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A NOTE ON STABLE JAMES NUMBERS OF PROJECTIVE SPACES

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In [4] and [5], the second named author has evaluated the stable James numbers

$$d_F(n) = \#[\text{cokernel of } \{FP^n, FP^1\} \xrightarrow{i^*} \{FP^1, FP^1\}]$$

of the n -dimensional F -projective spaces FP^n for $F=C$ (complex numbers) or H (quaternions), and especially decided $d_C(n)=k_2^{n \cdot 2}$ for $n \leq 4$, the odd components of $d_C(n)$ and $d_H(m)$ for $m \leq 2$. The purpose of this note is to prove the following theorem.

We will use the notations introduced in [4] and [5] without notice.

- Theorem.** (i) $d_C(5)=2^2 \cdot 3 \cdot 5$, $v_2(d_C(6)) \leq 3$,
 (ii) $d_C(7)=d_C(8)$, $d_C(15)=d_C(16)$,
 (iii) $v_2(d_C(9)) \leq v_2(d_C(7)) + 1 \leq 5$,
 (iv) $d_H(3)=2^3 \cdot 3^2 \cdot 5$, $d_H(4)=2^5 \cdot 3^2 \cdot 5 \cdot 7$, $d_H(5)=2^{5+\varepsilon} \cdot 3^2 \cdot 5^2 \cdot 7$ and $d_H(6)=2^{5+\varepsilon+\tau} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11$, where $0 \leq \varepsilon \leq 1$ and $0 \leq \tau \leq 2$.

Proof. (ii) follows from [4] for the odd components and the following fact for the 2-component that

$$(\#) \quad \text{if } v_2(|G_{2j-1}|) = 0, \text{ then } v_2(d_C(j)) = v_2(d_C(j+1)),$$

where $|G_{2j-1}| = \min \{k > 0; kG_{2j-1} = 0\}$. Then (ii) follows, since $G_{13} = Z_3$ [6] and $v_2(|G_{29}|) = 0$ [3]. We shall prove (#). Let $f: CP^j \rightarrow S^2 = CP^1$ be a stable map such that the degree of the composition $S^2 \subset CP^j \xrightarrow{f} S^2$ is $d_C(j)$. Then $f \circ p: S^{2j+1} \xrightarrow{p} CP^j \xrightarrow{f} S^2$ represents an element of the $(2j-1)$ -stem G_{2j-1} and so that $d_C(j+1)$ is a divisor of $|G_{2j-1}|d_C(j)$, where p denotes a natural projection. Hence $v_2(d_C(j+1)) \leq v_2(|G_{2j-1}|) + v_2(d_C(j))$. But $v_2(d_C(j)) \leq v_2(d_C(j+1))$, since $d_C(j+1)$ is a multiple of $d_C(j)$ [4]. Therefore (#) follows.

Recall that $K(FP^n) = Z[\mu_F(n)]/\mu_F(n)^{n+1}$, where $\mu_F(n)$ indicates the stable bundle of the underlying complex vector bundle of the canonical F -line bundle over FP^n .

For $f \in \{FP^n, FP^1\}$, we consider the following commutative diagram

$$\begin{array}{ccc} K(FP^n) & \xleftarrow{f^*} & K(FP^1) \\ \psi^2 \downarrow & & \downarrow \psi^2 \\ K(FP^n) & \xleftarrow{f^*} & K(FP^1) \end{array}$$

where ψ^2 denotes the Adams operation. Let $f^*(\mu_F(1)) = \sum_{i=1}^n a_i \mu_F(n)^i$ and $d_F = 2$ (if $F=C$) or 4 (if $F=H$). Then

$$f^* \psi^2(\mu_F(1)) = f^*(d_F \mu_F(1)) = d_F \sum_{i=1}^n a_i \mu_F(n)^i$$

and this equals

$$\begin{aligned} \psi^2 f^*(\mu_F(1)) &= \sum_{i=1}^n a_i \{\psi^2(\mu_F(n))\}^i = \sum_{i=1}^n a_i \{\mu_F(n)^2 + d_F \mu_F(n)\}^i \\ &= \sum_{i=1}^n \sum_{j=1}^i a_i \binom{i}{j-i} d_F^{2i-j} \mu_F(n)^j, \end{aligned}$$

since $\psi^2(\mu_C(n)) = \{1 + \mu_C(n)\}^2 - 1 = \mu_C(n)^2 + 2\mu_C(n)$ and $\psi^2(\mu_H(n)) = \mu_H(n)^2 + 4\mu_H(n)$ (see e.g. [2]). Comparing the coefficients of $\mu_F(n)^j$, we have

$$(1)_F \quad d_F a_j = \sum_{i=1}^n a_i \binom{i}{j-i} d_F^{2i-j}.$$

Notice that $(1)_C$ implies $a_1 = (-1)^{j+1} j a_j$ if $F=C$ (cf. [4]).

For $f \in \{FP^n, FP^1\}$, we consider the following commutative diagram of fibrations

$$\begin{array}{ccccccc} S^{(n+1)d_F-1} & \xrightarrow{p} & FP^n & \longrightarrow & FP^{n+1} & \longrightarrow & S^{(n+1)d_F} \\ \downarrow = & & \downarrow f & & \downarrow \bar{f} & & \downarrow = \\ S^{(n+1)d_F-1} & \xrightarrow{g} & FP^1 & \xrightarrow{j} & C_g & \xrightarrow{h} & S^{(n+1)d_F} \end{array}$$

where p denotes the canonical projection, $g = f \circ p$ and C_g the mapping cone of g . Then we have the commutative diagram of the short exact sequences

$$\begin{array}{ccccccc} 0 & \longleftarrow & \tilde{K}(FP^n) & \longleftarrow & \tilde{K}(FP^{n+1}) & \longleftarrow & \tilde{K}(S^{(n+1)d_F}) \longleftarrow 0 \\ & & \uparrow f^* & & \uparrow j^* & & \uparrow = \\ 0 & \longleftarrow & \tilde{K}(FP^1) & \longleftarrow & \tilde{K}(C_g) & \longleftarrow & \tilde{K}(S^{(n+1)d_F}) \longleftarrow 0. \end{array}$$

Let $x \in \tilde{K}(C_g)$ be such that $j^*(x) = \mu_F(1) = g_C^{d_F/2}$. Let $y = h^*(g_C^{(n+1)d_F/2})$ and let $\bar{f}^*(x) = \sum_{i=1}^{n+1} a_i \mu_F(n+1)^i$. Then

$$f^*(\mu_F(1)) = \sum_{i=1}^n a_i \mu_F(n)^i$$

and

$$\psi^2(x) = d_F x + \lambda y \quad \text{for some } \lambda \in Z$$

and

$$e(g) = \frac{\lambda}{(2^{nd_F/2} - 1)2^{d_F/2}} = \frac{\lambda}{(d_F^n - 1)d_F} \in Q/Z,$$

where e denotes the e -invariant (see e.g. [1]). Now

$$f^* \psi^2(x) = f^*(d_F x + \lambda y) = d_F \sum_{i=1}^{n+1} a_i \mu_F(n+1)^i + \lambda \mu_F(n+1)^{n+1}$$

and this equals

$$\psi^2 f^*(x) = \psi^2 \left(\sum_{i=1}^{n+1} a_i \mu_F(n+1)^i \right) = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_i \binom{i}{j-i} d_F^{2i-j} \mu_F(n+1)^j.$$

Comparing the coefficients of $\mu_F(n+1)^{n+1}$, we have

$$d_F a_{n+1} + \lambda = \sum_{i=1}^{n+1} a_i \binom{i}{n+1-i} d_F^{2i-n-1}$$

and so

$$(2)_F \quad e(g) = \frac{\lambda}{(d_F^n - 1)d_F} = \frac{\sum_{i=1}^n a_i \binom{i}{n+1-i} d_F^{2i-n-1}}{(d_F^n - 1)d_F}.$$

Consider the case with $F=C$ and $n=4$. Suppose that the degree of the composition $S^2 \subset CP^4 \xrightarrow{f} S^2$ is $d_C(4)=12$ [4]. Then $(1)_C$ implies

$$a_1 = 12, \quad a_2 = -6, \quad a_3 = 4 \quad \text{and} \quad a_4 = -3,$$

and then by $(2)_C$

$$e(g) = -\frac{12}{5}$$

and so

$$e(5g) = 0.$$

But $e: G_7 \rightarrow Q/Z$ is monomorphic [1, §7], so that

$$5g = 0.$$

Then there exists $f': CP^5 \rightarrow S^2$ such that the composition $CP^4 \subset CP^5 \xrightarrow{f'} S^2$ coincides with $5f$. Clearly the degree of the composition $S^2 \subset CP^5 \xrightarrow{f'} S^2$ is $5d_C(4)=2^2 \cdot 3 \cdot 5$ [4]. Hence $d_C(5)$ is a divisor of $2^2 \cdot 3 \cdot 5$. But, by [4], $d_C(5)$ is a multiple of $2^2 \cdot 3 \cdot 5$. Therefore we have that $d_C(5)=2^2 \cdot 3 \cdot 5$. This implies the half of (i).

By the same method as the proof of (ii), we have that $d_C(6)$ is a divisor of $2d_C(5)$, since $G_9 = Z_2 + Z_2 + Z_2$ [6]. Hence we have

$$v_2(d_C(6)) \leq v_2(2d_C(5)) = 3.$$

This completes the proof of (i).

When $F=C$ and $n=8$, the same computations as the proof of (i) imply that

$$3e(g) = 0.$$

But the kernel of $e: G_{15} \rightarrow Q/Z$ is Z_2 [1, §7], so that

$$2 \cdot 3g = 0$$

and then $d_C(9)$ is a divisor of $2 \cdot 3d_C(8)$. Then (ii) and [4, Th. A] imply

$$v_2(d_C(9)) \leq v_2(2 \cdot 3d_C(8)) = v_2(d_C(8)) + 1 \leq 5.$$

This implies (iii).

By the same computations as the proof of (i) using the fact $d_H(2) = 2^3 \cdot 3$ [4], we have

$$d_H(3) = 2^3 \cdot 3^2 \cdot 5.$$

And when $F=H$ and $n=3$, we have

$$e(g) = \frac{3^2}{2 \cdot 7}.$$

But $e = 2e'_R: G_{11} \rightarrow Q/Z$ and the real e -invariant e'_R is monomorphic in this case [1, §7] so that

$$2^2 \cdot 7g = 0$$

and then $d_H(4)$ is a divisor of $2^2 \cdot 7d_H(3) = 2^5 \cdot 3^2 \cdot 5 \cdot 7$. But, by [5], $d_H(4)$ is a multiple of $2^5 \cdot 3^2 \cdot 5 \cdot 7$. Thus

$$d_H(4) = 2^5 \cdot 3^2 \cdot 5 \cdot 7.$$

By the same methods as above, we can prove the remaining parts of (iv).

Q.E.D.

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