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STABLE OPERATIONS IN MOD p COHOMOLOGY THEORIES

Dedicated to Prof. Atuo Komatu for his 60th birthday

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Introduction

By a cohomology theory we understand throughout the present work, a general reduced cohomology theory defined on the category of finite CW-complexes with base vertices (cf. [10]).

In this paper we consider the stable operations in mod p cohomology theories, where the stable operation means the natural linear operation which commutes with the suspension isomorphism.

Maunder [6] considered the stable operations in the mod p $\boldsymbol{K}$-theory, by making use of a duality map ω: $SM\wedge SM\rightarrow S^5$, where M is a co-Moore space of type $(Z_p, 2)$. We shall also use this map. For the completeness we summarize some known results on duality maps which owe to Spanier [7, 8] in section 1.

In section 2, we construct a natural transformation

$$\Gamma(\omega)\colon h^*(\ ;\boldsymbol{E}\wedge M)\rightarrow h^*(\ ;\boldsymbol{E}\ mod\ p)$$

for any spectrum $\boldsymbol{E}$, which is of degree 1, stable and isomorphic. This transformation is an essential tool in the present work.

And in section 3, we consider the relation between $\boldsymbol{O}^*(\boldsymbol{E})$, the algebra of the stable operations in the cohomology theory $h^*(\ ;\boldsymbol{E})$, and $\boldsymbol{O}^*(\boldsymbol{E}; Z_p)$, the one in the mod p cohomology theory associated with $h^*(\ ;\boldsymbol{E})$. As an application, we shall study the stable operations in the mod p $\boldsymbol{U}$-cobordism theory, by making use of Landweber's result [5].

Throughout this paper we shall use the terms "space", "CW-complex" and "map" to refer to space with a base point, CW-complex with a base vertex and continuous map preserving base points.

1. Known results on duality maps

In this section we summarize some basic properties of duality maps which owe to E.H. Spanier [7, 8].

1.1. First we shall fix some notations:

$X\wedge Y$ the reduced join of two spaces X and Y,
$f\wedge g$ the reduced join of two maps f and g,
$SX=X\wedge S^1$ the reduced suspension of X,
$1=1_A;\ A\to A$ an identity map of A into itself,
$T=T(A, B)\colon A\wedge B\to B\wedge A$ a map switching factors,
$S^nA=S(S^{n-1}A)=A\wedge S^{n-1}\wedge S^1=A\wedge S^n$ an n-fold suspension of A,
$S^nf=f\wedge 1_{S^n}$ an n-fold suspension of a map f,
$[X, Y]$ the set of homotopy classes of maps of X into Y,
$\{X, Y\}$ the stable homotopy group of X into Y,
$p\colon S^1\to S^1$ for any integer p to denote a map of degree p given by $p\{t\}=\{pt\}$ for $\{t \bmod 1\}\in S^1$.

1.2. Let X, X' be finite CW-complexes and $u\colon X\wedge X'\to S^n$ be a map. Such a map induces a homomorphism

$$\delta = \delta^z(u)_w\colon \{Z, W\wedge X\} \to \{Z\wedge X', W\wedge S^n\},$$

by the relation $\delta(\{f\})=\{(1\wedge u)(f\wedge 1)\}$ for any spaces Z and W.

A map $u\colon X\wedge X'\to S^n$ is called a semi-duality map provided $\delta^z(u)_w$ are isomorphisms for $W=S^0$ and $Z=S^k$, $k=1, 2, 3, \cdots$. If u is a duality map in the Spanier sense, then u is a semi-duality map ([7], Lemma 5.8).

From the definition of semi-duality map, we obtain the following results.

(1.2.1) Let $u\colon X\wedge X'\to S^n$ and $v\colon Y\wedge Y'\to S^n$ be maps, and let $f\colon Y\to X$ and $g\colon X'\to Y'$ be maps such that

$$\{u(f\wedge 1)\} = \{v(1\wedge g)\} \quad \text{in} \quad \{Y\wedge X', S^n\}.$$

Then the following diagram is commutative for any spaces Z and W:

$$\begin{array}{ccc} \{Z, W\wedge Y\} & \xrightarrow{\delta} & \{Z\wedge Y', W\wedge S^n\} \\ \downarrow f_* & & \downarrow g^* \\ \{Z, W\wedge X\} & \xrightarrow{\delta} & \{Z\wedge X', W\wedge S^n\}. \end{array}$$

(1.2.2) Let $u\colon X\wedge X'\to S^n$ be a semi-duality map.
Then the homomorphism $\delta^z(u)_w$ is an isomorphism for any finite CW-complexes Z and W.

(1.2.3) Let $u\colon X\wedge X'\to S^n$ be a semi-duality map.
Then two maps

$$u_{0,1}\colon X\wedge SX' = X\wedge X'\wedge S^1 \xrightarrow{u\wedge 1} S^n\wedge S^1 = S^{n+1},$$

$$u_{1,0}\colon SX\wedge X' = X\wedge S^1\wedge X' \xrightarrow{1\wedge T} X\wedge X'\wedge S^1 \xrightarrow{u\wedge 1} S^{n+1}$$

are also semi-duality maps.

1.3. Let X, X', Y and Y' be finite CW-complexes and let $u: X\wedge X'\to S^n$, $v: Y\wedge Y'\to S^n$, $f: Y\to X$ and $g: X'\to Y'$ be maps such that f and g are cellular and $u(f\wedge 1)$ and $v(1\wedge g)$ are homotopic maps from $Y\wedge X'$ into S^n.

We consider the following sequences:

$$\begin{array}{c} Y \xrightarrow{f} X \xrightarrow{i} C_f \xrightarrow{p} SY \xrightarrow{Sf} SX\,, \\ SY' \xleftarrow{Sg} SX' \xleftarrow{q} C_g \xleftarrow{j} Y' \xleftarrow{g} X'\,, \end{array} \tag{1.3.1}$$

where C_f and C_g are mapping cones of f and g respectively. Then there exists a map $\omega: C_f\wedge C_g\to S^{n+1}$ such that the following diagrams are homotopy commutative ([7], §6):

$$\begin{array}{ccc} C_f\wedge Y' & \xrightarrow{1\wedge j} & C_f\wedge C_g \\ \downarrow{\scriptstyle p\wedge 1} & & \downarrow{\scriptstyle \omega} \\ SY\wedge Y' & \xrightarrow[v_{1,0}]{} & S^{n+1} \end{array} \qquad \begin{array}{ccc} X\wedge C_g & \xrightarrow{1\wedge q} & X\wedge SX' \\ \downarrow{\scriptstyle i\wedge 1} & & \downarrow{\scriptstyle u_{0,1}} \\ C_f\wedge C_g & \xrightarrow[\omega]{} & S^{n+1} \end{array} \tag{1.3.2}$$

With an application of the "five lemma," we obtain the following result from (1.2.1), (1.2.3), (1.3.1) and (1.3.2).

(1.3.3) Let $u: X\wedge X'\to S^n$ and $v: Y\wedge Y'\to S^n$ be semi-duality maps, and let $f: Y\to X$ and $g: X'\to Y'$ be cellular maps such that $u(f\wedge 1)$ and $v(1\wedge g)$ are homotopic. Then the above map $\omega: C_f\wedge C_g\to S^{n+1}$ is a semi-duality map.

1.4. Let $u: S^2\wedge S^2\to S^4$ be a canonical identification, and let $Sp: S^2\to S^2$ be a suspension of the map $p: S^1\to S^1$. Then $u(Sp\wedge 1)$ and $u(1\wedge Sp)$ are homotopic and u is a semi-duality map. Thus we obtain a semi-duality map

$$\omega: SM_p\wedge SM_p \to S^5\,,$$

where $M_p=S^1\cup_p e^2$ is a co-Moore space of type $(Z_p, 2)$.

2. Stable natural transformation $\Gamma(\omega)$

By a spectrum $\boldsymbol{E}=\{E_k, \varepsilon_k \mid k\in Z\}$, we shall mean a sequence of CW-complexes E_k and maps $\varepsilon_k: SE_k\to E_{k+1}$ for any integer k.

Throughout this section, let M, N be fixed finite CW-complexes.

2.1. For any finite CW-complex X, and any integers l, k, we have homomorphisms

$$[S^kX\wedge N, E_{k+l}\wedge M] \xrightarrow{S} [S^{k+1}X\wedge N, SE_{k+l}\wedge M] \xrightarrow{(\varepsilon_{k+l})_\#} [S^{k+1}X\wedge N, E_{k+l+1}\wedge M]\,,$$

$$\sigma_l^k\colon [S^k(SX)\wedge N,\ E_{k+l}\wedge M] \underset{\approx}{\longrightarrow} [S^{k+1}X\wedge N,\ E_{k+l}\wedge M]\,,$$

where $S=S(M, N)$ is defined by the suspension, $(\varepsilon_{k+l})_\#=\varepsilon_{k+l}(M, N)_\#$ is an induced homomorphism and $\sigma_l^k=\sigma_l^k(M, N)$ is induced from the identification $S^{k+1}X=X\wedge S^{k+1}=X\wedge S^1\wedge S^k=S^k(SX)$.

Let $h^l(X;\, \boldsymbol{E}\wedge M \bmod N)=\lim_k \operatorname{dir}\{[S^kX\wedge N, E_{k+l}\wedge M], (\varepsilon_{k+l})_\# S\}$ and $\sigma=\sigma(M, N)\colon h^l(SX;\, \boldsymbol{E}\wedge M \bmod N)\underset{\approx}{\longrightarrow} h^{l-1}(X;\, \boldsymbol{E}\wedge M \bmod N)$ be the direct limit of maps $\{\sigma_l^k\}$.

Then, $\{h^*(\ ;\, \boldsymbol{E}\wedge M \bmod N), \sigma(M, N)\}$ becomes a cohomology theory [10]. In particular, we define

$$\begin{aligned} h^*(\ ;\, \boldsymbol{E}\wedge M) &= h^*(\ ;\, \boldsymbol{E}\wedge M \bmod S^0), \quad \sigma^M = \sigma(M, S^0)\,,\\ h^*(\ ;\, \boldsymbol{E} \bmod N) &= h^*(\ ;\, \boldsymbol{E}\wedge S^0 \bmod N), \quad \sigma_N = \sigma(S^0, N)\,,\\ h^*(\ ;\, \boldsymbol{E}) &= h^*(\ ;\, \boldsymbol{E}\wedge S^0 \bmod S^0), \quad \sigma = \sigma(S^0, S^0)\,,\end{aligned}$$

and
$$h^k(\ ;\, \boldsymbol{E} \bmod p) = h^{k+2}(\ ;\, \boldsymbol{E} \bmod M_p), \quad \sigma_p = \sigma(S^0, M_p)\,,$$

where $M_p=S^1\cup_p e^2$ be a co-Moore space of type $(Z_p, 2)$, then the third cohomology theory is just one defined by G.W. Whitehead [10], and the last is just a mod p cohomology theory associated with $h^*(\ ;\boldsymbol{E})$ defined by A. Dold [3] and considered by S. Araki and H. Toda [1].

2.2. Let $\{h_1^*, \sigma_1\}$ and $\{h_2^*, \sigma_2\}$ be cohomology theories and

$$t\colon h_1^k \to h_2^{k+s} \qquad \text{for any integer} \quad k\,,$$

be a linear natural transformation of degree s. If $\sigma_2 t=(-1)^s t\sigma_1$, then we call t is a stable natural transformation of degree s. In particular, if $\{h_1^*, \sigma_1\}=\{h_2^*, \sigma_2\}$, then we call t is a stable operation of degree s.

2.3. Let $\omega\colon M\wedge N\to S^n$ be a map, then ω induces a homomorphism

$$\gamma(\omega)_l^k\colon [S^kX,\ E_{k+l}\wedge M] \to [S^kX\wedge N,\ E_{k+l}\wedge S^n]\,,$$

by the relation $\gamma(\omega)\,([f])=[(1\wedge\omega)\,(f\wedge 1)]$ for any spectrum $\boldsymbol{E}=\{E_k, \varepsilon_k\}$ and any finite CW-complex X.

Proposition 2.3. *For any spectrum $\boldsymbol{E}=\{E_k, \varepsilon_k\}$, a map $\omega\colon M\wedge N\to S^n$ induces a stable natural transformation*

$$\gamma(\omega)\colon h^*(\ ;\, \boldsymbol{E}\wedge M) \to h^*(\ ;\, \boldsymbol{E}\wedge S^n \bmod N)$$

of degree 0, *where $\gamma(\omega)$ is the direct limit of homomorphisms $\{\gamma(\omega)_l^k\}$.*

Proof. For any integers l, k,

$$\varepsilon_{k+l}(S^n, N)_\# S(S^n, N)\gamma(\omega)_l^k = \gamma(\omega)_l^{k+1}\varepsilon_{k+l}(M, S^0)_\# S(M, S^0)\,,$$

so we can define a natural transformation $\gamma(\omega)$ of degree 0. Moreover, for any integers l, k,

$$\sigma_l^k(S^n, N)\gamma(\omega)_l^k = \gamma(\omega)_{l-1}^{k+1}\sigma_l^k(M, S^0),$$

thus

$$\sigma(S^n, N)\gamma(\omega) = \gamma(\omega)\sigma(M, S^0).$$

Therefore $\gamma(\omega)$ is stable. q.e.d.

2.4. From some switching map T, we can define the following homomorphisms:

$$[S^kX\wedge N, E_{l+k}\wedge SM] \xrightarrow[\approx]{T(M, S^1)_\#} [S^kX\wedge N, SE_{l+k}\wedge M],$$

$$[S^kX\wedge SN, E_{l+k}\wedge M] \xrightarrow[\approx]{T(N, S^1)^\#} [S^{k+1}X\wedge N, E_{l+k}\wedge M]$$

for any spectrum $\boldsymbol{E}=\{E_k, \varepsilon_k\}$ and any finite CW-complex X.

Proposition 2.4. *For any spectrum $\boldsymbol{E}=\{E_k, \varepsilon_k\}$, T induces stable natural transformations*

$$T_*: h^*(\ ; \boldsymbol{E}\wedge SM \text{ mod } N) \to h^*(\ ; \boldsymbol{E}\wedge M \text{ mod } N),$$
$$T^*: h^*(\ ; \boldsymbol{E}\wedge M \text{ mod } SN) \to h^*(\ ; \boldsymbol{E}\wedge M \text{ mod } N),$$

such that degree $T_*=1$ and degree $T^*=-1$, where T_* is the direct limit of homomorphisms $\{(-1)^k\varepsilon_{k+l}(M, N)_\# T(M, S^1)_\#\}$ and T^* is the direct limit of homomorphisms $\{(-1)^kT(N, S^1)^\#\}$. Moreover, for any finite CW-complex X and any integer l,

$$T_*: h^l(X; \boldsymbol{E}\wedge SM \text{ mod } N) \to h^{l+1}(X; \boldsymbol{E}\wedge M \text{ mod } N),$$

and

$$T^*: h^l(X; \boldsymbol{E}\wedge M \text{ mod } SN) \to h^{l-1}(X; \boldsymbol{E}\wedge M \text{ mod } N)$$

are isomorphisms.

Proof. For any integers l, k,

$$\begin{aligned}&\varepsilon_{k+l+1}(M, N)_\# S(M, N)\varepsilon_{k+l}(M, N)_\# T(M, S^1)_\# \\ &= -\varepsilon_{k+l+1}(M, N)_\# T(M, S^1)_\#\varepsilon_{k+l}(SM, N)_\# S(SM, N),\end{aligned}$$

and

$$\begin{aligned}&\sigma_{l+1}^k(M, N)\varepsilon_{k+l}(M, N)_\# T(M, S^1)_\# \\ &= \varepsilon_{k+l}(M, N)_\# T(M, S^1)_\#\sigma_l^k(SM, N),\end{aligned}$$

so we can define a stable natural transformation T_* of degree 1 induced from the sequence $\{(-1)^k\varepsilon_{k+l}(M, N)_\# T(M, S^1)_\#\}$. Similarly, we can define a stable natural transformation T^* of degree -1 induced from the sequence $\{(-1)^kT(N, S^1)^\#\}$. Since $T(N, S^1)^\#$ is an isomorphism, T^* is an isomorphism.

Next we consider the following homomorphism

$$[S^kX\wedge N, E_{l+k}\wedge M]\xrightarrow{S}[S^{k+1}X\wedge N, SE_{l+k}\wedge M]\xrightarrow[\approx]{T(S^1, M)_\#}$$
$$[S^{k+1}X\wedge N, E_{l+k}\wedge SM],$$

then the sequence $\{(-1)^{k+1}T(S^1, M)_\# S\}$ induces a stable natural transformation

$$S_*: h^*(\ ;\boldsymbol{E}\wedge M\ mod\ N)\to h^*(\ ;\boldsymbol{E}\wedge SM\ mod\ N)$$

of degree -1 and clearly S_* is the inverse transformation of T_*. Therefore T_* is an isomorphism. q.e.d.

2.5. Let ω: $S^aM\wedge S^bN\to S^{n+a+b}$ be a map. For any spectrum $\boldsymbol{E}$, we obtain a stable natural transformation

$$\Gamma(\omega): h^*(\ ;\boldsymbol{E}\wedge M)\to h^*(\ ;\boldsymbol{E}\ mod\ N)$$

of degree n, which is defined by $\Gamma(\omega)=(T^*)^b(T_*)^{n+a+b}\gamma(\omega)(T_*)^{-a}$.

Theorem 2.5. *If* ω: $S^aM\wedge S^bN\to S^{n+a+b}$ *is a semi-duality map. Then, for any spectrum* $\boldsymbol{E}$, *the stable natural transformation*

$$\Gamma(\omega): h^k(X;\boldsymbol{E}\wedge M)\to h^{k+n}(X;\boldsymbol{E}\ mod\ N)$$

is an isomorphism for any integer k and any finite CW-complex X.

Proof. It is sufficient to prove that $\gamma(\omega)$ is an isomorphism, by Proposition 2.4. For any finite CW-complex X, there exists a canonical isomorphism

$$\iota(A, B): h^k(X;\boldsymbol{E}\wedge A\ mod\ B)\to \lim_l \text{dir}\ \{\{S^lX\wedge B, E_{k+l}\wedge A\}, \{\varepsilon_{k+l}\}_\#\}$$

induced from canonical homomorphisms

$$[S^lX\wedge B, E_{k+l}\wedge A]\to\{S^lX\wedge B, E_{k+l}\wedge A\},$$

where A, B are any finite CW-complexes.

From (1.2.2), the semi-duality map ω: $S^aM\wedge S^bN\to S^{n+a+b}$ induces an isomorphism

$$\delta(\omega)^l_k: \{S^lX, E_{k+l}\wedge S^aM\}\to\{S^lX\wedge S^bN, E_{k+l}\wedge S^{n+a+b}\}$$

and the sequence $\{\delta(\omega)^l_k\}$ defines an isomorphism

$$\delta(\omega): \lim_l \text{dir}\ \{\{S^lX, E_{k+l}\wedge S^aM\}, \{\varepsilon_{k+l}\}_\#\}$$
$$\to \lim_l \text{dir}\ \{\{S^lX\wedge S^bN, E_{k+l}\wedge S^{n+a+b}\}, \{\varepsilon_{k+l}\}_\#\}.$$

Since the relation $\iota(S^0, S^bN)\gamma(\omega)=\delta(\omega)\iota(S^aM, S^0)$ holds, the homomorphism $\gamma(\omega)$ is an isomorphism. And therefore $\Gamma(\omega)$ is an isomorphism. q.e.d.

REMARK. Let $X=S^0$ in Theorem 2.5. Since $h^{-k}(S^0; \boldsymbol{E}\wedge M)=h_k(M; \boldsymbol{E})$, a reduced homology group of M, and $h^k(S^0; \boldsymbol{E}\ mod\ N)=h^k(N; \boldsymbol{E})$, we obtain a duality isomorphism

$$\Gamma(\omega)\colon h_k(M; \boldsymbol{E}) \to h^{n-k}(N; \boldsymbol{E}),$$

whenever $\omega\colon S^aM\wedge S^aN\to S^{n+a+b}$ is a semi-duality map.

3. Stable operations

Throughout this section, let p be a fixed prime, and let $M=S^1\cup_p e^2$ be the co-Moore space of type $(Z_p, 2)$. Denote by $i\colon S^1\to M$ and $\pi\colon M\to S^2$, the canonical inclusion and the map collapsing S^1 to a point.

3.1. Let $\{h^*, \sigma\}$ be a cohomology theory. The mod p cohomology theory (cf. [1]), $\{h^*(\ ; Z_p), \sigma_p\}$ is defined by

$$h^k(X; Z_p) = h^{k+2}(X\wedge M) \qquad \text{for all} \quad k,$$

and the suspension isomorphism

$$\sigma_p\colon h^k(SX; Z_p) \to h^{k-1}(X; Z_p) \qquad \text{for all} \quad k,$$

is defined as the composition

$$h^k(SX; Z_p) = h^{k+2}(X\wedge S^1\wedge M) \xrightarrow[\approx]{(1\wedge T)^*} h^{k+2}(X\wedge M\wedge S^1)$$

$$\xrightarrow[\approx]{\sigma} h^{k+1}(X\wedge M) = h^{k-1}(X; Z_p),$$

where $T=T(S^1, M)$. If $\{h^*, \sigma\}$ is defined by a spectrum $\boldsymbol{E}$, then $\{h^*(\ ; Z_p), \sigma_p\}$ is equivalent to the cohomology theory defined in section 2.

Making use of maps $i\colon S^1\to M$ and $\pi\colon M\to S^2$, we put

$$\rho_p\colon h^k(X) \xleftarrow[\approx]{\sigma^2} h^{k+2}(X\wedge S^2) \xrightarrow{(1\wedge\pi)^*} h^{k+2}(X\wedge M) = h^k(X; Z_p)$$

and $$\delta_p\colon h^k(X; Z_p) = h^{k+2}(X\wedge M) \xrightarrow{(1\wedge i)^*} h^{k+2}(X\wedge S^1)$$

$$\xleftarrow[\approx]{\sigma} h^{k+3}(X\wedge S^2) \xrightarrow{(1\wedge\pi)^*} h^{k+3}(X\wedge M) = h^{k+1}(X; Z_p),$$

which are natural and called as the reduction "mod p" and the "mod p" Bockstein homomorphism. The following relations are easily seen.

(3.1) $$\sigma_p\rho_p = \rho_p\sigma,\ \sigma_p\delta_p = -\delta_p\sigma_p,\ \delta_p\rho_p = 0 \quad \text{and} \quad \delta_p\delta_p = 0.$$

In particular, the Bockstein homomorphism δ_p is a stable operation of degree 1 in mod p cohomology theory.

3.2. Let θ be a stable operation of degree n in the cohomology theory $\{h^*, \sigma\}$, i.e., $\sigma\theta=(-1)^n\theta\sigma$. We put

$$\theta_p: h^k(X; Z_p) = h^{k+2}(X\wedge M) \xrightarrow{\theta} h^{k+n+2}(X\wedge M) = h^{k+n}(X; Z_p) \quad \text{for all} \quad k,$$

which is a stable operation of degree n in the mod p cohomology theory $\{h^*(\ ; Z_p), \sigma_p\}$, i.e., $\sigma_p\theta_p=(-1)^n\theta_p\sigma_p$, and called as the "mod p" reduction of θ. From the definitions of ρ_p, δ_p, θ_p and a stable operation of degree n, we obtain the following relations.

$$\delta_p\theta_p = (-1)^n\theta_p\delta_p \quad \text{and} \quad \theta_p\rho_p = \rho_p\theta. \tag{3.2}$$

REMARK. In the "mod 2" singular cohomology theory. Since $\delta_2=S_q^1$ and $S_q^1S_q^{2n}=S_q^{2n+1}+S_q^{2n}S_q^1$ for any $n=1, 2, 3, \cdots$, S_q^{2n} ($n=1, 2, 3, \cdots$) are not the "mod 2" reduction of any stable operations in the integral cohomology theory.

3.3. Let $\{h^*, \sigma\}$ be a cohomology theory. Let X be a CW-complex with a base vertex x_0, and let $\{X_\alpha\}$ be the family of all finite subcomplexes with base vertex x_0. Then $\{h^*(X_\alpha)\}$ becomes an inverse system with respect to the homomorphisms induced from the inclusion maps, and we can define

$$h^*(X) = \lim_{\alpha} \operatorname{inv} h^*(X_\alpha).$$

3.4. In the rest of this paper, we consider a fixed spectrum $\boldsymbol{E}=\{E_k, \varepsilon_k\}$ and denote by $\{h^*, \sigma\}$, the cohomology theory $\{h^*(\ ; \boldsymbol{E}), \sigma\}$ associated with $\boldsymbol{E}$. And we consider a fixed semi-duality map ω: $SM\wedge SM\to S^5$.

Denote by $\iota^k\in h^k(E_k)=h^k(E_k; \boldsymbol{E})$ and $\iota_M^k\in h^k(E_k\wedge M; \boldsymbol{E}\wedge M)$, the classes represented by the identity maps of E_k and $E_k\wedge M$ respectively. And denote by $\omega^k\in h^{k+1}(E_k\wedge M; Z_p)=h^{k+1}(E_k\wedge M; \boldsymbol{E} \bmod p)$, the class $\Gamma(\omega)(\iota_M^k)$, where

$$\Gamma(\omega): h^*(\ ; \boldsymbol{E}\wedge M) \to h^*(\ ; \boldsymbol{E} \bmod p)$$

is an isomorphic stable natural transformation of degree 1.

We put

$$\varepsilon_{k,M}: S(E_k\wedge M) = E_k\wedge M\wedge S^1 \xrightarrow{1\wedge T} E_k\wedge S^1\wedge M \xrightarrow{\varepsilon_k\wedge 1} E_{k+1}\wedge M,$$

and consider the following sequences:

$$h_*(E_{k+1}) \xrightarrow{\varepsilon_k^*} h^*(SE_k) \underset{\approx}{\xrightarrow{\sigma}} h^*(E_k),$$

$$h^*(E_{k+1}\wedge M; \boldsymbol{E}\wedge M) \xrightarrow{\varepsilon_{k,M}^*} h^*(S(E_k\wedge M); \boldsymbol{E}\wedge M) \underset{\approx}{\xrightarrow{\sigma}} h^*(E_k\wedge M; \boldsymbol{E}\wedge M),$$

$$h^*(E_{k+1}\wedge M;Z_p)\xrightarrow{\mathcal{E}^*_{k,M}} h^*(S(E_k\wedge M);Z_p)\underset{\approx}{\xrightarrow{\sigma_p}} h^*(E_k\wedge M;Z_p)\,,$$

$$h^*(E_k\wedge M;Z_p)\xrightarrow{(1\wedge i)^*} h^*(E_k\wedge S^1;Z_p)\underset{\approx}{\xrightarrow{\sigma_p}} h^*(E_k;Z_p)\,,$$

$$h^*(E_k;Z_p)\underset{\approx}{\xleftarrow{\sigma_p^2}} h^*(E_k\wedge S^2;Z_p)\xrightarrow{(1\wedge\pi)^*} h^*(E_k\wedge M;Z_p)\,.$$

Proposition 3.4. *There are relations*:

(i) $\iota^k=\sigma\mathcal{E}^*_k(\iota^{k+1})$, (ii) $\iota^k_M=\sigma\mathcal{E}^*_{k,M}(\iota^{k+1}_M)$,

(iii) $\omega^k=-\sigma_p\mathcal{E}^*_{k,M}(\omega^{k+1})$, (iv) $\rho_p(\iota^k)=\sigma_p(1\wedge i)^*(\omega^k)$.

(v) $\delta_p(\omega^k)=-(1\wedge\pi)^*\sigma_p^{-2}\rho_p(\iota^k)$.

Proof. Relations (i) and (ii) are trivial, and (iii) is a consequence of $\omega^k=\Gamma(\omega)(\iota^k_M)$ and the fact that $\Gamma(\omega)$ is a stable natural transformation of degree 1, i.e., $\sigma_p\Gamma(\omega)=-\Gamma(\omega)\sigma$ and $\mathcal{E}^*_{k,M}\Gamma(\omega)=\Gamma(\omega)\mathcal{E}^*_{k,M}$.

Relations (iv) and (v) follow from the diagrams below (cf. (1.3.2)), the definition of ρ_p and δ_p, and the fact that $\omega^k\in h^{k+1}(E_k\wedge M;Z_p)=h^{k+3}(E_k\wedge M\wedge M;\boldsymbol{E})$ is represented by the composition:

$$S^2(E_k\wedge M\wedge M)=E_k\wedge M\wedge M\wedge S^1\wedge S^1\xrightarrow{1\wedge 1\wedge T\wedge 1} E_k\wedge SM\wedge SM$$
$$\xrightarrow{1\wedge\omega} E_k\wedge S^5\xrightarrow{\mathcal{E}} E_{k+5}\,,$$

where $\mathcal{E}$ is the composition:

$$E_k\wedge S^5=S^4(SE_k)\xrightarrow{S^4\mathcal{E}_k} S^4E_{k+1}\xrightarrow{S^3\mathcal{E}_{k+1}} S^3E_{k+2}\to\cdots$$
$$\to SE_{k+4}\xrightarrow{\mathcal{E}_{k+4}} E_{k+5}\,.$$

Because, by making use of the following homotopy commutative diagram:

$$\begin{array}{ccc} S^2\wedge SM & \xrightarrow{Si\wedge 1} & SM\wedge SM \\ \big\downarrow{\scriptstyle 1\wedge S\pi} & & \big\downarrow{\scriptstyle\omega} \\ S^2\wedge S(S^2) & = & S^5 \end{array}$$

the class $(1\wedge i)^*(\omega^k)$ is represented by the composition:

$$S^2(E_k\wedge S^1\wedge M)=E_k\wedge S^1\wedge M\wedge S^2\xrightarrow{1\wedge 1\wedge\pi\wedge 1} E_k\wedge S^1\wedge S^2\wedge S^2$$
$$=E_k\wedge S^5\xrightarrow{\mathcal{E}} E_{k+5}\,.$$

Therefore, the class $\sigma_p(1\wedge i)^*(\omega^k)$ is represented by the composition:

$$S^3(E_k \wedge M) = E_k \wedge M \wedge S^3 \xrightarrow{1\wedge\pi\wedge 1} E_k \wedge S^2 \wedge S^3 = E_k \wedge S^5 \xrightarrow{\varepsilon} E_{k+5}.$$

And also this map represents $\rho_p(\iota^k)$. Thus the relation (iv) is obtained.

The relation (v) is obtained by the similar way from the following homotopy commutative diagram:

$$\begin{array}{ccc} SM\wedge S^2 & \xrightarrow{S\pi\wedge 1} & S(S^2)\wedge S^2 \\ \downarrow 1\wedge Si & & \| \\ SM\wedge SM & \xrightarrow{\omega} & S^5. \end{array} \qquad \text{q.e.d.}$$

Now, we can define

$$h^s(\boldsymbol{E}) = \lim_k \operatorname{inv} \{h^{k+s}(E_k),\ (-1)^s\sigma\varepsilon_k^*\},$$
$$h^s(\boldsymbol{E}; Z_p) = \lim_k \operatorname{inv} \{h^{k+s}(E_k; Z_p),\ (-1)^s\sigma_p\varepsilon_k^*\},$$
$$h^s(\boldsymbol{E}\wedge M; \boldsymbol{E}\wedge M) = \lim_k \operatorname{inv} \{h^{k+s}(E_k\wedge M; \boldsymbol{E}\wedge M),\ (-1)^s\sigma\varepsilon_{k,M}^*\},$$
$$h^s(\boldsymbol{E}\wedge M; Z_p) = \lim_k \operatorname{inv} \{h^{k+s}(E_k\wedge M; Z_p),\ (-1)^s\sigma_p\varepsilon_{k,M}^*\},$$
$$\rho_p = \{\rho_p\}: h^s(\boldsymbol{E}) \to h^s(\boldsymbol{E}; Z_p),$$
$$i^{**} = \{\sigma_p(1\wedge i)^*\}: h^s(\boldsymbol{E}\wedge M; Z_p) \to h^{s-1}(\boldsymbol{E}; Z_p),$$
$$\pi^{**} = \{(1\wedge\pi)^*\sigma_p^{-2}\}: h^s(\boldsymbol{E}; Z_p) \to h^{s+2}(\boldsymbol{E}\wedge M; Z_p),$$

and we can denote

$$\tilde\iota = \{\iota^k\} \in h^0(\boldsymbol{E}),\quad \tilde\iota_p = \{\rho_p(\iota^k)\} \in h^0(\boldsymbol{E}; Z_p),$$
$$\tilde\iota_M = \{\iota_M^k\} \in h^0(\boldsymbol{E}\wedge M; \boldsymbol{E}\wedge M),\quad \tilde\omega = \{\omega^k\} \in h^1(\boldsymbol{E}\wedge M; Z_p),$$
$$\delta_p(\tilde\omega) = \{\delta_p(\omega^k)\} \in h^2(\boldsymbol{E}\wedge M; Z_p),$$

which are well-defined from (3.1) and Proposition 3.4. Then we obtain the following relations.

(3.4.1) (i) $\tilde\iota_p = \rho_p(\tilde\iota)$, (ii) $\tilde\iota_p = i^{**}(\tilde\omega)$,
(iii) $\delta_p(\tilde\omega) = -\pi^{**}(\tilde\iota_p)$.

REMARK. Making use of the cofibration

$$S^1 \xrightarrow{i} M \xrightarrow{\pi} S^2,$$

we have the following exact sequence.

$$\text{(3.4.2)}\qquad \xrightarrow{\times p} h^*(X) \xrightarrow{(1\wedge\pi)^*\sigma^{-2}} h^*(X\wedge M) \xrightarrow{\sigma(1\wedge i)^*} h^*(X) \xrightarrow{\times p}$$

for any cohomology theory $\{h^*, \sigma\}$ and any finite CW-complex X. But, in general the limit sequence of an inverse system of exact sequences need not be exact (cf. [4], Chap. 8). So the following sequence need not be exact.

$$(3.4.3) \qquad \xrightarrow{\times p} h^*(\boldsymbol{E};\, Z_p) \xrightarrow{\pi^{**}} h^*(\boldsymbol{E}\wedge M;\, Z_p) \xrightarrow{i^{**}} h^*(\boldsymbol{E};\, Z_p) \xrightarrow{\times p} .$$

3.5. Denote by $\boldsymbol{O}^n(\boldsymbol{E})$, $\boldsymbol{O}^n(\boldsymbol{E}\wedge M)$ and $\boldsymbol{O}^n(\boldsymbol{E};Z_p)$, the modules of the stable operations of degree n in the cohomology theories $h^*(\ ;\boldsymbol{E})$, $h^*(\ ;\boldsymbol{E}\wedge M)$ and $h^*(\ ;\boldsymbol{E}\ mod\ p)$ respectively, where the addition is defined by pointwise operation.

Let $\theta\in\boldsymbol{O}^n(\boldsymbol{E})$. Since $\theta(\iota^k)=\theta(\sigma\,\varepsilon_k^*(\iota^{k+1}))=(-1)^n\sigma\,\varepsilon_k^*(\theta(\iota^{k+1}))$ in $h^{n+k}(E_k;\boldsymbol{E})$, we can define $\theta(\tilde{\iota})=\{\theta(\iota^k)\}\in h^n(\boldsymbol{E};\boldsymbol{E})$. Similarly, we can define $\theta(\tilde{\iota}_M)=\{\theta(\iota_M^k)\}\in h^n(\boldsymbol{E}\wedge M;\boldsymbol{E}\wedge M)$ for $\theta\in\boldsymbol{O}^n(\boldsymbol{E}\wedge M)$, and $\theta(\tilde{\omega})=\{\theta(\omega^k)\}\in h^{n+1}(\boldsymbol{E}\wedge M;\boldsymbol{E}\ mod\ p)$ for $\theta\in\boldsymbol{O}^n(\boldsymbol{E};Z_p)$.

Theorem 3.5. *The following homomorphisms are isomorphisms.*

(i) $\Phi\colon \boldsymbol{O}^n(\boldsymbol{E}) \to h^n(\boldsymbol{E}) = h^n(\boldsymbol{E};\boldsymbol{E})$ *defined by* $\Phi(\theta)=\theta(\tilde{\iota})$,

(ii) $\Phi_M\colon \boldsymbol{O}^n(\boldsymbol{E}\wedge M) \to h^n(\boldsymbol{E}\wedge M;\boldsymbol{E}\wedge M)$ *defined by* $\Phi_M(\theta)=\theta(\tilde{\iota}_M)$,

(iii) $\Phi_p\colon \boldsymbol{O}^n(\boldsymbol{E};Z_p) \to h^{n+1}(\boldsymbol{E}\wedge M;Z_p) = h^{n+1}(\boldsymbol{E}\wedge M;\boldsymbol{E}\ mod\ p)$ *defined by* $\Phi_p(\theta)=\theta(\tilde{\omega})$.

Proof. Let $\alpha\in h^k(X;\boldsymbol{E})$ be a class represented by a map

$$f\colon S^lX \to E_{l+k},$$

then $\alpha=\sigma^l f^*(\iota^{l+k})$, and also α is represented by the composition:

$$S^{l+1}X \xrightarrow{Sf} SE_{l+k} \xrightarrow{\varepsilon_{l+k}} E_{l+k+1}.$$

Thus, for $\theta\in\boldsymbol{O}^n(\boldsymbol{E})$,

$$\begin{aligned}\theta(\alpha) &= \theta(\sigma^l f^*(\iota^{l+k})) = (-1)^{ln}\sigma^l f^*\theta(\iota^{l+k})\\ &= (-1)^{(l+1)n}\sigma^l f^*\sigma\,\varepsilon_{l+k}^*\theta(\iota^{l+k+1})\\ &= (-1)^{(l+1)n}\sigma^{l+1}(Sf)^*\varepsilon_{l+k}^*\theta(\iota^{l+k+1})\\ &= \theta(\sigma^{l+1}(Sf)^*\varepsilon_{l+k}^*(\iota^{l+k+1})),\end{aligned}$$

since $\sigma(Sf)^*=f^*\sigma$. And this assures (i). (ii) is similarly proved, because $h^*(\ ;\boldsymbol{E}\wedge M)$ is the cohomology theory defined on the spectrum $\{E_k\wedge M,\ \varepsilon_{k,M}\}$.

Let $\theta\in\boldsymbol{O}^n(\boldsymbol{E};Z_p)$, then $\Gamma(\omega)^{-1}\theta\Gamma(\omega)\in\boldsymbol{O}^n(\boldsymbol{E}\wedge M)$ and this correspondence of θ to $\Gamma(\omega)^{-1}\theta\Gamma(\omega)$ induces an isomorphism of $\boldsymbol{O}^n(\boldsymbol{E};Z_p)$ to $\boldsymbol{O}^n(\boldsymbol{E}\wedge M)$, because $\Gamma(\omega)$ is an isomorphic stable natural transformation. Since $\tilde{\omega}=\Gamma(\omega)(\tilde{\iota}_M)$, we obtain (iii). q.e.d.

Because of the above theorem, we study $h^*(\boldsymbol{E}\wedge M;Z_p)$ for the investigation of the graded algebra $O^*(E; Z_p)=\sum_n O^n(E; Z_p)$, where the multiplication is defined by the composition.

We obtain the following relations from (3.2), (3.4) and (3.4.1).

(3.5.1) (i) $i^{**}\phi(\tilde{\omega})=(-1)^n\phi(\tilde{\iota}_p)$ and $\pi^{**}\phi(\tilde{\iota}_p)=-\phi\delta_p(\tilde{\omega})$ for $\phi\in O^n(E; Z_p)$,

(ii) $i^{**}\theta_p(\tilde{\omega})=(-1)^n\rho_p(\theta(\tilde{\iota}))$ and $\pi^{**}\rho_p(\theta(\tilde{\iota}))=-\theta_p\delta_p(\tilde{\omega})$ for $\theta\in O^n(E)$, where θ_p is the mod p reduction of θ.

3.6. Now, we consider some conditions on the spectrum $E=\{E_k, \varepsilon_k\}$ under which the sequence (3.4.3) becomes exact.

Let $\{E_{k,l}, \varepsilon_{k,l} \mid k, l\in Z\}$ be a family of finite CW-complexes $E_{k,l}$ and maps $\varepsilon_{k,l}$: $SE_{k,l}\to E_{k+1,l}$, where the set $\{E_{k,l} \mid l\in Z\}$ is a family of subcomplexes of E_k with the common base vertex as one of E_k for any integer k, such that

$$\text{(i)}\quad E_{k,l}\subset E_{k,l+1}, \qquad \text{(ii)}\quad E_k=\bigcup_l E_{k,l}$$

and (iii) the following diagrams are commutative:

$$\begin{array}{ccc} SE_{k,l} & \xrightarrow{\varepsilon_{k,l}} & E_{k+1,l} \\ \downarrow & & \downarrow \\ SE_k & \xrightarrow{\varepsilon_k} & E_{k+1} \end{array}$$

where the vertical arrows are inclusion maps.

Then,
$$h^*(E_k; E)=\lim_l \text{inv}\,\{h^*(E_{k,l}; E), \iota_{k,l}^*\},$$
and
$$\begin{aligned} h^*(E; E) &= \lim_k \text{inv}\,\{h^*(E_k; E), (-1)^*\sigma\varepsilon_k^*\} \\ &= \lim_k \text{inv}\,\{h^*(E_{k,k}; E), (-1)^*\sigma\varepsilon_{k,k}^*\iota_{k+1,k}^*\} \end{aligned}$$
where $\iota_{k,l}$; $E_{k,l}\to E_{k,l+1}$ is an inclusion map.

From (3.4.2), the following results are easy consequence of the properties of the inverse limit.

(3.6.1) If $h^*(E_{k,k})$ have no p-torsion for any $k\in Z$.

Then the following sequences are exact:

$$0\to h^*(E; Z_p)\xrightarrow{\pi^{**}} h^*(E\wedge M; Z_p)\xrightarrow{i^{**}} h^*(E; Z_p),$$
$$0\to h^*(E)\xrightarrow{\times p} h^*(E)\xrightarrow{\rho_p} h^*(E; Z_p).$$

(3.6.2) If $h^*(E_{k,k})$ are free abelian groups and the maps $\sigma\varepsilon_{k,k}^*\iota_{k+1,k}^*$: $h^*(E_{k+1,k+1})\to h^*(E_{k,k})$ are onto for any $k\in Z$. Then ρ_p: $h^*(E)\to h^*(E; Z_p)$ is an onto homomorphism and therefore $h^*(E)\otimes Z_p\approx h^*(E; Z_p)$.

From (3.6.1) and (3.6.2), we obtain

(3.6.3) Under the condition of (3.6.2) on $h^*(E_{k,k})$, if the order of $\tilde{\omega}\in h^1(\boldsymbol{E}\wedge M;\,Z_p)$ is p, i.e., $p\tilde{\omega}=0$. Then the following sequence is a split exact sequence:

$$0\rightarrow h^*(\boldsymbol{E};\,Z_p)\xrightarrow{\pi^{**}}h^*(\boldsymbol{E}\wedge M;\,Z_p)\xrightarrow{i^{**}}h^*(\boldsymbol{E};\,Z_p)\rightarrow 0$$

and $h^*(\boldsymbol{E}\wedge M;\,Z_p)$ is a free $\boldsymbol{O}^*(\boldsymbol{E})/p\boldsymbol{O}^*(\boldsymbol{E})$-module with generators $\tilde{\omega}$ and $\delta_p(\tilde{\omega})$.

Proof. We consider a correspondence of $\rho_p(\theta(\tilde{\iota}))$ to $\theta_p(\tilde{\omega})$. Since $p\tilde{\omega}=0$ and $h^*(\boldsymbol{E};\,Z_p)=\rho_p h^*(\boldsymbol{E})\approx h^*(\boldsymbol{E})\otimes Z_p$, this correspondence is a well-defined homomorphism of $h^*(\boldsymbol{E};\,Z_p)$ to $h^*(\boldsymbol{E}\wedge M;\,Z_p)$ and this is a right inverse of i^{**} from (3.5.1), thus the above sequence is a split exact sequence. Since $h^*(\boldsymbol{E};\,Z_p)$ is a free $\boldsymbol{O}^*(\boldsymbol{E})/p\boldsymbol{O}^*(\boldsymbol{E})$-module with one generator $\tilde{\iota}_p$ from (3.2), (3.6.2) and Theorem 3.5, the final part follows from (3.5.1). q.e.d.

REMARK. If p is an odd prime, then the relation $p\tilde{\omega}=0$ is always true, and if $p=2$, this is true under some condition connecting with the Hopf map η: $S^3\rightarrow S^2$ ([1]).

As a corollary of (3.2), (3.5.1), (3.6.3) and Theorem 3.5, we obtain

Theorem 3.6. *If $p\tilde{\omega}=0$, $h^*(E_{k,k})$ are free abelian groups and the maps $\sigma\varepsilon^*_{k,k}\iota^*_{k+1,k}$: $h^*(E_{k+1,k+1})\rightarrow h^*(E_{k,k})$ are onto for any $k\in Z$. Then, there exists an isomorphism*

$$\boldsymbol{O}^*(\boldsymbol{E};\,Z_p)\approx(\boldsymbol{O}^*(\boldsymbol{E})/p\boldsymbol{O}^*(\boldsymbol{E}))\otimes\Lambda_p(\delta_p)$$

as graded algebras over Z_p, where $\Lambda_p(\delta_p)$ is the exterior algebra generated by the Bockstein homomorphism δ_p. Moreover, $\boldsymbol{O}^*(\boldsymbol{E})/p\boldsymbol{O}^*(\boldsymbol{E})$ is identified with the mod p reduction of $\boldsymbol{O}^*(\boldsymbol{E})$, a subalgebra of $\boldsymbol{O}^*(\boldsymbol{E};\,Z_p)$.

3.7. As an application of Theorem 3.6, we consider the stable operations in mod p $\boldsymbol{U}$-cobordism theory.

Denote by $\xi_{k,l}$ the canonical complex k-plane bundle over the complex Grassmann manifold $G_{k,l}$ of k-planes in C^{k+l}, and denote by $M(\xi_{k,l})$ the Thom complex of $\xi_{k,l}$.

Let $\varepsilon_{k,l}$: $S^2M(\xi_{k,l})\rightarrow M(\xi_{k+1,l})$ be a map induced from the canonical bundle map $\xi_{k,l}\oplus C^1\rightarrow\xi_{k+1,l}$, and let $\iota_{k,l}$: $M(\xi_{k,l})\rightarrow M(\xi_{k,l+1})$ be a canonical inclusion. Then the Thom spectrum $\boldsymbol{MU}=\{MU(k),\,\varepsilon_k\}$ is defined by

$$MU(k)=\lim_{l}\text{ind}\,\{M(\xi_{k,l}),\,\iota_{k,l}\}\quad\text{and}\quad\varepsilon_k|M(\xi_{k,l})=\varepsilon_{k,l}\,.$$

And $\boldsymbol{U}$-cobordism theory is the cohomology theory associated with $\boldsymbol{MU}$.

The family $\{M(\xi_{k,l}),\,\varepsilon_{k,l}\}$ satisfies the hypothesis of Theorem 3.6 (cf. [5]) and the order of $\tilde{\omega}\in h^1(\boldsymbol{MU}\wedge M;\,Z_p)$ is p for any prime p (cf. [1], Th. 2.3).

Therefore, Theorem 3.6 is applicable.

On the other hand, Landweber [5] shows that there exists an isomorphism

$$O^*(MU) \approx Z[\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots] \otimes h^*(S^0; MU)$$

as modules, where γ_n is the n-th U-cobordism characteristic class defined by Conner & Floyd [2], which corresponds to a stable U-cobordism operation of degree $2n$. Therefore, we obtain

Theorem 3.7. *There exists an isomorphism*

$$O^*(MU; Z_p) \approx Z_p[\gamma_1, \gamma_2, \cdots, \gamma_n, \cdots] \otimes \Lambda_p(\delta_p) \otimes (h^*(S^0; MU)/ph^*(S^0; MU))$$

as modules over Z_p.

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