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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 8(1) P.135-P.149</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1971</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/6417">https://doi.org/10.18910/6417</a></td>
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<tr>
<td>DOI</td>
<td>10.18910/6417</td>
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CONTRIBUTIONS TO THE THEORY OF INTERPOLATION OF OPERATIONS

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(Received August 31, 1970)

1. Introduction. Let $(R, \mu)$ and $(S, \nu)$ be two measure spaces of totally $\sigma$-finite in the sense of P. Halmos [7]. Let us consider operation $T$ which transforms measurable functions on $R$ to those on $S$. The operation $T$ is called quasi-linear if:

(i) $T(f_1+f_2)$ is uniquely defined whenever $Tf_1$ and $Tf_2$ are defined and

$$|T(f_1+f_2)| \leq \kappa(|Tf_1|+|Tf_2|)$$

where $\kappa$ is a constant independent of $f_1$ and $f_2$;

(ii) $T(cf)$ is uniquely defined whenever $Tf$ is defined and

$$|T(cf)| = |c| |Tf|$$

for all scalars $c$.

We say that

$$\bar{f} = Tf$$

is an operation of type $(a, b)$, $1 \leq a \leq b \leq \infty$, if:

(i) $Tf$ is defined for each $f \in L^a(R)$, that is for each $f$ measurable with respect to $\mu$ such that

$$\|f\|_{a, \mu} = \left(\int_R |f|^a d\mu\right)^{1/a}$$

is finite, the right side being interpreted as the essential upper bound (with respect to $\mu$) of $|f|$ if $a = \infty$;

(ii) for every $f \in L^a(R)$, $\bar{f} = Tf$ is in $L^b(S)$ and

$$\|\bar{f}\|_{b, \nu} \leq M \|f\|_{a, \mu},$$

where $M$ is a constant independent of $f$.

The least admissible value of $M$ in (1.1) is called the $(a, b)$-norm of operation $T$.  

Next let us define the weak type \((a, b)\) of operations. Suppose first that \(1 \leq b < \infty\). Given any \(y > 0\) denote by \(E_y = E_y[\tilde{f}]\) the set of points of the space \(S\) where

\[|\tilde{f}(x)| > y,\]

and write \(\nu(E_y)\) for the \(\nu\)-measure of the set \(E_y\). An immediate consequence of (1.1) is that

\[(1.2) \quad \nu(E_y[\tilde{f}]) \leq \left(\frac{M}{y} \|f\|_{\alpha, \mu}\right)^b.\]

An operation \(T\) which satisfies (1.2) will be called to be of weak type \((a, b)\). The least admissible value of \(M\) in (1.2) is called the weak type \((a, \infty)\)-norm of \(T\).

We define weak type \((a, \infty)\) as identical with type \((a, \infty)\). Hence \(T\) is the weak type \((\alpha, \infty)\) if

\[\text{ess. sup } |\tilde{f}| \leq M \|f\|_{\alpha, \mu}.\]

If no confusion arises we omit the symbols \(\mu\) and \(\nu\) in the notation for norms.

In a number of problems we are led to consider integrals of type

\[\int_R \varphi(|f|) d\mu\]

where \(\varphi\) is not necessarily a power.

The interpolation of operation on the type of space with finite measure has been considered firstly by J. Marcinkiewicz [12] and A. Zygmund [15]. In the previous paper [10], the author treated an extension to the space with totally \(\sigma\)-finite measure. We intend further extension and refinement of those theorems to the space which is closely related to the intermediate space. The intermediate between a pair of Banach spaces was firstly introduced by A.J. Luxemburg [11].

Let us consider two continuous increasing functions \(\varphi_1(u)\) and \(\varphi_2(u)\). The former is defined on the interval \(0 \leq u \leq \gamma\) and the latter is on \(\frac{1}{\gamma} \leq u < \infty\), and \(\gamma\) is a constant larger than 1. Those satisfy the following properties:

(i) \(\varphi_1(0) = 0\) and \(\varphi_1(2u) = 0(\varphi_1(u))\)

\[\int_u^{2u} \frac{\varphi_1(t)}{t^{b+1}} dt = 0\left(\frac{\varphi_1(u)}{u^b}\right)\]

\[\int_0^u \frac{\varphi_1(t)}{t^{a+1}} dt = 0\left(\frac{\varphi_1(u)}{u^a}\right)\]

for \(u \to 0\), Here and in what follows it is assumed that \(a < b\);

(ii) \(\varphi_2(2u) = 0(\varphi_2(u))\)
\[ \int_0^\infty \varphi_2(t) \, dt = 0 \left( \frac{\varphi_2(u)}{u^a} \right) \]
\[ \int_1^\infty \varphi_2(t) \, dt = 0 \left( \frac{\varphi_2(u)}{u^a} \right) \]

for \( u \to \infty \);

(iii) \( \varphi_1(1) = \varphi_2(1) \) and so necessarily \( \varphi_1(u) \sim \varphi_2(u) \) on an appropriate interval containing the unity, say \( \frac{1}{\gamma} \leq u \leq \gamma \), \( \gamma > 1 \). It means that there exist positive constants \( A, B \) such that

\[ A \leq \frac{\varphi_1(u)}{\varphi_2(u)} \leq B \quad \text{if} \quad \frac{1}{\gamma} \leq u \leq \gamma, \quad \gamma > 1. \]

Let us join \( \varphi_1 \) with \( \varphi_2 \) and introduce a new function \( \varphi \), that is

\[ \varphi(u) = \begin{cases} \varphi_1(u), & \text{if } 0 \leq u \leq 1 \\ \varphi_2(u), & \text{if } 1 < u < \infty \end{cases} \]

The typical example is

\[ \varphi(u) = \begin{cases} u^{c_1} \psi_1(u), & \text{if } 0 \leq u \leq 1 \\ u^{c_2} \psi_2(u), & \text{if } 1 < u < \infty \end{cases} \]

where \( a < c_1, c_2 < b \) and \( \psi_1, \psi_2 \) are slowly varying function (c.f. A. Zygmund [16]).

**Theorem 1.** Suppose that a quasi-linear operation \( T \) is of weak type \((a, a)\) and \((b, b)\) with norms \( M_a \) and \( M_b \), where \( 1 \leq a < b < \infty \). Then \( T f \) is defined for every \( f \) with \( \mu \)-integrable \( \varphi(|f|) \), \( \varphi(|T f|) \) is \( \nu \)-integrable and we have

\[ \int_S \varphi(|T f|) \, d\nu \leq K \int_R \varphi(|f|) \, d\mu \]

where \( K = 0(M_a \lor M_b), M_a \lor M_b \) meaning the maximum value of \( M_a, M_b \).

Let us consider another pair of continuous increasing functions \( \chi_i(u) \) and \( \chi_2(u) \) which satisfy the following properties:

(i) \( \chi_i(0) = 0, \quad \chi_i(2u) = 0(\chi_i(u)) \)

\[ \int_u^{b+1} \chi_i(t) \, dt = 0 \left( \frac{\chi_i(u)}{u^a} \right) \]

\[ \int_0^{b+1} \chi_i(t) \, dt = 0 \left( \frac{\chi_i(u)}{u^a} \right) \]

for \( u \to 0 \);

(ii) \( \chi_2(2u) = 0(\chi_2(u)) \)
\[
\int_u^\infty \frac{X_i(t)}{t^{b+1}} \, dt = 0 \left( \frac{X_i(u)}{u^b} \right)
\]
for \( u \to \infty \);

(iii) \( \chi_i(1) = \chi_i(1) \) and so necessarily \( \chi_i(u) \sim \chi_i(u) \) on the interval \( \frac{1}{\gamma} \leq u \leq \gamma \) for some \( \gamma > 1 \).

Write

\[
\chi^*_i(u) = u^a \int_1^u \frac{X_i(t)}{t^{b+1}} \, dt \quad \text{if } u > 1
\]

and let us join \( \chi_1 \) with \( \chi_2 \) and \( \chi_2^* \) and introduce new functions \( \chi \) and \( \chi^* \), that is

\[
\chi(u) = \begin{cases} 
\chi_1(u), & \text{if } 0 \leq u \leq 1 \\
\chi_2(u), & \text{if } 1 < u < \infty
\end{cases}
\]

\[
\chi^*(u) = \begin{cases} 
\chi_1(u), & \text{if } 0 \leq u \leq 1 \\
\chi_2(u) + \chi^*_2(u), & \text{if } 1 < u < \infty
\end{cases}
\]

The typical example is

\[
\chi_i(u) = u^a \psi_i(u), \quad \text{if } 0 \leq u \leq 1
\]
\[
\chi_i(u) = u^a, \quad \chi^*_i(u) = u^a \log^+ u, \quad \text{if } 1 < u < \infty
\]

where \( a < c < b \), \( \psi_i(u) \) is a slowly varying function.

**Theorem 2.** Suppose that a quasi-linear operation \( T \) is of weak type \((a, a)\) and \((b, b)\) with norms \( M_a \) and \( M_b \), where \( 1 \leq a < b < \infty \). Then \( Tf \) is defined for every \( \mu \)-integrable \( \chi^* (|f|) \), \( \chi(|Tf|) \) is \( \nu \)-integrable and we have

\[
\int_S (|Tf|) \, d\nu \leq K \int_R \chi^*(|f|) \, d\mu
\]

where \( K = O(M_a \vee M_b) \).

We shall prove those theorems in § 2. In § 3, we shall add some remarks which are useful on a certain case. In § 4, we shall prove the following theorem.

**Theorem 3.** Suppose that a quasi-linear operation \( T \) is of weak type \((1, 1)\) and type \((p, p)\) for some \( p > 1 \). Then we have

\[
\int_{|f| \leq 1} |Tf|^p \, d\nu + \int_{|f| > 1} |Tf| \, d\nu 
\]

\[
\leq K \left\{ \int_{|f| \leq 1} |f|^p \, d\mu + \int_{|f| > 1} |f| (1 + \log^+ |f|) \, d\mu \right\}
\]
where $K$ is a constant independent of $f$.

In § 5, we shall state some applications to singular integral operators. Here the present author thanks to the referee for his kind advices.

2. Proofs of Theorems 1 and 2. Firstly we intend to prove Theorem 2. The $\chi_1(u)$ has the following properties

$$Bu^a \leq \chi_1(u) \leq Au^a \quad (0 \leq u \leq 1)$$

where we shall use letters $A$, $B$, etc. as absolute constants.

If we denote by $f^*$ equi-measurable, non-increasing rearrangement of $|f|$, and by $R_t$ the subset of the space $R$ where $|f| \leq 1$, then

$$\int_{R_t} |f|^b \, d\mu = \int_t^\infty (f^*)^b \, dx < B^{-1} \int_t^\infty \chi_1(f^*) \, dx = B^{-1} \int_{R_t} \chi_1(|f|) \, d\mu,$$

where $t$ denotes the $\mu$-measure of set $\{x \mid |f(x)| > 1\}$.

The $\chi_2(u)$ and $\chi^*_2(u)$ have the following properties. The $\chi^*_2(u)$ is continuous, non-decreasing function for $u > 1$ and

$$\chi^*_2(2u) = 0(\chi^*_2(u))$$

for $u \to \infty$. Because for $u' > u > 1$, we have

$$\chi^*_2(u') - \chi^*_2(u) = (u')^a \int_{t^a+1}^{u'} \chi_2(t) \, dt - u^a \int_{t^a+1}^{u} \chi_2(t) \, dt > u^a \int_{t^a+1}^{u'} \chi_2(t) \, dt > 0$$

and since $\chi_2(2u) = 0(\chi_2(u))$ for $u \to \infty$, we have

$$\chi^*_2(2u) = (2u)^a \int_{t^a+1}^{2u} \phi_2(t) \, dt = (2u)^a \left\{ \int_{t^a+1}^{u} \chi_2(t) \, dt + \int_{u}^{2u} \chi_2(t) \, dt \right\}$$

$$= A \chi^*_2(u) + A'(2u)^a \int_{t^a+1}^{u} \chi_2(t) \, dt$$

$$\leq A \chi^*_2(u) + A' u^a \int_{t^a+1}^{u} \chi_2(t) \, dt \leq A'' \chi^*_2(u).$$

By similar arguments read

$$\chi_2(u) \leq A \chi_2^*(u)$$

$$u^a \leq A \chi^*_2(u)$$

and
respectively. We have
\[
\int_u^\infty \frac{\chi_2^*(t)}{t^{b+1}} dt = 0 \left( \frac{\chi_2^*(u)}{u^b} \right)
\]
for \( u \to \infty \). Because we have by the definition of \( \chi_2^* \),
\[
\int_u^\infty \frac{\chi_2^*(t)}{t^{b+1}} dt = \int_u^\infty \frac{dt}{t^{b+1}} \int_t^\infty \frac{ds}{s^{b+1}}
\]
\[
= \int_1^\infty \frac{dt}{t^{b+1}} \int_t^\infty \frac{ds}{s^{b+1}} + \int_1^\infty \frac{ds}{s^{b+1}} \int_s^\infty \frac{dt}{t^{b+1}}
\]
\[
= \frac{1}{(b-a)u^{b-a}} \int_1^\infty \frac{ds}{s^{b+1}} + \frac{1}{(b-a)} \int_u^\infty \frac{ds}{s^{b+1}}
\]
\[
\leq A \frac{\chi_2^*(u)}{u^b} + A' \frac{\chi_2^*(u)}{u^b} \leq A' \frac{\chi_2^*(u)}{u^b}.
\]
If we denote by \( R_2 \) the sub-set of \( R \) where \( |f| > 1 \), then
\[
\int_{R_2} |f| d\mu = \int_{R_2} (f^*)^a dx < A \int_{R_2} \chi_2^*(f^*) dx
\]
\[
= A \int_{R_2} \chi_2^*(|f|) d\mu,
\]
where \( t \) denotes the \( \mu \)-measure of \( \{x | \mu(x) > 1\} \). Under those preparations, let \( f \in L_\mu^{a*}(R) \) and write
\[
f = f' + f''
\]
where \( f' = f \) whenever \( |f| \leq 1 \) and \( f' = 0 \) otherwise; \( f'' = f - f' \). Since \( f' \in L_\mu^a \) and \( f' \in L_\mu^b \), \( f'' \in L_\mu^3 \) and so \( f'' \in L_\mu^a \). Hence \( Tf' \) and \( Tf'' \) are defined, by hypothesis, and so \( Tf - Tf'' \). Let \( n_r(y) \) by the distribution function \( |Tf| \). We have
\[
\int_s \chi(|Tf|) d\nu = -\int_0^\infty \chi(y) dn_r(y)
\]
\[
= \int_0^\infty n_r(y)d\chi(y) \leq \sum_{j=0}^\infty \eta_i \delta_j
\]
where \( \delta_j = \chi(\lambda 2^{j+1}) - \chi(\lambda 2^j) \) and \( \eta_1 = \nu(E_{j+1})[|Tf|], \lambda = 3\xi^2 \). The passage from the second to the third integral is justified as in A. Zygmund [15, Vol. I, p. 112 (4.8)].

For each fixed \( j \geq 0 \), we write \( f = f_1 + f_2 + f_3 \), where \( f_1 \) equals \( f \) or 0 according as \( 1 < |f| \leq 2^j \) or else; \( f_2 \) does \( f \) or 0 according as \( |f| > 2^j \) or else; and so \( f_3 \) does \( f \) or 0 according as \( |f| \leq 1 \) or else. Since \( f_1 \in L_\mu^a \cap L_\mu^b, f_2 \in L_\mu^a \) and
If $|Tf_i| < y$, for all $i = 1, 2, 3$ and any positive real number $y$, then $|Tf| < \lambda y$ with $\lambda = 3\kappa^2$. Therefore we have

$$\{x : |Tf| > \lambda y\} \subset \bigcup_{i=1}^{3} \{x : |Tf_i| > y\}$$

and if we take $y = 2^j$, we get the following formula,

$$\eta_j \leq C \left\{ 2^{-jb} \int_{R_2} |f_i|^b d\mu + 2^{-ja} \int_{R_2} |f_s|^a d\mu + 2^{-jb} \int_{R_1} |f_s|^b d\mu \right\}$$

and then

$$\sum_{j=0}^{\infty} \eta_j \delta_j \leq C \left\{ 2^{-jb} \delta_j \int_{R_2} |f_i|^b d\mu + \sum_{j=0}^{\infty} 2^{-ja} \delta_j \int_{R_2} |f_s|^a d\mu + \sum_{j=0}^{\infty} 2^{-jb} \delta_j \int_{R_1} |f_s|^b d\mu \right\}$$

$$= I_s + I_s + I_s, \text{ say.}$$

By $\varepsilon_i (i=1, 2, \cdots)$, we denote the $\mu$-measure of the set where $2^{i-1} < |f| \leq 2^i$, then then if we interchange the order of summation and substitute above estimates we are led to

$$I_1 = C \sum_{j=0}^{\infty} 2^{-jb} \delta_j \int_{R} |f_i|^b d\mu \leq C \sum_{j=0}^{\infty} 2^{-jb} \delta_j \sum_{i=1}^{j} 2^{ib} \varepsilon_i$$

$$= C \sum_{j=1}^{\infty} 2^{ib} \varepsilon_i \sum_{j=0}^{\infty} 2^{-jm} \delta_j \leq C' \sum_{j=1}^{\infty} 2^{ib} \varepsilon_i \int_{\lambda^2+1}^{\infty} \frac{\chi_2(u)}{u^{b+1}} du$$

$$\leq C'' \sum_{i=1}^{\infty} \chi_2(2^i) \varepsilon_i \leq C'' \int_{R_2} \chi_2(|f|) d\mu$$

$$I_2 = C \sum_{j=0}^{\infty} 2^{-ja} \delta_j \int_{R} |f_s|^a d\mu \leq C \sum_{j=0}^{\infty} 2^{-ja} \delta_j \sum_{j=1}^{\infty} 2^{ja} \varepsilon_i$$

$$= C \sum_{i=1}^{\infty} 2^{ja} \varepsilon_i \sum_{j=0}^{\infty} 2^{-ja} \delta_j \leq C' \sum_{i=1}^{\infty} 2^{ja} \varepsilon_i \int_{1}^{\infty} \frac{\chi_2(u)}{u^{a+1}} du$$

$$\leq C'' \sum_{j=0}^{\infty} \chi_2^a(2^j) \varepsilon_i \leq C'' \int_{R_2} \chi_2^a(|f|) d\mu$$

$$I_3 = C \sum_{j=0}^{\infty} 2^{-jb} \delta_j \int_{R} |f_s|^b d\mu = C \int_{R_1} |f|^b d\mu \sum_{j=0}^{\infty} 2^{-jb} \delta_j$$

$$\leq C \int_{R_1} |f|^b d\mu \sum_{j=0}^{\infty} \frac{\chi_2(u)}{u^{b+1}} du \leq C \int_{R_1} \chi_1(|f|) d\mu.$$
Therefore we have
\[ \sum_{j=-\infty}^{\infty} \eta_j \delta_j \leq C \int_{R_2} \chi_2(|f|)d\mu + C' \int_{R_2} \chi_2(|f|)d\mu + C'' \int_{R_1} \chi_1(|f|)d\mu \]
\[ \leq C \int_{R} \chi^*(|f|)d\mu. \]

Similarly, for each fixed \( j < 0 \), we write \( f = f_4 + f_5 + f_6 \), where \( f_4 \) equals \( f \) or 0 according as \( 2^j < |f| \leq 1 \) or else; \( f_5 \) does \( f \) or 0 according as \( 0 \leq |f| \leq 2^j \) or else; and so \( f_6 \) does \( f \) or 0 according as \( 1 < |f| \) or else. Since \( f_4 \in L^a_{\mu} \cap L^b_{\mu} \), \( f_5 \in L^a_{\mu} \) and \( f_6 \in L^a_{\mu} \) respectively, we have
\[ \eta_j \leq D \left\{ 2^{-ja} \int_{R_1} |f_4|^a d\mu + 2^{-ja} \int_{R_1} |f_5|^b d\mu + 2^{-ja} \int_{R_2} |f_6|^a d\mu \right\} \]

We can estimate the summation \( \sum_{j=-\infty}^{\infty} \eta_j \delta_j \) just the same as \( \sum_{j=0}^{\infty} \eta_j \delta_j \) and we have
\[ \sum_{j=-\infty}^{\infty} \eta_j \delta_j \leq D \int_{R_1} \chi_1(|f|)d\mu + D' \int_{R_2} \chi_2^*(|f|)d\mu \]
and hence we attain the desired inequality
\[ \int_S \chi(|Tf|)d\nu \leq K \int_R \chi^*(|f|)d\mu \]

The proof of Theorem 1 is a rather easy repetition of that of Theorem 2 and need not be gone into.

3. Some remarks. (1). If the operation \( T \) is linear, then we can present theorems 1 and 2 as more general forms which are useful on a certain case (c.f. E.M. Stein - G. Weiss [13]).

We say that the operation \( T \) is of restricted weak type \((a, b)\), if for every simple function \( f \) on \( R \), \( Tf \) is \( v \)-measurable function on \( S \) and satisfies
\[ \nu(E_f[|Tf|]) \leq \left( \frac{M}{y} \|f\|_{a,\mu} \right)^b \]
where \( M \) is a constant independent of \( f \). We can state

**Corollary 1.** In Theorem 1, if the operation \( T \) is linear and of restricted weak type \((a, a)\) and \((b, b)\) where \( 1 \leq a < b < \infty \) respectively. Then we have for every simple function \( f \) on \( R \),
\[ \int_S \varphi(|Tf|)d\nu \leq K \int_R \varphi(|f|)d\mu \]
and moreover we can extend the operation \( T \) to the whole space \( L^a_\mu \) preserving the
norm of operation. 

Proof. We need only to prove the process of extension. Take any \( f \) in \( L^p_\mu \). Let us write

\[
\eta_n = \begin{cases} 
\text{(sign } f \text{)} \frac{k-1}{n}, & \text{if } \frac{k-1}{n} \leq |f| < \frac{k}{n} \\
\text{(sign } f \text{)} n, & \text{if } |f| > n
\end{cases}
\]

for \( k = 1, 2, \ldots, n; n = 1, 2, \ldots \). Then \( \eta_n \) tends to \( f \) monotone increasingly for a.e. \( x \) and so \( \varphi(|\eta_n|) \) does to \( \varphi(|f|) \). By the Lebesgue convergence theorem we have

\[
\lim_{n \to \infty} \int_R \varphi(|\eta_n|) d\mu = \int_R \varphi(|f|) d\mu
\]

and

\[
\lim_{n, m \to \infty} \int_R \varphi(|f_m - \eta_n|) d\mu = 0.
\]

If we write \( \tilde{f}_n = Tf_n \), then by hypothesis we have

\[
\int_S \varphi(|\tilde{f}_n|) dv \leq K \int_R \varphi(|f_n|) d\mu
\]

and since \( T \) is of linear type \( \varphi(\int f - f_n) d\mu \).

The least formula shows that \( \{\tilde{f}_n\} \) is a sequence of fundamental in measure and so there exist a limit function \( \tilde{f} \) uniquely except a set of \( \nu \)-measure zero and subsequence \( (n_k) \) of \( (n) \) such that \( \tilde{f}_{n_k} \) converges to \( \tilde{f} \) for a.e. \( x \). Applying the Fatou lemma we have the desired result.

The same argument leads to

**Corollary 2.** In Theorem 2, if the operation \( T \) is linear and of restricted weak type \((a, a)\) and \((b, b)\) where \( 1 \leq a < b < \infty \), respectively. Then we have for every simple function \( f \) on \( R \),

\[
\int_S \chi(|Tf|) dv \leq K \int_R \chi^*(|f|) d\mu
\]

and moreover we can extend the operation \( T \) to the whole space \( L^p_\nu \) preserving the norm of operation.

(2) Next we meet the \( \varphi(u) \) which is continuous and not necessarily increasing on the whole interval. If we suppose that \( \varphi \) is ultimately increasing for the value of \( u \) near zero and infinity; in the middle interval, say \((\frac{1}{\gamma}, \gamma)\) with
\(\gamma > 1\), is of bounded variation, then we can find an increasing function \(\varphi^*\) such that
\[
\varphi(u) \leq \varphi^*(u) \leq A_\gamma \varphi(u), \quad \text{for all } u \geq 0.
\]

For example a construction of \(\varphi^*\) is as follows:
\[
\varphi^*(u) = \begin{cases} 
\varphi(u), & \text{if } 0 \leq u < \frac{1}{\gamma} \\
\frac{1}{\gamma} + \int_{1/\gamma}^{u} |d\varphi|, & \text{if } \frac{1}{\gamma} \leq u < \gamma \\
\varphi(u) + \int_{\gamma}^{\infty} |d\varphi| + (\varphi(u) - \varphi(\gamma)), & \text{if } \gamma \leq u < \infty
\end{cases}
\]

The simple calculation shows that the inequality is satisfied
\[
A_\gamma = \frac{\varphi(1/\gamma) + \int_{1/\gamma}^{\gamma} |d\varphi|}{\min_{1/\gamma \leq u \leq \gamma} \varphi(u)}
\]

**Corollary 3.** In Theorem 1, if the \(\varphi(u)\) is ultimately increasing for the value of \(u\) near zero and infinity; in the middle interval, is of bounded variation. The same conclusion is also true.

The same argument leads to

**Corollary 4.** In Theorem 2, if the \(\kappa(u)\) is ultimately increasing for the value of \(u\) near zero and infinity; in the middle interval, is of bounded variation. The same conclusion is also true.

**4. Proof of Theorem 3.** Let us suppose that \(f \in L^p + L \log^+ L\). Write \(f = g + h:\)

\[
g = \begin{cases} 
f, & \text{if } |f| \leq 1 \\
0, & \text{if } |f| > 1
\end{cases}
\]

\(h = f - g\)

We have \(g \in L^p\) and \(h \in L \log^+ L\) respectively. Since the operation \(T\) is of type \((p, p)\) by hypothesis, we have
\[
\int_{|f| \leq 1} |Th|^p d\nu = -n_e(1) + p \int_{0}^{1} n_e(y)y^{p-1} dy
\]
\[
\leq p \int_{0}^{1} M_1 ||h||_1, y^p y^{p-1} dy = \frac{pM_1}{p-1} \int _R |h| d\mu,
\]
and therefore
\[
\text{(1)} \quad \int_{|f| \leq 1} |Th|^p d\nu \leq 0 \left( \frac{pM_1}{p-1} \right) \int_{|f| > 1} |f| d\mu
\]
Next if we follow carefully on the lines of proof of Theorem 2, we have

\[
\int_{|\tau|>1} Th \, dv = -\int_{1}^{\infty} y \, dn_{\tau}(y) = n_{\tau}(1) + \int_{1}^{\infty} n_{\tau}(y) \, dy
\]

\[
\leq n_{\tau}(1) + 0(1) \sum_{j=0}^{\infty} \eta_{j} \delta_{j}
\]

\[
\leq 0(M_{1}) \int_{|\tau|>1} |h| \, d\mu + 0 \left(\frac{M_{p}}{p-1} + M_{1}\right) \int_{|\tau|>1} |f| \, (1 + \log^{+} |f|) \, d\mu
\]

Therefore

\[
(2) \quad \int_{|\tau|>1} |Th| \, dv \leq 0 \left(\frac{M_{p}}{p-1} + M_{1}\right) \int_{|\tau|>1} |f| \, (1 + \log^{+} |f|) \, d\mu
\]

We have immediately

\[
(3) \quad \int_{\Sigma} |Tg|^{p} \, dv \leq M_{p}^{p} \int_{\sigma} |g|^{p} \, d\mu = M_{p}^{p} \int_{|\tau|>1} |f|^{p} \, d\mu
\]

and also

\[
\int_{|\tau|>1} |Tg| \, dv = n_{\tau}(1) + \int_{1}^{\infty} n_{\tau}(y) \, dy
\]

\[
\leq M_{p}^{p} \|g\|_{p, \mu}^{p} + \int_{1}^{\infty} \left(\frac{M_{p}^{p} \|g\|_{p, \mu}}{y^{p}}\right)^{p} \, dy
\]

and therefore

\[
(4) \quad \int_{|\tau|>1} |Tg| \, dv \leq 0 \left(\frac{M_{p}}{p-1}\right)^{p} \int_{|\tau|>1} |f|^{p} \, d\mu
\]

We need the following lemma

**Lemma.** From an inequality

\[A \leq \kappa(B+C), \ A, B, C \geq 0, \ \kappa \geq 1\]

we have (i) if 0 \leq A \leq 1

\[A \leq \begin{cases} \kappa(B+C), & \text{if } 0 \leq C \leq 1 \\ \kappa(B+C^{1/p}), & \text{if } C > 1 \end{cases}\]

(ii) if A > 1

\[A \leq \begin{cases} (2\kappa)^{p}(B^{p}+C^{p}), & \text{if } 0 \leq C \leq 1 \\ (2\kappa)^{p}(B^{p}+C^{p}), & \text{if } C > 1. \end{cases}\]

**Proof.** (i) Suppose that 0 \leq A \leq 1. If 0 \leq C \leq 1, it is trivial; if C<1

\[A \leq 1 < C^{1/p} \leq \kappa(B+C^{1/p}).\]

(ii) Suppose that A > 1. From an inequality A \leq \kappa(B+C), one of the relation
$B > \frac{A}{2\kappa}$ and $C > \frac{A}{2\kappa}$ always holds. If $B > \frac{A}{2\kappa}$,

$$A \leq 2\kappa B \leq (2\kappa B)^p$$

$$\leq \begin{cases} (2\kappa)^p(B^p+C^p), & \text{if } 0 \leq C \leq 1 \\ (2\kappa)^p(B^p+C), & \text{if } C > 1. \end{cases}$$

If $C > \frac{A}{2\kappa}$,

$$A \leq 2\kappa C$$

$$\leq \begin{cases} (2\kappa)^p(B^p+C^p), & \text{if } 0 \leq C \leq 1 \\ (2\kappa)(B^p+C), & \text{if } C > 1. \end{cases}$$

Let us estimate $T\psi$ on the set $S_1 = \{x : |T\psi| \leq 1\}$. Applying Lemma (i) such as $A = |T\psi|$, $B = |Tg|$ and $C = |Th|$ and the Minkowsky inequality, we have

$$\left(\int_{S_1} |T\psi|^p d\nu \right)^{1/p} \leq \kappa \left(\int_{S_1 \cap \{|T\psi| > 1\}} (|Tg| + |Th|^{1/p})^p d\nu \right)^{1/p}$$

$$+ \kappa \left(\int_{S_1 \cap \{|T\psi| \leq 1\}} (|Tg| + |Th|)^p d\nu \right)^{1/p}$$

$$\leq 2\kappa \left(\int_{S_1} |Tg|^p d\nu \right)^{1/p} + \kappa \left(\int_{|T\psi| > 1} |Th|^p d\nu \right)^{1/p}$$

Substituting (1) (2) and (3),

$$\left(\int_{S_1} |T\psi|^p d\nu \right)^{1/p} \leq 0(M^p) \int_{|T\psi| \leq 1} |f|^p d\mu$$

$$+ 0(M^p + M^i) \int_{|T\psi| > 1} |f| (1 + \log^+ |f|) d\mu$$

Let us estimate $T\psi$ on the set $S_2 = \{x : |T\psi| > 1\}$, we have

$$\int_{S_2} |T\psi| d\nu \leq (2\kappa)^p \int_{S_2 \cap \{|T\psi| > 1\}} (|Tg|^p + |Th|^p) d\nu + (2\kappa)^p \int_{S_2 \cap \{|T\psi| \leq 1\}} (|Tg|^p + |Th|^p) d\nu$$

$$\leq 2(2\kappa)^p \int_{S_2} |Tg|^p d\nu + (2\kappa)^p \int_{|T\psi| > 1} |Th|^p d\nu$$

Substituting (1) (2) and (3)

$$\left(\int_{|T\psi| > 1} |T\psi|^p d\nu \right)^{1/p} \leq 0(M^p) \int_{|T\psi| \leq 1} |f|^p d\mu$$

$$+ 0(M^p + M^i) \int_{|T\psi| > 1} |f| (1 + \log^+ |f|) d\mu$$
The formulas (5) and (6) complete the proof of Theorem 3.

5. Applications. Let \( x=(x_1, x_2, \ldots, x_n) \), \( y=(y_1, y_2, \ldots, y_n) \), by points of the real \( n \)-dimensional space \( E_n \). A.P. Calderon-A. Zygmund [2] studied the singular integral operator:

\[
\tilde{f}(x) = (f * K)(x) = \text{P.V.} \int_{E_n} f(x-y) K(y) \, dy
\]

where kernel \( K(x) \) has the form

\[
K(x) = \frac{1}{|x|^{n}}, \quad x' = \frac{x}{|x|}.
\]

Let us denote by \( \Sigma \) the unit sphere on which the \( \Omega(x') \) is defined. Let us denote by \( \omega(\delta) \) the modulus of continuity of \( \Omega(x') \),

\[
|\Omega(x') - \Omega(y')| \leq \omega(x' - y').
\]

Let us suppose that

(a) \( \int_\Omega \Omega(x') \, dx' = 0 \)

(b) \( \Omega(x') \in L^1(\Sigma) \) and its modulus of continuity \( \omega(\delta) \) satisfy the Dini condition,

\[
\int_0^\delta \frac{\omega(\delta)}{\delta} \, d\delta < \infty.
\]

Then they proved that the operations \( Tf=\tilde{f} \) and \( Tsf=\tilde{f}_s \) are both linear and of type \( (p, p) \) for every \( p>1 \) and of weak type \( (1,1) \) respectively. Applying our theorem 3, we have for example

\[
\left( \int |f|^p \right)^{\frac{1}{p}} \leq K \left\{ \int |f| \, dx + \int |f| \, dx \right\},
\]

where \( K \) is a constant depending on \( p \) and not on \( f \).

A.P. Calderon-M. Weiss-A. Zygmund [4] proved that the condition (b) of \( \Omega(x') \) can be replaced by the (rotational) integrated modulus of continuity \( \omega_i(\delta) \) instead of \( \omega(\delta) \). That is, the \( \omega_i(\delta) \) is defined as follows

\[
\omega_i(\delta) = \sup_{|\rho| \leq \delta} \int |\Omega(\rho x') - \Omega(x')| \, dx'
\]

where \( \rho \) is any rotation of \( \Sigma \) and \( |\rho| \) its magnitude.

Furthermore the maximal operation \( \tilde{Tf} = \tilde{f} \)
\[ \tilde{T}f = \tilde{f} = \sup |\tilde{f}_x| \]
satisfy the same assumptions as the operations \( Tf = \tilde{f} \) and \( \tilde{T}f = \tilde{f}_x \) and so necessarily the same conclusions. See, L. Hörmander [8], A.P. Calderon-A. Zygmund [3] and A.P. Calderon-M. Weiss-A. Zygmund [4].

As a special case, the one-dimensional Hilbert transform
\[ Hf(x) = P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} \, dy \]
and the Riesz transform
\[ R_jf(x) = P.V. \frac{1}{C_n} \int_{E_n} f(y) \frac{x_j-y_j}{|x-y|^{n+1}} \, dy \quad (j = 1, 2, \ldots n) \]
where
\[ C_n = \frac{\pi^{(n+1)/2}}{\Gamma \left( \frac{n+1}{2} \right)} \]
and also the unified operator of Hilbert transform and ergodic operator belong to our category. See J. Horváth [9], M. Cotlar [5] and E.M. Stein [14].

On the other hand let us consider
\[ \tilde{f}_a(x) = P.V. \int_{E_n} \frac{f(x-y)}{|y|^{n-a}} \, dy, \quad 0 < \alpha < n; \]
then the following is known according to G.H. Hardy-J.E. Littlewood [6] and A. Zygmund [15] (c.f. also, E.M. Stein [14]):
(i) it is of type \((r, s)\)
\[ \|\tilde{f}_a\|_r \leq M_{rs} \|f\|_r, \]
where \(1 < r < s < \infty, \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{n}, \)
(ii) it is of weak type \((1, \frac{1}{n-\alpha})\).
Thus the potential operator is beyond the scope of Theorem 3. We shall give a conjecture.

Let us write \( \alpha_i = \frac{1}{a_i}, \beta_i = \frac{1}{b_i} \) \((i=1, 2)\). Let (\(\alpha_i, \beta_i\)) and (\(\alpha_2, \beta_2\)) be any two points of the triangle
\[ \Delta: 0 \leq \beta \leq \alpha \leq 1 \]
such that \(\beta_1 \neq \beta_2\). If \(\alpha_i > \alpha_2\), let us suppose that a quasilinear operation \(\tilde{f} = Tf\) is of weak type \(\left(\frac{1}{\alpha_1}, \frac{1}{\beta_1}\right)\) and type \(\left(\frac{1}{\alpha_2}, \frac{1}{\beta_2}\right)\), then we have
where $k_i = \frac{b_i}{a_i}$, $K$ is a constant independent of $f$.

We shall have an analogous result in the case $\alpha_1 < \alpha_2$.

References
