu(\gamma_t)\} \supset G_{\gamma_t}\ .$$

Then

(1) *G/K_t is a flag manifold for any $t \in \mathbb{R}$;*
(2) *for $t \in A = \gamma^{-1}(M_{\text{reg}})$, L is a subgroup of codimension one in K_t.*

Furthermore, if G acts almost effectively, then the center $Z(G)$ is at most one dimensional.

Proof. (1) follows from the fact that $\mu(G \cdot \gamma_t) = G/K_t$ is a coadjoint orbit in $\mathfrak{g}^*$ (for this property and others regarding the flag manifolds, see [4]). For (2) we only need to evaluate the dimension of the fiber of the mapping $\mu_p\colon G/L \to G/K_t$ for any $p = \gamma_t$ with $t \in A$.

If $v \in \ker d\mu_p \cap T_p(G/L)$, we may find $Y \in \mathfrak{g}$ with $\hat{Y}_p = v$ and we have, for all $Z \in \mathfrak{g}$

$$\begin{aligned} 0 = \left.\frac{d}{ds}\right|_{s=0} \mu(\exp(sY)p)(Z) = \left.\frac{d}{ds}\right|_{s=0} f_Z(\exp(sY)p) = \\ = df_Z|_p(Y) = (i_{\hat{Z}}\omega)|_p(Y) = \omega_p(Z, Y). \end{aligned}$$

Since the pull back of ω to the orbit $G \cdot p = G/L$ has one-dimensional kernel, it follows that $\dim \ker d\mu_p = 1$, and hence $\dim K_t/L = 1$.

Finally, we note that if G acts (almost) effectively on M, then so does it on each regular orbit G/L; if $\dim Z(G) > 1$, since $Z(G) \subset K_t$ for all t, we would have that $\dim Z(G) \cap L > 0$ and this contradicts the (almost) effectiveness of the G-action on G/L. □

From the previous result, G is either semisimple or it admits a one dimensional center. We will discuss the second case in a separate paper and from now on, we will suppose that *the group G is semisimple*. Recall that being G/K_t a flag manifold, K_t is a maximal rank subgroup of G. This puts strong restrictions on the pair (G, L). In fact

Lemma 2.2. *Let G be a compact, semisimple Lie group G and let L be a closed Lie subgroup which is a codimension one subgroup of a maximal rank Lie subgroup K of G. Then the centralizer $C_{\mathfrak{g}}(\mathfrak{l})$ of the Lie algebra $\mathfrak{l}$ in $\mathfrak{g}$ admits a decomposition*

$$C_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{a}$$

where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$ and $\mathfrak{a}$ is either a 1-dimensional Lie subalgebra or a rank one simple Lie algebra with $\mathfrak{a}^{\mathbb{C}} = \mathfrak{g}(\alpha) = span_{\mathbb{C}} < E_{\pm\alpha}, H_\alpha >$, for some root α of a Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$.

Proof. Let $\mathfrak{l}$ and $\mathfrak{k}$ the Lie algebras of L and K and let $\mathfrak{l}^\perp$ the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{g}$ w.r.t. the Cartan-Killing form $\mathcal{B}$. By hypotheses, we may decompose the Lie algebra $\mathfrak{k}$ of K as

$$\mathfrak{k} = \mathfrak{l} + \mathbb{R}Z,$$

for some non zero $Z \in (\mathfrak{k} \cap \mathfrak{l}^\perp)$. Obviously $[\mathfrak{l}, Z] = 0$. Moreover, if we fix a maximal torus $\mathfrak{t}$ of $\mathfrak{l}$, we have that $\hat{\mathfrak{t}} = \mathfrak{t} + \mathbb{R}Z$ is a maximal torus for $\mathfrak{k}$ and hence also for $\mathfrak{g}$. It then follows that $\hat{\mathfrak{t}}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ and we have the standard decomposition

$$\mathfrak{g}^{\mathbb{C}} = \hat{\mathfrak{t}}^{\mathbb{C}} + \sum_{\alpha \in R} \mathbb{C}E_\alpha ,$$

where R denotes the root system of the pair $(\mathfrak{g}^{\mathbb{C}}, \hat{\mathfrak{t}}^{\mathbb{C}})$. Moreover we have

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{l}^{\mathbb{C}} + \mathbb{C}Z = \mathfrak{t}^{\mathbb{C}} + \sum_{\beta \in R_K} \mathbb{C}E_\beta,$$

for some root subsystem $R_K \subset R$.

Now, it is clear that the centralizer $C_{\mathfrak{g}}(\mathfrak{l})$ contains $\mathfrak{z}(\mathfrak{l}) + \mathbb{R}Z$ and, if it is strictly bigger, then there is at least one $\alpha \in R \setminus R_K$ with $E_\alpha \in C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{l}^{\mathbb{C}})$. It then follows that α vanishes on $\mathfrak{t}^{\mathbb{C}}$; since $\mathfrak{t}^{\mathbb{C}}$ has codimension one in the CSA, there are at most two roots, say $\{\pm\alpha\}$, with this property and the centralizer $C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{l}^{\mathbb{C}})$ is given by

$$C_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{l}^{\mathbb{C}}) = \mathfrak{z}(\mathfrak{l}^{\mathbb{C}}) + \mathfrak{a}^{\mathbb{C}}, \quad \mathfrak{a}^{\mathbb{C}} = span_{\mathbb{C}} < Z, E_\alpha, E_{-\alpha} > .$$

Note that $[E_\alpha, E_{-\alpha}]$ is $\mathcal{B}$-orthogonal to $\mathfrak{t}^{\mathbb{C}}$, hence is a multiple of Z and we have that $\dim C_G(L) - \dim Z(L) = 3$. In particular, the sum $\mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$ coincides with $C_{\mathfrak{g}}(\mathfrak{l})$ and it is a direct sum. □

In the hypotheses of Lemma 2.2, it can be proved also the follwing fact (see [3]): $\mathfrak{a}$ *is a rank one simple Lie algebra if and only if G is simple, $L = C_G(\mathfrak{g}(\alpha))$ and the root α verifies the following property:* for any root β which is orthogonal to α, $\alpha \pm \beta \notin R$.

In particular, from this it can be checked that the pairs (G, L), with $G \neq G_2$ and for which $\mathfrak{a}$ is a 3-dimensional simple Lie algebra are in one to one correspondence with the homogeneous quaternionic Kähler manifolds of simple Lie groups (the so called *Wolf spaces*).

DEFINITION 2.3. Let G be a compact semisimple Lie group and let L be a compact subgroup which is contained in a subgroup of maximal rank as a subgroup of codimension one. We will say that the pair (G, L) is *ordinary* if

$$\dim C_G(L) = \dim Z(L) + 1$$

where $C_G(L)$ and $Z(L)$ are the centralizer of L in G and the center of L, respectively. It is called *extra-ordinary* otherwise. Similarly, we will also say that the action of G on M is *ordinary* (resp. *extra-ordinary*) if the regular orbits are of codimension one and G-equivalent to G/L, for some ordinary (resp. extra-ordinary) pair (G, L).

We conclude this section with the following immediate Corollary of Proposition 2.1.

Corollary 2.4. *Let G be a compact semisimple Lie group acting on M with ordinary action and let γ be a normal geodesic. Then for any $t \in A = \gamma^{-1}(M_{\mathrm{reg}})$, the moment mapping μ fibers the orbit $G \cdot \gamma_t = G/L$ over the same flag manifold G/K.*

3. The invariants of cohomogeneity one compact Kahler manifolds

The purpose of this section is to determine a set of invariants which can be defined on the family of flag manifold G/K_t of Proposition 2.1 and by which the complex and the Riemannian structure of M can be uniquely characterized.

We keep the same notations as in the previous section; we also fix the following $\mathcal{B}$-orthogonal decomposition of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{n} = \mathfrak{l} + \mathfrak{a} + \mathfrak{m} \,, \tag{3.1}$$

where $\mathfrak{n} = \mathfrak{a} + \mathfrak{m}$ and $\mathfrak{a}$ is the orthogonal complement to $\mathfrak{z}(\mathfrak{l})$ in $C_{\mathfrak{g}}(\mathfrak{l})$, as in Lemma 2.2. In case the action of G is ordinary, we set $\mathfrak{a} = \mathbb{R}Z$ for a fixed element Z with $\mathcal{B}(Z, Z) = -1$.

By means of the decomposition (3.1), for each $t \in A = \gamma^{-1}(M_{\mathrm{reg}})$, we have an identification map $\phi_t : \mathfrak{n} \to T_{\gamma_t}(G \cdot \gamma_t)$ given by

$$\phi_t(X) = \hat{X}_{\gamma_t} \,. \tag{3.2}$$

3.1 The invariant associated to the complex structure. The complex structure J of M induces an integrable G-invariant CR-structure on each regular orbit G/L. For any $p \in G/L$, we denote by $\mathcal{D}_p$ the maximal J_p-invariant subspace of $T_p(G/L)$ so that the CR structure on G/L is uniquely determined by the pair $(\mathcal{D}, J|_{\mathcal{D}})$.

By means of the identification (3.2), for any $t \in A$ we define the ad($\mathfrak{l}$)-invariant subspace $\mathfrak{m}_t \subset \mathfrak{n}$

$$\mathfrak{m}_t = \phi_t^{-1}(\mathcal{D}_{\gamma_t}) \,. \tag{3.3}$$

Note that for each $\mathfrak{m}_t$ there exists a unique (up to sign) ad($\mathfrak{l}$)-invariant element $Z_{\mathcal{D}}(t) \in \mathfrak{n}$ so that

$$Z_{\mathcal{D}}(t) \in \mathfrak{n} \cap \mathfrak{m}_t^{\perp} \,, \qquad \mathcal{B}(Z_{\mathcal{D}}(t), Z_{\mathcal{D}}(t)) = -1 \,. \tag{3.4}$$

Clearly, $Z_{\mathcal{D}}(t) \in C_{\mathfrak{g}}(\mathfrak{l})$ and hence, if the action of G is ordinary, we may assume that

$$Z_{\mathcal{D}}(t) \equiv Z\,. \tag{3.5}$$

By (3.2) we also define a complex structure J_t on $\mathfrak{m}_t$ for any $t \in A$

$$J_t = \phi_t^*(J|_{\mathcal{D}_{\gamma_t}})\,.$$

In the following, we will consider J_t as trivially extended to an endomorphism of $\mathfrak{n}$ for any t. It is clear that each J_t is ad($\mathfrak{l}$)-invariant. From the integrability of the CR-structures $(\mathcal{D}, J|_{\mathcal{D}})$ it can also easily proved that, for all $X, Y \in \mathfrak{m}_t$,

$$[J_tX, Y]_{\mathfrak{n}} + [X, J_tY]_{\mathfrak{n}} \in \mathfrak{m}_t \tag{3.6}$$

$$J_t([J_tX, Y]_{\mathfrak{n}} + [X, J_tY]_{\mathfrak{n}}) = ([J_tX, J_tY] - [X, Y])_{\mathfrak{n}}. \tag{3.7}$$

Finally, for any $t \in A$, we may consider the unique (up to sign) element

$$Z_{J\xi}(t) \in \mathbb{R}\phi_t^{-1}(J\xi_{\gamma_t})\,, \qquad \mathcal{B}(Z_{J\xi}(t), Z_{J\xi}(t)) = -1\,. \tag{3.8}$$

We note that even the vector $Z_{J\xi}(t)$ is in $\mathfrak{a}$, because the tangent vector $J\xi_{\gamma_t}$ is left fixed under the isotropy representation of L. Therefore if the action of G is ordinary, we may always choose

$$Z_{J\xi}(t) \equiv Z\,. \tag{3.9}$$

It is immediate to realize that the map

$$\mathcal{J}\colon A \to \mathfrak{a} \times End(\mathfrak{n}) \times \mathfrak{a}$$

$$\mathcal{J}(t) \stackrel{\text{def}}{=} (Z_{\mathcal{D}}(t), J_t, Z_{J\xi}(t))$$

determines uniquely the complex structure J at all tangent spaces $T_{\gamma_t}M$ and, being J G-invariant, it characterizes completely J over M_{reg}. Furthermore, it is unique up to G-equivalence (i.e. G-invariant biholomorphic isometry) in the following sense: *if (M, J, g) and (M', J', g') are G-equivalent and γ and γ' are two normal geodesics of M and M' which are mapped one onto the other by a G-equivariant isometric biholomorphism, then the corresponding maps $\mathcal{J}$ and $\mathcal{J}'$ coincide (up to the signs of $Z_{\mathcal{D}}$ and of $Z_{J\xi}$).*

We will call $\mathcal{J}$ the *invariant associated to the complex structure.*

Finally, let us consider the following definition: we say that the complex structure J of M is *projectable* if

$$Ad(K_t) \cdot J_t = J_t \tag{3.10}$$

for any $t \in A$. This means that each CR structures J_t descends to a G-invariant almost complex structure J_{K_t} on the flag manifold G/K_t and such that the natural map $\pi\colon G/L \to G/K_t$ verifies

$$\pi_*(J_t(X)) = J_{K_t}\pi_*(X) \qquad \forall X \in \mathcal{D}$$

Note that (3.7) implies that each J_{K_t} is an *integrable* complex structure on G/K_t.

Now let us limit ourselves to considering ordinary actions. In this case, we have already seen that $Z_{\mathcal{D}}(t) \equiv Z_{J\xi}(t) \equiv Z$ and hence that $\mathcal{J}$ is uniquely given by the family of complex structures J_t on the subspace $\mathfrak{m} \equiv \mathfrak{m}_t$. If furthermore J is projectable, each J_t is mapped onto an invariant complex structure of the same flag manifold $G/K_t \equiv G/K$ (by Corollary 2.4). Since the invariant complex structures on a flag manifold are a discrete finite set (see e.g. [4]), we have that the complex structures J_t are constant on each connected interval of A. Hence we conclude that *if the action of G is ordinary and J is projectable, then the invariant $\mathcal{J}$ is uniquely determined by a corresponding G-invariant complex structure J_K on the flag manifold G/K.*

REMARK 3.1. It can be proved that, for a *generic* compact Kähler manifold with ordinary, cohomogeneity one action, the complex structure J *is* projectable (see [AS]). However, a useful criterion which guarantees the projectability of the complex structure is given by the following lemma of straightforward proof.

Lemma 3.2. *Suppose that the action of G on M is ordinary and consider the decomposition $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$ in* (3.1). *If every ad($\mathfrak{l}$)-irreducible submodule of $\mathfrak{m}$ appears with multiplicity one, then the complex structure J is projectable.*

EXAMPLE 3.3. Consider the Hermitian symmetric space $M = SO(n+2)/SO(2) \times SO(n)$, which is the complex quadric Q_n of $\mathbb{C}P^{n+1}$ and which provides one compactification of the tangent bundle of a sphere S^{n+1}. The group $G = SO(n+1)$ acts with cohomogeneity one on M with principal isotropy subgroup $L = SO(n-1)$. A normal geodesic is explicity given by

$$\gamma_t = \left[\begin{pmatrix} \cos t & 0 & -\sin t & \\ 0 & 1 & 0 & \mathbb{O} \\ \sin t & 0 & \cos t & \\ & \mathbb{O} & & \mathbb{I}_{n-1} \end{pmatrix}\right],$$

where $[\cdot]$ denotes the projection of an element of $SO(n+2)$ onto the quotient space M. The pair (G, L) is ordinary, but the complex structure is not projectable. In fact, keeping the notation as above, if we write $\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$, then $\mathfrak{m}$ splits as the sum of two equivalent, irreducible ad($\mathfrak{l}$)-submodules $\mathfrak{m}_1, \mathfrak{m}_2 \cong \mathbb{R}^{n-1}$; moreover an easy computation shows that the complex structure $J_t \in End(\mathfrak{m})$ is given, for $(v_1, v_2) \in \mathfrak{m}_1 + \mathfrak{m}_2$, by

$$J_t((v_1, v_2)) = (v_2/\sin t, -(\sin t)v_1).$$

These endomorphisms do not commute with ad(Z) and hence with ad($\mathfrak{k}$). Note that ad(Z) represents the standard ad($\mathfrak{l}$)-invariant complex structure $(v_1, v_2) \mapsto (-v_2, v_1)$ on $\mathfrak{m}$.

3.2 The invariants associated to the Kähler form and to the metric tensor. Let us consider the usual normal geodesic γ. For any $t \in \mathbb{R}$, consider the following 2-form $\{\omega_t\}_{t\in\mathbb{R}}$ on $\mathfrak{g}$:

$$\omega_t(X, Y) \stackrel{\text{def}}{=} \omega_{\gamma_t}(\hat{X}, \hat{Y}), \quad X, Y \in \mathfrak{g}.$$

Since the Kahler form ω is closed and every Killing vector field preserves it, we have that

$$0 = 3d\omega(\hat{X}, \hat{Y}, \hat{W}) = \omega(\hat{X}, [\hat{Y}, \hat{W}]) + \omega(\hat{W}, [\hat{X}, \hat{Y}]) + \omega(\hat{Y}, [\hat{W}, \hat{X}]) ,$$

so that for all $t \in \mathbb{R}$ and for all $X, Y, W \in \mathfrak{g}$ we have

$$\omega_t([X, Y], W) + \omega_t([W, X], Y) + \omega_t([Y, W], X) = 0 . \tag{3.11}$$

Since $\mathfrak{g}$ is semisimple, we may find $F_t \in End(\mathfrak{g})$ with $\omega_t(X, Y) = \mathcal{B}(F_t(X), Y)$ for all $X, Y \in \mathfrak{g}$. Now, from (3.11) and the non degenerancy of $\mathcal{B}$, we get that

$$F_t([X, Y]) = [F_t(X), Y] + [X, F_t(Y)],$$

that is each F_t is a derivation of $\mathfrak{g}$. But then for each $t \in \mathbb{R}$ there exists an unique Z_ω such that

$$F_t = ad(Z_\omega(t)) .$$

Note that, for all $t \in A$, we may say that $\ker F_t = \mathfrak{l} + \mathbb{R}v_t$, where $v_t \in \mathfrak{l}^\perp$ and $\hat{v}_t|_{\gamma_t} = J\xi$. Therefore $\ker F_t$ coincides with the Lie algebra of the subgroup K_t defined in Proposition 2.1 and the flag manifold G/K_t is G-equivalent to the G-orbit of $Z_\omega(t)$ in $\mathfrak{g}$. In particular, for any $t \in A$, $Z_\omega(t)$ is $ad(\mathfrak{l})$-invariant and $Z_\omega(t) \in \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$.

In conclusion, to the G-equivalence class of (M, g, J) we may uniquely associate the map

$$Z_\omega \colon \mathbb{R} \to \mathfrak{g} \quad , \quad Z_\omega|_A \colon A \to \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$$

$$\mathcal{B}([Z_\omega(t), X], Y) = \omega_{\gamma_t}(\hat{X}, \hat{Y}) \tag{3.12}$$

We will call $Z_\omega(t)$ *the invariant associated to the Kahler form.*

REMARK 3.4. If we decompose $\mathfrak{g} = \mathfrak{k}_t + \mathfrak{m}_t$ with $\mathfrak{m}_t$ $\mathcal{B}$-orthogonal and ad($\mathfrak{k}_t$)-invariant subspace, the restriction $\omega_t|_{\mathfrak{m}_t}$ is ad($\mathfrak{k}_t$)-invariant and hence it defines a symplectic form on the flag manifold G/K_t.

Note that, by the previous observations, $Z_\omega(t) \neq 0$ for all $t \in A$; it can be checked that $Z_\omega(t) = 0$ only when the orbit $G \cdot \gamma_t$ is a point or a totally real submanifold.

Now, let us use the identification (3.2) to determine an invariant which can be used to characterize the metric tensor. For each $t \in A$, let us consider the ad($\mathfrak{l}$)-invariant, positively definite bilinear form on $\mathfrak{n}$ given by the pull back of the Riemannian metric:

$$g_t \stackrel{\text{def}}{=} \phi_t^*(g|_{T_{\gamma_t}G(\gamma_t)}).$$

Note that $g_t|_{\mathfrak{m}_t}$ is completely determined by the invariants $\mathcal{J}$ and Z_ω: in fact for all $t \in A$

$$g_t|_{\mathfrak{m}_t} = -\phi_t^*(\omega(\cdot, J\cdot)|_{\mathcal{D}_{\gamma_t}}) = -\mathcal{B}([Z_\omega(t), \cdot], J_t\cdot)|_{\mathfrak{m}_t}$$

Therefore, once $\mathcal{J}$ and Z_ω are given, the only ingredient that is necessary to recover the whole tensor g_t is the function $g_t(Z, Z) = ||\hat{Z}_{\mathcal{D}}(t)||^2$. We define as *invariant associated to the metric tensor* the following map $a\colon \mathbb{R} \to \mathbb{R}$

$$a(t) = \begin{cases} (-1)^k ||\hat{Z}_{\mathcal{D}}(t)||_{\gamma(t)} & t \in]k\mathrm{d}, (k+1)\mathrm{d}[\\ 0 & t \in \mathbb{Z}\mathrm{d} \end{cases} \tag{3.13}$$

where d is the distance between the two singular orbits and the parametrization of γ is assumed to satisfy (2.1). The definition has been motivated by the fact that, if the action is ordinary and $Z_{\mathcal{D}} \equiv Z$, such a function is indeed $\mathcal{C}^\infty$ over $\mathbb{R}$, as we will show in the next section.

4. The properties of the invariants $\mathcal{J}$, Z_ω and a on the K.O.P. manifolds

From now on, we will assume that the action of G is ordinary and that the complex structure is projectable, i.e. that M is a compact *K.O.P.* manifold. The first fact is that $\mathcal{J} \equiv (Z, J, Z)$ where J is a CR structure which is projectable onto an invariant complex structure on G/K. We continue with the following

Proposition 4.1. *Let Z_ω and a be the invariants associated to the Kähler form and the metric tensor of M. Then they are both smooth functions of t and they verify the ordinary differential equation*

$$\frac{d}{dt} Z_\omega(t) = -a(t) Z\ . \tag{4.1}$$

Moreover, if we denote by H_1, H_2 the two singular stability subgroups of the singular points γ_0 and γ_d respectively, then:

(1) $H_1 \cap H_2 = K$;
(2) *the singular orbits G/H_i are complex submanifolds of M;*
(3) $H_i = C_G(Z_\omega(t_i))$, $t_i = 0$ *or* d, *and* $\mu(G/H_i) \simeq G/H_i$;
(4) *any homogeneous space H_i/K is diffeomorphic to a complex projective space;*

In particular the geodesic symmetries σ_1, σ_2 at γ_0 and γ_d belong to K.

Proof. For the proof, we only need to show that (4.1) is verified at all $t \in \mathbb{R}$, so that the smoothness of a will follow immediately from the smoothness of Z_ω.

In order to do this, we write down the Kähler condition $d\omega = 0$ and find the corresponding condition on Z_ω. Let us consider two arbitrary vectors $v_1, v_2 \in \mathfrak{m}$, so that $\phi_t(v_i) \in \mathcal{D}_{\gamma_t}$ for all $t \in A$. The condition $d\omega = 0$ imply that

$$\begin{aligned}
0 &= 3d\omega(\xi, \hat{v}_1, \hat{v}_2) = \xi\omega(\hat{v}_1, \hat{v}_2) - \hat{v}_1\omega(\xi, \hat{v}_2) + \hat{v}_2\omega(\xi, \hat{v}_1)\\
&\quad - \omega([\xi, \hat{v}_1], \hat{v}_2) + \omega([\xi, \hat{v}_2], \hat{v}_1) - \omega([\hat{v}_1, \hat{v}_2], \xi)\\
&= \xi\omega(\hat{v}_1, \hat{v}_2) - \omega(\xi, [\hat{v}_1, \hat{v}_2]) + \omega(\xi, [\hat{v}_2, \hat{v}_1]) - \omega([\hat{v}_1, \hat{v}_2], \xi)\\
&= \xi\omega(\hat{v}_1, \hat{v}_2) + \omega([\hat{v}_1, \hat{v}_2], \xi),
\end{aligned} \tag{4.2}$$

where we have used the fact that $\mathcal{L}_{\hat{v}_i}\omega = 0$ and $[\hat{v}_i, \xi] = 0$ for $i = 1, 2$. Now (4.2) reduces to the following for $t \in]k\mathrm{d}, (k+1)\mathrm{d}[$

$$(-1)^k \frac{d}{dt}\mathcal{B}([Z_\omega(t), v_1], v_2) - g_t([v_1, v_2]_{\mathfrak{n}}, (-1)^k \frac{1}{a(t)} Z) = 0. \tag{4.3}$$

Since now Z and $\mathfrak{m}$ are both g_t- and $\mathcal{B}$-orthogonal, (4.3) becomes

$$(-1)^k \frac{d}{dt}\mathcal{B}(Z_\omega(t), [v_1, v_2]) + (-1)^k \frac{1}{a(t)} g_t(Z, Z)\mathcal{B}([v_1, v_2]_{\mathfrak{n}}, Z) = 0,$$

or equivalently

$$(-1)^k \frac{d}{dt}\mathcal{B}(Z_\omega(t), [v_1, v_2]) + (-1)^k a(t)\mathcal{B}([v_1, v_2], Z) = 0,$$

so that the vector $\frac{d}{dt}Z_\omega(t) + a(t)Z \in \mathfrak{z}(\mathfrak{k})$ must be $\mathcal{B}$-orthogonal to $[\mathfrak{m}, \mathfrak{m}]$. We now recall the decompositions

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in R_K} \mathbb{C}E_\alpha, \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in R} \mathbb{C}E_\alpha = \mathfrak{k}^{\mathbb{C}} + \sum_{\alpha \in R \setminus R_K} \mathbb{C}E_\alpha,$$

where $\mathfrak{t}$ is the Lie algebra of a maximal torus of K, R is the root system of the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ and R_K is the root subsystem given by $R_K = \{\alpha \in R; \alpha|_{\mathfrak{z}(\mathfrak{k})} = 0\}$. Now if

$v = \frac{d}{dt}Z_\omega(t) + a(t)Z \neq 0$, we choose a root $\beta \in R$ such that $\beta(v) \neq 0$; then $\beta \notin R_K$ and therefore $E_\beta, E_{-\beta} \in \mathfrak{m}^{\mathbb{C}}$; so

$$0 \neq \beta(v) = \mathcal{B}(v, [E_\beta, E_{-\beta}]) \in \mathcal{B}(v, [\mathfrak{m}^{\mathbb{C}}, \mathfrak{m}^{\mathbb{C}}]).$$

This contradiction shows that $v = 0$ and equation (4.2) is equivalent to (4.1) at all $t \in A$.

It remains to show that (4.1) holds also when $t \in \mathbb{Z}\mathrm{d}$. By definition of a, this amounts to show that a is everywhere continuous, or that $\hat{Z}$ vanishes at the singular points $\gamma_{k\mathrm{d}}$ for $k \in \mathbb{Z}$. We show this for $t = 0$, since for all other $t = k\mathrm{d}$, the argument would be the same.

Let $H_1 \supset L$ be the stability sugroup of the singular point γ_0 and let $\mathfrak{h}_1$ be its Lie algebra. Consider the decomposition $\mathfrak{h}_1 = \mathfrak{l} + \mathfrak{p}$, with $\mathfrak{p} = \mathfrak{h} \cap \mathfrak{l}^\perp$ and $[\mathfrak{l}, \mathfrak{p}] \subseteq \mathfrak{p}$.

Assume that $Z \notin \mathfrak{h}_1$. Then $Z \notin \mathfrak{p}$, so that $Z = Z_1 + Z_2$ with $Z_1 \in \mathfrak{p}$ and $0 \neq Z_2 \in \mathfrak{p}^\perp \cap \mathfrak{l}^\perp$. Since Z centralizes $\mathfrak{l}$, we have that $Z_2 \in C_{\mathfrak{g}}(\mathfrak{l}) \cap \mathfrak{l}^\perp$ and hence $Z_2 \in \mathbb{R}Z$. Therefore we conclude that $Z \in \mathfrak{p}^\perp$ and that $\mathfrak{p} \subset \mathfrak{m}$.

We now take $v \in \mathfrak{p} \setminus \{0\}$ and consider the function $\varphi(t) = ||\hat{v}||^2_{\gamma_t}$ with $t \in \mathbb{R}$. Since $v \notin \mathfrak{l}$, the function φ has a minimum for $t = 0$, so that $\varphi'(0) = 0$. On the other hand, for $t \in]0, \mathrm{d}[$, we have that

$$\varphi(t) = \omega(J\hat{v}, \hat{v})_{\gamma_t} = \mathcal{B}([Z_\omega(t), Jv], v) ,$$

and therefore, by (4.1) on $]0, \mathrm{d}[$,

$$\frac{d}{dt}\varphi(t) = -||\hat{Z}||_{\gamma_t}\mathcal{B}([Z, Jv], v) ,$$

with $\mathcal{B}([Z, Jv], v) \neq 0$, since the function φ is not constant. This implies that $\hat{Z}_{\gamma_0} = 0$, hence $Z \in \mathfrak{h}_1$. This contradiction concludes the proof of (4.1). Furthermore we also obtained that $K \subseteq H_1 \cap H_2$.

Next, we prove (2), i.e. the singular orbits are complex submanifolds.

We recall the decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and observe that, being G/K a flag manifold, the $\mathfrak{k}$-module $\mathfrak{m}$ splits as sum of irreducible and inequivalent $\mathfrak{k}$-submodules $\mathfrak{m}_i, i = 1, \ldots, s$, each of them stable under the projectable complex structure J. Since $\mathfrak{k} \subseteq \mathfrak{h}_i$, then the $\mathcal{B}$-orthogonal complement $\mathfrak{h}_i^\perp$ in $\mathfrak{g}$ is sum of some of the submodules $\mathfrak{m}_j$ and therefore is J-stable. Let $v \in \mathfrak{h}_i^\perp$ and consider the vector field $\hat{v}_{\gamma_t}$ for $t \in [0, \mathrm{d}[$; along the normal geodesic γ_t with $t \in]0, \mathrm{d}[$, we have

$$J\hat{v}_{\gamma_t} = \widehat{Jv}_{\gamma_t}.$$

Since both members are continuous vector fields, we get that $J\hat{v}_p = \widehat{Jv}_p \in T_pG/H_i$, where $p = \gamma_0$ and this proves that each singular orbit is complex.

It is now easy to recognize that each $\mathfrak{h}_i$ is the centralizer in $\mathfrak{g}$ di $Z_\omega(0)$: this proves (3) and the fact that each H_i is connected by Hopf's theorem. Moreover from the inclusion $L \subset K \subseteq H$, we get a fibration

$$S^1 \hookrightarrow H/L \to H/K.$$

Since H/L is a sphere (see e.g. [1]), we get that H/K must be diffeomorphic to some complex projective space and this proves (4).

We now want to prove that $H_1 \cap H_2 = K$. Indeed, we first note that H_1, H_2, K share a common maximal torus T of G and $Z(H_i) \subseteq Z(K) \subset T$; now it is easy to see that $H_1 \cap H_2$ is the centralizer in G of the torus $Z(H_1) \cdot Z(H_2) \subset T$, hence it is connected with Lie algebra $\mathfrak{h}_1 \cap \mathfrak{h}_2$. If $K \neq H_1 \cap H_2$, we may find a non zero vector $v \in \mathfrak{m} \cap \mathfrak{h}_1 \cap \mathfrak{h}_2$; for such v, we consider the function $\varphi(t) = ||\hat{v}||^2_{\gamma_t}$ for $t \in [0, \mathrm{d}]$. Since $\varphi(0) = \varphi(\mathrm{d}) = 0$, by Rolle theorem we get that $\varphi'(c) = 0$ for some $c \in (0, \mathrm{d})$; but, by

$$\varphi'(t) = -a(t)\mathcal{B}([Z, Jv], v) = \pm||\hat{Z}||_{\gamma_t}\mathcal{B}([Z, Jv], v)$$

$\varphi'(t)$ cannot vanish in the open interval $]0, \mathrm{d}[$ and we are done.

We now come to our last claim about geodesic symmetries. It is known (see [1]) that at any singular point $\gamma_{k\mathrm{d}}$ we can find an element σ belonging to the stabilizer $G_{\gamma_{k\mathrm{d}}}$ such that σ reverses the normal geodesic γ; it is clear that σ normalizes L. In our case we have shown that each singular orbit is a complex submanifold; hence the slice representation of, say, H_1 must preserve some complex structure on the normal space. It then follows that, if ν_1 denotes the slice representation, then $\nu_1(H_1)$ is one of the following groups: $SU(m), U(m), Sp(m), T^1 \cdot Sp(m)$. Now, in each of these cases, the geodesic symmetry lies in the connected component $N^o_H(L)$ and therefore $\sigma \in N^o_G(L) = K$. □

REMARK 4.2. The fact that the singular orbits are complex submanifolds relies on the projectability of the complex structure J. Indeed the cohomogeneity one action of $G = SO(n+1)$ on the complex quadric Q_n admits a totally real singular orbit. Note that a singular orbit could be neither complex nor totally real; such an example is given by a cohomogeneity one action of the group $G = U(n+1)$ on Q_{2n} (see [7]).

It is also known (see e.g. [11]) that, given a cohomogeneity one Kahler G-manifold, we can always blow up the singular orbits and reduce ourselves to the case when both singular orbits have complex codimension one. However, as far as we know, there is no control on the differential-geometric properties of the metric when we blow up. At this regard, we refer the reader also to the paper by Koiso-Sakane ([13]).

We want now to determine a fundamental relation between the invariants Z_ω and a. We begin with a technical lemma.

Lemma 4.3. *Let γ be parametrized so that (2.1) holds and let a be defined as in (3.13). Then (up to a reparametrization $t \to -t$) for any $t \in A$,*

$$\frac{1}{a(t)} \hat{Z}_{\gamma_t} = J\gamma_t' \tag{4.4}$$

Proof. First of all, observe that both $\frac{1}{a(t)} \hat{Z}_{\gamma_t}$ and $J\gamma_t'$ are ad($\mathfrak{l}$)-invariant vectors in $T_{\gamma_t}(G \cdot \gamma_t)$. By means of the identification (3.2), they have to correspond to two ad($\mathfrak{l}$)-invariant vectors in $\mathfrak{n}$ and hence, since the action of G is ordinary, they both belong to $\mathbb{R}Z$. Furthermore, $|a(t)| = ||\hat{Z}||_{\gamma_t}$ and hence both vectors are of unit length. So we may assume that (4.4) holds for all $t \in]-\mathrm{d}, 0[$. On the other hand, for $t \in]0, \mathrm{d}[$, we may consider the geodesic symmetry $\sigma \in H_1$, which lies in K by Prop. 4.1. So,

$$J\gamma_t' = -\sigma_*(J\gamma'(-t)) = \frac{1}{||\hat{Z}||_{\gamma(-t)}} \sigma_*(\hat{Z}_{\gamma(-t)}) = \frac{1}{||\hat{Z}||_{\gamma_t}} \widehat{Ad(\sigma)Z}_{\gamma_t} = \frac{1}{||\hat{Z}||_{\gamma_t}} \hat{Z}_{\gamma_t}$$

where we have used the fact that $Z \in \mathfrak{z}(\mathfrak{k})$. This proves (4.4) in the interval $]0, \mathrm{d}[$. By reflecting the geodesic at all singular points, the same argument proves (4.4) at all $t \in A$. □

We continue with an important proposition which will bring to a complete description of the invariant Z_ω.

Proposition 4.4. *Let γ parametrized so that γ_0 and γ_{d} are the intersections with the two singular orbits. Then $Z_\omega(t)$ is even with respect to the reflections of the parameter $t \to -t$ and $t \to 2d - t$. In particular, $Z_\omega(t)$ and $a(t)$ are both periodic of period $T = 2d$.*

Proof. To see that Z_ω is invariant with respect to the reflection $t \to -t$ (or $t \to 2\mathrm{d}-t$), consider the geodesic symmetry σ at the singular point γ_0 (or γ_{d}) such that $\gamma(-t) = \sigma \cdot \gamma_t$. We know that $\sigma \in K$ by Prop. 4.1. Moreover, using the definition of Z_ω we obtain that

$$Z_\omega(-t) = Ad(\sigma) Z_\omega(t) = Z_\omega(t)$$

because $Z_\omega(t) \in \mathfrak{z}(\mathfrak{l}) + \mathfrak{a}$. This proves the claim. □

We conclude by summing up the results of Propositions 4.1, 4.2 and 4.3 in order to get the following straightforward characterization of the image $Z_\omega(\mathbb{R})$. For the statement, we need to recall a few basic fact on the flag manifold G/K (see e.g. [4]). An element $X \in \mathfrak{z}(\mathfrak{k})$ is called *regular* if $C_G(X) = K$. The set of regular elements is open and dense in $\mathfrak{z}(\mathfrak{k})$ and each connected component is called *T-chamber* of the flag manifold. Let $\mathfrak{t}$ be a maximal torus for $\mathfrak{k}$ (and hence also for $\mathfrak{g}$) and R (resp. R_K) be the root system of the pair $(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ (resp. of $(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$). Then $X \in \mathfrak{z}(\mathfrak{k})$ is regular if and only if $\alpha(X) \neq 0$ for all

$\alpha \in R \setminus R_K$. Finally, recall that for any G-invariant complex structure J on G/K there exists an ordering of the roots so that, for any $\alpha \in R \setminus R_K$, $JE_\alpha = +iE_\alpha$ if and only if $\alpha \in R^+$. The T-chamber of the elements $\{iX \in \mathfrak{z}(\mathfrak{k}) : \alpha(X) > 0\}$ is uniquely associated to J and it is called the *positive T-chamber corresponding to J.*

Corollary 4.5. *Let G/K be the flag manifold which is the image by the moment map of the regular orbit G/L of M. Then the trace $\ell_Z = Z_\omega(\mathbb{R})$ is a segment parallel to Z, which verifies the following properties:*

i) *ℓ_Z is contained in the closure of the positive T-chamber corresponding to J;*
ii) *the centralizers H_1 and H_2 of the endpoints Z_1 and Z_2 of ℓ_Z are such that $H_1 \cap H_2 = K$ and $H_i/K \simeq \mathbb{C}P^{m_i}$ for some m_i (in case Z_i is inner to the T-chamber, H_i/K is point and we consider $m_i = 0$);*
iii) *there exist two invariant complex structures J_i, $i = 1, 2$ on the flag manifolds G/H_i such that the projections*

$$\pi_1 \colon (G/K, J_K) \to (G/H_i, J_i)$$

are holomorphic;

iv) *the length of ℓ_Z is equal to $\int_0^{\mathrm{d}} ||\hat{Z}||_{\gamma_t} dt$.*

Proof. The only claim that it is not an immediate corollary of the previous results is the fact that ℓ_Z lies in the positive T-chamber of J; but this is a consequence of the fact that the symmetric two form on $\mathfrak{m}$

$$g_t(X,Y) \stackrel{\text{def}}{=} \mathcal{B}([Z_\omega(t), J_K X], Y)$$

is positive definite at all $t \in \mathbb{R}$. $\square$

5. Abstract models: proof of the Theorem 1.2

In the following, G is a compact semisimple Lie group, G/K is a flag manifold and L is a closed subgroup of codimension one in K. As usual we will consider the decomposition

$$\mathfrak{g} = \mathfrak{l} + \mathfrak{n} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m}$$

with the meaning of the symbols as in the previous sections. Assume also that the pair (G, L) is ordinary. We may now give the definition of abstract model:

DEFINITION 5.1. Let J_K be a complex structure on G/K and let ℓ_Z an oriented segment in $\mathfrak{z}(\mathfrak{k})$ which is parallel to Z. We say that it is an *admissible segment for* (G, L, J_K) if it verifies i), ii) and iii) of Corollary 4.5.

Denote by Z_i, $i = 1, 2$, the endpoints of ℓ_Z: we call *degree* of Z_i the integer $\deg(Z_i) = m_i + 1$, where m_i is the complex dimension of $H_i/K \simeq \mathbb{C}P^{m_i}$ (note: in case Z_i is inner to the T-chamber, we set $\deg(Z_i) = 1$).

If $C = ||Z_2 - Z_1||$, w.r.t. $\mathcal{B}$, a smooth function $f : \mathbb{R} \to [0, C]$ is said *admissible parametrization of* ℓ_Z if:

a) $f(0) = 0$, $f(\mathrm{d}) = C$ and it is monotone increasing in the interval $]0, \mathrm{d}[$, for some d;

b) it is invariant by the symmetries $t \to -t$ and $t \to 2\mathrm{d} - t$;

c) $f''(0) = 1 = -f''(\mathrm{d})$.

A pentuple of the form $\mathcal{K} = (G, L, J_K, \ell_Z, f)$ where ℓ_Z is an admissible segment and f is an admissible parametrization of ℓ_Z is called *abstract model for a compact K.O.P. manifold* (= Kähler manifold with Ordinary and Projectable cohomogeneity one action).

We say that two abstract models (G, L, J_K, ℓ_Z, f) and $(G', L', J_{K'}, \ell_{Z'}, f')$ are *equivalent* if and only if: i) $G = G'$ and $(G/K, J_K)$ and $(G'/K', J_{K'})$ are biholomorphic via an automorphism φ of G; ii) $\varphi(L) = L'$; iii) $\varphi_*(Z) = Z'$ and $\varphi_*(\ell_Z) = \ell_{Z'}$ with the same orientation, or $\varphi_*(Z) = -Z'$ and $\varphi_*(\ell_Z)$ coincides with $\ell_{Z'}$ but with opposite orientation; iv) $f = f'$.

The proof of Theorem 1.2 coincides with the proof of following Theorem 5.2, since the second claim of Theorem 1.2 is a direct consequence of the definitions.

Theorem 5.2. *For any abstract model $\mathcal{K} = (G, L, J_K, \ell_Z, f)$, there exists a compact K.O.P. G-manifold (M, g, J) such that:*

a) *the invariant associated to the complex structure is $\mathcal{J}(t) = (Z, J_t, Z)$, where J_t coincide for all $t \in A$ with the unique complex structure which projects onto the complex structure J_K of G/K;*

b) *the invariant associated to the Kähler form is $Z_\omega(t) = Z_1 - f(t)Z$, where Z_1 is the first endpoint of ℓ_Z;*

c) *the invariant ociated to the metric tensor is the function $a(t) = f'(t)$*

We call such a (M, g, J) a realization of the abstract model $\mathcal{K}$.

Any compact K.O.P. manifold (M, g, J) is G-equivalent to a realization of an abstract model $\mathcal{K}$ and this corresponding abstract model is the same (up to equivalence of abstract models) for any other manifold in the same G-equivalence class.

Proof. We only prove the existence of a realization for any abstract model, since the proof of uniqueness up G-equivalence is straightforward. As usual, we will denote by $H_i = C_G(Z_i)$ the centralizers of the endpoints of ℓ_Z.

By definitions, the triple (H_1, L, H_2) defines a G-manifold which is of cohomogeneity one w.r.t. G, with regular orbits equivalent to G/L and with two singular orbits G/H_1 and G/H_2 respectively (see [1] for a detailed exposition). More precisely, each subgroup H_i, $i = 1, 2$, has an orthogonal representation $\rho_i : H_i \to O(V_i)$ in some vector space

V_i such that $\rho_i(H_i)$ acts transitively on the unit sphere $S(V_i)$ and $S(V_i) \cong H_i/L$; the manifold M is then obtained by glueing together two disk bundles over the two singular orbits G/H_i. Since the two singular orbits are flag manifolds, hence simply connected, by Seifert-Van Kampen Theorem, the resulting manifold M will be simply connected.

We begin with a Lemma which will be useful later on.

Lemma 5.3. *For $i = 1, 2$, we have that $\rho_i(H_i)$ is $SU(k_i), U(k_i)$ or $T^1 \cdot Sp(k_i)$ for some $k_i \geq 1$.*

Proof. The proof is obtained by inspection of Borel's list of compact Lie groups H acting transitively on some sphere with isotropy L, when we consider only ordinary pairs (H, L). □

We further recall that each singular orbit G/H_i, $i = 1, 2$ has a G-invariant tubular neighborhood which is G-diffeomorphic to the total space of the vector bundle $G \times_{H_i} V_i$ over G/H_i.

We may identify the regular part M_{reg} with $G/L \times]0, \mathrm{d}[$ and we shall consider the curve $\gamma :]0, \mathrm{d}[\to M_{\text{reg}}$ given by $\gamma(s) = (eL, s)$. Given the splitting

$$\mathfrak{g} = \mathfrak{l} + \mathbb{R}Z + \mathfrak{m},$$

we can identify the tangent space $T_p M_{\text{reg}}$ with $\mathbb{R}\frac{\partial}{\partial s} + \mathbb{R}Z + \mathfrak{m}$; moreover we choose a basis $\{e_1, \ldots, e_m\}$ of $\mathfrak{m}$ so that $\{\frac{\partial}{\partial s}, \hat{Z}, \hat{e}_j\}_{\gamma(s)}$ is a frame along γ. We shall denote by ds and η the 1-forms which are dual to $\frac{\partial}{\partial s}, \hat{Z}$ respectively.

We may now define a G-invariant metric g on M_{reg} by

$$g = ds^2 + f'(s)^2 \eta^2 + g|_{\hat{\mathfrak{m}} \times \hat{\mathfrak{m}}}, \tag{5.1}$$

where, for $X, Y \in \mathfrak{m}$,

$$g|_{\hat{\mathfrak{m}} \times \hat{\mathfrak{m}}}(\hat{X}, \hat{Y}) = \mathcal{B}([Z_\omega(s), J_K X], Y), \tag{5.2}$$

where $Z_\omega(s) = Z_1 - f(s)Z$. We now prove that g extends smoothly to a G-invariant Riemannian metric on the whole M. In order to do that, we restrict g to $M_1 = G \times_{H_1} V_1$.

First of all, we note that $\mathfrak{m}$ admits a further splitting as $\mathfrak{m} = \mathfrak{m}_1 + \tilde{\mathfrak{m}}$, where $\mathfrak{h}_1 = \mathfrak{k} + \mathfrak{m}_1$; the summand $\tilde{\mathfrak{m}}$ defines a smooth distribution $\mathcal{H}$ on the whole M_1, so that (5.2) and property (i) in Cor. 4.5 allow to extend smoothly $g|_{\mathcal{H} \times \mathcal{H}}$.

We are therefore left with proving that the restriction of g on the slice $\mathbb{V}$ in M_1 defined as

$$\mathbb{V} = \{[(e; v)] \; v \in V_1 \} \subset G \times_{H_1} V_1,$$

extends smoothly on the whole $\mathbb{V}$.

We denote by g_o the standard Euclidean metric on V_1, which is invariant by the linear group $\rho_1(H_1)$ and we express it as

$$g_o = ds^2 + s^2\eta^2 + g_o|_{\hat{\mathfrak{m}}_1 \times \hat{\mathfrak{m}}_1}.$$

We observe that both $g|_{\hat{\mathfrak{m}}_1 \times \hat{\mathfrak{m}}_1}$ and $g_o|_{\hat{\mathfrak{m}}_1 \times \hat{\mathfrak{m}}_1}$ are Ad(K)-invariant; according to Lemma 5.3, the Ad(K)-module $\mathfrak{m}_1$ is either irreducible or it splits as the sum of two irreducible, inequivalent submodules $\mathfrak{m}_1^{(0)}, \mathfrak{m}_1^{(1)}$. Therefore we my write that

$$g|_{\hat{\mathfrak{m}}_1 \times \hat{\mathfrak{m}}_1} = \lambda^2(s) g_o|_{\hat{\mathfrak{m}}_1 \times \hat{\mathfrak{m}}_1} \ ,$$

if $\mathfrak{m}_1$ is Ad(K)-irreducible, or otherwise

$$g|_{\hat{\mathfrak{m}}_1 \times \hat{\mathfrak{m}}_1} = \lambda^2(s) g_o|_{\widehat{\mathfrak{m}_1^{(0)}} \times \widehat{\mathfrak{m}_1^{(0)}}} + \mu^2(s) g_o|_{\widehat{\mathfrak{m}_1^{(1)}} \times \widehat{\mathfrak{m}_1^{(1)}}}$$

for suitable positive functions λ, μ. From (5.2) and the properties of $Z_\omega = Z_1 - fZ$, it is clear that $\lambda(s)$ and $\mu(s)$ admit smooth and even C^∞-extensions over $\mathbb{R}$.

Using the results in [16], we have that $g|_{T\mathbb{V}}$ extends smoothly as a Riemannian metric on the whole $T\mathbb{V}$ if and only if the functions $\frac{f'(s)}{s}, \lambda(s), \mu(s)$ extend to smooth, even and positive functions over $\mathbb{R}$ with $\lim_{s \to 0} \frac{f'(s)}{s} = 1$ and these conditions are satisfied in view of the previous remarks and our hypothesis on $f(s)$.

In the same way we prove that the metric g extends smoothly on the whole $M_2 = G \times_{H_2} V_2$.

We now want to define an almost complex structure $\mathbb{J}$ on M. We first define it on the regular part by putting, along the curve γ,

$$\mathbb{J}(\frac{\partial}{\partial s}) = \frac{1}{f'(s)}\hat{Z}, \quad \mathbb{J}(\hat{Z}) = -f'(s)\frac{\partial}{\partial s}, \quad \mathbb{J}|_{\hat{m}} = J_K|_{\hat{m}} \ .$$

It is clear that the metric g is $\mathbb{J}$-Hermitian. We want to prove that $\mathbb{J}$ extends to a smooth almost complex structure on the whole M. Using property (iii) in Cor. 4.5 and the same arguments as in the proof of extendability of the metric, we are left with proving that $\mathbb{J}$ extends smoothly on the whole slice $\mathbb{V}$ in M_1. The restriction of $\mathbb{J}$ on $T\mathbb{V}$ takes the form $\mathbb{J}(\frac{\partial}{\partial s}) = \frac{1}{f'(s)}\hat{Z}, \mathbb{J}(\hat{Z}) = -f'(s)\frac{\partial}{\partial s}, \mathbb{J}|_{\hat{m}_1} = J_K|_{\hat{m}_1}$. Following Sakane ([15]), we change the parameter s by considering a new function

$$r(s) = \exp(\int_{s_o}^{s} \frac{1}{f'(u)} du), \quad s \in]0, \mathrm{d}[$$

where $s_o \in]0, \mathrm{d}[$ is a fixed value of the parameter.

We have

Lemma 5.4. *The function r admits a smooth and odd extension to $]-d, d[$.*

Proof. Indeed, by assumption, we know that the function $f'(u)$ admits an odd extension on the whole $\mathbb{R}$, so that we may write $f'(u) = u(1 + u^2 b(u))$, for some C^∞, even function b. Now, for $s \in]0, \mathrm{d}[$ we have

$$r(s) = \frac{s}{s_o} \exp\left(-\int_{s_o}^{s} \frac{ub(u)}{1 + u^2 b(u)} du\right) \tag{5.3}$$

Since f' does not vanish on $]-\mathrm{d}, 0[$, we have that the integral $\int_{s_o}^{s} \frac{ub(u)}{1+u^2b(u)} du$ has a natural C^∞ extension on the whole $]-\mathrm{d}, \mathrm{d}[$ as an even function. Then (5.3) shows that r extends as claimed. □

If we now define a map ϕ on $U^* = \{v \in \mathbb{V} \setminus \{[e, 0]\};\ ||v|| < \mathrm{d}\}$, where the norm $|| \cdot ||$ refers to the Euclidean metric g_o, by

$$\phi(v) = \frac{r(||v||)}{||v||} v,$$

then the property of the function r established in Lemma 5.4, shows that ϕ extends to a diffeomorphism of $U = U^* \cup \{[e, 0]\}$ onto some symmetric neighborhood of 0 in $\mathbb{V}$. It is now easy to verify that

$$\begin{aligned}(\phi_* \mathbb{J})\left(\frac{\partial}{\partial s}\right) &= \frac{1}{s} \hat{Z} \\ (\phi_* \mathbb{J})|_{\hat{m}_1} &= J_K|_{\hat{\mathfrak{m}}_1}.\end{aligned}$$

It then follows that the almost complex structure $\phi_* \mathbb{J}$ can be extended to the whole $T\mathbb{V}$, since it coincides with the unique (up to sign) complex structure on V_1 which is invariant by the linear group $\rho_1(H_1)$ (see Lemma 5.3).

Again, the same arguments can be applied to the submanifold $M_2 = G \times_{H_2} V_2$, proving that $\mathbb{J}$ extends smoothly to an almost complex structure on M. We now prove that $\mathbb{J}$ is integrable.

Lemma 5.5. *The almost complex structure $\mathbb{J}$ is integrable*

Proof. We will check this, proving that the Nijenhuis tensor N of $\mathbb{J}$ vanishes on the regular part, which is dense in M.

The regular part M_{reg} fibers as

$$\pi : M_{\text{reg}} \cong G/L \times]0, \mathrm{d}[\to G/K,$$

with typical fibre $F \cong S^1 \times]0, \mathrm{d}[$; since the vertical space $\mathcal{V}$ of π is $\mathbb{J}_1$-invariant and the fibres are 2-dimensional, we have that $N(\mathcal{V}, \mathcal{V}) = 0$.

Moreover, the equation $N(\hat{\mathfrak{m}}, \hat{\mathfrak{m}}) = 0$ is automatic from the fact that $\mathbb{J}|_{\hat{\mathfrak{m}}}$ projects down to the integrable complex structure J_K on the flag manifold G/K. We are therefore left with proving that $N(\mathcal{V}, \hat{\mathfrak{m}}) = 0$.

If we denote by ξ the vector field $\frac{\partial}{\partial s}$, where $s \in]0, \mathrm{d}[$, then it is enough to prove that

$$N(\xi, \hat{\mathfrak{m}}) = 0 \tag{5.4}$$

We choose a basis $\{e_1, \dots, e_m\}$ of $\mathfrak{m}$ and compute

$$\begin{aligned} N(\xi, \hat{e}_i) &= [\mathbb{J}\xi, \mathbb{J}\hat{e}_i] - [\xi, \hat{e}_i] - \mathbb{J}[\xi, \mathbb{J}\hat{e}_i] - \mathbb{J}[\mathbb{J}\xi, \hat{e}_i] \\ &= [\mathbb{J}\xi, \mathbb{J}\hat{e}_i] - \mathbb{J}[\xi, \mathbb{J}\hat{e}_i] . \end{aligned}$$

Now, we fix a regular point $p \in M_{1\,\mathrm{reg}}$ with $G_p = L$ and we write locally around p

$$\mathbb{J}\hat{e}_i = a\xi + b\hat{Z} + \sum_{k=1}^{m} c_k \hat{e}_k,$$

for some C^∞ functions a, b, c_k; from the construction of $\mathbb{J}$ it follows that, if γ denotes the integral curve of ξ through p, then $a \circ \gamma = b \circ \gamma = 0$ and the $c_k \circ \gamma$ are constant functions for $k = 1, \dots, m$.

It is then straightforward to see that $[\xi, \mathbb{J}\hat{e}_i]_p = 0$. Therefore,

$$N(\xi, \hat{e}_i)_p = [\mathbb{J}\xi, \mathbb{J}\hat{e}_i]_p.$$

If we now write locally

$$\mathbb{J}\xi = \beta\hat{Z} + \sum_{k=1}^{m} \mu_k \hat{e}_k$$

for some C^∞ functions β, μ_k with $\mu_k(p) = 0$, we can compute

$$\begin{aligned} [\beta\hat{Z} + \sum_{k=1}^{m} \mu_k \hat{e}_k, \mathbb{J}\hat{e}_i]_p &= (-(\mathbb{J}\hat{e}_i)(\beta)\hat{Z} + \beta[\hat{Z}, \mathbb{J}\hat{e}_i] - (\mathbb{J}\hat{e}_i)(\mu_k)\hat{e}_k)_p = \\ &= (-(\widehat{J_K e_i})(\beta)\hat{Z} + \beta[\hat{Z}, \mathbb{J}\hat{e}_i] - (\widehat{J_K e_i})(\mu_k)\hat{e}_k)_p \end{aligned} \tag{5.5}$$

We now prove the following

Sublemma 5.6. *We have*

$$[\hat{Z}, \mathbb{J}\hat{e}_i]_p = [\hat{Z}, \widehat{J_K e_i}]_p.$$

Proof. Indeed

$$[\hat{Z}, \mathbb{J}\hat{e}_i]_p = \mathbb{J}[\hat{Z}, \hat{e}_i]_p = -\mathbb{J}\widehat{[Z, e_i]}_p = \\ = -\widehat{J_K[Z, e_i]}_p = -\widehat{[Z, J_K e_i]}_p = [\hat{Z}, \widehat{J_K e_i}]_p,$$

where we have used the fact that J_K is ad(Z)-invariant. □

From Sublemma 5.6 and (5.5), we conclude that

$$N(\xi, \hat{e}_i)_p = [\mathbb{J}\xi, \mathbb{J}\hat{e}_i]_p = [\mathbb{J}\xi, \widehat{J_K e_i}]_p = 0\,. \qquad \square$$

We are left with proving that the metric g is Kähler, that is the associated fundamental form ω is closed. Again, we will check this on the regular part. The condition $d\omega = 0$ on M_{reg} is equivalent to the following four equations along the normal geodesic $\gamma(s)$:

i) $d\omega(\hat{v}_1, \hat{v}_2, \hat{v}_3) = 0$;
ii) $d\omega(\xi, \hat{v}_1, \hat{v}_2) = 0$;
iii) $d\omega(\hat{Z}, \hat{v}_1, \hat{v}_2) = 0$;
iv) $d\omega(\xi, \hat{Z}, \hat{v}_1) = 0$,

where v_1, v_2, v_3 belong to $\mathfrak{m}$ and $\xi = \frac{\partial}{\partial s}$.

We note that equations (i) and (iii) are equivalent to condition (3.11). Equation (ii) is equivalent to the equation (4.1) satisfied by the invariant $Z_\omega(s)$.

As for the last equation (iv), using again the fact that $\mathcal{L}_{\hat{v}_1}\omega = \mathcal{L}_{\hat{Z}}\omega = 0$, we get

$$\begin{aligned} d\omega(\xi, \hat{Z}, \hat{v}_1) &= \xi\omega(\hat{Z}, \hat{v}_1) - \hat{Z}\omega(\xi, \hat{v}_1) + \hat{v}_1\omega(\xi, \hat{Z}) \\ &\quad - \omega([\xi, \hat{Z}], \hat{v}_1) + \omega([\xi, \hat{v}_1], \hat{Z}) - \omega([\hat{Z}, \hat{v}_1], \xi) = \\ &= \xi\omega(\hat{Z}, \hat{v}_1) + \omega(\xi, [\hat{v}_1, \hat{Z}]) = \\ &= \frac{d}{ds}\mathcal{B}([Z_\omega(s), Z], v_1) + g_s(Z, [v_1, Z]) = 0, \end{aligned}$$

since $Z_\omega(s)$ and Z commute and Z is g_s-orthogonal to $\mathfrak{m}$.

This concludes the proof of the theorem. □

6. The Ricci tensor of a realization of $\mathcal{K} = (G, L, J_K, \ell_Z, f)$

In this section, we will compute the Ricci tensor of the Riemannian manifold (M, g) corresponding to an abstract model $\mathcal{K} = (G, L, J_K, \ell_Z, f)$.

We shall keep the same notations as in the previous section. Moreover, we denote by Z_1, Z_2 the endpoints of ℓ_Z, so that $Z_\omega(t) = Z_1 - f(t)Z$. Also, we fix a basis $\{v_1, \ldots, v_p\}$

for $\mathfrak{m}$ which is orthonormal with respect to $\mathcal{B}$. If g is the Riemannian metric determined by Z_ω, the norms $||\hat{v}_i||$ can be determined by the formula

$$||\hat{v}_i||^2 = \mathcal{B}([Z_1, Jv_i], v_i) - f(t)\mathcal{B}([Z, Jv_i], v_i) \stackrel{\text{def}}{=} h_i - f(t)k_i \; .$$

Along the geodesic $\gamma(t)$, at all the regular points, we will denote by e_i the normalized vectors parallel to the $\hat{v}_i|_t$, that is

$$e_i = \frac{1}{||\hat{v}_i||}\hat{v}_i = \frac{1}{\sqrt{h_i - fk_i}}\hat{v}_i \; .$$

We recall that $||\hat{Z}||^2 = a(t)^2 = f'(t)^2$. Finally, recall also that the Riemannian metric g verifies $g(\hat{Z}, \hat{v}_i) \equiv 0$ for all v_i. We will also denote by g_t the Riemannian metric induced on the regular orbit $G \cdot \gamma_t$.

Let us now start the computation of the Ricci curvature at the regular points. First of all we need the following Lemma (see also [15]):

Lemma 6.1. *Let r be the Ricci tensor of the G-invariant metric g on a realization M of $\mathcal{K}$. Then $r(\xi, \hat{\mathfrak{m}})|_{\gamma_t} = 0$.*

Proof. We recall that $[\mathfrak{l}, \mathfrak{m}] = \mathfrak{m}$. If $v \in \mathfrak{m}$, then $v = \sum_i [X_i, Y_i]$ for some $X_i \in \mathfrak{l}$ and $Y_i \in \mathfrak{m}$. Then

$$r(\xi, [\hat{X}_i, \hat{Y}_i])|_{\gamma_t} = \hat{X}_i r(\xi, \hat{Y}_i)|_{\gamma_t} - r([\hat{X}_i, \xi], \hat{Y}_i)|_{\gamma_t} = 0$$

since $X_i \in \mathfrak{l}$. □

Lemma 6.1 is not the only vanishing property of r. Indeed, note that the bilinear forms induced by g_t and r on $\mathfrak{m}$, via the identification map ϕ_t in (3.2), are both $ad(\mathfrak{k})$-invariant. Now, if $\mathfrak{m} = \mathfrak{m}_1 + \cdots + \mathfrak{m}_p$ is the ad($\mathfrak{k}$)-invariant decomposition of $\mathfrak{m}$ into irreducible submodules, we recall that the $\mathfrak{m}_i$'s are mutually non-equivalent, since K is the centralizer of a torus in G (see e.g. [17]). Therefore if $\mathfrak{a}_t$ is an ad($\mathfrak{k}$)-invariant, symmetric bilinear form on $\mathfrak{m}$, we have that $\mathfrak{a}_t(\mathfrak{m}_i, \mathfrak{m}_j) \equiv 0$ for $i \neq j$ and that $\mathfrak{a}_t|_{\mathfrak{m}_i}$ is a multiple of $\mathcal{B}|_{\mathfrak{m}_i}$ for all $i = 1, \ldots, p$. From such observations, it follows immediately that

$$g_t(\hat{v}_i, \hat{v}_j) = 0 \quad , \quad r(\hat{v}_i, \hat{v}_j)|_{\gamma_t} = 0 \qquad \forall i \neq j.$$

In conclusion, $r|_{\gamma_t}$ is uniquely determined by the values of $r(e_i, e_i)$ and $r(\xi, \xi)$.

We may now compute the Ricci tensor by considering the Riemannian submersions $\pi\colon (M_{\text{reg}}, g) \to (\mathbb{R}, dt)$ and $\kappa\colon (G \cdot \gamma(t), g_t) \to (G/K, g_t^{\kappa})$, where g_t^{κ} is the G-invariant metric on G/K induced by g_t. The reader interested in detailed computations is referred to [14].

We will also denote by A^π, A^κ, T^π, T^κ ... the usual O'Neill tensors (see e.g. [4]), which are related to the submersions π and κ, respectively.

By O'Neill's formulae for the submersion κ, we compute $r_t(e_i, e_i)$:

$$r_t(e_i, e_i) = r_t^\kappa(e_i, e_i) - 2g_t(A^\kappa_{e_i} J\xi, A^\kappa_{e_i} J\xi) =$$

$$= r_t^\kappa(e_i, e_i) - \frac{(f')^2}{2}\frac{k_i^2}{(h_i - fk_i)^2}$$

Now, in order to compute $r(e_i, e_i)$ we use O'Neill's formulae for the submersion π:

$$r(e_i, e_i) = r_t(e_i, e_i) - g(N^\pi, T^\pi_{e_i} e_i) + (\tilde{\delta} T^\pi)(e_i, e_i).$$

A lenghty but straightforward computation shows that

$$r(e_i, e_i) = r_t^\kappa(e_i, e_i) + \left[f'' - \frac{(f')^2}{4}\sum_m \frac{k_m}{h_m - fk_m}\right]\frac{\mathcal{B}([Z, Jv_i], v_i)}{||\hat{v}_i||^2},$$

so that for all $X, Y \in \mathfrak{m}$

$$\boxed{r(\hat{X}, \hat{Y}) = r_t^\kappa(\hat{X}, \hat{Y}) + \left[f'' - \frac{(f')^2}{4}\sum_i \frac{k_i}{h_i - fk_i}\right]\mathcal{B}([Z, JX], Y)} \tag{6.1}$$

The evaluation of $r(\xi, \xi)$ can also be computed by using O'Neill's formulas for the submersion π. So we get that

$$\boxed{r(\xi, \xi) = -\frac{f'''}{f'} + \frac{1}{2}f''\sum_i \frac{k_i}{h_i - fk_i} + \frac{1}{4}(f')^2\sum_i \frac{k_i^2}{(h_i - fk_i)^2}} \tag{6.2}$$

$$\left(= -\frac{1}{f'}\frac{d}{dt}\left[f'' - \frac{(f')^2}{4}\sum_i \frac{k_i}{h_i - fk_i}\right]\right) \tag{6.2$_1$}$$

We remark here that, using the fact that the 2-form ρ is closed and Lemma 6.1, if X and Y belong to $\mathfrak{m}$, then

$$\xi\rho(\hat{X}, \hat{Y}) = -g([\hat{X}, \hat{Y}], J\xi)r(\xi, \xi). \tag{6.3}$$

It then follows that, if the Einstein equation $r(\hat{X}, \hat{Y}) = cg(\hat{X}, \hat{Y})$ is fulfilled for some $c \in \mathbb{R}^+$ and for all $X, Y \in \mathfrak{m}$, then also the Einstein equation $r(\xi, \xi) = c$ is satisfied, by (6.3) and (4.1).

7. Einstein-Kahler manifolds of cohomogeneity one

Recall that if $(G/K, J_K)$ is a flag manifold with invariant complex structure J_K, then for any invariant Kahler matric g^κ compatible with J_K, the Ricci form ρ^κ can be written as

$$\rho^\kappa(\hat{X}, \hat{Y})_{eK} = \mathcal{B}([Z^\kappa, J_K X], Y), \quad X, Y \in \mathfrak{m} \tag{7.1}$$

where $Z^\kappa \in \mathfrak{z}(\mathfrak{k})$ *and it does not depend on the metric g but only on the complex structure* J_K (see e.g. [6]). We will call it the *Ricci invariant of* $(G/K, J_K)$ (the explicit expression of Z^κ is given in the Introduction).

Using this fact and (6.1) and (6.3) we have that a realization of $\mathcal{K}$ is a Kahler-Einstein manifold with Einstein constant c (i.e. $r = cg$) if and only if the invariant $Z_\omega(t) = Z_1 - f(t)Z$ verifies the following ordinary differential equation

$$\left[f'' - \frac{(f')^2}{4}\sum_i \frac{k_i}{h_i - fk_i} + cf\right] Z = cZ_1 - Z^\kappa \tag{7.2}$$

We will now look for admissible segments and parametrizations which solve (7.2) for $c = 1$.

First of all, there is a constant D such that

$$Z_1 - Z^\kappa = -DZ \tag{7.3}$$

$$f'' - \frac{(f')^2}{4}\sum_i \frac{k_i}{h_i - fk_i} + f + D = 0\,. \tag{7.4}$$

Let us call S_1 and S_2 the two singular orbits of a realization of $\mathcal{K}$ and recall that $S_i = G/H_i$, where $H_i = C_G(Z_i)$. Note also that the complex codimension of each S_i coincide with $m_i = \deg(Z_i)$ (see Def. 5.1 and Cor. 4.5). Furthermore, this implies that $2(m_1 - 1)$ equals the number of constants h_i which vanish and that $2(m_2 - 1)$ is the number of vanishing values $h_i - f(\mathrm{d})k_i$, where d is the distance between the two singular orbits.

Now if f is a solution of (7.4) for some open interval $]0, t_o[$, then it represents an admissible parametrization only if

$$\lim_{t\to 0^+} f(t) = 0 = \lim_{t\to 0^+} f'(t) \quad \lim_{t\to 0^+} f''(t) = 1$$

and this implies that

$$D = -(\lim_{t\to 0^+} f''(t) - \lim_{t\to 0^+} \frac{(f')^2}{4}\sum_{i=1} \frac{k_i}{h_i - fk_i}) = -m_1 \tag{7.5}$$

The analogous conditions for a solution defined in an open interval $]t_o, \mathrm{d}[$ are

$$\lim_{t\to \mathrm{d}^-} f(t) = f_{\mathrm{d}} > 0 \quad \lim_{t\to \mathrm{d}^-} f'(t) = 0 \quad \lim_{t\to \mathrm{d}^-} f''(t) = -1$$

and from this we get that

$$f_{\mathrm{d}} = -(\lim_{t\to \mathrm{d}^-} f''(t) - \lim_{t\to \mathrm{d}^-} \frac{(f')^2}{4}\sum_{i=1} \frac{k_i}{h_i - fk_i} - m_1) = m_1 + m_2 \tag{7.6}$$

Note that (7.3), (7.5) and (7.6) determine completely the endpoints Z_1 and $Z_2 = Z_1 - f_{\mathrm{d}} Z$ which are

$$Z_1 = m_1 Z + Z^{\kappa} \qquad Z_2 = -m_2 Z + Z^{\kappa}\ . \tag{7.7}$$

We may now prove the last main theorem.

Proof of Theorem 1.3. The necessity of (1.3) follows immediately from (7.6) and (7.7).

To prove (1.4), let us find an admissible parametrization f which solves (7.4) with $D = -m_1$. Let us introduce the notation

$$\mathcal{F}(f) \stackrel{\mathrm{def}}{=} \frac{1}{4}\sum_i \frac{k_i}{h_i - fk_i} \qquad \mathcal{H}(f) \stackrel{\mathrm{def}}{=} f - m_1$$

and consider the function p which verifies the differential equation $p(f) = f'$. Then (7.4) implies that p verifies $p'p - p^2\mathcal{F}(f) + \mathcal{H}(f) = 0$. So, if we set $p^2 = u$, then

$$\frac{1}{2}u' - u\mathcal{F}(f) + \mathcal{H}(f) = 0 \tag{7.8}$$

The general solution of (7.8) is

$$u = -2e^{2\int \mathcal{F}(v)dv}\left[\int_0^f \mathcal{H}(v)e^{-2\int \mathcal{F}(s)ds}dv + C\right] =$$

$$= -2\frac{1}{\sqrt{\prod_i |h_i - k_i f|}}\left[\int_0^f \sqrt{\prod_i |h_i - k_i v|}(v - m_1)dv + C\right] \tag{7.9}$$

where the constant C must be determined by the initial conditions. Since we want that $\lim_{t\to 0^+} f = 0$, we may immediately set $C = 0$

Moreover, since the solution of our original problem must verify $f(\mathrm{d}) = f_{\mathrm{d}} = m_1 + m_2$, $f'(t) > 0$ for $t \in]0, \mathrm{d}[$ and $f'(0) = f'(\mathrm{d}) = 0$, the solutions of (7.9) we are interested in verify the conditions

$$u(f) > 0 \quad f \in]0, f_{\mathrm{d}}[\,, \qquad \lim_{f \to 0^+} u(f) = 0 = \lim_{f \to f_{\mathrm{d}}^-} u(f)$$

This implies that $f_{\mathrm{d}} = m_1 + m_2$ is the first value after the 0 for which the following integral vanishes

$$\int_0^{m_1+m_2} \sqrt{\prod_i |h_i - k_i v|}(v - m_1)dv = 0 \tag{7.10}$$

Note that if this occurs, the only values for f on which u vanishes are exactly 0 and $m_1 + m_2$.

Changing the variable $y = v - m_1$, (7.10) reduces to

$$\int_{-m_1}^{m_2} y\sqrt{\prod_i |(h_i + m_1 k_i) - k_i y|}dy = 0. \tag{7.11}$$

Now, consider the root system defined in §1 and take as orthonormal basis $\{v_i\}$ the one formed by the vectors $\frac{1}{2}(E_\alpha + E_{-\alpha})$ and $\frac{1}{2i}(E_\alpha - E_{-\alpha})$. Then (7.7) implies immediately the necessity of (1.4).

Now, recalling that $u = p^2 = (f')^2$, from (7.9) we obtain that

$$t(f) = \int_0^f \frac{\sqrt[4]{\prod_i |h_i - k_i s|}}{\sqrt{-2\int_0^s \sqrt{\prod_i |h_i - k_i v|}(v - m_1)dv}} ds\,. \tag{7.12}$$

Then set $\mathrm{d} \stackrel{\mathrm{def}}{=} t(f_{\mathrm{d}}) = t((m_1 + m_2))$. The restriction of the desired function $f(t)$ on the interval $]0, \mathrm{d}[$ is the inverse function of (7.12). We now should verify that the function f extends to a C^∞-function on $\mathbb{R}$, which is invariant by the reflections at 0 and d. It is not difficult, by direct computation, to show that f extends to a C^3 function which is invariant by the reflections at 0 and d. Now we note that the function $a(t) = f'(t)$ gives rise to a C^2 Einstein metric g; by a result of DeTurck and Kazdan (see [4]), the metric g is real analytic in geodesic normal coordinates, so that the function f is C^∞.

The sufficiency of conditions (1), (2) and (3) is clear. □

References

[1] A.V. Alekseevsky, D.V.Alekseevsky: *G-manifolds with one dimensional orbit space*, Adv. in Sov. Math. **8** (1992), 1–31.

[2] ______: *Riemannian G-manifolds with one dimensional orbit space*, Ann. Glob. Anal. and Geom. **11** (1993), 197–211.

[3] D.V. Alekseevsky, A. Spiro: *Invariant CR-structures on compact homogeneous manifolds* (1997), preprint E.S.I..

[4] A.L. Besse: Einstein manifolds, Springer-Verlag, 1986.

[5] G.E. Bredon: Introduction to compact transformation groups, Acad. Press N.Y. London, 1972.

[6] M. Bordemann, M. Forger, H. Romer: *Paving the way towards supersymmetric sygma-models*, Comm. Math. Physics (1986), 103–145.

[7] B.Y. Chen, T. Nagano: *Totally geodesic submanifolds of symmetric spaces, I*, Duke Math. J. **44** (1977), 745–755.

[8] A. Dancer, M. Wang: *Kähler-Einstein metrics of cohomogeneity one and bundle constructions for Einstein Hermitian metrics* (1997), preprint.

[9] V. Guillemin, S. Sternberg: Symplectic techniques in physics, Cambridge Univ. Press, 1984.

[10] W.Y. Hsiang, H.B. Lawson: *Minimal submanifolds of low cohomogeneity*, J.Diff.Geom. **5** (1971), 1–38.

[11] A. Huckleberry, D. Snow: *Almost-homogeneous Kähler manifolds with hypersurface orbits*, Osaka J. math. **19** (1982), 763–786.

[12] N. Koiso, Y. Sakane: *Non-homogeneous Kähler-Einstein manifolds on compact complex manifolds*, Lecture Notes in Math., Springer Verlag **1201** (1986), 165–179.

[13] N. Koiso, Y. Sakane: *Non-homogeneous Kähler-Einstein metrics on compact complex manifolds II*, Osaka J. Math. **25** (1988), 933–959.

[14] F.Podesta, A. Spiro: *Kähler manifolds with large isometry group*, preprint (1997).

[15] Y. Sakane: *Examples of compact Kähler-Einstein manifolds with positive Ricci curvature*, Osaka Math. J. **23** (1986), 585–616.

[16] L. Verdiani: *Invariant metrics on cohomogeneity one manifolds*, preprint (1996).

[17] J.A. Wolf: Spaces of Constant Curvature, Univ. of California Press, Berkeley, 1972.

F. Podestà
Math. Institute, Via dell'Agnolo, I- Firenze
e-mail:podesta@ udini.math.unifi.it

A. Spiro
Math. Department, Via delle Brecce Bianche, I- Ancona
spiro@ anvax1.cineca.it