FINITE GROUPS WHICH ACT FREELY ON SPHERES

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We will study the problem: Let G be a finite group which acts freely (and topologically) on the sphere S^{2t-1}. Can G act freely and orthogonally on S^{2t-1}?

The result of T. Petrie [5] shows that the answer is no for t odd prime. As is easily seen, the answer is yes for $t=1$. The problem for $t=2$ is unsolved at present (see [2], [3], [4]). In this note it will be shown that the answer is yes for $t=4$, and also for $t=2^{\nu}$ ($\nu \geq 3$) if G is solvable.

1. Preliminary theorems

By J. Milnor [3] we have

(1.1) *If G is a group which acts freely on S^n, then G satisfies the following properties*:
  i) *Any element of order 2 in G belongs to the center of G.*
  ii) *G has at most one element of order 2.*
The following (1.2) and (1.3) are shown in [1].

(1.2) *If G acts freely on S^n, the cohomology of G has period $n+1$.*

(1.3) *For a finite group G, the following two conditions are equivalent*:
  i) *G has periodic cohomology.*
  ii) *Every abelian subgroup of G is cyclic.*
A complete classification of finite groups satisfying the condition ii) of (1.3) is known by H. Zassenhaus [11] and M. Suzuki [6]. For future reference we reproduce it below after J. Wolf [10] and C.B. Thomas-C.T.C. Wall [8].

(1.4) *Let G be a finite group satisfying the condition* ii) *of* (1.3). *If G is solvable, it is one of the following groups*:

Type	Generators	Relations	Conditions	Order
I	A, B	$A^m = B^n = 1,$ $BAB^{-1} = A^r$	$m \geq 1, \; n \geq 1,$ $(n(r-1), m) = 1,$ $r^n \equiv 1 \; (m)$	mn
II	A, B, R	As in I; also $R^2 = B^{n/2},$ $RAR^{-1} = A^l,$ $RBR^{-1} = B^k$	As in I; also $l^2 \equiv r^{k-1} \equiv 1 \; (m),$ $n = 2^u v, \; u \geq 2,$ $k \equiv -1 \; (2^u),$ $k^2 \equiv 1 \; (n)$	$2\,mn$
III	A, B, P, Q	As in I; also $P^4 = 1, \; P^2 = Q^2 = (PQ)^2,$ $AP = PA, \; AQ = QA,$ $BPB^{-1} = Q,$ $BQB^{-1} = PQ$	As in I; also $n \equiv 1 \; (2),$ $n \equiv 0 \; (3)$	$8\,mn$
IV	A, B, P, Q, R	As in III; also $R^2 = P^2, \; RPR^{-1} = QP$ $RQR^{-1} = Q^{-1},$ $RAR^{-1} = A^l,$ $RBR^{-1} = B^k$	As in III; also $k^2 \equiv 1 \; (n),$ $k \equiv -1 \; (3),$ $r^{k-1} \equiv l^2 \equiv 1 \; (m)$	$16\,mn$

If G is non-solvable, it is one of the following groups.

V. $G = K \times SL(2, p)$, *where p is a prime* ≥ 5, *and K is a group of type* I *and order prime to* $|SL(2, p)| = p(p^2 - 1)$.

VI.   *G is generated by a group of type* V *and an element S such that*

$$S^2 = -1 \in SL(2, p), \quad SAS^{-1} = A^{-1},$$
$$SBS^{-1} = B, \quad SLS^{-1} = \theta(L) \; (L \in SL(2, p)).$$

Here, $SL(2, p)$ denotes the multiplicative group of 2×2 matrices of determinant 1 with entries in the field Z_p, and θ is an automorphism of $SL(2, p)$ given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & \omega^{-1} \\ -\omega & 0 \end{pmatrix},$$

ω being a generator of the multiplicative group in Z_p.

Let G be any finite group, and p a prime. Then the *p-period* of G is defined to be the least positive integer q such that the Tate cohomology groups $\hat{H}^i(G; A)$ and $\hat{H}^{i+q}(G; A)$ have isomorphic p-primary components for all i and all A. The *period* of G is the least common multiple of all the p-periods. R.G. Swan [7] gave a method to calculate the p-period as follows:

(1.5)   (i)   *If a 2-Sylow subgroup of a finite group G is cyclic, the 2-period of G is 2. If a 2-Sylow subgroup of G is a generalized quaternion group, the 2-peirod of G is 4.*

(ii) *Suppose p is odd and a p-Sylow subgroup G_p of G is cyclic. Let Φ_p denote the group of automorphisms of G_p induced by inner automorphisms of G. Then the p-period of G is $2|\Phi_p|$.*

If $N(G_p)$, $C(G_p)$ denote the normalizer and centralizer of G_p, it holds $\Phi_p \cong N(G_p)/C(G_p)$. From this we have the following (see [8]).

(1.6) *If a 3-Sylow subgroup of G is cyclic, the 3-period of G divides 4.*

We shall next consider free orthogonal actions on S^n. A representation ρ of a group G is said to be *fixed point free* if $1 \neq g \in G$ implies that $\rho(g)$ does not have $+1$ for an eigenvalue.

With the notations of (1.4), let d denote the order of r in the multiplicative group of residues modulo m of integers prime to m. Modifying the work of G. Vincent [9], J. Wolf proves the following (1.7), (1.8) in [10].

(1.7) *For a finite group G, the following two conditions are equivalent*:
  i) *G has a fixed point free complex representation.*
  ii) *G is of type* I, II, III, IV, V *for $p=5$, or* VI *for $p=5$, with the additional condition*: *n/d is divisible by every prime divisor of d.*

(1.8) *Let G be a finite group satisfying the conditions in* (1.7). *Then each fixed point free, irreducible complex representation of G has the degree $\delta(G)$ which is given as follows*:

Type	I	II	III	IV	V	VI
$\delta(G)$	d	d or $2d$	$2d$	$2d$ or $4d$	$2d$	$4d$

If $|G| > 2$, G acts freely and orthogonally on S^{2q-1} if and only if q is divisible by $\delta(G)$.

REMARK. Wolf states in 7.2.18 of [10] that $\delta(G)=2d$ for G of type II. This mistake is revised in the errata sheet of [10].

2. Finite groups acting freely on $S^{2^\nu-1}$

We shall consider the following conditions for a finite group G:
(A_ν) *G can act freely and orthogonally on $S^{2^\nu-1}$.*
(B_ν) *G can act freely on $S^{2^\nu-1}$.*
(C_ν) *G has the cohomology of period 2^ν and has at most one element of order 2.*

$(A_\nu) \Rightarrow (B_\nu)$ is trivial, and $(B_\nu) \Rightarrow (C_\nu)$ holds by (1.2) and (1.3). We shall study whether $(C_\nu) \Rightarrow (A_\nu)$ holds.

Let G be a finite group satisfying (C_ν). Then, by (1.3) and (1.4), G is of type I, II, III, IV, V or VI. We shall retain the notations in §1.

Case 1: $m \neq 1$.

It follows from the conditions of type I that m is odd. Put $m = \Pi p_i^{c_i}$, where $\{p_i\}$ are distinct odd primes and $c_i \geq 1$. Then the subgroup generated by $A_i = A^{m/m_i}$ $(m_i = p_i^{c_i})$ is a p_i Sylow-subgroup of G. Let d_i denote the order of r in the multiplicative group of residues modulo m_i of integers prime to m_i. It follows that

$$B^j A_i B^{-j} = A_i^{r^j} \qquad (j = 0, 1, \cdots, d_i - 1)$$

are distinct. Therefore, by (1.5) the p_i-period of G is a multiple of $2d_i$. Let d' denote the least common multiple of $\{d_i\}$. Then it follows that d divides d', and that $2d'$ divides the period of G. Thus 2^ν is a multiple of $2d$, and so d is a divisor of $2^{\nu-1}$. Since $m = 1$ is equivalent to $d = 1$, we have

$$d = 2^\alpha \quad \text{with} \quad \alpha = 1, 2, \cdots, \nu - 1 \,.$$

Since n is a multiple of d, n is even. Therefore G can not be of type III, IV, V or VI. If G is of type II and $d = 2^\alpha$ with $\alpha \geq 2$, the conditions on k yield a contradiction. Thus G is of type I with $d = 2^\alpha$ $(\alpha = 1, 2, \cdots, \nu - 1)$, or of type II with $d = 2$.

Since the order of $B^{n/2}$ is 2, by (1.1) we have

$$B^{n/2} A B^{-n/2} = A \,.$$

Since $BAB^{-1} = A^r$, we have also

$$B^{n/2} A B^{-n/2} = A^{r^{n/2}}.$$

Hence $r^{n/2} \equiv 1(m)$, and $n/2$ is a multiple of $d = 2^\alpha$. This shows that n/d is divisible by every prime divisor of d. Therefore it follows from (1.7) and (1.8) that G has a fixed point free complex representation whose degree is 2^α if G is of type I with $d = 2^\alpha$, and is 2 or 4 if G is of type II with $d = 2$. Thus if $\nu \geq 3$, G acts freely and orthogonally on $S^{2^\nu - 1}$. If $\nu = 2$, so does G of type I with $d = 2$. However (1.8) shows that some groups G of type II with $d = 2$ can not act freely and orthogonally on S^3.

Case 2: $m = 1$, G is solvable.

In this case we have $d = 1$. Therefore it follows from (1.7) and (1.8) that G has a fixed point free complex representation whose degree is 1, 2 or 4. Thus if $\nu \geq 3$, G acts freely and orthogonally on $S^{2^\nu - 1}$. If $\nu = 2$, so does G of type I, II, or III. However (1.8) shows that some groups G of type IV can not act freely and orthogonally on S^3.

Case 3: $m = 1$, G is non-solvable.

For

$$X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, p)$$

we have

$$X^i = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \qquad (i = 0, 1, \cdots, p-1).$$

Therefore X generates a cyclic group of order p. If we observe the order of G, it follows that this cyclic group is a p-Sylow subgroup of G. For

$$Y_i = \begin{pmatrix} \omega^i & 0 \\ 0 & \omega^{-i} \end{pmatrix}, \quad Z_i = \begin{pmatrix} 0 & -\omega^i \\ \omega^{-i} & 0 \end{pmatrix}$$

we have

$$Y_i X Y_i^{-1} = \begin{pmatrix} 1 & \omega^{2i} \\ 0 & 1 \end{pmatrix},$$

$$Z_i S X S^{-1} Z_i^{-1} = \begin{pmatrix} 1 & \omega^{2i+1} \\ 0 & 1 \end{pmatrix}.$$

Therefore it follows from (1.5) that 2^ν is a multiple of $p-1$ if G is of type V, and that 2^ν is a multiple of $2(p-1)$ if G is of type VI. Thus G is of the following type $V_\alpha^* (2 \leq \alpha \leq \nu)$ or $VI_\alpha^* (2 \leq \alpha \leq \nu-1)$.

V_α^*. $G = Z_n \times SL(2, p)$, where p is a prime of the form $2^\alpha + 1$, and $(n, p(p^2-1)) = 1$.

VI_α^*. G is generated by a group of type V_α^* and an element S satisfying the conditions in VI.

In particular, if $\nu = 2$, G is of type V_2^* and it acts freely and orthogonally on S^3 by (1.7) and (1.8). If $\nu = 3$, G is of type V_2^* or VI_2^*, and it acts freely and orthogonally on S^7 by (1.7) and (1.8). If $\nu = 4$, G is of type V_2^*, V_4^* or VI_2^*. The groups of type V_2^* or VI_2^* act freely and orthogonally on S^{15}, but (1.7) shows that the groups of type V_4^* can not do so.

REMARK. A prime of the form $2^\alpha + 1$ is called the *Fermat number*, and α is known to be of a power 2^β. But the converse is not true; for example $2^{32} + 1$ is divisible by 641.

Summing up the above arguments, we have proved the following two theorems.

(2.1) **Theorem.** *The conditions (A_3), (B_3), (C_3) are mutually equivalent for any finite group G, and the following is a list of all finite groups satisfying these conditions*:
1) *The groups of types* I, II, III, IV *with $d=1$.*
2) *The groups of type* I *with $d = 2^\alpha$ and $n \equiv 0$ mod $2^{\alpha+1}$ $(\alpha = 1, 2)$.*
3) *The groups of type* II *with $d = 2$.*
4) *The groups of types* V, VI *with $d = 1$ and $p = 5$.*

(2.2) **Theorem.** *If $\nu \geqq 3$, the conditions (A_ν), (B_ν), (C_ν) are mutually equivalent for any finite solvable group G, and the following is a list of all finite solvable groups satisfying these conditions:*

  1) *The groups of types* I, II, III, IV *with $d=1$.*
  2) *The groups of type* I *with $d=2^\alpha$ and $n \equiv 0 \ mod \ 2^{\alpha+1}$ $(\alpha=1, 2, \cdots, \nu-1)$.*
  3) *The groups of type* II *with $d=2$.*

For $\nu=4$ we have also

(2.3) **Theorem.** *The following two conditions for a finite group G are equivalent:*

  i) *G satisfies the condition (C_4) but does not satisfy (A_4).*
  ii) *$G = \boldsymbol{Z}_n \times SL(2, 17)$ with $(n, 2 \cdot 3 \cdot 17)=1$.*

Proof. It has been proved in the arguments above that i) implies ii) and the groups of type V_4^* do not satisfy (A_4). It is easily seen that the groups of type V_4^* has only one element of order 2. We shall prove that each group G of type V_4^* has period 16.

If $UXU^{-1}=X^i$ for some $U \in SL(2, p)$, then it is easily seen that i is an even power of ω. Therefore it follows that the p-period of $SL(2, p)$ is $p-1$. This shows that the 17-period of G is 16. By (1.5) and (1.6), the 2-period and the 3-period of G divide 4. If q is a prime dividing n, the q-period of G is 2. Since $|G|=2^5 \cdot 3^2 \cdot 17 \cdot n$, it holds that the period of G is 16.

Here is a problem: Can $SL(2, 17)$ act freely on the sphere S^{16t-1}?

In his study on finite groups acting freely on S^3, Milnor [3] introduces the finite groups presented as follows:

  (1) $D'(2^t(2s+1))=\{A, B; A^{2s+1}=B^{2^t}=1, BAB^{-1}=A^{-1}\}$, where $s \geqq 1$, $t \geqq 1$.
  (2) $Q(8t, s_1, s_2)=\{A, B, R; A^{s_1 s_2}=1, R^2=B^{2t}, BAB^{-1}=A^{-1}, RAR^{-1}=A^l,$ $RBR^{-1}=B^{-1}\}$, where $8t, s_1, s_2$ are pairwise relatively prime positive integers, and $l \equiv -1 \ mod \ s_1$, $l \equiv +1 \ mod \ s_2$.
  (3) $T'(8 \cdot 3^t)=\{B, P, Q; B^{3^t}=1, P^2=Q^2=(PQ)^2, BPB^{-1}=Q, BQB^{-1}=PQ\}$, where $t \geqq 1$.
  (4) $O'(48t)=\{B, P, Q, R; B^{3t}=1, P^2=Q^2=R^2=(PQ)^2, BPB^{-1}=Q, BQB^{-1}$ $=PQ, RPR^{-1}=QP, RQR^{-1}=Q^{-1}, RBR^{-1}=B^{-1}\}$, where t is a positive odd integer.

These groups are generalizations of the binary polyhedral groups. In fact, the binary dihedral group $Q(4n)$ is $D'(4(2s+1))$ if $n=2s+1$ and is $Q(8t, 1, 1)$ if $n=2t$; the binary tetrahedral group T^* and the binary octahedral group O^* are $T'(24)$, $O'(48)$ respectively.

We shall generalize the group $D'(2^t(2s+1))$ as follows:

  (1)$_\alpha$ $D^{(\alpha)}(2^t(2s+1))=\{A, B; A^{2s+1}=B^{2^t}=1, BAB^{-1}=A^r\}$,
where s, t, α are positive integers, $\alpha \leqq t$, and $r^{2^{\alpha-1}} \equiv -1 \ mod \ 2s+1$. Note that $D^{(1)}(2^t(2s+1))=D'(2^t(2s+1))$.

(2.4) **Theorem.** *The following is a list of all finite solvable groups which act freely (and orthogonally) on* $S^{2\nu-1}$ ($\nu \geqq 3$).

(1) *The groups* 1, $Q(8t, s_1, s_2)$, $T'(8 \cdot 3^t)$, $O'(48t)$.

(2) *The groups* $D^{(\omega)}(2^t(2s+1))$ *with* $t \geqq \alpha+1$, *where* $\alpha = 1, 2, \cdots, \nu-1$.

(3) *The direct product of any of these groups with a cyclic group of relatively prime order.*

Proof. If G is of type I with $d=1$, we have $G = Z_n$.

Let G be of type II with $d=1$. It follows that there are B_i of order n_i ($i=1, 2$) such that

$$\{B\} = \{B_1\} \times \{B_2\}, \quad n = n_1 n_2, \quad (2n_1, n_2) = 1,$$
$$RB_1 R^{-1} = B_1^{-1}, \quad RB_2 R^{-1} = B_2$$

(see p. 203 of [10]). Then $n_1 = 4t$, and G is the product of $Z_{n_2} = \{B_2\}$ and $Q(8t) = \{B_1, R\}$.

Let G be of type III with $d=1$. Put $n = 3^t n'$, $(n', 3) = 1$. Then G is the product of $Z_{n'} = \{B^{3^t}\}$ and $T'(8 \cdot 3^t) = \{B''', P, Q\}$.

Let G be of type IV with $d=1$. Then $n_1 = 3t$ with t odd, and G is the product of $Z_{n_2} = \{B_2\}$ and $O'(48t) = \{B_1, P, Q, R\}$.

Let G be of type I with $d=2^\alpha$ and $n \equiv 0 \bmod 2^{\alpha+1} (\alpha \geqq 1)$. Put $n = 2^t n'$, $(2, n') = 1$, $m = 2s+1$. Then $t \geqq \alpha+1$ and G is the product of $Z_{n'} = \{B^{2^t}\}$ and $D^{(\omega)}(2^t(2s+1)) = \{A, B''\}$.

Let G be of type II with $d=2$. Then $n = 4t$, $m > 1$, and there exist positive integers s_1, s_2 such that $m = s_1 s_2$, $l \equiv -1 \bmod s_1$, $l \equiv +1 \bmod s_2$. G is the product of $Z_{n_2} = \{B_2\}$ and $Q(8t, s_1, s_2) = \{A, B_1, R\}$.

Consequently the desired result is only a restatement of Theorem (2.2).

From (2.1) and (2.4), we have

(2.5) **Theorem.** *The following is a list of all finite groups which act freely (and orthogonally) on* S^7.

(1) *The groups* 1, $Q(8t, s_1, s_2)$, $T'(8 \cdot 3^t)$, $O'(48t)$.

(2) *The groups* $D^{(\omega)}(2^t(2s+1))$ *with* $t \geqq \alpha+1$, *where* $\alpha = 1, 2$.

(3) *The binary icosahedral group* $I^* = SL(2, 5)$.

(4) *The group generated by* $SL(2, 5)$ *and* S, *where* $S^2 = -1 \in SL(2, 5)$ *and* $SLS^{-1} = \theta(L)$ ($L \in SL(2, 5)$).

(5) *The direct product of any of these groups with a cyclic group of relatively prime order.*

REMARK. A necessary and sufficient condition for G of type II (or IV) to have $\delta(G) = 2d$ (or $4d$) is given in [10] (see the errata sheet of [10]). If we use these results, the above arguments for $\nu = 2$ yield theorems 2 and 3 of [3].

3. Finite groups acting freely on S^{2n-1} (n: odd prime)

Let $Z_{m,n}$ be a group of type I with m odd, n odd prime and $d=n$.
By the arguments similar to §2 but simpler, we have

(3.1) **Theorem.** *Let n be an odd prime. Then the following two conditions for a finite group G are equivalent*:

    i) *G has cohomology of period $2n$, has at most one element of degree 2, and can not act freely and orthogonally on S^{2n-1}.*

    ii) *G is of type $Z_h \times Z_{m,n}$ with $(h, mn)=1$ and $h \geq 1$.*

REMARK. It is known by T. Petrie [5] that the group $Z_{m,n}$ can act freely on S^{2n-1}. Here is a problem: If $h>1$, can the group $Z_h \times Z_{m,n}$ act freely on S^{2n-1}? (R. Lee states in a letter to the author that if h, m are odd primes the problem has an affirmative answer.)

REMARK. Consideration of groups satisfying the condition i) of (3.1) for $n=6$ yields the following problem: Can the group $SL(2, 7)$ act freely on S^{11}?

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