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STABLE OPERATIONS IN MOD $p$ COHOMOLOGY THEORIES

Dedicated to Prof. Atuo Komatu for his 60th birthday

FUICHI UCHIDA

(Received September 10, 1968)

Introduction

By a cohomology theory we understand throughout the present work, a general reduced cohomology theory defined on the category of finite CW-complexes with base vertices (cf. [10]).

In this paper we consider the stable operations in mod $p$ cohomology theories, where the stable operation means the natural linear operation which commutes with the suspension isomorphism.

Maunder [6] considered the stable operations in the mod $p$ $K$-theory, by making use of a duality map $\omega: SM \wedge SM \rightarrow S^5$, where $M$ is a co-Moore space of type $(\mathbb{Z}_p, 2)$. We shall also use this map. For the completeness we summarize some known results on duality maps which owe to Spanier [7, 8] in section 1.

In section 2, we construct a natural transformation

$$\Gamma(\omega): h^*( \ ; E \wedge M) \rightarrow h^*( \ ; E \mod p)$$

for any spectrum $E$, which is of degree 1, stable and isomorphic. This transformation is an essential tool in the present work.

And in section 3, we consider the relation between $O^*(E)$, the algebra of the stable operations in the cohomology theory $h^*( \ ; E)$, and $O^*(E; \mathbb{Z}_p)$, the one in the mod $p$ cohomology theory associated with $h^*( \ ; E)$. As an application, we shall study the stable operations in the mod $p$ $U$-cobordism theory, by making use of Landweber's result [5].

Throughout this paper we shall use the terms "space", "CW-complex" and "map" to refer to space with a base point, CW-complex with a base vertex and continuous map preserving base points.

1. Known results on duality maps

In this section we summarize some basic properties of duality maps which owe to E.H. Spanier [7, 8].
1.1. First we shall fix some notations:

- $X \wedge Y$ the reduced join of two spaces $X$ and $Y$,
- $f \wedge g$ the reduced join of two maps $f$ and $g$,
- $SX = X \wedge S^i$ the reduced suspension of $X$,
- $1 = 1_A; A \to A$ an identity map of $A$ into itself,
- $T = T(A, B): A \wedge B \to B \wedge A$ a map switching factors,
- $S^n A = S(S^{n-1} A) = A \wedge S^{n-1} \wedge S^1 = A \wedge S^n$ an $n$-fold suspension of $A$,
- $S^n f = f \wedge 1_S^*$ an $n$-fold suspension of a map $f$,
- $[X, Y]$ the set of homotopy classes of maps of $X$ into $Y$,
- $\{X, Y\}$ the stable homotopy group of $X$ into $Y$,
- $p: S^1 \to S^1$ for any integer $p$ to denote a map of degree $p$ given by $p\{t\} = \{pt\}$ for $\{t \mod 1\} \in S^1$.

1.2. Let $X, X'$ be finite CW-complexes and $u: X \wedge X' \to S^n$ be a map. Such a map induces a homomorphism

$$\delta = \delta^*(u)_w: \{Z, W \wedge X\} \to \{Z \wedge X', W \wedge S^n\},$$

by the relation $\delta(\{f\}) = \{(1 \wedge u)(f \wedge 1)\}$ for any spaces $Z$ and $W$.

A map $u: X \wedge X' \to S^n$ is called a semi-duality map provided $\delta^*(u)_w$ are isomorphisms for $W = S^k$ and $Z = S^k$, $k = 1, 2, 3, \ldots$. If $u$ is a duality map in the Spanier sense, then $u$ is a semi-duality map ([7], Lemma 5.8).

From the definition of semi-duality map, we obtain the following results.

1.2.1 Let $u: X \wedge X' \to S^n$ and $v: Y \wedge Y' \to S^n$ be maps, and let $f: Y \to X$ and $g: X' \to Y'$ be maps such that

$$\{u(f \wedge 1)\} = \{v(1 \wedge g)\} \quad \text{in} \quad \{Y \wedge X', S^n\}.$$

Then the following diagram is commutative for any spaces $Z$ and $W$:

$$\begin{array}{ccc}
\{Z, W \wedge Y\} & \xrightarrow{\delta} & \{Z \wedge Y', W \wedge S^n\} \\
\downarrow f_* & & \downarrow g_* \\
\{Z, W \wedge X\} & \xrightarrow{\delta} & \{Z \wedge X', W \wedge S^n\}.
\end{array}$$

1.2.2 Let $u: X \wedge X' \to S^n$ be a semi-duality map. Then the homomorphism $\delta^*(u)_w$ is an isomorphism for any finite CW-complexes $Z$ and $W$.

1.2.3 Let $u: X \wedge X' \to S^n$ be a semi-duality map. Then two maps

$$u_{0,1}: X \wedge SX' = X \wedge X' \wedge S^1 \xrightarrow{u \wedge 1} S^n \wedge S^1 = S^{n+1},$$

$$u_{1,0}: SX \wedge X' = X \wedge S^1 \wedge X' \xrightarrow{1 \wedge T} X \wedge X' \wedge S^1 \xrightarrow{u \wedge 1} S^{n+1}$$
are also semi-duality maps.

1.3. Let $X, X', Y$ and $Y'$ be finite CW-complexes and let $u: X \wedge X' \to S^n$, $v: Y \wedge Y' \to S^n$, $f: Y \to X$ and $g: X' \to Y'$ be maps such that $f$ and $g$ are cellular and $u(f \wedge 1)$ and $v(1 \wedge g)$ are homotopic maps from $Y \wedge X'$ into $S^n$.

We consider the following sequences:

\[
\begin{align*}
Y \xrightarrow{f} X & \xrightarrow{i} C_f \xrightarrow{p} SY \xrightarrow{Sf} SX, \\
SY' & \xleftarrow{Sg} SX' \xleftarrow{g} C_g \xleftarrow{j} Y' \xrightarrow{g} X',
\end{align*}
\]

where $C_f$ and $C_g$ are mapping cones of $f$ and $g$ respectively. Then there exists a map $\omega: C_f \wedge C_g \to S^{n+1}$ such that the following diagrams are homotopy commutative ([7], §6):

\[
\begin{align*}
C_f \wedge Y' & \xrightarrow{1 \wedge j} C_f \wedge C_g \\
SY \wedge Y' & \xrightarrow{\omega} S^{n+1},
\end{align*}
\]

With an application of the "five lemma," we obtain the following result from (1.2.1), (1.2.3), (1.3.1) and (1.3.2).

(1.3.3) Let $u: X \wedge X' \to S^n$ and $v: Y \wedge Y' \to S^n$ be semi-duality maps, and let $f: Y \to X$ and $g: X' \to Y'$ be cellular maps such that $u(f \wedge 1)$ and $v(1 \wedge g)$ are homotopic. Then the above map $\omega: C_f \wedge C_g \to S^{n+1}$ is a semi-duality map.

1.4. Let $u: S^2 \wedge S^2 \to S^4$ be a canonical identification, and let $Sp: S^2 \to S^2$ be a suspension of the map $p: S^1 \to S^1$. Then $u(Sp \wedge 1)$ and $u(1 \wedge Sp)$ are homotopic and $u$ is a semi-duality map. Thus we obtain a semi-duality map

$\omega: SM_p \wedge SM_p \to S^5$, where $M_p = S^1 \cup e^2$ is a co-Moore space of type $(Z_p, 2)$.

2. Stable natural transformation $\Gamma(\omega)$

By a spectrum $E = \{E_k, \varepsilon_k | k \in Z\}$, we shall mean a sequence of CW-complexes $E_k$ and maps $\varepsilon_k: SE_k \to E_{k+1}$ for any integer $k$.

Throughout this section, let $M, N$ be fixed finite CW-complexes.

2.1. For any finite CW-complex $X$, and any integers $l, k$, we have homomorphisms

\[
[S^k X \wedge N, E_{k+l} \wedge M] \xrightarrow{S} [S^{k+1} X \wedge N, SE_{k+1} \wedge M] \xrightarrow{(\varepsilon_{k+1})_k} [S^{k+1} X \wedge N, E_{k+l+1} \wedge M],
\]
\[ \sigma^t: [S^k(SX) \wedge N, E_{k+l} \wedge M] \to [S^{k+1}X \wedge N, E_{k+l} \wedge M], \]

where \( S = S(M, N) \) is defined by the suspension, \((\varepsilon_{k+l})_* = \varepsilon_{k+l}(M, N)_* \) is an induced homomorphism and \( \sigma^t = \sigma^t(M, N) \) is induced from the identification \( S^{k+1}X = X \wedge S^{k+1} = X \wedge S^1 \wedge S^k = S^k(SX) \).

Let \( h'(X; E \wedge M \mod N) = \lim_k \text{dir} \{ [S^kX \wedge N, E_{k+l} \wedge M], (\varepsilon_{k+l})_* S \} \) and \( \sigma = \sigma(M, N) \): \( h'(SX; E \wedge M \mod N) \to h^{-1}(X; E \wedge M \mod N) \) be the direct limit of maps \( \{ \sigma^t \} \).

Then, \( \{ h^*(E \wedge M \mod N), \sigma(M, N) \} \) becomes a cohomology theory \([10]\).

In particular, we define

\[ h^*(E \wedge M) = h^*(E \wedge M \mod S^0), \quad \sigma^M = \sigma(M, S^0), \]
\[ h^*(E \mod N) = h^*(E \wedge S^0 \mod N), \quad \sigma^N = \sigma(S^0, N), \]
\[ h^*(E) = h^*(E \wedge S^0 \mod S^0), \quad \sigma = \sigma(S^0, S^0), \]

and

\[ h^k(E \mod p) = h^{k+2}(E \mod M_p), \quad \sigma^p = \sigma(S^0, M_p), \]

where \( M_p = S^1 \cup \mathbb{C}^2 \) be a co-Moore space of type \((Z_p, 2)\), then the third cohomology theory is just one defined by G.W. Whitehead \([10]\), and the last is just a mod \( p \) cohomology theory associated with \( h^*(E) \) defined by A. Dold \([3]\) and considered by S. Araki and H. Toda \([1]\).

2.2. Let \( \{ h^*_1, \sigma_1 \} \) and \( \{ h^*_2, \sigma_2 \} \) be cohomology theories and

\[ t: h^*_1 \to h^*_2 \]

be a linear natural transformation of degree \( s \). If \( \sigma_2 t = (-1)^t \sigma_1 \), then we call \( t \) is a stable natural transformation of degree \( s \). In particular, if \( \{ h^*_1, \sigma_1 \} = \{ h^*_2, \sigma_2 \} \), then we call \( t \) is a stable operation of degree \( s \).

2.3. Let \( \omega: M \wedge N \to S^n \) be a map, then \( \omega \) induces a homomorphism

\[ \gamma(\omega): [S^kX, E_{k+l} \wedge M] \to [S^kX \wedge N, E_{k+l} \wedge S^n], \]

by the relation \( \gamma(\omega)([f]) = [(1 \wedge \omega)(f \wedge 1)] \) for any spectrum \( E = \{ E_k, \varepsilon_k \} \) and any finite CW-complex \( X \).

**Proposition 2.3.** For any spectrum \( E = \{ E_k, \varepsilon_k \} \), a map \( \omega: M \wedge N \to S^n \) induces a stable natural transformation

\[ \gamma(\omega): h^*(E \wedge M) \to h^*(E \wedge S^n \mod N) \]

of degree 0, where \( \gamma(\omega) \) is the direct limit of homomorphisms \( \{ \gamma(\omega)_t \} \).

**Proof.** For any integers \( l, k \),

\[ \varepsilon_{k+l}(S^n, N) \wedge S(S^n, N) \gamma(\omega)_t = \gamma(\omega)_t \varepsilon_{k+l}(M, S^n) \wedge S(M, S^n), \]
so we can define a natural transformation $\gamma(\omega)$ of degree 0. Moreover, for any integers $l$, $k$,

$$\sigma^l(S^n, N)\gamma(\omega)^l = \gamma(\omega)^l\sigma^l(M, S^n),$$

thus

$$\sigma(S^n, N)\gamma(\omega) = \gamma(\omega)\sigma(M, S^n).$$

Therefore $\gamma(\omega)$ is stable. q.e.d.

2.4. From some switching map $T$, we can define the following homomorphisms:

$$[S^kX \wedge N, E_{l+k} \wedge SM] \xrightarrow{T(M, S^l)^l} [S^kX \wedge N, SE_{l+k} \wedge M],$$

$$[S^kX \wedge SN, E_{l+k} \wedge M] \xrightarrow{T(N, S^l)^l} [S^{k+1}X \wedge N, E_{l+k} \wedge M]$$

for any spectrum $E=\{E_k, \varepsilon_k\}$ and any finite CW-complex $X$.

**Proposition 2.4.** For any spectrum $E=\{E_k, \varepsilon_k\}$, $T$ induces stable natural transformations

$$T_\#: h^*(; E \wedge SM \text{ mod } N) \rightarrow h^*(; E \wedge M \text{ mod } N),$$

$$T^\#: h^*(; E \wedge M \text{ mod } SN) \rightarrow h^*(; E \wedge M \text{ mod } N),$$

such that degree $T_\# = 1$ and degree $T^\# = -1$, where $T_\#$ is the direct limit of homomorphisms $\{(-1)^{k+l}(M, N)_hT(M, S^l)\}$ and $T^\#$ is the direct limit of homomorphisms $\{(-1)^kT(N, S^l)\}$. Moreover, for any finite CW-complex $X$ and any integer $l$,

$$T_\#: h^l(X; E \wedge SM \text{ mod } N) \rightarrow h^{l+1}(X; E \wedge M \text{ mod } N),$$

and

$$T^\#: h^l(X; E \wedge M \text{ mod } SN) \rightarrow h^{l-1}(X; E \wedge M \text{ mod } N)$$

are isomorphisms.

**Proof.** For any integers $l$, $k$,

$$\varepsilon_{k+l+1}(M, N)_hS(M, N)\varepsilon_{k+l}(M, N)_hT(M, S^l)$$

$$= -\varepsilon_{k+l+1}(M, N)_hT(M, S^l)\varepsilon_{k+l}(SM, N)_hS(M, N),$$

and

$$\sigma^l_{k+l}(M, N)\sigma_{k+l}(M, N)_hT(M, S^l),$$

$$= \varepsilon_{k+l}(M, N)_hT(M, S^l)\sigma_{k+l}(SM, N),$$

so we can define a stable natural transformation $T_\#$ of degree 1 induced from the sequence $\{(-1)^{k+l}(M, N)_hT(M, S^l)\}$. Similarly, we can define a stable natural transformation $T^\#$ of degree $-1$ induced from the sequence $\{(-1)^kT(N, S^l)\}$. Since $T(N, S^l)$ is an isomorphism, $T^\#$ is an isomorphism.
Next we consider the following homomorphism

\[ [S^k X \wedge N, E_{t+k} \wedge M] \xrightarrow{S} [S^{k+1}X \wedge N, SE_{t+k} \wedge M] \xrightarrow{T(S', M)^t} [S^{k+1}X \wedge N, E_{t+k} \wedge SM] , \]

then the sequence \{(-1)^{k+1}T(S', M)_kS\} induces a stable natural transformation

\[ S_*: h^* ( ; E \wedge M \mod N ) \to h^* ( ; E \wedge SM \mod N ) \]

of degree \(-1\) and clearly \(S_*\) is the inverse transformation of \(T_*\). Therefore \(T_*\) is an isomorphism. q.e.d.

2.5. Let \(\omega: S^a M \wedge S^b N \to S^{a+b} \) be a map. For any spectrum \(E\), we obtain a stable natural transformation

\[ \Gamma(\omega): h^* ( ; E \wedge M ) \to h^* ( ; E \mod N ) \]

of degree \(n\), which is defined by \(\Gamma(\omega) = (T^*)^b(T_*)^a(\omega)(T^*)^{-a}\).

**Theorem 2.5.** If \(\omega: S^a M \wedge S^b N \to S^{a+b}\) is a semi-duality map. Then, for any spectrum \(E\), the stable natural transformation

\[ \Gamma(\omega): h^k(X; E \wedge M) \to h^{k+n}(X; E \mod N) \]

is an isomorphism for any integer \(k\) and any finite CW-complex \(X\).

Proof. It is sufficient to prove that \(\gamma(\omega)\) is an isomorphism, by Proposition 2.4. For any finite CW-complex \(X\), there exists a canonical isomorphism

\(\iota(A, B): h^k(X; E \wedge A \mod B) \to \lim \dir \{ \{S^t X \wedge A, E_{k+t} \wedge A \}, \{\varepsilon_{k+t}\}_t\} \)

induced from canonical homomorphisms

\[ [S^t X \wedge B, E_{k+t} \wedge A] \to \{S^t X \wedge B, E_{k+t} \wedge A\}, \]

where \(A, B\) are any finite CW-complexes.

From (1.2.2), the semi-duality map \(\omega: S^a M \wedge S^b N \to S^{a+b}\) induces an isomorphism

\[ \delta(\omega)_t: \{S^t X, E_{k+t} \wedge S^a M\} \to \{S^t X \wedge S^b N, E_{k+t} \wedge S^{a+b}\} \]

and the sequence \(\{\delta(\omega)_t\}_t\) defines an isomorphism

\[ \delta(\omega): \lim \dir \{ \{S^t X, E_{k+t} \wedge S^a M\}, \{\varepsilon_{k+t}\}_t\} \]

\[ \to \lim \dir \{ \{S^t X \wedge S^b N, E_{k+t} \wedge S^{a+b}\}, \{\varepsilon_{k+t}\}_t\} . \]

Since the relation \(\iota(S^0, S^0)\gamma(\omega) = \delta(\omega)\iota(S^a M, S^b)\) holds, the homomorphism \(\gamma(\omega)\) is an isomorphism. And therefore \(\Gamma(\omega)\) is an isomorphism. q.e.d.
Remark. Let \( X = S^0 \) in Theorem 2.5. Since \( h^{-k}(S^0; E \wedge M) = h_k(M; E) \), a reduced homology group of \( M \), and \( h^k(S^a; E \text{ mod } N) = h^k(N; E) \), we obtain a duality isomorphism
\[
\Gamma(\omega): h_k(M; E) \to h^{n-k}(N; E),
\]
whenever \( \omega: S^a M \wedge S^a N \to S^{n+a+b} \) is a semi-duality map.

3. Stable operations

Throughout this section, let \( p \) be a fixed prime, and let \( M = S^1 \cup e^2 \) be the co-Moore space of type \((\mathbb{Z}_p, 2)\). Denote by \( i: S^1 \to M \) and \( \pi: M \to S^2 \), the canonical inclusion and the map collapsing \( S^1 \) to a point.

3.1. Let \( \{h^*, \sigma\} \) be a cohomology theory. The mod \( p \) cohomology theory (cf. [1]), \( \{\lambda^*(\mathbb{Z}_p), \sigma_p\} \), is defined by
\[
h^k(X; \mathbb{Z}_p) = h^{k+2}(X \wedge M) \quad \text{for all } k,
\]
and the suspension isomorphism
\[
\sigma_p: h^k(SX; \mathbb{Z}_p) \to h^{k-1}(X; \mathbb{Z}_p) \quad \text{for all } k,
\]
is defined as the composition
\[
h^k(SX; \mathbb{Z}_p) = h^{k+2}(X \wedge S^1) \xrightarrow{(1 \wedge T)^*} h^{k+3}(X \wedge M \wedge S^1) \xrightarrow{\sigma} h^{k+1}(X \wedge M) = h^{k-1}(X; \mathbb{Z}_p),
\]
where \( T = T(S^1, M) \). If \( \{h^*, \sigma\} \) is defined by a spectrum \( E \), then \( \{\lambda^*(\mathbb{Z}_p), \sigma_p\} \) is equivalent to the cohomology theory defined in section 2.

Making use of maps \( i: S^1 \to M \) and \( \pi: M \to S^2 \), we put
\[
\rho_p: h^k(X) \xleftarrow{\sigma_p^2} h^{k+2}(X \wedge S^1) \xrightarrow{(1 \wedge \pi)^*} h^{k+3}(X \wedge M) = h^k(X; \mathbb{Z}_p)
\]
and
\[
\delta_p: h^k(X; \mathbb{Z}_p) = h^{k+2}(X \wedge M) \xrightarrow{(1 \wedge i)^*} h^{k+3}(X \wedge S^2) \xrightarrow{\sigma} h^{k+1}(X \wedge S^1) \xrightarrow{\approx} h^{k+2}(X \wedge S^2) \xrightarrow{(1 \wedge \pi)^*} h^{k+3}(X \wedge M) = h^{k+1}(X; \mathbb{Z}_p),
\]
which are natural and called as the reduction "mod \( p \)" and the "mod \( p \)" Bockstein homomorphism. The following relations are easily seen,
\[
(3.1) \quad \sigma_p \rho_p = \rho_p \sigma, \quad \sigma_p \delta_p = -\delta_p \sigma_p, \quad \delta_p \rho_p = 0 \quad \text{and} \quad \delta_p \delta_p = 0.
\]

In particular, the Bockstein homomorphism \( \delta_p \) is a stable operation of degree 1 in mod \( p \) cohomology theory.
3.2. Let $\theta$ be a stable operation of degree $n$ in the cohomology theory $\{h^*, \sigma\}$, i.e., $\sigma\theta = (-1)^n \theta \sigma$. We put
\[
\theta_\rho: h^k(X; Z_\rho) = h^{k+n}(X \wedge M) \xrightarrow{\theta} h^{k+n}(X \wedge M) = h^{k+n}(X; Z_\rho)
\]
for all $k$, which is a stable operation of degree $n$ in the mod $p$ cohomology theory $\{h^* (\mod p), \sigma\}$, i.e., $\sigma_\rho \theta_\rho = (-1)^n \theta_\rho \sigma_\rho$, and called as the "mod $p$" reduction of $\theta$. From the definitions of $\rho_\rho, \delta_\rho, \theta_\rho$ and a stable operation of degree $n$, we obtain the following relations.

\[
(3.2) \quad \delta_\rho \theta_\rho = (-1)^n \theta_\rho \delta_\rho \quad \text{and} \quad \theta_\rho \rho_\rho = \rho_\rho \theta.
\]

Remark. In the "mod 2" singular cohomology theory. Since $S_2 = S_1$ and $S_{q,n} = S_{q,n+1} = S_{q,n+2}$ for any $n = 1, 2, 3, \ldots$, $S_{q,n} (n = 1, 2, 3, \ldots)$ are not the "mod 2" reduction of any stable operations in the integral cohomology theory.

3.3. Let $\{h^*, \sigma\}$ be a cohomology theory. Let $X$ be a CW-complex with a base vertex $x_0$, and let $\{X_\alpha\}$ be the family of all finite subcomplexes with base vertex $x_0$. Then $\{h^*(X_\alpha)\}$ becomes an inverse system with respect to the homomorphisms induced from the inclusion maps, and we can define
\[
h^*(X) = \lim\text{inv} \ h^*(X_\alpha).
\]

3.4. In the rest of this paper, we consider a fixed spectrum $E = \{E_k, \varepsilon_k\}$ and denote by $\{h^*, \sigma\}$, the cohomology theory $\{h^* (\mod E), \sigma\}$ associated with $E$. And we consider a fixed semi-duality map $\omega: SM \wedge SM \to S^3$.

Denote by $\iota^k \in h^k(E_k) = h^k(E_k; E)$ and $\iota^k_M \in h^k(E_k \wedge M; E \wedge M)$, the classes represented by the identity maps of $E_k$ and $E_k \wedge M$ respectively. And denote by $\omega^k \in h^{k+1}(E_k \wedge M; Z_\rho) = h^{k+1}(E_k \wedge M; E \mod p)$, the class $\Gamma(\omega)(\iota^k_M)$, where
\[
\Gamma(\omega): h^*(\mod E \wedge M) \to h^*(\mod E \wedge M)
\]
is an isomorphic stable natural transformation of degree 1.

We put
\[
\varepsilon_{k,M}: S(E_k \wedge M) = E_k \wedge M \wedge S^1  \xrightarrow{1 \wedge \eta} E_k \wedge S^1 \wedge M \xrightarrow{\varepsilon_k \wedge 1} E_{k+1} \wedge M,
\]
and consider the following sequences:
\[
h^*(E_{k+1}) \xrightarrow{\varepsilon_k^*} h^*(SE_k) \xrightarrow{\sigma} h^*(E_k),
\]
\[
h^*(E_{k+1} \wedge M; E \wedge M) \xrightarrow{\varepsilon_{k,M}^*} h^*(S(E_k \wedge M); E \wedge M) \xrightarrow{\sigma} h^*(E_k \wedge M; E \wedge M),
\]
\[ h^*(E_{k+1} \wedge M; Z_p) \xrightarrow{\xi^*} h^*(S(E_k \wedge M); Z_p) \xrightarrow{\sigma_p} h^*(E_k \wedge M; Z_p), \]

\[ h^*(E_k \wedge M; Z_p) \xrightarrow{(1 \wedge \iota)^*} h^*(E_k \wedge S^1; Z_p) \xrightarrow{\sigma_p} h^*(E_k; Z_p), \]

\[ h^*(E_k; Z_p) \xrightarrow{\sigma^2_p} h^*(E_k \wedge S^1; Z_p) \xrightarrow{(1 \wedge \pi)^*} h^*(E_k \wedge M; Z_p). \]

**Proposition 3.4.** There are relations:

(i) \( \iota^k = \sigma_p \xi^*(\iota^{k+1}) \),  
(ii) \( \iota_M^k = \sigma_p \xi^*_M(\iota^{k+1}_M) \),  
(iii) \( \omega^k = -\sigma_p \xi^*_M(\omega^{k+1}) \),  
(iv) \( \rho_p(\iota^k) = \sigma_p(1 \wedge \iota)^*(\omega^k) \).

(v) \( \delta_p(\omega^k) = -(1 \wedge \pi)^* \sigma_p^2 \rho_p(\iota^k) \).

Proof. Relations (i) and (ii) are trivial, and (iii) is a consequence of \( \omega^k = \Gamma(\omega)(\iota^k) \) and the fact that \( \Gamma(\omega) \) is a stable natural transformation of degree 1, i.e., \( \sigma_p \Gamma(\omega) = -\Gamma(\omega) \sigma \) and \( \xi^*_M \Gamma(\omega) = \Gamma(\omega) \xi^*_M \).

Relations (iv) and (v) follow from the diagrams below (cf. (1.3.2)), the definition of \( \rho_p \) and \( \delta_p \), and the fact that \( \omega^k \in h^{k+1}(E_k \wedge M; Z_p) = h^{k+1}(E_k \wedge M; E) \) is represented by the composition:

\[ S^2(E_k \wedge M \wedge M) = E_k \wedge M \wedge M \wedge S^1 \wedge S^1 \xrightarrow{1 \wedge 1 \wedge T \wedge 1} E_k \wedge SM \wedge SM \]

\[ \xrightarrow{1 \wedge \omega} E_k \wedge S^5 \xrightarrow{\xi} E_{k+5}, \]

where \( \xi \) is the composition:

\[ E_k \wedge S^5 = S'(SE_k) \xrightarrow{S^i \xi_k} S'E_{k+1} \xrightarrow{S^i \xi_{k+1}} S'E_{k+2} \xrightarrow{} \]

\[ \xrightarrow{SE_{k+4} \xi_{k+4}} E_{k+5}. \]

Because, by making use of the following homotopy commutative diagram:

\[ S^2 \wedge SM \xrightarrow{S^i \wedge 1} SM \wedge SM \]

\[ \xrightarrow{1 \wedge S \pi} S^2 \wedge S(S^2) \xrightarrow{\omega} S^5 \]

the class \( (1 \wedge i)^*(\omega^k) \) is represented by the composition:

\[ S^2(E_k \wedge S^1 \wedge M) = E_k \wedge S^1 \wedge M \wedge S^2 \xrightarrow{1 \wedge 1 \wedge \pi \wedge 1} E_k \wedge S^1 \wedge S^2 \wedge S^2 \]

\[ = E_k \wedge S^5 \xrightarrow{\xi} E_{k+5}. \]

Therefore, the class \( \sigma_p(1 \wedge i)^*(\omega^k) \) is represented by the composition:
\[ S^3(E_k \wedge M) = E_k \wedge M \wedge S^3 \xrightarrow{1 \wedge \pi \wedge 1} E_k \wedge S^6 \wedge S^3 = E_k \wedge S^5 \xrightarrow{\varepsilon} E_{k+5}. \]

And also this map represents \( \rho_p(i^k) \). Thus the relation (iv) is obtained.

The relation (v) is obtained by the similar way from the following homotopy commutative diagram:

\[
\begin{array}{c}
SM \wedge S^2 \\
\downarrow 1 \wedge \pi \\
SM \wedge SM \\
\end{array} \xrightarrow{\omega} S^5.
\]

Now, we can define

\[
\begin{align*}
\hat{h}(E) &= \lim \text{inv} \{ h^{k+s}(E_k), (-1)^s \varepsilon^*_p \}, \\
h^*(E; Z_p) &= \lim \text{inv} \{ h^{k+s}(E_k; Z_p), (-1)^s \varepsilon^*_p \}, \\
h^*(E \wedge M; E \wedge M) &= \lim \text{inv} \{ h^{k+s}(E_k \wedge M; E \wedge M), (-1)^s \varepsilon^*_p \}, \\
h^*(E \wedge M; Z_p) &= \lim \text{inv} \{ h^{k+s}(E_k \wedge M; Z_p), (-1)^s \varepsilon^*_p \}, \\
\rho_p &= \{ \rho_p \} : h^*(E) \to h^*(E; Z_p), \\
i^{**} &= \{ \varepsilon^*_p(1 \wedge i)^* \} : h^*(E \wedge M; Z_p) \to h^{*+1}(E; Z_p), \\
\pi^{**} &= \{ (1 \wedge \pi)^* \varepsilon^*_p \} : h^*(E; Z_p) \to h^{*+1}(E \wedge M; Z_p),
\end{align*}
\]

and we can denote

\[
\begin{align*}
\tilde{i} &= \{ i^k \} \in h^*(E), \\
\tilde{i}_p &= \{ \rho_p(i^k) \} \in h^*(E; Z_p), \\
\tilde{i}_M &= \{ i^k_M \} \in h^*(E \wedge M; E \wedge M), \\
\tilde{\omega} &= \{ \omega^k \} \in h^*(E \wedge M; Z_p), \\
\delta_p(\tilde{\omega}) &= \{ \delta_p(\omega^k) \} \in h^*(E \wedge M; Z_p),
\end{align*}
\]

which are well-defined from (3.1) and Proposition 3.4. Then we obtain the following relations.

\[(3.4.1) \quad (i) \quad \tilde{i}_p = \rho_p(\tilde{i}), \quad (ii) \quad \tilde{i}_p = i^{**}(\tilde{\omega}), \quad (iii) \quad \delta_p(\tilde{\omega}) = -\pi^{**}(\tilde{i}_p).\]

Remark. Making use of the cofibration

\[
S^1 \xrightarrow{i} M \xrightarrow{\pi} S^5,
\]

we have the following exact sequence.

\[(3.4.2) \quad \begin{array}{c}
\times \pi \xrightarrow{h^*(X)} (1 \wedge \pi)^* \sigma^{-2} \xrightarrow{h^*(X \wedge M)} \sigma(1 \wedge i)^* \xrightarrow{h^*(X)} \times \pi
\end{array}
\]

for any cohomology theory \{ h^*, \sigma \} and any finite CW-complex \( X \). But, in
general the limit sequence of an inverse system of exact sequences need not be exact (cf. [4], Chap. 8). So the following sequence need not be exact.

\[(3.4.3) \quad \times p \rightarrow h^*(E; Z_p) \xrightarrow{\pi**} h^*(E \wedge M; Z_p) \xrightarrow{i**} h^*(E; Z_p) \times p \rightarrow .\]

3.5. Denote by $O^+(E)$, $O^+(E \wedge M)$ and $O^+(E; Z_p)$, the modules of the stable operations of degree $n$ in the cohomology theories $h^*(E), h^*(E \wedge M)$ and $h^*(E \mod p)$ respectively, where the addition is defined by pointwise operation.

Let $\theta \in O^+(E)$. Since $\theta(\iota^k) = \theta(\sigma \varepsilon^k(\iota^{k+1})) = (-1)^n \sigma \varepsilon^k(\theta(\iota^{k+1}))$ in $h^{n+k}(E_k; E)$, we can define $\theta(\iota) = \{\theta(\iota^k)\} \in h^n(E \wedge M; E \wedge M)$ for $\theta \in O^+(E \wedge M)$, and $\theta(\omega) = \{\theta(\omega^k)\} \in h^{n+1}(E \wedge M; E \mod p)$ for $\theta \in O^+(E; Z_p)$.

**Theorem 3.5.** The following homomorphisms are isomorphisms.

(i) $\Phi: O^+(E) \rightarrow h^*(E) = h^n(E; E)$ defined by $\Phi(\theta) = \theta(\iota)$,

(ii) $\Phi_M: O^+(E \wedge M) \rightarrow h^*(E \wedge M; E \wedge M)$ defined by $\Phi_M(\theta) = \theta(\iota_M)$,

(iii) $\Phi_p: O^+(E; Z_p) \rightarrow h^{n+1}(E \wedge M; Z_p) = h^{n+1}(E \wedge M; E \mod p)$ defined by $\Phi_p(\theta) = \theta(\omega)$.

**Proof.** Let $\alpha \in h^*(X; E)$ be a class represented by a map

$$f: S^I X \rightarrow E_{l+k},$$

then $\alpha = \sigma f^*(\iota^{l+k})$, and also $\alpha$ is represented by the composition:

$$S^{I+1} X \xrightarrow{Sf} SE_{l+k} \xrightarrow{\varepsilon_{l+k}} E_{l+k+1}.$$

Thus, for $\theta \in O^+(E)$,

\[
\theta(\alpha) = \theta(\sigma f^*(\iota^{l+k})) = (-1)^n \sigma f^* \theta(\iota^{l+k}) \\
= (-1)^n \sigma f^* \theta(\iota^{l+k}) \\
= (-1)^n \sigma f^* \theta(\iota^{l+k}) \\
= (-1)^n \sigma f^* \theta(\iota^{l+k}) \\
= (-1)^n \sigma f^* \theta(\iota^{l+k}) \\
= (-1)^n \sigma f^* \theta(\iota^{l+k}) \]

since $\sigma(Sf)^* = f^* \sigma$. And this assures (i). (ii) is similarly proved, because $h^*(E \wedge M)$ is the cohomology theory defined on the spectrum $E_k \wedge M, \varepsilon_{k,M}$.

Let $\theta \in O^+(E; Z_p)$, then $\Gamma(\omega)^{-1} \theta \Gamma(\omega) \in O^+(E \wedge M)$ and this correspondence of $\theta$ to $\Gamma(\omega)^{-1} \theta \Gamma(\omega)$ induces an isomorphism of $O^+(E; Z_p)$ to $O^+(E \wedge M)$, because $\Gamma(\omega)$ is an isomorphic stable natural transformation. Since $\bar{\omega} = \Gamma(\omega)(\iota_M)$, we obtain (iii). q.e.d.

Because of the above theorem, we study $h^*(E \wedge M; Z_p)$ for the investigation
of the graded algebra $O^*(E; Z_p) = \sum_n O^*(E; Z_p)$, where the multiplication is defined by the composition.

We obtain the following relations from (3.2), (3.4) and (3.4.1).

(3.5.1)  
(i) $i^\ast \phi(\omega) = (-1)^n \phi(\iota_p)$ and $\pi^\ast \phi(\iota_p) = -\phi \delta_p(\bar{\omega})$ for $\phi \in O^*(E; Z_p)$,  
(ii) $i^\ast \theta_p(\omega) = (-1)^r \theta_p(\bar{\iota})$ and $\pi^\ast \theta_p(\bar{\iota}) = -\theta_p \delta_p(\bar{\omega})$ for $\theta \in O^*(E)$,  
where $\theta_p$ is the mod $p$ reduction of $\theta$.

3.6. Now, we consider some conditions on the spectrum $E = \{E_k, \varepsilon_k\}$ under which the sequence (3.4.3) becomes exact.

Let $\{E_{k,l}, \varepsilon_{k,l} | k, l \in \mathbb{Z}\}$ be a family of finite CW-complexes $E_{k,l}$ and maps $\varepsilon_{k,l} : SE_{k,l} \to E_{k+1,l}$, where the set $\{E_{k,l} | l \in \mathbb{Z}\}$ is a family of subcomplexes of $E_k$ with the common base vertex as one of $E_k$ for any integer $k$, such that

(i) $E_{k,l} \subset E_{k,l+1}$,  
(ii) $E_k = \bigcup_i E_{k,i}$  
and (iii) the following diagrams are commutative:

$$
\begin{array}{ccc}
SE_{k,l} & \xrightarrow{\varepsilon_{k,l}} & E_{k+1,l} \\
\downarrow & & \downarrow \\
SE_k & \xrightarrow{\varepsilon_k} & E_{k+1}
\end{array}
$$

where the vertical arrows are inclusion maps.

Then, $h^\ast(E_k; E) = \lim \text{inv} \{h^\ast(E_{k,l}; E), \varepsilon_{k,l}^\ast\}$,  
and $h^\ast(E; E) = \lim \text{inv} \{h^\ast(E_k; E), (-1)^r \sigma \varepsilon_k^\ast\}$

$$= \lim \text{inv} \{h^\ast(E_k; E), (-1)^r \sigma \varepsilon_k^\ast \varepsilon_{k+1,k}^\ast\}$$

where $\varepsilon_{k,l} : E_{k,l} \to E_{k,l+1}$ is an inclusion map.

From (3.4.2), the following results are easy consequence of the properties of the inverse limit.

(3.6.1)  
If $h^\ast(E_{k,h})$ have no $p$-torsion for any $k \in \mathbb{Z}$.  

Then the following sequences are exact:

$$
0 \to h^\ast(E; Z_p) \xrightarrow{\pi^\ast} h^\ast(E \wedge M; Z_p) \xrightarrow{i^\ast} h^\ast(E; Z_p),
$$

$$
0 \to h^\ast(E) \xrightarrow{p} h^\ast(E) \xrightarrow{\rho_p} h^\ast(E; Z_p).
$$

(3.6.2)  
If $h^\ast(E_{k,h})$ are free abelian groups and the maps $\sigma \varepsilon_k^\ast \varepsilon_{k+1,k}^\ast : h^\ast(E_{k+1,h+1}) \to h^\ast(E_{k,h})$ are onto for any $k \in \mathbb{Z}$. Then $\rho_p : h^\ast(E) \to h^\ast(E; Z_p)$ is an onto homomorphism and therefore $h^\ast(E) \otimes Z_p \approx h^\ast(E; Z_p)$.  

From (3.6.1) and (3.6.2), we obtain

(3.6.3) Under the condition of (3.6.2) on \(h^*(E_{k,k})\), if the order of \(\bar{\omega} \in h^*(E \wedge M; Z_p)\) is \(p\), i.e., \(p\bar{\omega} = 0\). Then the following sequence is a split exact sequence:

\[
0 \to h^*(E; Z_p) \overset{\pi^*}{\to} h^*(E \wedge M; Z_p) \overset{i^*}{\to} h^*(E; Z_p) \to 0
\]

and \(h^*(E \wedge M; Z_p)\) is a free \(O^*(E)/pO^*(E)\)-module with generators \(\bar{\omega}\) and \(\delta_p(\bar{\omega})\).

**Proof.** We consider a correspondence of \(p\bar{\omega}(\theta(\bar{i}))\) to \(\theta(\bar{i})\). Since \(p\bar{\omega} = 0\) and \(h^*(E; Z_p) = \rho h^*(E) \cong h^*(E) \otimes Z_p\), this correspondence is a well-defined homomorphism of \(h^*(E; Z_p)\) to \(h^*(E \wedge M; Z_p)\) and this is a right inverse of \(i^*\) from (3.5.1), thus the above sequence is a split exact sequence. Since \(h^*(E; Z_p)\) is a free \(O^*(E)/pO^*(E)\)-module with one generator \(\bar{\omega}\) from (3.2), (3.6.2) and Theorem 3.5, the final part follows from (3.5.1). q.e.d.

**REMARK.** If \(p\) is an odd prime, then the relation \(p\bar{\omega} = 0\) is always true, and if \(p=2\), this is true under some condition connecting with the Hopf map \(\eta: S^3 \to S^2\) ([1]).

As a corollary of (3.2), (3.5.1), (3.6.3) and Theorem 3.5, we obtain

**Theorem 3.6.** If \(p\bar{\omega} = 0\), \(h^*(E_{k,k})\) are free abelian groups and the maps \(\sigma \in \xi_{k,1} \xi_{k+1,1}: h^*(E_{k+1,k+1}) \to h^*(E_{k,k})\) are onto for any \(k \in Z\). Then, there exists an isomorphism

\[
O^*(E; Z_p) \cong (O^*(E)/pO^*(E)) \otimes \Lambda_p(\delta_p)
\]

as graded algebras over \(Z_p\), where \(\Lambda_p(\delta_p)\) is the exterior algebra generated by the Bockstein homomorphism \(\delta_p\). Moreover, \(O^*(E)/pO^*(E)\) is identified with the mod \(p\) reduction of \(O^*(E)\), a subalgebra of \(O^*(E; Z_p)\).

**3.7.** As an application of Theorem 3.6, we consider the stable operations in mod \(p\) \(U\)-cobordism theory.

Denote by \(\xi_{k,l}\) the canonical complex \(k\)-plane bundle over the complex Grassmann manifold \(G_{k,l}\) of \(k\)-planes in \(C^{k+l}\), and denote by \(M(\xi_{k,l})\) the Thom complex of \(\xi_{k,l}\).

Let \(\xi_{k,l}^*: S^2 M(\xi_{k,l}) \to M(\xi_{k+1,l})\) be a map induced from the canonical bundle map \(\xi_{k,l}^* \oplus C \to \xi_{k+1,l}\), and let \(\bar{e}_{k,l}: M(\xi_{k,l}) \to M(\xi_{k+1,l})\) be a canonical inclusion. Then the Thom spectrum \(MU = \{MU(k), e_k\}\) is defined by

\[
MU(k) = \lim \text{ind}_i \{M(\xi_{k,i}), \bar{e}_{k,l}\} \quad \text{and} \quad e_k \mid M(\xi_{k,l}) = \bar{e}_{k,l}.
\]

And \(U\)-cobordism theory is the cohomology theory associated with \(MU\).

The family \(\{M(\xi_{k,l}), \bar{e}_{k,l}\}\) satisfies the hypothesis of Theorem 3.6 (cf. [5]) and the order of \(\bar{\omega} \in h^*(MU \wedge M; Z_p)\) is \(p\) for any prime \(p\) (cf. [1], Th. 2.3).
Therefore, Theorem 3.6 is applicable.

On the other hand, Landweber [5] shows that there exists an isomorphism

\[ O^*(MU) \cong Z[\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots] \otimes h^*(S^0; MU) \]

as modules, where \( \gamma_n \) is the \( n \)-th \( U \)-cobordism characteristic class defined by Conner & Floyd [2], which corresponds to a stable \( U \)-cobordism operation of degree \( 2n \). Therefore, we obtain

**Theorem 3.7.** There exists an isomorphism

\[ O^*(MU; Z_p) \cong Z_p[\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots] \otimes \Lambda_p(\delta_p) \]

\[ \otimes (h^*(S^0; MU)/\phi^*(S^0; MU)) \]

as modules over \( Z_p \).

**References**


