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Author(s)	Koizumi, Sumiyuki
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Osaka University

CONTRIBUTIONS TO THE THEORY OF INTERPOLATION OF OPERATIONS

SUMIYUKI KOIZUMI

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1. Introduction. Let (R, μ) and (S, ν) be two measure spaces of totally σ -finite in the sense of P. Halmos [7]. Let us consider operation T which transforms measurable functions on R to those on S . The operation T is called quasi-linear if:

(i) $T(f_1+f_2)$ is uniquely defined whenever Tf_1 and Tf_2 are defined and

$$|T(f_1+f_2)| \leq \kappa(|Tf_1| + |Tf_2|)$$

where κ is a constant independent of f_1 and f_2 ;

(ii) $T(cf)$ is uniquely defined whenever Tf is defined and

$$|T(cf)| = |c| |Tf|$$

for all scalars c .

We say that

$$\tilde{f} = Tf$$

is an operation of type (a, b) , $1 \leq a \leq b \leq \infty$, if :

(i) Tf is defined for each $f \in L^a_\mu(R)$, that is for each f measurable with respect to μ such that

$$\|f\|_{a, \mu} = \left(\int_R |f|^a d\mu \right)^{1/a}$$

is finite, the right side being interpreted as the essential upper bound (with respect to μ) of $|f|$ if $a = \infty$;

(ii) for every $f \in L^a_\mu(R)$, $\tilde{f} = Tf$ is in $L^b_\nu(S)$ and

$$(1.1) \quad \|\tilde{f}\|_{b, \nu} \leq M \|f\|_{a, \mu},$$

where M is a constant independent of f .

The least admissible value of M in (1.1) is called the (a, b) -norm of operation T .

Next let us define the weak type (a, b) of operations.

Suppose first that $1 \leq b < \infty$. Given any $y > 0$ denote by $E_y = E_y[\tilde{f}]$ the set of points of the space S where

$$|\tilde{f}(x)| > y,$$

and write $\nu(E_y)$ for the ν -measure of the set E_y . An immediate consequence of (1.1) is that

$$(1.2) \quad \nu(E_y[\tilde{f}]) \leq \left(\frac{M}{y} \|f\|_{a, \mu}\right)^b.$$

An operation T which satisfies (1.2) will be called to be of weak type (a, b) . The least admissible value of M in (1.2) is called the weak type (a, b) -norm of T .

We define weak type (a, ∞) as identical with type (a, ∞) . Hence T is the weak type (a, ∞) if

$$\text{ess. sup } |\tilde{f}| \leq M \|f\|_{a, \mu}.$$

If no confusion arises we omit the symbols μ and ν in the notation for norms.

In a number of problems we are led to consider integrals of type

$$\int_R \varphi(|f|) d\mu$$

where φ is not necessarily a power.

The interpolation of operation on the type of space with finite measure has been considered firstly by J. Marcinkiewicz [12] and A. Zygmund [15]. In the previous paper [10], the author treated an extension to the space with totally σ -finite measure. We intend further extension and refinement of those theorems to the space which is closely related to the intermediate space. The intermediate between a pair of Banach spaces was firstly introduced by A.J. Luxemburg [11].

Let us consider two continuous increasing functions $\varphi_1(u)$ and $\varphi_2(u)$. The former is defined on the interval $0 \leq u \leq \gamma$ and the latter is on $\frac{1}{\gamma} \leq u < \infty$, and γ is a constant larger than 1. Those satisfy the following properties:

$$(i) \quad \begin{aligned} \varphi_1(0) = 0 \quad \text{and} \quad \varphi_1(2u) = 0(\varphi_1(u)) \\ \int_u^1 \frac{\varphi_1(t)}{t^{b+1}} dt = 0\left(\frac{\varphi_1(u)}{u^b}\right) \\ \int_0^u \frac{\varphi_1(t)}{t^{a+1}} dt = 0\left(\frac{\varphi_1(u)}{u^a}\right) \end{aligned}$$

for $u \rightarrow 0$, Here and in what follows it is assumed that $a < b$;

$$(ii) \quad \varphi_2(2u) = 0(\varphi_2(u))$$

$$\int_u^\infty \frac{\varphi_2(t)}{t^{b+1}} dt = O\left(\frac{\varphi_2(u)}{u^b}\right)$$

$$\int_1^u \frac{\varphi_2(t)}{t^{a+1}} dt = O\left(\frac{\varphi_2(u)}{u^a}\right)$$

for $u \rightarrow \infty$;

(iii) $\varphi_1(1) = \varphi_2(1)$ and so necessarily $\varphi_1(u) \sim \varphi_2(u)$ on an appropriate interval containing the unity, say $\frac{1}{\gamma} \leq u \leq \gamma$, $\gamma > 1$. It means that there exist positive constants A, B such that

$$A \leq \frac{\varphi_1(u)}{\varphi_2(u)} \leq B \quad \text{if } \frac{1}{\gamma} \leq u \leq \gamma, \gamma > 1.$$

Let us join φ_1 with φ_2 and introduce a new function φ , that is

$$\varphi(u) = \begin{cases} \varphi_1(u), & \text{if } 0 \leq u \leq 1 \\ \varphi_2(u), & \text{if } 1 < u < \infty \end{cases}$$

The typical example is

$$\varphi(u) = \begin{cases} u^{c_1} \psi_1(u), & \text{if } 0 \leq u \leq 1 \\ u^{c_2} \psi_2(u), & \text{if } 1 < u < \infty \end{cases}$$

where $a < c_1, c_2 < b$ and ψ_1, ψ_2 are slowly varying function (*c.f.* *A. Zygmund* [16]).

Theorem 1. *Suppose that a quasi-linear operation T is of weak type (a, a) and (b, b) with norms M_a and M_b , where $1 \leq a < b < \infty$. Then Tf is defined for every f with μ -integrable $\varphi(|f|)$, $\varphi(|Tf|)$ is ν -integrable and we have*

$$\int_S \varphi(|Tf|) d\nu \leq K \int_R \varphi(|f|) d\mu$$

where $K = O(M_a \vee M_b)$, $M_a \vee M_b$ meaning the maximum value of M_a, M_b .

Let us consider another pair of continuous increasing functions $\chi_1(u)$ and $\chi_2(u)$ which satisfy the following properties:

(i) $\chi_1(0) = 0, \chi_1(2u) = O(\chi_1(u))$

$$\int_u^1 \frac{\chi_1(t)}{t^{b+1}} dt = O\left(\frac{\chi_1(u)}{u^b}\right)$$

$$\int_0^u \frac{\chi_1(t)}{t^{a+1}} dt = O\left(\frac{\chi_1(u)}{u^a}\right)$$

for $u \rightarrow 0$;

(ii) $\chi_2(2u) = O(\chi_2(u))$

$$\int_u^\infty \frac{\chi_2(t)}{t^{b+1}} dt = o\left(\frac{\chi_2(u)}{u^b}\right)$$

for $u \rightarrow \infty$;

(iii) $\chi_1(1) = \chi_2(1)$ and so necessarily $\chi_1(u) \sim \chi_2(u)$ on the interval $\frac{1}{\gamma} \leq u \leq \gamma$ for some $\gamma > 1$.

Write

$$\chi_2^*(u) = u^a \int_1^u \frac{\chi_2(t)}{t^{a+1}} dt \quad \text{if } u > 1$$

and let us join χ_1 with χ_2 and χ_2^* and introduce new functions χ and χ^* , that is

$$\chi(u) = \begin{cases} \chi_1(u), & \text{if } 0 \leq u \leq 1 \\ \chi_2(u), & \text{if } 1 < u < \infty \end{cases}$$

$$\chi^*(u) = \begin{cases} \chi_1(u), & \text{if } 0 \leq u \leq 1 \\ \chi_2(u) + \chi_2^*(u), & \text{if } 1 < u < \infty \end{cases}$$

The typical example is

$$\chi_1(u) = u^c \psi_1(u), \quad \text{if } 0 \leq u \leq 1$$

$$\chi_2(u) = u^a, \quad \chi_2^*(u) = u^a \log^+ u, \quad \text{if } 1 < u < \infty$$

where $a < c < b$, $\psi_1(u)$ is a slowly varying function.

Theorem 2. *Suppose that a quasi-linear operation T is of weak type (a, a) and (b, b) with norms M_a and M_b , where $1 \leq a < b < \infty$. Then Tf is defined for every μ -integrable $\chi^*(|f|)$, $\chi(|Tf|)$ is ν -integrable and we have*

$$\int_S (|Tf|) d\nu \leq K \int_R \chi^*(|f|) d\mu$$

where $K = O(M_a \vee M_b)$.

We shall prove those theorems in § 2. In § 3, we shall add some remarks which are useful on a certain case. In § 4, we shall prove the following theorem.

Theorem 3. *Suppose that a quasi-linear operation T is of weak type $(1, 1)$ and type (p, p) for some $p > 1$. Then we have*

$$\int_{|Tf| \leq 1} |Tf|^p d\nu + \int_{|Tf| > 1} |Tf| d\nu$$

$$\leq K \left\{ \int_{|f| \leq 1} |f|^p d\mu + \int_{|f| > 1} |f| (1 + \log^+ |f|) d\mu \right\}$$

where K is a constant independent of f .

In § 5, we shall state some applications to singular integral operators. Here the present author thanks to the referee for his kind advices.

2. Proofs of Theorems 1 and 2. Firstly we intend to prove Theorem 2. The $\chi_1(u)$ has the following properties

$$Bu^b \leq \chi_1(u) \leq Au^a \quad (0 \leq u \leq 1)$$

where we shall use letters A, B , etc. as absolute constants.

If we denote by f^* equi-measurable, non-increasing rearrangement of $|f|$, and by R_1 the sub-set of the space R where $|f| \leq 1$, then

$$\begin{aligned} \int_{R_1} |f|^b d\mu &= \int_t^\infty (f^*)^b dx < B^{-1} \int_t^\infty \chi_1(f^*) dx \\ &= B^{-1} \int_{R_1} \chi_1(|f|) d\mu, \end{aligned}$$

where t denotes the μ -measure of set $\{x \mid |f(x)| > 1\}$.

The $\chi_2(u)$ and $\chi_2^*(u)$ have the following properties. The $\chi_2^*(u)$ is continuous, non-decreasing function for $u > 1$ and

$$\chi_2^*(2u) = 0(\chi_2^*(u))$$

for $u \rightarrow \infty$. Because for $u' > u > 1$, we have

$$\begin{aligned} \chi_2^*(u') - \chi_2^*(u) &= (u')^a \int_1^{u'} \frac{\chi_2(t)}{t^{a+1}} dt - u^a \int_1^u \frac{\chi_2(t)}{t^{a+1}} dt \\ &> u^a \int_u^{u'} \frac{\chi_2(t)}{t^{a+1}} dt > 0 \end{aligned}$$

and since $\chi_2(2u) = 0(\chi_2(u))$ for $u \rightarrow \infty$, we have

$$\begin{aligned} \chi_2^*(2u) &= (2u)^a \int_1^{2u} \frac{\chi_2(t)}{t^{a+1}} dt = (2u)^a \left\{ \int_1^u \frac{\chi_2(t)}{t^{a+1}} dt + \int_u^{2u} \frac{\chi_2(t)}{t^{a+1}} dt \right\} \\ &= A \chi_2^*(u) + A'(2u)^a \int_{u/2}^u \frac{\chi_2(2t)}{t^{a+1}} dt \\ &\leq A \chi_2^*(u) + A'u^a \int_1^u \frac{\chi_2(t)}{t^{a+1}} dt \leq A'' \chi_2^*(u). \end{aligned}$$

By similar arguments read

$$\begin{aligned} \chi_2(u) &\leq A \chi_2^*(u) \\ u^a &\leq A \chi_2^*(u) \end{aligned}$$

and

$$\chi_2(u) \leq Bu^b$$

respectively. We have

$$\int_u^\infty \frac{\chi_2^*(t)}{t^{b+1}} dt = O\left(\frac{\chi_2^*(u)}{u^b}\right)$$

for $u \rightarrow \infty$. Because we have by the definition of χ_2^* ,

$$\begin{aligned} \int_u^\infty \frac{\chi_2^*(t)}{t^{b+1}} dt &= \int_u^\infty \frac{dt}{t^{b+1}} t^a \int_1^t \frac{\chi_2(s)}{s^{a+1}} ds \\ &= \int_1^u \frac{\chi_2(s)}{s^{a+1}} ds \int_u^\infty \frac{dt}{t^{b-a+1}} + \int_u^\infty \frac{\chi_2(s)}{s^{a+1}} ds \int_s^\infty \frac{dt}{t^{b-a+1}} \\ &= \frac{1}{(b-a)u^{b-a}} \int_1^u \frac{\chi_2(s)}{s^{a+1}} ds + \frac{1}{(b-a)} \int_u^\infty \frac{\chi_2(s)}{s^{b+1}} ds \\ &\leq A \frac{\chi_2^*(u)}{u^b} + A' \frac{\chi_2(u)}{u^b} \leq A'' \frac{\chi_2^*(u)}{u^b}. \end{aligned}$$

If we denote by R_2 the sub-set of R where $|f| > 1$, then

$$\begin{aligned} \int_{R_2} |f|^a d\mu &= \int_0^t (f^*)^a dx < A \int_0^t \chi_2^*(f^*) dx \\ &= A \int_{R_2} \chi_2^*(|f|) d\mu, \end{aligned}$$

where t denotes the μ -measure of $\{x \mid |f(x)| > 1\}$. Under those preparations, let $f \in L_\mu^{k*}(R)$ and write

$$f = f' + f''$$

where $f' = f$ whenever $|f| \leq 1$ and $f' = 0$ otherwise; $f'' = f - f'$. Since $f' \in L_\mu^k$ and so $f' \in L_\mu^b$, $f'' \in L_\mu^{k*}$ and so $f'' \in L_\mu^a$. Hence Tf' and Tf'' are defined, by hypothesis, and so $Tf = T(f' + f'')$. Let $n_\nu(y)$ by the distribution function $|Tf|$. We have

$$\begin{aligned} \int_S \chi(|Tf|) d\nu &= - \int_0^\infty \chi(y) dn_\nu(y) \\ &= \int_0^\infty n_\nu(y) d\chi(y) \leq \sum_{j=-\infty}^\infty \eta_j \delta_j \end{aligned}$$

where $\delta_j = \chi(\lambda 2^{j+1}) - \chi(\lambda 2^j)$ and $\eta_j = \nu(E_{\lambda 2^j}) [|Tf|]$, $\lambda = 3\kappa^2$. The passage from the second to the third integral is justified as in A. Zygmund [15, Vol. II, p. 112 (4.8)].

For each fixed $j \geq 0$, we write $f = f_1 + f_2 + f_3$, where f_1 equals f or 0 according as $1 < |f| \leq 2^j$ or else; f_2 does f or 0 according as $|f| > 2^j$ or else; and so f_3 does f or 0 according as $|f| \leq 1$ or else. Since $f_1 \in L_\mu^a \cap L_\mu^b$, $f_2 \in L_\mu^a$ and

$f_3 \in L_\mu^b$ respectively. In view of the inequality

$$\begin{aligned} |Tf| &\leq \kappa(|T(f_1+f_2)| + |Tf_3|) \\ &\leq \kappa^2(|Tf_1| + |Tf_2| + |Tf_3|) \quad (\kappa > 1) \end{aligned}$$

if $|Tf_i| < y$, for all $i = 1, 2, 3$ and any positive real number y , then $|Tf| < \lambda y$ with $\lambda = 3\kappa^2$. Therefore we have

$$\{x \mid |Tf| > \lambda y\} \subset \bigcup_{i=1}^3 \{x \mid |Tf_i| > y\}$$

and if we take $y = 2^j$, we get the following formula,

$$\eta_j \leq C \left\{ 2^{-jb} \int_{R_2} |f_1|^b d\mu + 2^{-ja} \int_{R_2} |f_2|^a d\mu + 2^{-jb} \int_{R_1} |f_3|^b d\mu \right\}$$

and then

$$\begin{aligned} \sum_{j=0}^{\infty} \eta_j \delta_j &\leq C \left\{ \sum_{j=0}^{\infty} 2^{-jb} \delta_j \int_{R_2} |f_1|^b d\mu + \sum_{j=0}^{\infty} 2^{-ja} \delta_j \int_{R_2} |f_2|^a d\mu \right. \\ &\quad \left. + \sum_{j=0}^{\infty} 2^{-jb} \delta_j \int_{R_1} |f_3|^b d\mu \right\} \\ &= I_1 + I_2 + I_3, \text{ say.} \end{aligned}$$

By ε_i ($i=1, 2, \dots$), we denote the μ -measure of the set where $2^{i-1} < |f| \leq 2^i$, then then if we interchange the order of summation and substitute above estimates we are led to

$$\begin{aligned} I_1 &= C \sum_{j=0}^{\infty} 2^{-jb} \delta_j \int_R |f_1|^b d\mu \leq C \sum_{j=1}^{\infty} 2^{-jb} \delta_j \sum_{i=1}^j 2^{ib} \varepsilon_i \\ &= C \sum_{i=1}^{\infty} 2^{ib} \varepsilon_i \sum_{j=i}^{\infty} 2^{-jb} \delta_j \leq C' \sum_{j=1}^{\infty} 2^{ib} \varepsilon_i \int_{\lambda 2^{i+1}}^{\infty} \frac{\chi_2(u)}{u^{b+1}} du \\ &\leq C'' \sum_{i=1}^{\infty} \chi_2(2^i) \varepsilon_i \leq C'' \int_{R_2} \chi_2(|f|) d\mu \\ I_2 &= C \sum_{j=0}^{\infty} 2^{-ja} \delta_j \int_R |f_2|^a d\mu \leq C \sum_{j=0}^{\infty} 2^{-ja} \delta_j \sum_{i=j+1}^{\infty} 2^{ia} \varepsilon_i \\ &= C \sum_{i=1}^{\infty} 2^{ia} \varepsilon_i \sum_{j=0}^{i-1} 2^{-ja} \delta_j \leq C' \sum_{i=1}^{\infty} 2^{ia} \varepsilon_i \int_1^{\lambda 2^i} \frac{\chi_2(u)}{u^{a+1}} du \\ &\leq C'' \sum_{i=0}^{\infty} \chi_2^*(2^i) \varepsilon_i \leq C'' \int_{R_2} \chi_2^*(|f|) d\mu \\ I_3 &= C \sum_{j=0}^{\infty} 2^{-jb} \delta_j \int_R |f_3|^b d\mu = C \int_{R_1} |f|^b d\mu \sum_{j=0}^{\infty} 2^{-jb} \delta_j \\ &\leq C \int_{R_1} |f|^b d\mu \int_1^{\infty} \frac{\chi_2(u)}{u^{b+1}} du \leq C' \int_{R_1} \chi_1(|f|) d\mu \end{aligned}$$

Therefore we have

$$\begin{aligned} \sum_{j=0}^{\infty} \eta_j \delta_j &\leq C \int_{R_2} \chi_2^*(|f|) d\mu + C' \int_{R_2} \chi_2(|f|) d\mu + C'' \int_{R_1} \chi_1(|f|) d\mu \\ &\leq C \int_R \chi^*(|f|) d\mu. \end{aligned}$$

Similarly, for each fixed $j < 0$, we write $f = f_4 + f_5 + f_6$, where f_4 equals f or 0 according as $2^j < |f| \leq 1$ or else; f_5 does f or 0 according as $0 \leq |f| \leq 2^j$ or else; and so f_6 does f or 0 according as $1 < |f|$ or else. Since $f_4 \in L_\mu^a \cap L_\mu^b$, $f_5 \in L_\mu^b$ and $f_6 \in L_\mu^a$ respectively, we have

$$\eta_j \leq D \left\{ 2^{-ja} \int_{R_1} |f_4|^a d\mu + 2^{-jb} \int_{R_1} |f_5|^b d\mu + 2^{-ja} \int_{R_2} |f_6|^a d\mu \right\}$$

We can estimate the summation $\sum_{j=-\infty}^{-1} \eta_j \delta_j$ just the same as $\sum_{j=0}^{\infty} \eta_j \delta_j$ and we have

$$\sum_{j=-\infty}^{-1} \eta_j \delta_j \leq D \int_{R_1} \chi_1(|f|) d\mu + D' \int_{R_2} \chi_2^*(|f|) d\mu$$

and hence we attain the desired inequality

$$\int_S \chi(|Tf|) d\nu \leq K \int_R \chi^*(|f|) d\mu$$

The proof of Theorem 1 is a rather easy repetition of that of Theorem 2 and need not be gone into.

3. Some remarks. (1). If the operation T is linear, then we can present theorems 1 and 2 as more general forms which are useful on a certain case (c.f. E.M. Stein - G. Weiss [13]).

We say that the operation T is of restricted weak type (a, b) , if for every simple function f on R , Tf is ν -measurable function on S and satisfies

$$\nu(E_y [|Tf|]) \leq \left(\frac{M}{y} \|f\|_{a,\mu} \right)^b$$

where M is a constant independent of f . We can state

Corollary 1. *In Theorem 1, if the operation T is linear and of restricted weak type (a, a) and (b, b) where $1 \leq a < b < \infty$ respectively. Then we have for every simple function f on R ,*

$$\int_S \varphi(|Tf|) d\nu \leq K \int_R \varphi(|f|) d\mu$$

and moreover we can extend the operation T to the whole space L_μ^a preserving the

norm of operation.

Proof. We need only to prove the process of extension. Take any f in L_μ^φ . Let us write

$$f_n = \begin{cases} (\text{sign } f) \frac{k-1}{n}, & \text{if } \frac{k-1}{n} \leq |f| < \frac{k}{n} \\ (\text{sign } f)n, & \text{if } |f| > n \end{cases}$$

$k = 1, 2, \dots, n; n = 1, 2, \dots$. Then f_n tends to f monotone increasingly for a.e. x and so $\varphi(|f_n|)$ does to $\varphi(|f|)$. By the Lebesgue convergence theorem we have

$$\lim_{n \rightarrow \infty} \int_R \varphi(|f_n|) d\mu = \int_R \varphi(|f|) d\mu$$

and

$$\lim_{m, n \rightarrow \infty} \int_R \varphi(|f_m - f_n|) d\mu = 0.$$

If we write $\tilde{f}_n = Tf_n$, then by hypothesis we have

$$\int_S \varphi(|\tilde{f}_n|) d\nu \leq K \int_R \varphi(|f_n|) d\mu$$

and since T is of linear

$$\int_S \varphi(|\tilde{f}_m - \tilde{f}_n|) d\nu \leq K \int_R \varphi(|f_m - f_n|) d\mu.$$

The least formula shows that $\{\tilde{f}_n\}$ is a sequence of fundamental in measure and so there exist a limit function \tilde{f} uniquely except a set of ν -measure zero and subsequence (n_k) of (n) such that \tilde{f}_{n_k} converges to \tilde{f} for a.e. x . Applying the Fatou lemma we have the desired result.

The same argument leads to

Corollary 2. *In Theorem 2, if the operation T is linear and of restricted weak type (a, a) and (b, b) where $1 \leq a < b < \infty$, respectively. Then we have for every simple function f on R ,*

$$\int_S \chi(|Tf|) d\nu \leq K \int_R \chi^*(|f|) d\mu$$

*and moreover we can extend the operation T to the whole space L_μ^{**} preserving the norm of operation.*

(2) Next we meet the $\varphi(u)$ which is continuous and not necessarily increasing on the whole interval. If we suppose that φ is ultimately increasing for the value of u near zero and infinity; in the middle interval, say $(\frac{1}{\gamma}, \gamma)$ with

$\gamma > 1$, is of bounded variation, then we can find an increasing function φ^* such that

$$\varphi(u) \leq \varphi^*(u) \leq A_\gamma \varphi(u), \quad \text{for all } u \geq 0.$$

For example a construction of φ^* is as follows:

$$\varphi^*(u) = \begin{cases} \varphi(u), & \text{if } 0 \leq u < \frac{1}{\gamma} \\ \varphi\left(\frac{1}{\gamma}\right) + \int_{1/\gamma}^u |d\varphi|, & \text{if } \frac{1}{\gamma} \leq u < \gamma \\ \varphi\left(\frac{1}{\gamma}\right) + \int_{1/\gamma}^\gamma |d\varphi| + (\varphi(u) - \varphi(\gamma)), & \text{if } \gamma \leq u < \infty \end{cases}$$

The simple calculation shows that the inequality is satisfied

$$A_\gamma = \frac{\varphi\left(\frac{1}{\gamma}\right) + \int_{1/\gamma}^\gamma |d\varphi|}{\min_{1/\gamma \leq u \leq \gamma} \varphi(u)}$$

Corollary 3. *In Theorem 1, if the $\varphi(u)$ is ultimately increasing for the value of u near zero and infinity; in the middle interval, is of bounded variation. The same conclusion is also true.*

The same argument leads to

Corollary 4. *In Theorem 2, if the $\chi(u)$ is ultimately increasing for the value of u near zero and infinity; in the middle interval, is of bounded variation. The same conclusion is also true.*

4. Proof of Theorem 3. Let us suppose that $f \in L^p + L \log^+ L$. Write $f = g + h$:

$$g = \begin{cases} f, & \text{if } |f| \leq 1 \\ 0, & \text{if } |f| > 1 \end{cases} \quad h = f - g$$

We have $g \in L^p$ and $h \in L \log^+ L$ respectively. Since the operation T is of type (p, p) by hypothesis, we have

$$\begin{aligned} \int_{|Th| \leq 1} |Th|^p d\nu &= -n_\nu(1) + p \int_0^1 n_\nu(y) y^{p-1} dy \\ &\leq p \int_0^1 \frac{M_1}{y} \|h\|_{1, \mu} y^{p-1} dy = \frac{pM_1}{p-1} \int_R |h| d\mu, \end{aligned}$$

and therefore

$$(1) \quad \int_{|Th| \leq 1} |Th|^p d\nu \leq 0 \left(\frac{pM_1}{p-1} \right) \int_{|f| > 1} |f| d\mu$$

Next if we follow carefully on the lines of proof of Theorem 2, we have

$$\begin{aligned} \int_{|x_h|>1} Th \, d\nu &= -\int_1^\infty y \, dn_\nu(y) = n_\nu(1) + \int_1^\infty n_\nu(y) dy \\ &\leq n_\nu(1) + 0(1) \sum_{j=0}^{\infty} \eta_j \delta_j \\ &\leq 0(M_1) \int_{|h|>1} |h| \, d\mu + 0\left(\frac{M_p^p}{p-1}\right) \int_{|h|>1} |h| \, d\mu + 0(M_1) \int_{|h|>1} |h| \log^+ |h| \, d\mu \end{aligned}$$

Therefore

$$(2) \quad \int_{|x_h|>1} |Th| \, d\nu \leq 0\left(\frac{M_p^p}{p-1} + M_1\right) \int_{|f|>1} |f| (1 + \log^+ |f|) \, d\mu$$

We have immediately

$$(3) \quad \int_S |Tg|^p \, d\nu \leq M_p^p \int_R |g|^p \, d\mu = M_p^p \int_{|f|\leq 1} |f|^p \, d\mu$$

and also

$$\begin{aligned} \int_{|x_g|>1} |Tg| \, d\nu &= n_\nu(1) + \int_1^\infty n_\nu(y) dy \\ &\leq M_p^p \|g\|_{p,\mu} + \int_1^\infty \left(\frac{M_p \|g\|_{p,\mu}}{y}\right)^p dy \end{aligned}$$

and therefore

$$(4) \quad \int_{|x_g|>1} |Tg| \, d\nu \leq 0\left(\frac{M_p^p}{p-1}\right) \int_{|f|\leq 1} |f|^p \, d\mu$$

We need the following lemma

Lemma. *From an inequality*

$$A \leq \kappa(B+C), \quad A, B, C \geq 0, \quad \kappa \geq 1$$

we have (i) if $0 \leq A \leq 1$

$$A \leq \begin{cases} \kappa(B+C), & \text{if } 0 \leq C \leq 1 \\ \kappa(B+C^{1/p}), & \text{if } C > 1 \end{cases}$$

(ii) if $A > 1$

$$A \leq \begin{cases} (2\kappa)^p (B^p + C^p), & \text{if } 0 \leq C \leq 1 \\ (2\kappa)^p (B^p + C), & \text{if } C > 1. \end{cases}$$

Proof. (i) Suppose that $0 \leq A \leq 1$. If $0 \leq C \leq 1$, it is trivial; if $C < 1$

$$A \leq 1 < C^{1/p} \leq \kappa(B+C^{1/p}).$$

(ii) Suppose that $A > 1$. From an inequality $A \leq \kappa(B+C)$, one of the relation

$B > \frac{A}{2\kappa}$ and $C > \frac{A}{2\kappa}$ always holds. If $B > \frac{A}{2\kappa}$,

$$\begin{aligned} A &\leq 2\kappa B \leq (2\kappa B)^p \\ &\leq \begin{cases} (2\kappa)^p(B^p + C^p), & \text{if } 0 \leq C \leq 1 \\ (2\kappa)^p(B^p + C), & \text{if } C > 1. \end{cases} \end{aligned}$$

If $C > \frac{A}{2\kappa}$,

$$\begin{aligned} A &\leq 2\kappa C \\ &\leq \begin{cases} (2\kappa C)^p \leq (2\kappa)^p(B^p + C^p), & \text{if } 0 \leq C \leq 1 \\ (2\kappa)(B^p + C), & \text{if } C > 1. \end{cases} \end{aligned}$$

Let us estimate Tf on the set $S_1 = \{x \mid |Tf| \leq 1\}$. Applying Lemma (i) such as $A = |Tf|$, $B = |Tg|$ and $C = |Th|$ and the Minkowsky inequality, we have

$$\begin{aligned} \left(\int_{S_1} |Tf|^p d\nu \right)^{1/p} &\leq \kappa \left(\int_{S_1 \cap \{|Th| > 1\}} (|Tg| + |Th|^{1/p})^p d\nu \right)^{1/p} \\ &\quad + \kappa \left(\int_{S_1 \cap \{|Th| \leq 1\}} (|Tg| + |Th|)^p d\nu \right)^{1/p} \\ &\leq 2\kappa \left(\int_{S_1} |Tg|^p d\nu \right)^{1/p} + \kappa \left(\int_{|Th| > 1} |Th| d\nu \right)^{1/p} + \kappa \left(\int_{|Th| \leq 1} |Th|^p d\nu \right)^{1/p} \end{aligned}$$

Substituting (1) (2) and (3),

$$\begin{aligned} (5) \quad \int_{|f| \leq 1} |Tf|^p d\nu &\leq 0(M_p^p) \int_{|f| \leq 1} |f|^p d\mu \\ &\quad + 0 \left(\frac{M_p^p + M_1}{p-1} \right) \int_{|f| > 1} |f| (1 + \log^+ |f|) d\mu \end{aligned}$$

Let us estimate Tf on the set $S_2 = \{x \mid |Tf| > 1\}$, we have

$$\begin{aligned} \int_{S_2} |Tf| d\nu &\leq (2\kappa)^p \int_{S_2 \cap \{|Th| > 1\}} (|Tg|^p + |Th|) d\nu + (2\kappa)^p \int_{S_2 \cap \{|Th| \leq 1\}} (|Tg| + |Th|^p) d\nu \\ &\leq 2(2\kappa)^p \int_{S_2} |Tg|^p d\nu + (2\kappa)^p \int_{|Th| > 1} |Th| d\nu + (2\kappa)^p \int_{|Th| \leq 1} |Th|^p d\nu \end{aligned}$$

Substituting (1) (2) and (3)

$$(6) \quad \int_{|Tf| > 1} |Tf| d\nu \leq 0(M_p^p) \int_{|f| \leq 1} |f|^p d\mu + 0 \left(\frac{M_p^p + M_1}{p-1} \right) \int_{|f| > 1} |f| (\log^+ |f|) d\mu$$

The formulas (5) and (6) complete the proof of Theorem 3.

5. Applications. Let $x=(x_1, x_2, \dots, x_n)$, $y=(y_1, y_2, \dots, y_n)$, by points of the real n -dimensional space E_n . A.P. Calderon-A. Zygmund [2] studied the singular integral operator:

$$\begin{aligned} \tilde{f}(x) &= (f * K)(x) = \text{P.V.} \int_{E_n} f(x-y) K(y) dy \\ &= \lim_{\varepsilon \rightarrow 0} \tilde{f}_\varepsilon(x) = \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} f(x-y) K(y) dy, \end{aligned}$$

where kernel $K(x)$ has the form

$$K(x) = |x|^{-n} \Omega(x'), \quad x' = \frac{x}{|x|}.$$

Let us denote by Σ the unit sphere on which the $\Omega(x')$ is denfied. Let us denote by $\omega(\delta)$ the modulus of continuity of $\Omega(x')$,

$$|\Omega(x') - \Omega(y')| \leq \omega(x' - y').$$

Let us suppose that

(a) $\int_{\Sigma} \Omega(x') dx' = 0$

(b) $\Omega(x') \in L^1(\Sigma)$ and its modulus of continuity $\omega(\delta)$ satisfy the Dini condition,

$$\int_0^1 \frac{\omega(\delta)}{\delta} d\delta < \infty.$$

Then they proved that the operations $Tf = \tilde{f}$ and $T_\varepsilon f = \tilde{f}_\varepsilon$ are both linear and of type (p, p) for every $p > 1$ and of weak type $(1, 1)$ respectively. Applying our theorem 3, we have for example

$$\begin{aligned} & \int_{|\tilde{f}| \leq 1} |\tilde{f}|^p dx + \int_{|\tilde{f}| > 1} |\tilde{f}| dx \\ & \leq K \left\{ \int_{|f| \leq 1} |f|^p dx + \int_{|f| > 1} |f| (1 + \log^+ |f|) dx \right\}, \end{aligned}$$

where K is a constant depending on p and not on f .

A.P. Calderon-M. Weiss-A. Zygmund [4] proved that the condition (b) of $\Omega(x')$ can be replaced by the (rotational) integrated modulus of continuity $\omega_1(\delta)$ instead of $\omega(\delta)$. That is, the $\omega_1(\delta)$ is defined as follows

$$\omega_1(\delta) = \sup_{|\rho| \leq \delta} \int_{\Sigma} |\Omega(\rho x') - \Omega(x')| dx'$$

where ρ is any rotation of Σ and $|\rho|$ its magnitude.

Furthermore the maximal operation $\bar{T}f = \bar{f}$

$$\bar{T}f = \bar{f} = \sup |\tilde{f}_\varepsilon|$$

satisfy the same assumptions as the operations $Tf = \tilde{f}$ and $T_\varepsilon f = \tilde{f}_\varepsilon$ and so necessarily the same conclusions. See, L. Hörmander [8], A.P. Calderon-A. Zygmund [3] and A.P. Calderon-M. Weiss-A. Zygmund [4].

As a special case, the one-dimensional Hilbert transform

$$Hf(x) = P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy$$

and the Riesz transform

$$R_j f(x) = P.V. \frac{1}{C_n} \int_{E_n} f(y) \frac{x_j - y_j}{|x - y|^{n+1}} dy \quad (j = 1, 2, \dots, n)$$

where

$$C_n = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$$

and also the unified operator of Hilbert transform and ergodic operator belong to our category. See J. Horváth [9], M. Cotlar [5] and E.M. Stein [14].

On the other hand let us consider

$$\tilde{f}_\alpha(x) = P.V. \int_{E_n} \frac{f(x-y)}{|y|^{n-\alpha}} dy, \quad 0 < \alpha < n;$$

then the following is known according to G.H. Hardy- J.E. Littlewood [6] and A. Zygmund [15] (c.f. also, E.M. Stein [14]):

(i) it is of type (r, s)

$$\|\tilde{f}_\alpha\|_s \leq M_{rs} \|f\|_r,$$

where $1 < r < s < \infty, \frac{1}{r} - \frac{1}{s} = \frac{\alpha}{n}$,

(ii) it is of weak type $\left(1, \frac{1}{n-\alpha}\right)$.

Thus the potential operator is beyond the scope of Theorem 3. We shall give a conjecture.

Let us write $\alpha_i = \frac{1}{a_i}, \beta_i = \frac{1}{b_i} (i=1, 2)$. Let (α_1, β_1) and (α_2, β_2) be any two points of the triangle

$$\Delta: 0 \leq \beta \leq \alpha \leq 1$$

such that $\beta_1 \neq \beta_2$. If $\alpha_1 > \alpha_2$, let us suppose that a quasilinear operation $\tilde{f} = Tf$ is of weak type $\left(\frac{1}{\alpha_1}, \frac{1}{\beta_1}\right)$ and type $\left(\frac{1}{\alpha_2}, \frac{1}{\beta_2}\right)$, then we have

$$\int_{|Tf| \leq 1} |Tf|^{b_2} d\nu + \int_{|Tf| > 1} |Tf|^{b_1} d\nu \\ \leq K \left\{ \int_{|f| \leq 1} |f|^{a_2} d\mu + \int_{|f| > 1} |f|^{a_1} \{1 + (\log^+ |f|)^{k_1}\} d\mu \right\}$$

where $k_1 = \frac{b_1}{a_1}$, K is a constant independent of f .

We shall have an analogous result in the case $\alpha_1 < \alpha_2$.

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