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Author(s)	Yamaki, Hiroyoshi
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Osaka University

ON THE JANKO'S SIMPLE GROUP OF ORDER 175560

HIROYOSHI YAMAKI

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1. Introduction

Let $\mathfrak{S}(11)$ be the Janko's simple group of order 175560 presented in [1] and \mathfrak{A}_m be the alternating group of degree m . In his papers [1], [2] Janko characterized the non-solvable group having the centralizer of an involution in the center of a Sylow 2-subgroup isomorphic to the splitting central extension of a group of order 2 by \mathfrak{A}_4 or \mathfrak{A}_5 . His result is that such a non-solvable group containing no normal subgroup of index 2 must be isomorphic to $P\Gamma L(2,8)$ or $\mathfrak{S}(11)$. The purpose of this note is to sharpen his results [1], [2]. Namely we want to prove the following theorem.

Theorem. *Let \mathfrak{G} be a finite non-solvable group with the following two properties:*

- a) \mathfrak{G} has no normal subgroup of index 2,
- b) \mathfrak{G} contains an involution J in the center of a Sylow 2-subgroup of \mathfrak{G} such that the centralizer $C_{\mathfrak{G}}(J) = \langle J \rangle \times \mathfrak{X}_m$, where \mathfrak{X}_m is isomorphic to \mathfrak{A}_m .

Then one of the following holds:

- 1) $m=4$ and \mathfrak{G} is isomorphic to $P\Gamma L(2, 8)$,
- 2) $m=5$ and \mathfrak{G} is isomorphic to $\mathfrak{S}(11)$.

REMARK. Our proof depends on Janko's theorems [1], [2] and by his results it is sufficient to prove that $m=4$ or 5.

2. Proof of the Theorem

Put $m=4n+r$, where $0 \leq r \leq 3$. Assume that n is greater than 1. Then the group \mathfrak{A}_m contains involutions $\tilde{X}_i, \tilde{X}'_i$ ($1 \leq i \leq n$) and \tilde{Y}_j ($1 \leq j \leq n-1$) with the cycle decompositions

$$\begin{aligned}\tilde{X}_i &= (4i-3, 4i-2) (4i-1, 4i) \\ \tilde{X}'_i &= (4i-3, 4i-1) (4i-2, 4i) \\ \tilde{Y}_j &= (4j-3, 4j-2) (4j+1, 4j+2).\end{aligned}$$

In the isomorphism from \mathfrak{A}_m to \mathfrak{X}_m , let the images of the elements $\tilde{X}_i, \tilde{X}'_i$ and

\tilde{Y}_j be X_i , X'_i and Y_j , respectively. Put $\mathfrak{X} = \langle X_i, X'_j \mid 1 \leq i, j \leq n \rangle$ and $\mathfrak{Y} = \langle Y_j \mid 1 \leq j \leq n-1 \rangle$. Then \mathfrak{X} and \mathfrak{Y} are 2-groups and \mathfrak{Y} normalizes \mathfrak{X} . Hence $\mathfrak{X}\mathfrak{Y}$ is a 2-group. By the definition we have $Y_j^{-1}X'_jY_j = X_jX'_j$ and $Y_j^{-1}X'_{j+1}Y_j = X_{j+1}X'_{j+1}$, and then $\langle X_i \mid 1 \leq i \leq n \rangle$ is the commutator subgroup $(\mathfrak{X}\mathfrak{Y})'$ of $\mathfrak{X}\mathfrak{Y}$. Put $C_i = X_1X_2 \cdots X_i$ for $1 \leq i \leq n$. Then we may assume that $\{C_i \mid 1 \leq i \leq n\}$ is the set of the representatives of the conjugacy classes of involutions in \mathfrak{X}_m . Let \mathfrak{D} be a Sylow 2-subgroup of \mathfrak{G} contained in $C_{\mathfrak{G}}(J)$ and containing $\langle J \rangle \times \mathfrak{X}\mathfrak{Y}$. Hence the group \mathfrak{D}' contains C_n and the center $Z(\mathfrak{D})$ of \mathfrak{D} contains J and C_n . These facts are also true if $n=1$ and $r=2$ or 3.

Assume by way of contradiction that n is greater than 1, or $n=1$ and $r=2$ or 3. For $1 \leq i \leq n-1$, C_i is the square of an element of order 4 in \mathfrak{X}_m . Since \mathfrak{G} has no normal subgroup of index 2, it follows from a transfer lemma of Thompson [3] that J must fuse with C_n in \mathfrak{G} . Note that J is not a square of an element of order 4. Therefore Burnside's argument implies that J must fuse with C_n in the normalizer $N_{\mathfrak{G}}(\mathfrak{D})$ of \mathfrak{D} . This is impossible because \mathfrak{D}' contains C_n but does not J . Thus we get a contradiction and hence $n=1$ and $r=0$ or 1, that is, $m=4$ or 5. Applying the results of Janko [1], [2], \mathfrak{G} is isomorphic to $PFL(2,8)$ or $\mathfrak{F}(11)$, respectively.

The proof of our theorem is complete.

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References

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