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<th>Title</th>
<th>On the Janko's simple group of order 175560</th>
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1. Introduction

Let \( (11) \) be the Janko's simple group of order 175560 presented in [1] and \( \mathcal{A}_m \) be the alternating group of degree \( m \). In his papers [1], [2] Janko characterized the non-solvable group having the centralizer of an involution in the center of a Sylow 2-subgroup isomorphic to the splitting central extension of a group of order 2 by \( \mathcal{A}_4 \) or \( \mathcal{A}_5 \). His result is that such a non-solvable group containing no normal subgroup of index 2 must be isomorphic to \( PΓL(2,8) \) or \( 3(11) \). The purpose of this note is to sharpen his results [1], [2]. Namely we want to prove the following theorem.

**Theorem.** Let \( G \) be a finite non-solvable group with the following two properties:

a) \( G \) has no normal subgroup of index 2,

b) \( G \) contains an involution \( J \) in the center of a Sylow 2-subgroup of \( G \) such that the centralizer \( CG(J) = \langle J \rangle \times \mathcal{X}_m \), where \( \mathcal{X}_m \) is isomorphic to \( \mathcal{A}_m \).

Then one of the following holds:

1) \( m=4 \) and \( G \) is isomorphic to \( PΓL(2,8) \),

2) \( m=5 \) and \( G \) is isomorphic to \( 3(11) \).

**Remark.** Our proof depends on Janko's theorems [1], [2] and by his results it is sufficient to prove that \( m=4 \) or 5.

2. Proof of the Theorem

Put \( m=4n+r \), where \( 0\leq r \leq 3 \). Assume that \( n \) is greater than 1. Then the group \( \mathcal{A}_m \) contains involutions \( \check{X}_i, \check{X}_i' \) (\( 1 \leq i \leq n \)) and \( \check{Y}_j \) (\( 1 \leq j \leq n-1 \)) with the cycle decompositions

\[
\begin{align*}
\check{X}_i &= (4i-3, 4i-2)(4i-1, 4i) \\
\check{X}_i' &= (4i-3, 4i-1)(4i-2, 4i) \\
\check{Y}_j &= (4j-3, 4j-2)(4j+1, 4j+2).
\end{align*}
\]

In the isomorphism from \( \mathcal{A}_m \) to \( \mathcal{X}_m \), let the images of the elements \( \check{X}_i, \check{X}_i' \) and
Let \( Y_j \) be \( X_i, X'_i \) and \( Y_j \), respectively. Put \( \mathfrak{x} = \langle X_i, X'_i | 1 \leq i, j \leq n \rangle \) and \( \mathfrak{y} = \langle Y_j | 1 \leq i \leq n-1 \rangle \). Then \( \mathfrak{x} \) and \( \mathfrak{y} \) are 2-groups and \( \mathfrak{y} \) normalizes \( \mathfrak{x} \). Hence \( \mathfrak{x} \mathfrak{y} \) is a 2-group. By the definition we have \( Y_j^{-1}X'_jY_j = X_jX'_j \) and \( Y_j^{-1}X'_j, Y_j = X_{j+1}X'_{j+1} \) and then \( \langle X_i | 1 \leq i \leq n \rangle \) is the commutator subgroup \((\mathfrak{x}\mathfrak{y})'\) of \( \mathfrak{x}\mathfrak{y} \).

Put \( C_i = X_iX_2 \cdots X_i \) for \( 1 \leq i \leq n \). Then we may assume that \( \{ C_i | 1 \leq i \leq n \} \) is the set of the representatives of the conjugacy classes of involutions in \( X_m \). Let \( \mathfrak{D} \) be a Sylow 2-subgroup of \( \mathfrak{S} \) contained in \( C_0(J) \) and containing \( \langle J \rangle \times \mathfrak{x} \mathfrak{y} \). Hence the group \( \mathfrak{D}' \) contains \( C_m \) and the center \( Z(\mathfrak{D}) \) of \( \mathfrak{D} \) contains \( J \) and \( C_m \). These facts are also true if \( n = 1 \) and \( r = 2 \) or 3.

Assume by way of contradiction that \( n \) is greater than 1, or \( n = 1 \) and \( r = 2 \) or 3. For \( 1 \leq i \leq n-1 \), \( C_i \) is the square of an element of order 4 in \( X_m \). Since \( \mathfrak{S} \) has no normal subgroup of index 2, it follows from a transfer lemma of Thompson [3] that \( J \) must fuse with \( C_m \) in \( \mathfrak{S} \). Note that \( J \) is not a square of an element of order 4. Therefore Burnside’s argument implies that \( J \) must fuse with \( C_m \) in the normalizer \( N_{\mathfrak{S}}(\mathfrak{D}) \) of \( \mathfrak{D} \). This is impossible because \( \mathfrak{D}' \) contains \( C_m \) but does not \( J \). Thus we get a contradiction and hence \( n = 1 \) and \( r = 0 \) or 1, that is, \( m = 4 \) or 5. Applying the results of Janko [1], [2], \( \mathfrak{S} \) is isomorphic to \( P\Gamma L(2, 8) \) or \( \mathfrak{S}(11) \), respectively.

The proof of our theorem is complete.

**References**

