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CONTRIBUTIONS TO THE THEORY OF INTERPOLATION OF OPERATIONS

SUMIYUKI KOIZUMI

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1. Introduction. Let (R, μ) and (S, ν) be two measure spaces of totally σ-finite in the sense of P. Halmos [7]. Let us consider operation T which transforms measurable functions on R to those on S. The operation T is called quasi-linear if:

(i) $T(f_1+f_2)$ is uniquely defined whenever Tf_1 and Tf_2 are defined and

$$|T(f_1+f_2)| \leqq \kappa(|Tf_1|+|Tf_2|)$$

where κ is a constant independent of f_1 and f_2;

(ii) $T(cf)$ is uniquely defined whenever Tf is defined
and

$$|T(cf)| = |c|\ |Tf|$$

for all scalars c.

We say that

$$\tilde{f} = Tf$$

is an operation of type (a, b), $1 \leqq a \leqq b \leqq \infty$, if :

(i) Tf is defined for each $f \in L^a_\mu(R)$, that is for each f measurable with respect to μ such that

$$||f||_{a,\mu} = \left(\int_R |f|^a d\mu\right)^{1/a}$$

is finite, the right side being interpreted as the essential upper bound (with respect to μ) of $|f|$ if $a = \infty$;

(ii) for every $f \in L^a_\mu(R)$, $\tilde{f} = Tf$ is in $L^b_\nu(S)$ and

$$||\tilde{f}||_{b,\nu} \leqq M\, ||f||_{a,\mu}, \tag{1.1}$$

where M is a constant independent of f.
The least admissible value of M in (1.1) is called the (a, b)-norm of operation T.

Next let us define the weak type (a, b) of operations.
Suppose first that $1 \leqq b < \infty$. Given any $y > 0$ denote by $E_y = E_y[\tilde{f}]$ the set of points of the space S where

$$|\tilde{f}(x)| > y,$$

and write $\nu(E_y)$ for the ν-measure of the set E_y. An immediate consequence of (1.1) is that

$$\nu(E_y[\tilde{f}]) \leqq \left(\frac{M}{y}\|f\|_{a,\mu}\right)^b. \tag{1.2}$$

An operation T which satisfies (1.2) will be called to be of weak type (a, b). The least admissible value of M in (1.2) is called the weak type (a, b)-norm of T.

We define weak type (a, ∞) as identical with type (a, ∞). Hence T is the weak type (a, ∞) if

$$\text{ess. sup } |\tilde{f}| \leqq M \|f\|_{a,\mu}.$$

If no confusion arises we omit the symbols μ and ν in the notation for norms.

In a number of problems we are led to consider integrals of type

$$\int_R \varphi(|f|)d\mu$$

where φ is not necessarily a power.

The interpolation of operation on the type of space with finite measure has been considered firstly by J. Marcinkiewicz [12] and A. Zygmund [15]. In the previous paper [10], the author treated an extension to the space with totally σ-finite measure. We intend further extension and refinement of those theorems to the space which is closely related to the intermediate space. The intermediate between a pair of Banach spaces was firstly introduced by A.J. Luxemburg [11].

Let us consider two continuous increasing functions $\varphi_1(u)$ and $\varphi_2(u)$. The former is defined on the interval $0 \leqq u \leqq \gamma$ and the latter is on $\frac{1}{\gamma} \leqq u < \infty$, and γ is a constant larger than 1. Those satisfy the following properties:

(i)
$$\varphi_1(0) = 0 \quad \text{and} \quad \varphi_1(2u) = 0(\varphi_1(u))$$
$$\int_u^1 \frac{\varphi_1(t)}{t^{b+1}}dt = 0\left(\frac{\varphi_1(u)}{u^b}\right)$$
$$\int_0^u \frac{\varphi_1(t)}{t^{a+1}}dt = 0\left(\frac{\varphi_1(u)}{u^a}\right)$$

for $u \to 0$, Here and in what follows it is assumed that $a < b$;

(ii)
$$\varphi_2(2u) = 0(\varphi_2(u))$$

$$\int_u^\infty \frac{\varphi_2(t)}{t^{b+1}}dt = 0\left(\frac{\varphi_2(u)}{u^b}\right)$$

$$\int_1^u \frac{\varphi_2(t)}{t^{a+1}}dt = 0\left(\frac{\varphi_2(u)}{u^a}\right)$$

for $u \to \infty$;

(iii) $\varphi_1(1) = \varphi_2(1)$ and so necessarily $\varphi_1(u) \sim \varphi_2(u)$ on an appropriate interval containing the unity, say $\frac{1}{\gamma} \leqq u \leqq \gamma,\ \gamma>1$. It means that there exist positive constants A, B such that

$$A \leqq \frac{\varphi_1(u)}{\varphi_2(u)} \leqq B \quad \text{if } \frac{1}{\gamma} \leqq u \leqq \gamma,\ \gamma>1.$$

Let us join φ_1 with φ_2 and introduce a new function φ, that is

$$\varphi(u) = \begin{cases} \varphi_1(u), & \text{if } 0 \leqq u \leqq 1 \\ \varphi_2(u), & \text{if } 1 < u < \infty \end{cases}$$

The typical example is

$$\varphi(u) = \begin{cases} u^{c_1}\psi_1(u), & \text{if } 0 \leqq u \leqq 1 \\ u^{c_2}\psi_2(u), & \text{if } 1 < u < \infty \end{cases}$$

where a $< c_1, c_2 < b$ and ψ_1, ψ_2 are slowly varying function (*c.f. A.* Zygmund [16]).

Theorem 1. *Suppose that a quasi-linear operation T is of weak type (a, a) and (b, b) with norms M_a and M_b, where $1 \leqq a < b < \infty$. Then Tf is defined for every f with μ-integrable $\varphi(|f|)$, $\varphi(|Tf|)$ is ν-integrable and we have*

$$\int_S \varphi(|Tf|)\, d\nu \leqq K\int_R \varphi(|f|)d\mu$$

where $K = 0(M_a \vee M_b)$, $M_a \vee M_b$ meaning the maximum value of M_a, M_b.

Let us consider another pair of continuous increasing functions $\chi_1(u)$ and $\chi_2(u)$ which satisfy the following properties:

(i)
$$\chi_1(0) = 0, \quad \chi_1(2u) = 0(\chi_1(u))$$

$$\int_u^1 \frac{\chi_1(t)}{t^{b+1}}dt = 0\left(\frac{\chi_1(u)}{u^b}\right)$$

$$\int_0^u \frac{\chi_1(t)}{t^{a+1}}dt = 0\left(\frac{\chi_1(u)}{u^a}\right)$$

for $u \to 0$;

(ii)
$$\chi_2(2u) = 0(\chi_2(u))$$

$$\int_u^\infty \frac{X_2(t)}{t^{b+1}}dt = 0\left(\frac{X_2(u)}{u^b}\right)$$

for $u \to \infty$;

(iii) $X_1(1) = X_2(1)$ and so necessarily $X_1(u) \sim X_2(u)$ on the interval $\frac{1}{\gamma} \leqq u \leqq \gamma$ for some $\gamma > 1$.

Write

$$X_2^*(u) = u^a \int_1^u \frac{X_2(t)}{t^{a+1}}dt \quad \text{if } u>1$$

and let us join X_1 with X_2 and X_2^* and introduce new functions X and X^*, that is

$$X(u) = \begin{cases} X_1(u), & \text{if } 0 \leqq u \leqq 1 \\ X_2(u), & \text{if } 1 < u < \infty \end{cases}$$

$$X^*(u) = \begin{cases} X_1(u), & \text{if } 0 \leqq u \leqq 1 \\ X_2(u)+X_2^*(u), & \text{if } 1 < u < \infty \end{cases}$$

The typical example is

$$X_1(u) = u^c \psi_1(u), \quad \text{if } 0 \leqq u \leqq 1$$
$$X_2(u) = u^a,\ X_2^*(u) = u^a \log^+ u, \quad \text{if } 1 < u < \infty$$

where $a < c < b$, $\psi_1(u)$ is a slowly varying function.

Theorem 2. *Suppose that a quasi-linear operation T is of weak type (a, a) and (b, b) with norms M_a and M_b, where $1 \leqq a < b < \infty$. Then Tf is defined for every μ-integrable $X^*(|f|)$, $X(|Tf|)$ is ν-integrable and we have*

$$\int_S (|Tf|)d\nu \leqq K \int_R X^*(|f|)d\mu$$

where $K = O(M_a \vee M_b)$.

We shall prove those theorems in § 2. In § 3, we shall add some remarks which are useful on a certain case. In § 4, we shall prove the following theorem.

Theorem 3. *Suppose that a quasi-linear opeation T is of weak type* $(1, 1)$ *and type (p, p) for some $p>1$. Then we have*

$$\int_{|Tf|\leqq 1} |Tf|^p d\nu + \int_{|Tf|>1} |Tf| d\nu$$
$$\leqq K\left\{ \int_{|f|\leqq 1} |f|^p d\mu + \int_{|f|>1} |f|(1+\log^+|f|)d\mu \right\}$$

where K is a constant independent of f.

In § 5, we shall state some applications to singular integral operators. Here the present author thanks to the referee for his kind advices.

2. Proofs of Theorems 1 and 2. Firstly we intend to prove Theorem 2. The $\chi_1(u)$ has the following properties

$$Bu^b \leqq \chi_1(u) \leqq Au^a \quad (0 \leqq u \leqq 1)$$

where we shall use letters A, B, etc. as absoute constants.

If we denote by f^* equi-measurable, non-increasing rearrangement of $|f|$, and by R_1 the sub-set of the space R where $|f| \leqq 1$, then

$$\int_{R_1} |f|^b d\mu = \int_t^\infty (f^*)^b dx < B^{-1}\int_t^\infty \chi_1(f^*)dx$$
$$= B^{-1}\int_{R_1} \chi_1(|f|)d\mu,$$

where t denotes the μ-measure of set $\{x \mid |f(x)| > 1\}$.
The $\chi_2(u)$ and $\chi_2^*(u)$ have the following properties. The $\chi_2^*(u)$ is continuous, non-decreasing function for $u > 1$ and

$$\chi_2^*(2u) = 0(\chi_2^*(u))$$

for $u \to \infty$. Because for $u' > u > 1$, we have

$$\chi_2^*(u') - \chi_2^*(u) = (u')^a\int_1^{u'} \frac{\chi_2(t)}{t^{a+1}}dt - u^a\int_1^u \frac{\chi_2(t)}{t^{a+1}}dt$$
$$> u^a\int_u^{u'} \frac{\chi_2(t)}{t^{a+1}}dt > 0$$

and since $\chi_2(2u) = 0(\chi_2(u))$ for $u \to \infty$, we have

$$\chi_2^*(2u) = (2u)^a\int_1^{2u} \frac{\varphi_2(t)}{t^{a+1}}dt = (2u)^a\left\{\int_1^u \frac{\chi_2(t)}{t^{a+1}}dt + \int_u^{2u} \frac{\chi_2(t)}{t^{a+1}}dt\right\}$$
$$= A\,\chi_2^*(u) + A'(2u)^a\int_{u/2}^u \frac{\chi_2(2t)}{t^{a+1}}dt$$
$$\leqq A\,\chi_2^*(u) + A'u^a\int_1^u \frac{\chi_2(t)}{t^{a+1}}dt \leqq A''\chi_2^*(u).$$

By similar arguments read

$$\chi_2(u) \leqq A\,\chi_2^*(u)$$
$$u^a \leqq A\,\chi_2^*(u)$$

and

$$\chi_2(u) \leqq Bu^b$$

respectively. We have

$$\int_u^\infty \frac{\chi_2{}^*(t)}{t^{b+1}}dt = 0\left(\frac{\chi_2{}^*(u)}{u^b}\right)$$

for $u \to \infty$. Because we have by the definition of $\chi_2{}^*$,

$$\begin{aligned}\int_u^\infty \frac{\chi_2{}^*(t)}{t^{b+1}}dt &= \int_u^\infty \frac{dt}{t^{b+1}}t^a \int_1^t \frac{\chi_2(s)}{s^{a+1}}ds \\ &= \int_1^u \frac{\chi_2(s)}{s^{a+1}}ds \int_u^\infty \frac{dt}{t^{b-a+1}} + \int_u^\infty \frac{\chi_2(s)}{s^{a+1}}ds \int_s^\infty \frac{dt}{t^{b-a+1}} \\ &= \frac{1}{(b-a)u^{b-a}} \int_1^u \frac{\chi_2(s)}{s^{a+1}}ds + \frac{1}{(b-a)} \int_u^\infty \frac{\chi_2(s)}{s^{b+1}}ds \\ &\leqq A\frac{\chi_2{}^*(u)}{u^b} + A'\frac{\chi_2(u)}{u^b} \leqq A''\frac{\chi_2{}^*(u)}{u^b}.\end{aligned}$$

If we denote by R_2 the sub-set of R where $|f|>1$, then

$$\begin{aligned}\int_{R_2} |f|^a d\mu = \int_0^t (f^*)^a dx < A\int_0^t \chi_2{}^*(f^*)dx \\ = A\int_{R_2} \chi_2{}^*(|f|)d\mu,\end{aligned}$$

where t denotes the μ-measure of se $\{x \mid |f(x)|>1\}$. Under those preparations, let $f \in L_\mu^{\kappa^*}(R)$ and write

$$f = f' + f''$$

where $f' = f$ whenever $|f| \leqq 1$ and $f' = 0$ otherwise ; $f'' = f - f'$. Since $f' \in L_\mu^{\kappa_1}$ and so $f' \in L_\mu^b$, $f'' \in L_\mu^{\chi_2^*}$ and so $f'' \in L_\mu^a$. Hence Tf' and Tf'' are defined, by hypothesis, and so $Tf = T(f'+f'')$. Let $n_\nu(y)$ by the distribution function $|Tf|$. We have

$$\begin{aligned}\int_S \chi(|Tf|)d\nu = -\int_0^\infty \chi(y)dn_\nu(y) \\ = \int_0^\infty n_\nu(y)d\chi(y) \leqq \sum_{j=-\infty}^\infty \eta_j \delta_j\end{aligned}$$

where $\delta_j = \chi(\lambda 2^{j+1}) - \chi(\lambda 2^j)$ and $\eta_j = \nu(E_{\lambda 2^j})\,[|Tf|])$, $\lambda = 3\kappa^2$. The passage from the second to the third integral is justified as in A. Zygmund [15, Vol. II, p. 112 (4.8)].

For each fixed $j \geqq 0$, we write $f = f_1 + f_2 + f_3$, where f_1 equals f or 0 according as $1 < |f| \leqq 2^j$ or elae; f_2 does f or 0 according as $|f| > 2^j$ or else; and so f_3 does f or 0 according as $|f| \leqq 1$ or else. Since $f_1 \in L_\mu^a \cap L_\mu^b$, $f_2 \in L_\mu^a$ and

$f_3 \in L^b_\mu$ respectively. In view of the inequality

$$\begin{aligned} |Tf| &\leqq \kappa(|T(f_1+f_2)|+|Tf_3|) \\ &\leqq \kappa^2(|Tf_1|+|Tf_2|+|Tf_3|) \qquad (\kappa>1) \end{aligned}$$

if $|Tf_i|<y$, for all $i=1, 2, 3$ and any positive real number y, then $|Tf|<\lambda y$ with $\lambda=3\kappa^2$. Therefore we have

$$\{x \mid |Tf|>\lambda y\} \subset \bigcup_{i=1}^{3} \{x \mid |Tf_i|>y\}$$

and if we take $y=2^j$, we get the following formula,

$$\eta_j \leqq C\left\{2^{-jb}\int_{R_2}|f_1|^b d\mu+2^{-ja}\int_{R_2}|f_2|^a d\mu+2^{-jb}\int_{R_1}|f_3|^b d\mu\right\}$$

and then

$$\begin{aligned} \sum_{j=0}^{\infty}\eta_j\delta_j &\leqq C\left\{\sum_{j=0}^{\infty}2^{-jb}\delta_j\int_{R_2}|f_1|^b d\mu+\sum_{j=0}^{\infty}2^{-ja}\delta_j\int_{R_2}|f_2|^a d\mu\right. \\ &\qquad\qquad \left.+\sum_{j=0}^{\infty}2^{-jb}\delta_j\int_{R_1}|f_3|^b d\mu\right\} \\ &= I_1+I_2+I_3, \text{ say.} \end{aligned}$$

By ε_i $(i=1, 2,\cdots)$, we denote the μ-measure of the set where $2^{i-1}<|f|\leqq 2^i$, then then if we interchange the order of summation and substitute above estimates we are led to

$$\begin{aligned} I_1 &= C\sum_{j=0}^{\infty}2^{-jb}\delta_j\int_R|f_1|^b d\mu \leqq C\sum_{j=1}^{\infty}2^{-jb}\delta_j\sum_{i=1}^{j}2^{ib}\varepsilon_i \\ &= C\sum_{i=1}^{\infty}2^{ib}\varepsilon_i\sum_{j=i}^{\infty}2^{-jb}\delta_j \leqq C'\sum_{j=1}^{\infty}2^{ib}\varepsilon_i\int_{\lambda 2^{i+1}}^{\infty}\frac{\chi_2(u)}{u^{b+1}}du \\ &\leqq C''\sum_{i=1}^{\infty}\chi_2(2^i)\varepsilon_i \leqq C''\int_{R_2}\chi_2(|f|)d\mu \end{aligned}$$

$$\begin{aligned} I_2 &= C\sum_{j=0}^{\infty}2^{-ja}\delta_j\int_R|f_2|^a d\mu \leqq C\sum_{j=0}^{\infty}2^{-ja}\delta_j\sum_{i=j+1}^{\infty}2^{ia}\varepsilon_i \\ &= C\sum_{i=1}^{\infty}2^{ia}\varepsilon_i\sum_{j=0}^{i-1}2^{-ja}\delta_j \leqq C'\sum_{i=1}^{\infty}2^{ia}\varepsilon_i\int_1^{\lambda 2^i}\frac{\chi_2(u)}{u^{a+1}}du \\ &\leqq C''\sum_{i=0}^{\infty}\chi_2{}^*(2^i)\varepsilon_i \leqq C''\int_{R_2}\chi_2{}^*(|f|)d\mu \end{aligned}$$

$$\begin{aligned} I_3 &= C\sum_{j=0}^{\infty}2^{-jb}\delta_j\int_R|f_3|^b d\mu = C\int_{R_1}|f|^b d\mu\sum_{j=0}^{\infty}2^{-jb}\delta_j \\ &\leqq C\int_{R_1}|f|^b d\mu\int_1^{\infty}\frac{\chi_2(u)}{u^{b+1}}du \leqq C'\int_{R_1}\chi_1|(f)|d\mu \end{aligned}$$

Therefore we have

$$\sum_{j=0}^{\infty} \eta_j \delta_j \leqq C \int_{R_2} \chi_2^*(|f|)d\mu + C' \int_{R_2} \chi_2(|f|)d\mu + C'' \int_{R_1} \chi_1(|f|)d\mu$$
$$\leqq C \int_R \chi^*(|f|)d\mu.$$

Similarly, for each fixed $j < 0$, we write $f = f_4 + f_5 + f_6$, where f_4 equals f or 0 according as $2^j < |f| \leqq 1$ or else; f_5 does f or 0 according as $0 \leqq |f| \leqq 2^j$ or else; and so f_6 does f or 0 according as $1 < |f|$ or else. Since $f_4 \in L_\mu^a \cap L_\mu^b$, $f_5 \in L_\mu^b$ and $f_6 \in L_\mu^a$ respectively, we have

$$\eta_j \leqq D\left\{2^{-ja} \int_{R_1} |f_4|^a d\mu + 2^{-jb} \int_{R_1} |f_5|^b d\mu + 2^{-ja} \int_{R_2} |f_6|^a d\mu\right\}$$

We can estimate the summation $\sum_{j=-\infty}^{-1} \eta_j \delta_j$ just the same as $\sum_{j=0}^{\infty} \eta_j \delta_j$ and we have

$$\sum_{j=-\infty}^{-1} \eta_j \delta_j \leqq D \int_{R_1} \chi_1(|f|) d\mu + D'' \int_{R_2} \chi_2^*(|f|) d\mu$$

and hence we attain the desired inequality

$$\int_S \chi(|Tf|)d\nu \leqq K \int_R \chi^*(|f|)d\mu$$

The proof of Theorem 1 is a rather easy repetition of that of Theorem 2 and need not be gone into.

3. Some remarks. (1). If the operation T is linear, then we can present theorems 1 and 2 as more general forms which are useful on a certain case (c.f. E.M. Stein - G. Weiss [13]).

We say that the operation T is of restricted weak type (a, b), if for every simple function f on R, Tf is ν-measurable function on S and satisfies

$$\nu(E_y [|Tf|]) \leqq \left(\frac{M}{y} \|f\|_{a,\mu}\right)^b$$

where M is a constant independent of f. We can state

Corollary 1. *In Theorem 1, if the operation T is linear and of restricted weak type (a, a) and (b, b) where $1 \leqq a < b < \infty$ respectively. Then we have for every simple function f on R,*

$$\int_S \varphi(|Tf|)d\nu \leqq K \int_R \varphi(|f|)d\mu$$

and moreover we can extend the operation T to the whole space L_φ^μ preserving the *norm of operation.*

Proof. We need only to prove the process of extension. Take any f in L^{φ}_{μ}. Let us write

$$f_n = \begin{cases} (\operatorname{sign} f)\dfrac{k-1}{n}, & \text{if } \dfrac{k-1}{n} \leqq |f| < \dfrac{k}{n} \\ (\operatorname{sign} f)n, & \text{if } |f| > n \end{cases}$$

$k = 1, 2, \cdots, n$; $n = 1, 2, \cdots$. Then f_n tends to f monotone increasingly for a.e. x and so $\varphi(|f_n|)$ does to $\varphi(|f|)$. By the Lebesgue convergence theorem we have

$$\lim_{n\to\infty}\int_R \varphi(|f_n|)d\mu = \int_R \varphi(|f|)d\mu$$

and

$$\lim_{m,n\to\infty}\int_R \varphi(|f_m - f_n|)d\mu = 0.$$

If we write $\tilde{f}_n = Tf_n$, then by hypothesis we have

$$\int_S \varphi(|\tilde{f}_n|)d\nu \leqq K\int_R \varphi(|f_n|)d\mu$$

and since T is of linear

$$\int_S \varphi(|\tilde{f}_m - \tilde{f}_n|)d\nu \leqq K\int_R \varphi(|f_m - f_n|)d\mu.$$

The least formula shows that $\{\tilde{f}_n\}$ is a sequence of fundamental in measure and so there exist a limit function $\tilde{f}$ uniquely except a set of ν-measure zero and subsequence (n_k) of (n) such that $\tilde{f}_{n_k}$ converges to $\tilde{f}$ for a.e. x. Applying the Fatou lemma we have the desired result.

The same argument leads to

Corollary 2. *In Theorem 2, if the operation T is linear and of restricted weak type (a, a) and (b, b) where $1 \leqq a < b < \infty$, respectively. Then we have for every simple function f on R,*

$$\int_S \chi(|Tf|)d\nu \leqq K\int_R \chi^*(|f|)d\mu$$

and moreover we can extend the operation T to the whole space $L^{\kappa^*}_{\mu}$ preserving the norm of operation.

(2) Next we meet the $\varphi(u)$ which is continuous and not necessarily increasing on the whole interval. If we suppose that φ is ultimately increasing for the value of u near zero and infinity; in the middle interval, say $\left(\dfrac{1}{\gamma}, \gamma\right)$ with

$\gamma>1$, is of bounded variation, then we can find an increasing function φ^* such that

$$\varphi(u) \leqq \varphi^*(u) \leqq A_\gamma \varphi(u), \quad \text{for all } u \geqq 0.$$

For example a construction of φ^* is as follows:

$$\varphi^*(u)=\begin{cases} \varphi(u), & \text{if } 0 \leqq u<\dfrac{1}{\gamma} \\ \varphi\left(\dfrac{1}{\gamma}\right)+\displaystyle\int_{1/\gamma}^{u}|d\varphi|, & \text{if } \dfrac{1}{\gamma} \leqq u<\gamma \\ \varphi\left(\dfrac{1}{\gamma}\right)+\displaystyle\int_{1/\gamma}^{\gamma}|d\varphi|+(\varphi(u)-\varphi(\gamma)), & \text{if } \gamma \leqq u<\infty \end{cases}$$

The simple calculation shows that the inequality is satisfied

$$A_\gamma=\frac{\varphi\left(\dfrac{1}{\gamma}\right)+\displaystyle\int_{1/\gamma}^{\gamma}|d\varphi|}{\displaystyle\min_{1/\gamma \leqq u \leqq \gamma} \varphi(u)}$$

Corollary 3. *In Theorem* 1, *if the* $\varphi(u)$ *is ultimately increasing for the value of* u *near zero and infinity; in the middle interval, is of bounded variation. The same conclusion is also true.*

The same argument leads to

Corollary 4. *In Theorem* 2, *if the* $\chi(u)$ *is ultimately increasing for the value of* u *near zero and infinity; in the middle interval, is of bounded variation. The same conclusion is also true.*

4. Proof of Theorem 3. Let us suppose that $f \in L^p+L \log^+ L$. Write $f=g+h$:

$$g=\begin{cases} f, & \text{if } |f| \leqq 1 \\ 0, & \text{if } |f|>1 \end{cases} \qquad h=f-g$$

We have $g \in L^p$ and $h \in L \log^+ L$ respectively. Since the operation T is of type (p, p) by hypothesis, we have

$$\begin{aligned} \int_{|Th|\leqq 1} |Th|^p d &= -n_\nu(1)+p\int_0^1 n_\nu(y) y^{p-1}\,dy \\ &\leqq p\int_0^1 \frac{M_1}{y}\|h\|_{1,\mu} y^{p-1}\,dy = \frac{pM_1}{p-1}\int_R |h|\,d\mu, \end{aligned}$$

and therefore

$$(1) \qquad \int_{|Th|\leqq 1} |Th|^p d\nu \leqq 0\left(\frac{pM_1}{p-1}\right)\int_{|f|>1} |f|\,d\mu$$

Next if we follow carefully on the lines of proof of Theorem 2, we have

$$\int_{|Th|>1} Th\, d\nu = -\int_1^\infty y\, dn_\nu(y) = n_\nu(1)+\int_1^\infty n_\nu(y)dy$$

$$\leqq n_\nu(1)+0(1)\sum_{j=0}^{\infty}\eta_j\delta_j$$

$$\leqq 0(M_1)\int_{|h|>1}|h|\,d\mu+0\Big(\frac{M_p^p}{p-1}\Big)\int_{|h|>1}|h|\,d\mu+0(M_1)\int_{|h|>1}|h|\log^+|h|\,d\mu$$

Therefore

$$(2)\qquad \int_{|Th|>1}|Th|\,d\nu \leqq 0\Big(\frac{M_p^p}{p-1}+M_1\Big)\int_{|f|>1}|f|(1+\log^+|f|)d\mu$$

We have immediately

$$(3)\qquad \int_S |Tg|^p d\nu \leqq M_p^p\int_R |g|^p d\mu = M_p^p\int_{|f|\leqq 1}|f|^p d\mu$$

and also

$$\int_{|Tg|>1}|Tg|\,d\nu = n_\nu(1)+\int_1^\infty n_\nu(y)dy$$

$$\leqq M_p^p\|g\|_{p,\mu}+\int_1^\infty\Big(\frac{M_p\|g\|_{p,\mu}}{y}\Big)^p dy$$

and therefore

$$(4)\qquad \int_{|Tg|>1}|Tg|\,d\nu \leqq 0\Big(\frac{M_p^p}{p-1}\Big)\int_{|f|\leqq 1}|f|^p d\mu$$

We need the following lemma

Lemma. *From an inequality*

$$A \leqq \kappa(B+C),\ A,\ B,\ C \geqq 0,\ \kappa \geqq 1$$

we have (i) *if* $0 \leqq \mathrm{A} \leqq 1$

$$A \leqq \begin{cases}\kappa(B+C), & \text{if } 0 \leqq C \leqq 1\\ \kappa(B+C^{1/p}), & \text{if } \mathrm{C} > 1\end{cases}$$

(ii) *if* $A > 1$

$$A \leqq \begin{cases}(2\kappa)^p(B^p+C^p), & \text{if } 0 \leqq C \leqq 1\\ (2\kappa)^p(B^p+C), & \text{if } C > 1.\end{cases}$$

Proof. (i) Suppose that $0 \leqq A \leqq 1$. If $0 \leqq C \leqq 1$, it is trivial; if $\mathrm{C}<1$

$$A \leqq 1 < C^{1/p} \leqq \kappa(B+C^{1/p}).$$

(ii) Suppose that $A>1$. From an inequality $A \leqq \kappa(B+C)$, one of the relation

$B > \frac{A}{2\kappa}$ and $C > \frac{A}{2\kappa}$ always holds. If $B > \frac{A}{2\kappa}$,

$$A \leqq 2\kappa B \leqq (2\kappa B)^p$$
$$\leqq \begin{cases} (2\kappa)^p(B^p+C^p), & \text{if } 0 \leqq C \leqq 1 \\ (2\kappa)^p(B^p+C), & \text{if } C > 1. \end{cases}$$

If $C > \frac{A}{2\kappa}$,

$$A \leqq 2\kappa C$$
$$\leqq \begin{cases} (2\kappa C)^p \leqq (2\kappa)^p(B^p+C^p), & \text{if } 0 \leqq C \leqq 1 \\ (2\kappa)(B^p+C), & \text{if } C > 1. \end{cases}$$

Let us estimate Tf on the set $S_1 = \{x \mid |Tf| \leqq 1\}$. Applying Lemma (i) such as $A = |Tf|$, $B = |Tg|$ and $C = |Th|$ and the Minkowsky inequality, we have

$$\left(\int_{S_1} |Tf|^p d\nu\right)^{1/p} \leqq \kappa\left(\int_{S_1 \cap \{x \mid |Th|>1\}} (|Tg| + |Th|^{1/p})^p d\nu\right)^{1/p}$$
$$+\kappa\left(\int_{S_1 \cap \{x \mid |Th| \leqq 1\}} (|Tg| + |Th|)^p d\nu\right)^{1/p}$$
$$\leqq 2\kappa\left(\int_{S_1} |Tg|^p d\nu\right)^{1/p} + \kappa\left(\int_{|Th|>1} |Th| d\nu\right)^{1/p} + \kappa\left(\int_{|Th| \leqq 1} |Th|^p d\nu\right)^{1/p}$$

Substituting (1) (2) and (3),

$$(5) \qquad \int_{|f| \leqq 1} |Tf|^p d\nu \leqq 0(M_p^p) \int_{|f| \leqq 1} |f|^p d\mu$$
$$+0\left(\frac{M_p^p + M_1}{p-1}\right) \int_{|f|>1} |f|(1+\log^+|f|) d\mu$$

Let us estimate Tf on the set $S_2 = \{x \mid\ |Tf| > 1\}$, we have

$$\int_{S_2} |Tf| d\nu \leqq (2\kappa)^p \int_{S_2 \cap \{x \mid |Th|>1\}} (|Tg|^p + |Th|) d\nu + (2\kappa)^p \int_{S_2 \cap \{x \mid |Th| \leqq 1\}} (|Tg| +^p |Th|^p) d\nu$$
$$\leqq 2(2\kappa)^p \int_{S_2} |Tg|^p d\nu + (2\kappa)^p \int_{|Th|>1} |Th| d\nu + (2\kappa)^p \int_{|Th| \leqq 1} |Th|^p d\nu$$

Substituting (1) (2) and (3)

$$(6) \int_{|Tf|>1} |Tf| d\nu \leqq 0(M_p^p) \int_{|f| \leqq 1} |f|^p d\mu + 0\left(\frac{M_p^p + M_1}{p-1}\right) \int_{|f|>1} |f|(\log^+|f|) d\mu$$

The formulas (5) and (6) complete the proof of Theorem 3.

5. Applications. Let $x=(x_1, x_2, \cdots, x_n)$, $y=(y_1\, y_2, \cdots, y_n)$, by points of the real n-dimensional space E_n. A.P. Calderon-A. Zygmund [2] studied the singular integral operator:

$$\tilde{f}(x) = (f*K)(x) = \text{P.V.}\int_{E_n} f(x-y)\, K(y)\, dy$$
$$= \lim_{\varepsilon\to 0} \tilde{f}_\varepsilon(x) = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} f(x-y)K(y)\, dy,$$

where kernel K (x) has the form

$$K(\mathrm{x}) = |x|^{-n}\Omega(x'),\ x' = \frac{x}{|x|}.$$

Let us denote by Σ the unit sphere on which the $\Omega(x')$ is denfied. Let us denote by $\omega(\delta)$ the modulus of continuity of $\Omega(x')$,

$$|\Omega(x') - \Omega(y')| \leqq \omega(x'-y').$$

Let us suppose that

(a) $\int_\Sigma \Omega(x')\, dx' = 0$

(b) $\Omega(x') \in L^1(\Sigma)$ and its modulus of continuity $\omega(\delta)$ satisfy the Dini condition,

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, \mathrm{d}\delta < \infty.$$

Then they proved that the operations $Tf=\tilde{f}$ and $T_\varepsilon f=\tilde{f}_\varepsilon$ *are* both linear and of type (p, p) for every $p>1$ and of weak type (1, 1) respectively. Applying our theorem 3, we have for example

$$\int_{|\tilde{f}|\leqq 1} |\tilde{f}|^p dx + \int_{|\tilde{f}|>1} |\tilde{f}|\, dx$$
$$\leqq K\left\{\int_{|f|\leqq 1} |f|^p dx + \int_{|f|>1} |f|(1+\log^+|f|)\, dx\right\},$$

where K is a constant depending on p and not on f.

A.P. Calderon-M. Weiss-A. Zygmund [4] proved that the condition (b) of $\Omega(x')$ can be replaced by the (rotational) integrated modulus of continuity $\omega_1(\delta)$ instead of $\omega(\delta)$. That is, the $\omega_1(\delta)$ is defined as follows

$$\omega_1(\delta) = \sup_{|\rho|\leqq\delta} \int_\Sigma |\Omega(\rho x') - \Omega(x')|\, dx'$$

where ρ is any rotation of Σ and $|\rho|$ its magnitude.

Furthermore the maximal operation $\bar{T}f=\bar{f}$

$$\bar{T}f = \bar{f} = \sup |\tilde{f}_\varepsilon|$$

satisfy the same assumptions as the operations $Tf=\tilde{f}$ and $T_\varepsilon f=\tilde{f}_\varepsilon$ and so necessarily the same conclusions. See, L. Hörmander [8], A.P. Calderon-A. Zygmund [3] and A.P. Calderon-M. Weiss-A. Zygmund [4].

As a special case,the one-dimensional Hilbert transform

$$Hf(x) = \text{P.V.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$$

and the Riesz transform

$$R_j f(x) = \text{P.V.} \frac{1}{C_n} \int_{E_n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}} dy \quad (j=1, 2, \cdots n)$$

where

$$C_n = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$$

and also the unified operator of Hilbert transform and ergodic operator belong to our category. See J. Horváth [9], M. Cotlar [5] and E.M. Stein[14].

On the other hand let us consider

$$\tilde{f}_\alpha(x) = P.V. \int_{E_n} \frac{f(x-y)}{|y|^{n-\alpha}} dy, \quad 0 < \alpha < n;$$

then the following is known according to G.H. Hardy- J.E. Littlewood [6] and A. Zygmund [15] (c.f. also, E.M. Stein [14]):

(i) it is of type (r, s)

$$\|\tilde{f}_\alpha\|_s \leqq M_{rs} \|f\|_r,$$

where $1 < r < s < \infty$, $\dfrac{1}{r} - \dfrac{1}{s} = \dfrac{\alpha}{n}$,

(ii) it is of weak type $\left(1, \dfrac{1}{n-\alpha}\right)$.

Thus the potential operator is beyond the scope of Theorem 3. We shall give a conjecture.

Let us write $\alpha_i = \dfrac{1}{a_i}$, $\beta_i = \dfrac{1}{b_i}$ $(i=1, 2)$. Let (α_1, β_1) and (α_2, β_2) be any two points of the triangle

$$\Delta:\ 0 \leqq \beta \leqq \alpha \leqq 1$$

such that $\beta_1 \neq \beta_2$. If $\alpha_1 > \alpha_2$, let us suppose that a quasilinear operation $\tilde{f} = Tf$ is of weak type $\left(\dfrac{1}{\alpha_1}, \dfrac{1}{\beta_1}\right)$ and type $\left(\dfrac{1}{\alpha_2}, \dfrac{1}{\beta_2}\right)$, then we have

$$\int_{|Tf|\leqq 1} |Tf|^{b_2}\, d\nu + \int_{|Tf|>1} |Tf|^{b_1}\, d\nu$$

$$\leqq K\left\{\int_{|f|\leqq 1} |f|^{a_2}\, d\mu + \int_{|f|>1} |f|^{a_1}\{1+(\log^+|f|)^{k_1}\}\, d\mu\right\}$$

where $k_1 = \dfrac{b_1}{a_1}$, K is a constant independent of f.

We shall have an analogous result in the case $\alpha_1 < \alpha_2$.

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