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A COUNTEREXAMPLE TO TWO CONJECTURES ABOUT HIGH ORDER DERIVATIONS AND REGULARITY

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Let R be the local ring of a point on a plane irreducible algebraic curve defined over a field of characteristic $p\neq 0$. We give an example in which all the R-modules of high order derivations are free and the R-algebra of derivations is generated by p^i-th order derivations even though R is not regular.

Let P be a point on a plane irreducible algebraic curve defined over an algebraically closed field k. Let R be the local ring at P.

For each $n=1, 2, \cdots$, we let $\mathrm{Der}_k^n(R, M)$ denote the R-module of all n-th order derivations of R to an R-module M which vanish on k. Thus, $\phi\in\mathrm{Der}_k^n(R, M)$ if and only if $\phi\in\mathrm{Hom}_k(R, M)$ and for all $r_0, \cdots, r_n\in R$ we have

$$(1)\qquad \phi(r_0\cdots r_n)=\sum_{s=1}^{n}(-1)^s\sum_{i_1<\cdots<i_s} r_{i_1}\cdots r_{i_s}\,\phi(r_0\cdots \check{r}_{i_1}\cdots \check{r}_{i_s}\cdots r_n)\,.$$

When $M=R$, we write $\mathrm{Der}_k^n(R)$ instead of $\mathrm{Der}_k^n(R, R)$. Let $\mathrm{Der}(R)=\bigcup_n \mathrm{Der}_n^k(R)$.

When k has characteristic zero, the following two results are known to hold for plane curves:

(I) Lipman's conjecture: R is a regular local ring if and only if $\mathrm{Der}_k^1(R)$ is a free R-module [Theorem 1; 3].

(II) Nakai's conjecture: R is a regular local ring if and only if $\mathrm{Der}(R)$ is generated as an R-algebra by first order derivations [4].

When k has characteristic $p\neq 0$, both of these results are false since a p-th order derivation cannot be represented by first order derivations. Hence we have the following conjectures which are generalizations of (I) and (II):

(I′) R is a regular local ring if and only if $\mathrm{Der}_k^n(R)$ is a free R-module for all n.

(II′) R is a regular local ring if and only if $\mathrm{Der}(R)$ is generated as an R-algebra by p^i-th order derivations, $i=0, 1, \cdots$.

(II′) appears as a conjecture in [2]. It is known that if R is regular, then $\mathrm{Der}_k^n(R)$

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is a free R-module for all n and $\mathrm{Der}(R)$ is generated by p^i-th order derivations [Theorem 16.11.2; 1 and Theorem 4.3.; 5]

We shall give an example which shows that the converses of both these conjectures are false. The following lemma will be used repeatedly in the example.

Lemma. *If* $\lambda\in\mathrm{Der}_k^{p^n}(R)$ *and* $r, s\in R$, *then*

(a) $\lambda(r^{p^n}s)=r^{p^n}\lambda(s)+s\lambda(r^{p^n})$

(b) $\lambda(r^{p^{n+i}}s)=r^{p^{n+i}}\lambda(s)$ *for* $i\geq 1$.

Proof. The proof of (a) follows immediately from the definition of a p^n-th order derivation; this is equation (1).

For (b), the previous part gives

$$\lambda(r^{p^{n+i}}s)=\lambda((r^{p^i})^{p^n}s)=r^{p^{n+i}}\lambda(s)+s\lambda(r^{p^{n+i}})\,.$$

Since $\lambda(r^{p^{n+i}})=0$ [I, Prop. 10; 6],λ $(r^{p^{n+i}}s)=r^{p^{n+i}}\lambda(s)$. *q.e.d.*

Theorem. *Let R be the local ring at* $(0, 0)$ *of* $\Gamma: f(X, Y)=X^2-Y^3$ *over an algebraically closed field k of characteristic* 2. *Then* $\mathrm{Der}_k^n(R)$ *is a free R-module for all n and* $\mathrm{Der}(R)$ *as an R-algebra is generated by 2^i-th order derivations.*

Proof. Let $A=k[x, y]=k[X, Y]/(X^2-Y^3)$. Then $R=(A)_{(x,y)}$. For $m=0, 1, \cdots$, we define $\gamma_{2^m}\in\mathrm{Der}_k^{2^m}(k[y])$ as follows:

$$\gamma_{2^m}(y^i)=\begin{cases}0 & i<2^m\\ 1 & i=2^m\end{cases}. \tag{2}$$

Thus, γ_{2^m} is a 2^m-th order derivation of $k[y]$ to $k[y]$. For any i, we define $\gamma_i=\gamma_{2^I}{}^{\alpha_I}\cdot\cdots\cdot\gamma_1{}^{\alpha_0}$ where the α_j's are the coefficients in the 2-adic expansion of i; that is, $i=\sum_{j=0}^{I}\alpha_j 2^j$. It is easily shown that $\gamma_i(y^j)=\begin{cases}0 & j<i\\ 1 & j=i\end{cases}$. Also, $\gamma_i\in\mathrm{Der}_k^i(k[y])$ [I, Theorem 6.1; 6].

We now define a 2^n-th order derivation on $k[x]$ to $k[x, y]$, $n=0, 1, \cdots$. For $n=0$, we define $\lambda_1\in\mathrm{Der}_k^1(k[x], k[x, y])$ by $\lambda_1(x)=1$. For $n>0$, we define $\lambda_{2^n}\in\mathrm{Der}_k^{2^n}(k[x], k[x, y])$ as follows:

$$\lambda_{2^n}(x^{2j})=\gamma_{2^{n-1}}(y^{3j}) \qquad j=0, \cdots, 2^{n-1} \tag{3}$$

$$\lambda_{2^n}(x^{2j+1})=x\lambda_{2^n}(x^{2j}) \qquad j=1, \cdots, 2^{n-1}-1\,. \tag{4}$$

For each n, λ_{2^n} extends uniquely to a 2^n-th order derivation of $k(x)$ to $k(x, y)$ [I, Theorem 15; 6]. We call this extension λ_{2^n}. Since $k(x, y)$ is separably algebraic over $k(x)$, λ_{2^n} extends in a unique way to a 2^n-th order derivation of $k(x, y)$ to $k(x, y)$ [Theorem 17; 7]. This extension is also called λ_{2^n}.

From the definition of λ_{2^n}, we have

$$\lambda_{2^n}(x^{2^n}) = y^{2^n}. \tag{5}$$

For by the lemma

$$\begin{aligned}\lambda_{2^n}(x^{2^n}) &= \lambda_{2^n}((x^2)^{2^{n-1}}) = \gamma_{2^{n-1}}((y^3)^{2^{n-1}}) = \gamma_{2^{n-1}}(y^{2^n+2^{n-1}}) \\ &= y^{2^n}\gamma_{2^{n-1}}(y^{2^{n-1}}) = y^{2^n}\end{aligned}$$

Using (5) and the lemma, it is easily shown that (3) holds for all values of j. This is done by writing $2j=2^{n+k}+2i$ with $i=1, \cdots, 2^{n+k-1}$ and using induction on k. Likewise, it can be shown that (4) holds for all values of j. Here, we write $2j+1=2^{n+k}+2i-1$ where $i=1, \cdots, 2^{n+k-1}-1$ and induct on $k=0, 1, \cdots$.

Using (3) and (4), we now show that $\lambda_{2^n} \in \mathrm{Der}_k^{2^n}(A)$. In order to show this, we compute $\lambda_{2^n}(y^i)$ and $\lambda_{2^n}(xy^i)$ and show that

$$\lambda_{2^n}(y^i) = \gamma_{2^{n-1}}(y^i) \tag{6}$$

$$\lambda_{2^n}(xy^i) = x\gamma_{2^{n-1}}(y^i) \tag{7}$$

where $i=1, 2, \cdots$ and $n=1, 2, \cdots$.

To show (6), there are several cases depending on whether $i\equiv 0, 1, 2 \pmod 3$.

Case 1: $i\equiv 0 \pmod 3$.

Let $i=3l$. Then (3) implies that

$$\lambda_{2^n}(y^i) = \lambda_{2^n}(y^{3l}) = \lambda_{2^n}(x^{2l}) = \gamma_{2^{n-1}}(y^{3l}) = \gamma_{2^{n-1}}(y^i)$$

Case 2: $i\equiv 1 \pmod 3$ and $2^n\equiv 1 \pmod 3$.

Case 3: $i\equiv 2 \pmod 3$ and $2^n\equiv 2 \pmod 3$.

These cases are considered together for in both $i+2^{n+1}\equiv 0 \pmod 3$. Let $i+2^{n+1}=3l$. Then $\lambda_{2^n}(y^i y^{2^{n+1}})=y^{2^{n+1}}\lambda_{2^n}(y^i)$ by the lemma. On the other hand,

$$\lambda_{2^n}(y^{3l}) = \lambda_{2^n}(x^{2l}) = \gamma_{2^{n-1}}(y^{3l}) = \gamma_{2^{n-1}}(y^i y^{2^{n+1}}) = y^{2^{n+1}}\gamma_{2^{n-1}}(y^i).$$

So, $y^{2^{n+1}}\lambda_{2^n}(y^i)=y^{2^{n+1}}\gamma_{2^{n-1}}(y^i)$ or $\lambda_{2^n}(y^i)=\gamma_{2^{n-1}}(y^i)$.

Case 4: $i\equiv 2 \pmod 3$ and $2^n\equiv 1 \pmod 3$.

Case 5: $i\equiv 1 \pmod 3$ and $2^n\equiv 2 \pmod 3$.

In both of these cases $i+2^{n+2}\equiv 0 \pmod 3$. Let $i+2^{n+2}=3l$. The computations are similar to those made in cases 2 and 3 and hence are omitted.

Equation (7) is proved using (5). The cases are the same as above. Since the calculations necessary to prove (7) are routine and similar to those given above, they are omitted.

Since every monomial in A can be written as $x^j y^i$ with j being 0 or 1 and $0\le i$, (6) and (7) imply that $\lambda_{2^n}(A)\subseteq A$ for $n=1, 2, \cdots$. To show $\lambda_1(A)\subseteq A$, we compute $\lambda_1(y)$. Since

$$0=\lambda_1(x^2)=\lambda_1(y^3)=3y^2\lambda_1(y)\,,$$

$\lambda_1(y)=0$. Thus, $\lambda_1\in\mathrm{Der}_k^1(A)$. Hence, $\lambda_{2^n}(A)\subseteq A$ for $n=0, 1, \cdots$.

By taking composites, we define an m-th order derivation for $m=1, 2, \cdots$. We write m in its 2-adic expansion as $m=\sum_{i=0}^{M}\alpha_i 2^i$ where α_i equals 0 or 1. We define $\lambda_m=\lambda_{2^M}^{\alpha_M}\cdot\cdots\cdot\lambda_1^{\alpha_0}$. Then $\lambda_m\in\mathrm{Der}_k^m(A)$. Hence $\lambda_m\in\mathrm{Der}_k^m(R)$ [I, Theorem 15; 6].

We now make some observations about λ_m. We first consider $\lambda_{2l}=\lambda_{2^L}^{\alpha_L}\cdot\cdots\cdot\lambda_2^{\alpha_1}$ where $2l=\sum_{i=1}^{L}\alpha_i 2^i$. Equation (6) shows that λ_{2^n} when restricted to $k[y]$ equals $\gamma_{2^{n-1}}$. Thus when λ_{2l} is restricted to $k[y]$, we have

$$\lambda_{2l}\,|\,k[y]=\gamma_{2^{L-1}}^{\alpha_L}\cdot\cdots\cdot\gamma_1^{\alpha_1}=\gamma_l\,.$$

Using the fact that $\lambda_{2^n}(x)=0$ for $n\geq 1$, equation (2), and equation (7), we see that the following hold:

$$\lambda_{2l}(x)=0 \tag{8a}$$

$$\lambda_{2l}(y^i)=\begin{cases}0 & i<l\\ 1 & i=l\end{cases} \tag{8b}$$

$$\lambda_{2l}(xy^i)=x\lambda_{2l}(y^i)\,. \tag{8c}$$

We now consider $\lambda_{2l+1}=\lambda_{2l}\circ\lambda_1$. Since $\lambda_1(x)=1$, $\lambda_1(y^i)=0$, and $\lambda_1(xy^i)=y^i$, we have the following:

$$\lambda_{2l+1}(x)=0 \tag{9a}$$

$$\lambda_{2l+1}(y^i)=0 \tag{9b}$$

$$\lambda_{2l+1}(xy^i)=\lambda_{2l}(y^i)=\begin{cases}0 & i<l\\ 1 & i=l\,.\end{cases} \tag{9c}$$

Using (8) and (9), we shall show that $\lambda_1, \cdots, \lambda_n$ are free over K, the quotient field of R. Suppose there exist elements $a_1, \cdots, a_n\in K$ such that

$$a_1\lambda_1+\cdots+a_n\lambda_n=0\,. \tag{10}$$

Evaluating (10) at x gives $a_1=0$. Suppose $a_1=\cdots=a_{2k-1}=0$ for $2k-1<n$. If we evaluate (10) at y^k, we get $a_{2k}=0$ since $\lambda_i(y^k)=0$ for $i>2k$. Now, evaluating (10) at xy^k gives $a_{2k+1}=0$. Hence, by induction, $a_i=0$ for $i=1, \cdots, n$. Therefore $\lambda_1, \cdots, \lambda_n$ are free over K. Since $\mathrm{Der}_k^n(K)$ has rank n over K, $\lambda_1, \cdots, \lambda_n$ constitute a free basis of $\mathrm{Der}_k^n(K)$.

We now wish to show that $\lambda_1, \cdots, \lambda_n$ are free generators of $\mathrm{Der}_k^n(R)$ over R. Since $\mathrm{Der}_k^n(R)\otimes_R K\simeq\mathrm{Der}_k^n(K)$, we know that if $\mathrm{Der}_k^n(R)$ is a free R-module, then it will have rank n. Thus we need only show that $\lambda_1, \cdots, \lambda_n$ generate

$\mathrm{Der}_k^n(R)$. Let $\lambda \in \mathrm{Der}_k^n(R)$. Passing to $\mathrm{Der}_k^n(K)$, we have that there exist elements $a_1, \cdots, a_n \in K$ such that

$$\lambda = \sum_{i=1}^{n} a_i \lambda_i \,. \tag{11}$$

We must show that $a_i \in R$. Evaluating λ at x gives $\lambda(x) = a_1 \in R$ since$\lambda(x) \in R$ and $\lambda_i(x) = 0$ for $i \geq 2$. Suppose $a_1, \cdots, a_{2k-1} \in R$ for $2k-1 < n$. We consider $\lambda(y^k)$. Now $\lambda(y^k) = \sum_{i=1}^{n} a_i \lambda_i(y^k) = \sum_{i=1}^{2k-1} a_i \lambda_i(y^k) + a_{2k}$. Thus $a_{2k} = \lambda(y^k) - \sum_{i=1}^{2k-1} a_i \lambda_i(y^k) \in R$. Likewise, evaluating λ at xy^k gives $a_{2k+1} \in R$. Hence by induction $a_i \in R$ for $i = 1, \cdots, n$. Thus $\mathrm{Der}_k^n(R)$ is a free R-module for all n.

By construction, each generator λ_n is a composite of 2^i-th order derivations. Hence $\mathrm{Der}(R)$ is generated as an R-algebra by 2^i-th order derivations. *q.e.d.*

Since Γ has a singular point at the origin, R is not regular. Thus this is a counterexample to the generalization of Lipman's conjecture, (I′), and to the conjecture by Nakai, (II′).

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