STABLE OPERATIONS IN MOD p COHOMOLOGY THEORIES

Dedicated to Prof. Atuo Komatu for his 60th birthday

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Introduction

By a cohomology theory we understand throughout the present work, a general reduced cohomology theory defined on the category of finite CW-complexes with base vertices (cf. [10]).

In this paper we consider the stable operations in mod p cohomology theories, where the stable operation means the natural linear operation which commutes with the suspension isomorphism.

Maunder [6] considered the stable operations in the mod p K-theory, by making use of a duality map $\omega: SM \wedge SM \to S^5$, where M is a co-Moore space of type $(Z_p, 2)$. We shall also use this map. For the completeness we summarize some known results on duality maps which owe to Spanier [7, 8] in section 1.

In section 2, we construct a natural transformation

$$\Gamma(\omega): h^*(\ ; E \wedge M) \to h^*(\ ; E \bmod p)$$

for any spectrum E, which is of degree 1, stable and isomorphic. This transformation is an essential tool in the present work.

And in section 3, we consider the relation between $O^*(E)$, the algebra of the stable operations in the cohomology theory $h^*(\ ; E)$, and $O^*(E; Z_p)$, the one in the mod p cohomology theory associated with $h^*(\ ; E)$. As an application, we shall study the stable operations in the mod p U-cobordism theory, by making use of Landweber's result [5].

Throughout this paper we shall use the terms "space", "CW-complex" and "map" to refer to space with a base point, CW-complex with a base vertex and continuous map preserving base points.

1. Known results on duality maps

In this section we summarize some basic properties of duality maps which owe to E.H. Spanier [7, 8].

1.1.   First we shall fix some notations:

$X \wedge Y$ the reduced join of two spaces X and Y,

$f \wedge g$ the reduced join of two maps f and g,

$SX = X \wedge S^1$ the reduced suspension of X,

$1 = 1_A; A \to A$ an identity map of A into itself,

$T = T(A, B): A \wedge B \to B \wedge A$ a map switching factors,

$S^n A = S(S^{n-1}A) = A \wedge S^{n-1} \wedge S^1 = A \wedge S^n$ an n-fold suspension of A,

$S^n f = f \wedge 1_{S^n}$ an n-fold suspension of a map f,

$[X, Y]$ the set of homotopy classes of maps of X into Y,

$\{X, Y\}$ the stable homotopy group of X into Y,

$p: S^1 \to S^1$ for any integer p to denote a map of degree p given by $p\{t\} = \{pt\}$ for $\{t \bmod 1\} \in S^1$.

1.2.   Let X, X' be finite CW-complexes and $u: X \wedge X' \to S^n$ be a map. Such a map induces a homomorphism

$$\delta = \delta^z(u)_w: \{Z, W \wedge X\} \to \{Z \wedge X', W \wedge S^n\},$$

by the relation $\delta(\{f\}) = \{(1 \wedge u)(f \wedge 1)\}$ for any spaces Z and W.

A map $u: X \wedge X' \to S^n$ is called a semi-duality map provided $\delta^z(u)_w$ are isomorphisms for $W = S^0$ and $Z = S^k$, $k = 1, 2, 3, \cdots$.   If u is a duality map in the Spanier sense, then u is a semi-duality map ([7], Lemma 5.8).

From the definition of semi-duality map, we obtain the following results.

(1.2.1)   Let $u: X \wedge X' \to S^n$ and $v: Y \wedge Y' \to S^n$ be maps, and let $f: Y \to X$ and $g: X' \to Y'$ be maps such that

$$\{u(f \wedge 1)\} = \{v(1 \wedge g)\} \quad \text{in} \quad \{Y \wedge X', S^n\}.$$

Then the following diagram is commutative for any spaces Z and W:

$$
\begin{array}{ccc}
\{Z, W \wedge Y\} & \xrightarrow{\ \delta\ } & \{Z \wedge Y', W \wedge S^n\} \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g^*} \\
\{Z, W \wedge X\} & \xrightarrow{\ \delta\ } & \{Z \wedge X', W \wedge S^n\}.
\end{array}
$$

(1.2.2)   Let $u: X \wedge X' \to S^n$ be a semi-duality map. Then the homomorphism $\delta^z(u)_w$ is an isomorphism for any finite CW-complexes Z and W.

(1.2.3)   Let $u: X \wedge X' \to S^n$ be a semi-duality map. Then two maps

$$u_{0,1}: X \wedge SX' = X \wedge X' \wedge S^1 \xrightarrow{\ u \wedge 1\ } S^n \wedge S^1 = S^{n+1},$$

$$u_{1,0}: SX \wedge X' = X \wedge S^1 \wedge X' \xrightarrow{\ 1 \wedge T\ } X \wedge X' \wedge S^1 \xrightarrow{\ u \wedge 1\ } S^{n+1}$$

are also semi-duality maps.

1.3. Let X, X', Y and Y' be finite CW-complexes and let $u: X \wedge X' \to S^n$, $v: Y \wedge Y' \to S^n$, $f: Y \to X$ and $g: X' \to Y'$ be maps such that f and g are cellular and $u(f \wedge 1)$ and $v(1 \wedge g)$ are homotopic maps from $Y \wedge X'$ into S^n.

We consider the following sequences:

$$(1.3.1) \qquad Y \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{p} SY \xrightarrow{Sf} SX,$$
$$SY' \xleftarrow{Sg} SX' \xleftarrow{q} C_g \xleftarrow{j} Y' \xleftarrow{g} X',$$

where C_f and C_g are mapping cones of f and g respectively. Then there exists a map $\omega: C_f \wedge C_g \to S^{n+1}$ such that the following diagrams are homotopy commutative ([7], §6):

$$(1.3.2)$$

$$
\begin{array}{ccc}
C_f \wedge Y' \xrightarrow{1 \wedge j} C_f \wedge C_g & \qquad & X \wedge C_g \xrightarrow{1 \wedge q} X \wedge SX' \\
\downarrow{p \wedge 1} \qquad \downarrow{\omega} & & \downarrow{i \wedge 1} \qquad \downarrow{u_{0,1}} \\
SY \wedge Y' \xrightarrow{\quad v_{1,0} \quad} S^{n+1} & & C_f \wedge C_g \xrightarrow{\quad \omega \quad} S^{n+1}
\end{array}
$$

With an application of the "five lemma," we obtain the following result from (1.2.1), (1.2.3), (1.3.1) and (1.3.2).

(1.3.3)   Let $u: X \wedge X' \to S^n$ and $v: Y \wedge Y' \to S^n$ be semi-duality maps, and let $f: Y \to X$ and $g: X' \to Y'$ be cellular maps such that $u(f \wedge 1)$ and $v(1 \wedge g)$ are homotopic. Then the above map $\omega: C_f \wedge C_g \to S^{n+1}$ is a semi-duality map.

1.4. Let $u: S^2 \wedge S^2 \to S^4$ be a canonical identification, and let $Sp: S^2 \to S^2$ be a suspension of the map $p: S^1 \to S^1$. Then $u(Sp \wedge 1)$ and $u(1 \wedge Sp)$ are homotopic and u is a semi-duality map. Thus we obtain a semi-duality map

$$\omega: SM_p \wedge SM_p \to S^5,$$

where $M_p = S^1 \cup_p e^2$ is a co-Moore space of type $(Z_p, 2)$.

2. Stable natural transformation $\Gamma(\omega)$

By a spectrum $E = \{E_k, \varepsilon_k \mid k \in Z\}$, we shall mean a sequence of CW-complexes E_k and maps $\varepsilon_k: SE_k \to E_{k+1}$ for any integer k.

Throughout this section, let M, N be fixed finite CW-complexes.

2.1. For any finite CW-complex X, and any integers l, k, we have homomorphisms

$$[S^k X \wedge N, E_{k+l} \wedge M] \xrightarrow{S} [S^{k+1} X \wedge N, SE_{k+l} \wedge M] \xrightarrow{(\varepsilon_{k+l})_\#}$$
$$[S^{k+1} X \wedge N, E_{k+l+1} \wedge M],$$

$$\sigma_l^k \colon [S^k(SX) \wedge N,\ E_{k+l} \wedge M] \xrightarrow[\approx]{} [S^{k+1} X \wedge N,\ E_{k+l} \wedge M]\,,$$

where $S = S(M, N)$ is defined by the suspension, $(\mathcal{E}_{k+l})_\# = \mathcal{E}_{k+l}(M, N)_\#$ is an induced homomorphism and $\sigma_l^k = \sigma_l^k(M, N)$ is induced from the identification $S^{k+1} X = X \wedge S^{k+1} = X \wedge S^1 \wedge S^k = S^k(SX)$.

Let $h^l(X;\ \boldsymbol{E} \wedge M\ mod\ N) = \lim_k \mathrm{dir}\ \{[S^k X \wedge N,\ E_{k+l} \wedge M],\ (\mathcal{E}_{k+l})_\# S\}$ and $\sigma = \sigma(M, N) \colon h^l(SX;\ \boldsymbol{E} \wedge M\ mod\ N) \xrightarrow{\approx} h^{l-1}(X;\ \boldsymbol{E} \wedge M\ mod\ N)$ be the direct limit of maps $\{\sigma_l^k\}$.

Then, $\{h^*(\ ;\ \boldsymbol{E} \wedge M\ mod\ N),\ \sigma(M, N)\}$ becomes a cohomology theory [10]. In particular, we define

$$h^*(\ ;\ \boldsymbol{E} \wedge M) = h^*(\ ;\ \boldsymbol{E} \wedge M\ mod\ S^0),\quad \sigma^M = \sigma(M, S^0)\,,$$

$$h^*(\ ;\ \boldsymbol{E}\ mod\ N) = h^*(\ ;\ \boldsymbol{E} \wedge S^0\ mod\ N),\quad \sigma_N = \sigma(S^0, N)\,,$$

$$h^*(\ ;\ \boldsymbol{E}) = h^*(\ ;\ \boldsymbol{E} \wedge S^0\ mod\ S^0),\quad \sigma = \sigma(S^0, S^0)\,,$$

and $\quad h^k(\ ;\ \boldsymbol{E}\ mod\ p) = h^{k+2}(\ ;\ \boldsymbol{E}\ mod\ M_p),\quad \sigma_p = \sigma(S^0, M_p)\,,$

where $M_p = S^1 \underset{p}{\cup} e^2$ be a co-Moore space of type $(Z_p, 2)$, then the third cohomology theory is just one defined by G.W. Whitehead [10], and the last is just a mod p cohomology theory associated with $h^*(\ ;\ \boldsymbol{E})$ defined by A. Dold [3] and considered by S. Araki and H. Toda [1].

2.2. Let $\{h_1^*,\ \sigma_1\}$ and $\{h_2^*,\ \sigma_2\}$ be cohomology theories and

$$t \colon h_1^k \to h_2^{k+s} \qquad \text{for any integer}\ \ k\,,$$

be a linear natural transformation of degree s. If $\sigma_2 t = (-1)^s t \sigma_1$, then we call t is a stable natural transformation of degree s. In particular, if $\{h_1^*,\ \sigma_1\} = \{h_2^*,\ \sigma_2\}$, then we call t is a stable operation of degree s.

2.3. Let $\omega \colon M \wedge N \to S^n$ be a map, then ω induces a homomorphism

$$\gamma(\omega)_l^k \colon [S^k X,\ E_{k+l} \wedge M] \to [S^k X \wedge N,\ E_{k+l} \wedge S^n]\,,$$

by the relation $\gamma(\omega)\ ([f]) = [(1 \wedge \omega)\ (f \wedge 1)]$ for any spectrum $\boldsymbol{E} = \{E_k,\ \mathcal{E}_k\}$ and any finite CW-complex X.

Proposition 2.3. *For any spectrum* $\boldsymbol{E} = \{E_k,\ \mathcal{E}_k\}$, *a map* $\omega \colon M \wedge N \to S^n$ *induces a stable natural transformation*

$$\gamma(\omega) \colon h^*(\ ;\ \boldsymbol{E} \wedge M) \to h^*(\ ;\ \boldsymbol{E} \wedge S^n\ mod\ N)$$

of degree 0, *where* $\gamma(\omega)$ *is the direct limit of homomorphisms* $\{\gamma(\omega)_l^k\}$.

Proof. For any integers l, k,

$$\mathcal{E}_{k+l}(S^n, N)_\# S(S^n, N) \gamma(\omega)_l^k = \gamma(\omega)_l^{k+1} \mathcal{E}_{k+l}(M, S^0)_\# S(M, S^0)\,,$$

so we can define a natural transformation $\gamma(\omega)$ of degree 0. Moreover, for any integers l, k,

$$\sigma_l^k(S^n, N)\gamma(\omega)_l^k = \gamma(\omega)_{l-1}^{k+1}\sigma_l^k(M, S^0),$$

thus

$$\sigma(S^n, N)\gamma(\omega) = \gamma(\omega)\sigma(M, S^0).$$

Therefore $\gamma(\omega)$ is stable.   q.e.d.

2.4. From some switching map T, we can define the following homomorphisms:

$$[S^k X \wedge N, E_{l+k} \wedge SM] \xrightarrow[\approx]{T(M, S^1)_\sharp} [S^k X \wedge N, SE_{l+k} \wedge M],$$

$$[S^k X \wedge SN, E_{l+k} \wedge M] \xrightarrow[\approx]{T(N, S^1)^\sharp} [S^{k+1} X \wedge N, E_{l+k} \wedge M]$$

for any spectrum $\boldsymbol{E} = \{E_k, \varepsilon_k\}$ and any finite CW-complex X.

Proposition 2.4. *For any spectrum $\boldsymbol{E} = \{E_k, \varepsilon_k\}$, T induces stable natural transformations*

$$T_*: h^*(\ ; \boldsymbol{E} \wedge SM \bmod N) \to h^*(\ ; \boldsymbol{E} \wedge M \bmod N),$$
$$T^*: h^*(\ ; \boldsymbol{E} \wedge M \bmod SN) \to h^*(\ ; \boldsymbol{E} \wedge M \bmod N),$$

such that degree $T_* = 1$ and degree $T^* = -1$, where T_* is the direct limit of homomorphisms $\{(-1)^k \varepsilon_{k+l}(M, N)_\sharp T(M, S^1)_\sharp\}$ and T^* is the direct limit of homomorphisms $\{(-1)^k T(N, S^1)^\sharp\}$. Moreover, for any finite CW-complex X and any integer l,

$$T_*: h^l(X; \boldsymbol{E} \wedge SM \bmod N) \to h^{l+1}(X; \boldsymbol{E} \wedge M \bmod N),$$

and
$$T^*: h^l(X; \boldsymbol{E} \wedge M \bmod SN) \to h^{l-1}(X; \boldsymbol{E} \wedge M \bmod N)$$

are isomorphisms.

Proof.   For any integers l, k,

$$\varepsilon_{k+l+1}(M, N)_\sharp S(M, N)\varepsilon_{k+l}(M, N)_\sharp T(M, S^1)_\sharp$$
$$= -\varepsilon_{k+l+1}(M, N)_\sharp T(M, S^1)_\sharp \varepsilon_{k+l}(SM, N)_\sharp S(SM, N),$$

and
$$\sigma_{l+1}^k(M, N)\varepsilon_{k+l}(M, N)_\sharp T(M, S^1)_\sharp$$
$$= \varepsilon_{k+l}(M, N)_\sharp T(M, S^1)_\sharp \sigma_l^k(SM, N),$$

so we can define a stable natural transformation T_* of degree 1 induced from the sequence $\{(-1)^k \varepsilon_{k+l}(M, N)_\sharp T(M, S^1)_\sharp\}$. Similarly, we can define a stable natural transformation T^* of degree -1 induced from the sequence $\{(-1)^k T(N, S^1)^\sharp\}$. Since $T(N, S^1)^\sharp$ is an isomorphism, T^* is an isomorphism.

Next we consider the following homomorphism

$$[S^k X \wedge N, E_{l+k} \wedge M] \xrightarrow{\ S\ } [S^{k+1} X \wedge N, SE_{l+k} \wedge M] \xrightarrow[\approx]{\ T(S^1, M)^{\sharp}\ }$$

$$[S^{k+1} X \wedge N, E_{l+k} \wedge SM] ,$$

then the sequence $\{(-1)^{k+1} T(S^1, M)_* S\}$ induces a stable natural transformation

$$S_*: h^*(\ ; E \wedge M \bmod N) \to h^*(\ ; E \wedge SM \bmod N)$$

of degree -1 and clearly S_* is the inverse transformation of T_*. Therefore T_* is an isomorphism.   q.e.d.

2.5.   Let $\omega: S^a M \wedge S^b N \to S^{n+a+b}$ be a map. For any spectrum E, we obtain a stable natural transformation

$$\Gamma(\omega): h^*(\ ; E \wedge M) \to h^*(\ ; E \bmod N)$$

of degree n, which is defined by $\Gamma(\omega) = (T^*)^b (T_*)^{n+a+b} \gamma(\omega)(T_*)^{-a}$.

Theorem 2.5.   *If* $\omega: S^a M \wedge S^b N \to S^{n+a+b}$ *is a semi-duality map. Then, for any spectrum* E, *the stable natural transformation*

$$\Gamma(\omega): h^k(X; E \wedge M) \to h^{k+n}(X; E \bmod N)$$

is an isomorphism for any integer k *and any finite CW-complex* X.

Proof.   It is sufficient to prove that $\gamma(\omega)$ is an isomorphism, by Proposition 2.4.   For any finite CW-complex X, there exists a canonical isomorphism

$$\iota(A, B): h^k(X; E \wedge A \bmod B) \to \lim_l \mathrm{dir}\ \{\{S^l X \wedge B, E_{k+l} \wedge A\}, \{\varepsilon_{k+l}\}_{\sharp}\}$$

induced from canonical homomorphisms

$$[S^l X \wedge B, E_{k+l} \wedge A] \to \{S^l X \wedge B, E_{k+l} \wedge A\} ,$$

where A, B are any finite CW-complexes.

From (1.2.2), the semi-duality map $\omega: S^a M \wedge S^b N \to S^{n+a+b}$ induces an isomorphism

$$\delta(\omega)_k^l: \{S^l X, E_{k+l} \wedge S^a M\} \to \{S^l X \wedge S^b N, E_{k+l} \wedge S^{n+a+b}\}$$

and the sequence $\{\delta(\omega)_k^l\}$ defines an isomorphism

$$\delta(\omega): \lim_l \mathrm{dir}\ \{\{S^l X, E_{k+l} \wedge S^a M\}, \{\varepsilon_{k+l}\}_{\sharp}\}$$
$$\to \lim_l \mathrm{dir}\ \{\{S^l X \wedge S^b N, E_{k+l} \wedge S^{n+a+b}\}, \{\varepsilon_{k+l}\}_{\sharp}\} .$$

Since the relation $\iota(S^0, S^b N)\gamma(\omega) = \delta(\omega)\iota(S^a M, S^0)$ holds, the homomorphism $\gamma(\omega)$ is an isomorphism.   And therefore $\Gamma(\omega)$ is an isomorphism.   q.e.d.

REMARK. Let $X = S^0$ in Theorem 2.5. Since $h^{-k}(S^0; E \wedge M) = h_k(M; E)$, a reduced homology group of M, and $h^k(S^0; E \bmod N) = h^k(N; E)$, we obtain a duality isomorphism

$$\Gamma(\omega): h_k(M; E) \to h^{n-k}(N; E),$$

whenever $\omega: S^a M \wedge S^a N \to S^{n+a+b}$ is a semi-duality map.

3. Stable operations

Throughout this section, let p be a fixed prime, and let $M = S^1 \underset{p}{\cup} e^2$ be the co-Moore space of type $(Z_p, 2)$. Denote by $i: S^1 \to M$ and $\pi: M \to S^2$, the canonical inclusion and the map collapsing S^1 to a point.

3.1. Let $\{h^*, \sigma\}$ be a cohomology theory. The mod p cohomology theory (cf. [1]), $\{h^*(\ ; Z_p), \sigma_p\}$ is defined by

$$h^k(X; Z_p) = h^{k+2}(X \wedge M) \qquad \text{for all} \quad k,$$

and the suspension isomorphism

$$\sigma_p: h^k(SX; Z_p) \to h^{k-1}(X; Z_p) \qquad \text{for all} \quad k,$$

is defined as the composition

$$h^k(SX; Z_p) = h^{k+2}(X \wedge S^1 \wedge M) \xrightarrow[\approx]{(1 \wedge T)^*} h^{k+2}(X \wedge M \wedge S^1)$$

$$\xrightarrow[\approx]{\sigma} h^{k+1}(X \wedge M) = h^{k-1}(X; Z_p),$$

where $T = T(S^1, M)$. If $\{h^*, \sigma\}$ is defined by a spectrum E, then $\{h^*(\ ; Z_p), \sigma_p\}$ is equivalent to the cohomology theory defined in section 2.

Making use of maps $i: S^1 \to M$ and $\pi: M \to S^2$, we put

$$\rho_p: h^k(X) \xleftarrow[\approx]{\sigma^2} h^{k+2}(X \wedge S^2) \xrightarrow{(1 \wedge \pi)^*} h^{k+2}(X \wedge M) = h^k(X; Z_p)$$

and $\qquad \delta_p: h^k(X; Z_p) = h^{k+2}(X \wedge M) \xrightarrow{(1 \wedge i)^*} h^{k+2}(X \wedge S^1)$

$$\xleftarrow[\approx]{\sigma} h^{k+3}(X \wedge S^2) \xrightarrow{(1 \wedge \pi)^*} h^{k+3}(X \wedge M) = h^{k+1}(X; Z_p),$$

which are natural and called as the reduction "mod p" and the "mod p" Bockstein homomorphism. The following relations are easily seen.

$$(3.1) \qquad \sigma_p \rho_p = \rho_p \sigma, \ \sigma_p \delta_p = -\delta_p \sigma_p, \ \delta_p \rho_p = 0 \quad \text{and} \quad \delta_p \delta_p = 0.$$

In particular, the Bockstein homomorphism δ_p is a stable operation of degree 1 in mod p cohomology theory.

3.2. Let θ be a stable operation of degree n in the cohomology theory $\{h^*, \sigma\}$, i.e., $\sigma\theta = (-1)^n\theta\sigma$. We put

$$\theta_p\colon h^k(X; Z_p) = h^{k+2}(X \wedge M) \xrightarrow{\ \theta\ } h^{k+n+2}(X \wedge M) = h^{k+n}(X; Z_p)$$
$$\text{for all}\quad k,$$

which is a stable operation of degree n in the mod p cohomology theory $\{h^*(\ ; Z_p), \sigma_p\}$, i.e., $\sigma_p\theta_p = (-1)^n\theta_p\sigma_p$, and called as the "mod p" reduction of θ. From the definitions of ρ_p, δ_p, θ_p and a stable operation of degree n, we obtain the following relations.

$$(3.2) \qquad\qquad \delta_p\theta_p = (-1)^n\theta_p\delta_p \quad\text{and}\quad \theta_p\rho_p = \rho_p\theta.$$

REMARK. In the "mod 2" singular cohomology theory. Since $\delta_2 = S_q^1$ and $S_q^1 S_q^{2n} = S_q^{2n+1} \neq S_q^{2n} S_q^1$ for any $n = 1, 2, 3, \cdots$, S_q^{2n} $(n = 1, 2, 3, \cdots)$ are not the "mod 2" reduction of any stable operations in the integral cohomology theory.

3.3. Let $\{h^*, \sigma\}$ be a cohomology theory. Let X be a CW-complex with a base vertex x_0, and let $\{X_\omega\}$ be the family of all finite subcomplexes with base vertex x_0. Then $\{h^*(X_\omega)\}$ becomes an inverse system with respect to the homomorphisms induced from the inclusion maps, and we can define

$$h^*(X) = \lim_{\omega} \text{inv}\ h^*(X_\omega).$$

3.4. In the rest of this paper, we consider a fixed spectrum $\boldsymbol{E} = \{E_k, \varepsilon_k\}$ and denote by $\{h^*, \sigma\}$, the cohomology theory $\{h^*(\ ; \boldsymbol{E}), \sigma\}$ associated with $\boldsymbol{E}$. And we consider a fixed semi-duality map $\omega\colon SM \wedge SM \to S^5$.

Denote by $\iota^k \in h^k(E_k) = h^k(E_k; \boldsymbol{E})$ and $\iota_M^k \in h^k(E_k \wedge M; \boldsymbol{E} \wedge M)$, the classes represented by the identity maps of E_k and $E_k \wedge M$ respectively. And denote by $\omega^k \in h^{k+1}(E_k \wedge M; Z_p) = h^{k+1}(E_k \wedge M; \boldsymbol{E} \text{ mod } p)$, the class $\Gamma(\omega)(\iota_M^k)$, where

$$\Gamma(\omega)\colon h^*(\ ; \boldsymbol{E} \wedge M) \to h^*(\ ; \boldsymbol{E} \text{ mod } p)$$

is an isomorphic stable natural transformation of degree 1.
We put

$$\varepsilon_{k,M}\colon S(E_k \wedge M) = E_k \wedge M \wedge S^1 \xrightarrow{\ 1 \wedge T\ } E_k \wedge S^1 \wedge M \xrightarrow{\ \varepsilon_k \wedge 1\ } E_{k+1} \wedge M,$$

and consider the following sequences:

$$h_*(E_{k+1}) \xrightarrow{\ \varepsilon_k^*\ } h^*(SE_k) \underset{\approx}{\xrightarrow{\ \sigma\ }} h^*(E_k),$$

$$h^*(E_{k+1} \wedge M; \boldsymbol{E} \wedge M) \xrightarrow{\ \varepsilon_{k,M}^*\ } h^*(S(E_k \wedge M); \boldsymbol{E} \wedge M)$$
$$\underset{\approx}{\xrightarrow{\ \sigma\ }} h^*(E_k \wedge M; \boldsymbol{E} \wedge M),$$

$$h^*(E_{k+1} \wedge M; Z_p) \xrightarrow{\mathcal{E}^*_{k,M}} h^*(S(E_k \wedge M); Z_p) \xrightarrow[\approx]{\sigma_p} h^*(E_k \wedge M; Z_p),$$

$$h^*(E_k \wedge M; Z_p) \xrightarrow{(1 \wedge i)^*} h^*(E_k \wedge S^1; Z_p) \xrightarrow[\approx]{\sigma_p} h^*(E_k; Z_p),$$

$$h^*(E_k; Z_p) \xleftarrow[\approx]{\sigma_p^2} h^*(E_k \wedge S^2; Z_p) \xrightarrow{(1 \wedge \pi)^*} h^*(E_k \wedge M; Z_p).$$

Proposition 3.4. *There are relations*:

(i) $\iota^k = \sigma \mathcal{E}^*_k(\iota^{k+1})$,   (ii) $\iota^k_M = \sigma \mathcal{E}^*_{k,M}(\iota^{k+1}_M)$,

(iii) $\omega^k = -\sigma_p \mathcal{E}^*_{k,M}(\omega^{k+1})$,   (iv) $\rho_p(\iota^k) = \sigma_p(1 \wedge i)^*(\omega^k)$.

(v) $\delta_p(\omega^k) = -(1 \wedge \pi)^* \sigma_p^{-2} \rho_p(\iota^k)$.

Proof. Relations (i) and (ii) are trivial, and (iii) is a consequence of $\omega^k = \Gamma(\omega)(\iota^k_M)$ and the fact that $\Gamma(\omega)$ is a stable natural transformation of degree 1, i.e., $\sigma_p \Gamma(\omega) = -\Gamma(\omega)\sigma$ and $\mathcal{E}^*_{k,M} \Gamma(\omega) = \Gamma(\omega) \mathcal{E}^*_{k,M}$.

Relations (iv) and (v) follow from the diagrams below (cf. (1.3.2)), the definition of ρ_p and δ_p, and the fact that $\omega^k \in h^{k+1}(E_k \wedge M; Z_p) = h^{k+3}(E_k \wedge M \wedge M; E)$ is represented by the composition:

$$S^2(E_k \wedge M \wedge M) = E_k \wedge M \wedge M \wedge S^1 \wedge S^1 \xrightarrow{1 \wedge 1 \wedge T \wedge 1} E_k \wedge SM \wedge SM$$

$$\xrightarrow{1 \wedge \omega} E_k \wedge S^5 \xrightarrow{\mathcal{E}} E_{k+5},$$

where $\mathcal{E}$ is the composition:

$$E_k \wedge S^5 = S^4(SE_k) \xrightarrow{S^4 \mathcal{E}_k} S^4 E_{k+1} \xrightarrow{S^3 \mathcal{E}_{k+1}} S^3 E_{k+2} \to \cdots$$

$$\to SE_{k+4} \xrightarrow{\mathcal{E}_{k+4}} E_{k+5}.$$

Because, by making use of the following homotopy commutative diagram:

$$\begin{array}{ccc}
S^2 \wedge SM & \xrightarrow{Si \wedge 1} & SM \wedge SM \\
\downarrow{\scriptstyle 1 \wedge S\pi} & & \downarrow{\scriptstyle \omega} \\
S^2 \wedge S(S^2) & =\!=\!=\!= & S^5
\end{array}$$

the class $(1 \wedge i)^*(\omega^k)$ is represented by the composition:

$$S^2(E_k \wedge S^1 \wedge M) = E_k \wedge S^1 \wedge M \wedge S^2 \xrightarrow{1 \wedge 1 \wedge \pi \wedge 1} E_k \wedge S^1 \wedge S^2 \wedge S^2$$

$$= E_k \wedge S^5 \xrightarrow{\mathcal{E}} E_{k+5}.$$

Therefore, the class $\sigma_p(1 \wedge i)^*(\omega^k)$ is represented by the composition:

$$S^3(E_k \wedge M) = E_k \wedge M \wedge S^3 \xrightarrow{1 \wedge \pi \wedge 1} E_k \wedge S^2 \wedge S^3 = E_k \wedge S^5 \xrightarrow{\varepsilon} E_{k+5}.$$

And also this map represents $\rho_p(\iota^k)$. Thus the relation (iv) is obtained.

The relation (v) is obtained by the similar way from the following homotopy commutative diagram:

$$\begin{array}{ccc}
SM \wedge S^2 & \xrightarrow{S\pi \wedge 1} & S(S^2) \wedge S^2 \\
\downarrow{\scriptstyle 1 \wedge Si} & & \| \\
SM \wedge SM & \xrightarrow{\omega} & S^5.
\end{array} \qquad \text{q.e.d.}$$

Now, we can define

$$h^s(E) = \lim_{k} \mathrm{inv}\, \{h^{k+s}(E_k),\ (-1)^s \sigma \varepsilon_k^*\},$$

$$h^s(E;Z_p) = \lim_{k} \mathrm{inv}\, \{h^{k+s}(E_k;Z_p),\ (-1)^s \sigma_p \varepsilon_k^*\},$$

$$h^s(E \wedge M;\ E \wedge M) = \lim_{k} \mathrm{inv}\, \{h^{k+s}(E_k \wedge M;\ E \wedge M),\ (-1)^s \sigma \varepsilon_{k,M}^*\},$$

$$h^s(E \wedge M;Z_p) = \lim_{k} \mathrm{inv}\, \{h^{k+s}(E_k \wedge M;Z_p),\ (-1)^s \sigma_p \varepsilon_{k,M}^*\},$$

$$\rho_p = \{\rho_p\} : h^s(E) \to h^s(E;Z_p),$$

$$i^{**} = \{\sigma_p(1 \wedge i)^*\} : h^s(E \wedge M;Z_p) \to h^{s-1}(E;Z_p),$$

$$\pi^{**} = \{(1 \wedge \pi)^* \sigma_p^{-2}\} : h^s(E;Z_p) \to h^{s+2}(E \wedge M;Z_p),$$

and we can denote

$$\tilde{\iota} = \{\iota^k\} \in h^0(E), \quad \tilde{\iota}_p = \{\rho_p(\iota^k)\} \in h^0(E;Z_p),$$

$$\tilde{\iota}_M = \{\iota_M^k\} \in h^0(E \wedge M;\ E \wedge M), \quad \tilde{\omega} = \{\omega^k\} \in h^1(E \wedge M;Z_p),$$

$$\delta_p(\tilde{\omega}) = \{\delta_p(\omega^k)\} \in h^2(E \wedge M;Z_p),$$

which are well-defined from (3.1) and Proposition 3.4. Then we obtain the following relations.

(3.4.1)   (i) $\tilde{\iota}_p = \rho_p(\tilde{\iota})$,   (ii) $\tilde{\iota}_p = i^{**}(\tilde{\omega})$,

(iii) $\delta_p(\tilde{\omega}) = -\pi^{**}(\tilde{\iota}_p)$.

REMARK. Making use of the cofibration

$$S^1 \xrightarrow{i} M \xrightarrow{\pi} S^2,$$

we have the following exact sequence.

(3.4.2) $\xrightarrow{\times p} h^*(X) \xrightarrow{(1 \wedge \pi)^* \sigma^{-2}} h^*(X \wedge M) \xrightarrow{\sigma(1 \wedge i)^*} h^*(X) \xrightarrow{\times p}$

for any cohomology theory $\{h^*, \sigma\}$ and any finite CW-complex X. But, in

general the limit sequence of an inverse system of exact sequences need not be exact (cf. [4], Chap. 8). So the following sequence need not be exact.

$$(3.4.3) \quad \xrightarrow{\times p} h^*(E; Z_p) \xrightarrow{\pi^{**}} h^*(E \wedge M; Z_p) \xrightarrow{i^{**}} h^*(E; Z_p) \xrightarrow{\times p} .$$

3.5. Denote by $O^n(E)$, $O^n(E \wedge M)$ and $O^n(E; Z_p)$, the modules of the stable operations of degree n in the cohomology theories $h^*(\; ; E)$, $h^*(\; ; E \wedge M)$ and $h^*(\; ; E \bmod p)$ respectively, where the addition is defined by pointwise operation.

Let $\theta \in O^n(E)$. Since $\theta(\iota^k) = \theta(\sigma \varepsilon_k^*(\iota^{k+1})) = (-1)^n \sigma \varepsilon_k^*(\theta(\iota^{k+1}))$ in $h^{n+k}(E_k; E)$, we can define $\theta(\tilde{\iota}) = \{\theta(\iota^k)\} \in h^n(E; E)$. Similarly, we can define $\theta(\tilde{\iota}_M) = \{\theta(\iota_M^k)\}$ $\in h^n(E \wedge M; E \wedge M)$ for $\theta \in O^n(E \wedge M)$, and $\theta(\tilde{\omega}) = \{\theta(\omega^k)\} \in h^{n+1}(E \wedge M; E \bmod p)$ for $\theta \in O^n(E; Z_p)$.

Theorem 3.5. *The following homomorphisms are isomorphisms.*

(i) $\Phi: O^n(E) \to h^n(E) = h^n(E; E) \quad$ *defined by* $\quad \Phi(\theta) = \theta(\tilde{\iota})$,

(ii) $\Phi_M: O^n(E \wedge M) \to h^n(E \wedge M; E \wedge M) \quad$ *defined by* $\quad \Phi_M(\theta) = \theta(\tilde{\iota}_M)$,

(iii) $\Phi_p: O^n(E; Z_p) \to h^{n+1}(E \wedge M; Z_p) = h^{n+1}(E \wedge M; E \bmod p)$

   defined by $\quad \Phi_p(\theta) = \theta(\tilde{\omega})$.

Proof. Let $\alpha \in h^k(X; E)$ be a class represented by a map

$$f: S^l X \to E_{l+k},$$

then $\alpha = \sigma^l f^*(\iota^{l+k})$, and also α is represented by the composition:

$$S^{l+1} X \xrightarrow{Sf} SE_{l+k} \xrightarrow{\varepsilon_{l+k}} E_{l+k+1}.$$

Thus, for $\theta \in O^n(E)$,

$$\begin{aligned}
\theta(\alpha) &= \theta(\sigma^l f^*(\iota^{l+k})) = (-1)^{ln} \sigma^l f^* \theta(\iota^{l+k}) \\
&= (-1)^{(l+1)n} \sigma^l f^* \sigma \varepsilon_{l+k}^* \theta(\iota^{l+k+1}) \\
&= (-1)^{(l+1)n} \sigma^{l+1} (Sf)^* \varepsilon_{l+k}^* \theta(\iota^{l+k+1}) \\
&= \theta(\sigma^{l+1} (Sf)^* \varepsilon_{l+k}^*(\iota^{l+k+1})),
\end{aligned}$$

since $\sigma(Sf)^* = f^* \sigma$. And this assures (i). (ii) is similarly proved, because $h^*(\; ; E \wedge M)$ is the cohomology theory defined on the spectrum $\{E_k \wedge M, \varepsilon_{k,M}\}$.

Let $\theta \in O^n(E; Z_p)$, then $\Gamma(\omega)^{-1} \theta \Gamma(\omega) \in O^n(E \wedge M)$ and this correspondence of θ to $\Gamma(\omega)^{-1} \theta \Gamma(\omega)$ induces an isomorphism of $O^n(E; Z_p)$ to $O^n(E \wedge M)$, because $\Gamma(\omega)$ is an isomorphic stable natural transformation. Since $\tilde{\omega} = \Gamma(\omega)(\tilde{\iota}_M)$, we obtain (iii). q.e.d.

Because of the above theorem, we study $h^*(E \wedge M; Z_p)$ for the investigation

of the graded algebra $O^*(E; Z_p)=\sum_n O^n(E; Z_p)$, where the multiplication is defined by the composition.

We obtain the following relations from (3.2), (3.4) and (3.4.1).

(3.5.1)   (i) $i^{**}\phi(\tilde{\omega})=(-1)^n\phi(\tilde{\iota}_p)$ and $\pi^{**}\phi(\tilde{\iota}_p)=-\phi\delta_p(\tilde{\omega})$ for $\phi\in O^n(E; Z_p)$,

          (ii) $i^{**}\theta_p(\tilde{\omega})=(-1)^n\rho_p(\theta(\tilde{\iota}))$ and $\pi^{**}\rho_p(\theta(\tilde{\iota}))=-\theta_p\delta_p(\tilde{\omega})$ for $\theta\in O^n(E)$,

          where θ_p is the mod p reduction of θ.

3.6.   Now, we consider some conditions on the spectrum $E=\{E_k, \varepsilon_k\}$ under which the sequence (3.4.3) becomes exact.

Let $\{E_{k,l}, \varepsilon_{k,l} | k, l\in Z\}$ be a family of finite CW-complexes $E_{k,l}$ and maps $\varepsilon_{k,l}: SE_{k,l}\rightarrow E_{k+1,l}$, where the set $\{E_{k,l} | l\in Z\}$ is a family of subcomplexes of E_k with the common base vertex as one of E_k for any integer k, such that

     (i) $E_{k,l}\subset E_{k,l+1}$,   (ii) $E_k = \cup_l \dot{E}_{k,l}$

and   (iii)   the following diagrams are commutative:

$$
\begin{array}{ccc}
SE_{k,l} & \xrightarrow{\varepsilon_{k,l}} & E_{k+1,l} \\
\downarrow & & \downarrow \\
SE_k & \xrightarrow{\varepsilon_k} & E_{k+1}
\end{array}
$$

where the vertical arrows are inclusion maps.

Then, $h^*(E_k; E) = \lim_l \mathrm{inv} \{h^*(E_{k,l}; E), \iota_{k,l}^*\}$,

and $h^*(E; E) = \lim_k \mathrm{inv} \{h^*(E_k; E), (-1)^*\sigma\varepsilon_k^*\}$

                         $= \lim_k \mathrm{inv} \{h^*(E_{k,k}; E), (-1)^*\sigma\varepsilon_{k,k}^*\iota_{k+1,k}^*\}$

where $\iota_{k,l}; E_{k,l}\rightarrow E_{k,l+1}$ is an inclusion map.

From (3.4.2), the following results are easy consequence of the properties of the inverse limit.

(3.6.1)   If $h^*(E_{k,k})$ have no p-torsion for any $k\in Z$.

Then the following sequences are exact:

$$
0 \rightarrow h^*(E; Z_p) \xrightarrow{\pi^{**}} h^*(E\wedge M; Z_p) \xrightarrow{i^{**}} h^*(E; Z_p),
$$

$$
0 \rightarrow h^*(E) \xrightarrow{\times p} h^*(E) \xrightarrow{\rho_p} h^*(E; Z_p).
$$

(3.6.2)   If $h^*(E_{k,k})$ are free abelian groups and the maps $\sigma\varepsilon_{k,k}^*\iota_{k+1,k}^*$: $h^*(E_{k+1,k+1})\rightarrow h^*(E_{k,k})$ are onto for any $k\in Z$. Then $\rho_p: h^*(E)\rightarrow h^*(E; Z_p)$ is an onto homomorphism and therefore $h^*(E)\otimes Z_p\approx h^*(E; Z_p)$.

From (3.6.1) and (3.6.2), we obtain

(3.6.3)   Under the condition of (3.6.2) on $h^*(E_{k,k})$, if the order of $\tilde{\omega} \in h^1(E \wedge M; Z_p)$ is p, i.e., $p\tilde{\omega}=0$. Then the following sequence is a split exact sequence:

$$0 \to h^*(E; Z_p) \xrightarrow{\pi^{**}} h^*(E \wedge M; Z_p) \xrightarrow{i^{**}} h^*(E; Z_p) \to 0$$

and $h^*(E \wedge M; Z_p)$ is a free $O^*(E)/pO^*(E)$-module with generators $\tilde{\omega}$ and $\delta_p(\tilde{\omega})$.

Proof.   We consider a correspondence of $\rho_p(\theta(\bar{\imath}))$ to $\theta_p(\tilde{\omega})$. Since $p\tilde{\omega}=0$ and $h^*(E; Z_p)=\rho_p h^*(E) \approx h^*(E) \otimes Z_p$, this correspondence is a well-defined homomorphism of $h^*(E; Z_p)$ to $h^*(E \wedge M; Z_p)$ and this is a right inverse of i^{**} from (3.5.1), thus the above sequence is a split exact sequence. Since $h^*(E; Z_p)$ is a free $O^*(E)/pO^*(E)$-module with one generator $\bar{\imath}_p$ from (3.2), (3.6.2) and Theorem 3.5, the final part follows from (3.5.1).   q.e.d.

REMARK.   If p is an odd prime, then the relation $p\tilde{\omega}=0$ is always true, and if $p=2$, this is true under some condition connecting with the Hopf map $\eta: S^3 \to S^2$ ([1]).

As a corollary of (3.2), (3.5.1), (3.6.3) and Theorem 3.5, we obtain

Theorem 3.6.   *If $p\tilde{\omega}=0$, $h^*(E_{k,k})$ are free abelian groups and the maps $\sigma \mathcal{E}^*_{k,k} \iota^*_{k+1,k}: h^*(E_{k+1,k+1}) \to h^*(E_{k,k})$ are onto for any $k \in Z$. Then, there exists an isomorphism*

$$O^*(E; Z_p) \approx (O^*(E)/pO^*(E)) \otimes \Lambda_p(\delta_p)$$

as graded algebras over Z_p, where $\Lambda_p(\delta_p)$ is the exterior algebra generated by the Bockstein homomorphism δ_p. Moreover, $O^*(E)/pO^*(E)$ is identified with the mod p reduction of $O^*(E)$, a subalgebra of $O^*(E; Z_p)$.

3.7.   As an application of Theorem 3.6, we consider the stable operations in mod p U-cobordism theory.

Denote by $\xi_{k,l}$ the canonical complex k-plane bundle over the complex Grassmann manifold $G_{k,l}$ of k-planes in C^{k+l}, and denote by $M(\xi_{k,l})$ the Thom complex of $\xi_{k,l}$.

Let $\mathcal{E}_{k,l}: S^2 M(\xi_{k,l}) \to M(\xi_{k+1,l})$ be a map induced from the canonical bundle map $\xi_{k,l} \oplus C^1 \to \xi_{k+1,l}$, and let $\iota_{k,l}: M(\xi_{k,l}) \to M(\xi_{k,l+1})$ be a canonical inclusion. Then the Thom spectrum $MU=\{MU(k), \mathcal{E}_k\}$ is defined by

$$MU(k) = \lim_{l} \text{ind} \{M(\xi_{k,l}), \iota_{k,l}\} \quad \text{and} \quad \mathcal{E}_k | M(\xi_{k,l}) = \mathcal{E}_{k,l}.$$

And U-cobordism theory is the cohomology theory associated with MU.

The family $\{M(\xi_{k,l}), \mathcal{E}_{k,l}\}$ satisfies the hypothesis of Theorem 3.6 (cf. [5]) and the order of $\tilde{\omega} \in h^1(MU \wedge M; Z_p)$ is p for any prime p (cf. [1], Th. 2.3).

Therefore, Theorem 3.6 is applicable.

On the other hand, Landweber [5] shows that there exists an isomorphism

$$O^*(MU) \approx Z[\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots] \otimes h^*(S^0; MU)$$

as modules, where γ_n is the n-th U-cobordism characteristic class defined by Conner & Floyd [2], which corresponds to a stable U-cobordism operation of degree $2n$. Therefore, we obtain

Theorem 3.7. *There exists an isomorphism*

$$O^*(MU; Z_p) \approx Z_p[\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots] \otimes \Lambda_p(\delta_p)$$
$$\otimes (h^*(S^0; MU)/ph^*(S^0; MU))$$

as modules over Z_p.

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