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Author(s)	Kobayashi, Ryoichi
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KÄHLER-EINSTEIN METRIC ON AN OPEN ALGEBRAIC MANIFOLD

RYOICHI KOBAYASHI

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0. Introduction

In [10], S.-T. Yau proved that if M is a compact complex manifold with negative first Chern class, then there is a unique Kähler-Einstein metric with negative Ricci curvature up to a constant multiple. The condition "with negative first Chern class" is, by definition, to assume that there is a negative definite real closed (1,1)-form in the de Rham cohomology class of the first Chern class $c_1(M)$. By the fact that for a holomorphic line bundle E on a compact complex manifold M, any real closed (1,1)-form on M belonging to the first Chern class $c_1(E)$ is the curvature form of a Hermitian metric for E multiplied by $1/2\pi$ (See [6], pp. 148–150.), it is equivalent to assuming the existence of a volume form on M with negative definite Ricci form. Therefore, it is natural to suspect that in the non-compact version of Yau's theorem, the condition "with negative first Chern class" should be replaced by the existence of a volume form with negative Ricci form ω with some additional conditions to control the behavior of ω at infinity: for example, $-\omega$ defines a complete Kähler metric with bounded curvature on noncompact manifold under consideration. (In this paper, a Kähler metric is identified with its Kähler form.) In fact, in [2], S.-Y. Cheng and Yau proved that if Ω is a smooth bounded strongly pseudoconvex domain in C^n, then there is a complete Kähler-Einstein metric with negative Ricci curvature, which is invariant under biholomorphisms of Ω; the strong pseudoconvexity of Ω implies the existence of a volume form with the above properties. In this case, a model of such manifolds is the unit ball B^n in C^n with Poincaré-Bergman metric:

$$\sqrt{-1}\,\partial\bar{\partial}\log(1-|z|^2)\,.$$

The purpose of this paper is to prove the existence of a complete Kähler-Einstein metric with negative Ricci curvature on the complement of hypersurfaces of projective algebraic manifolds. In fact, we prove the following theorem.

Theorem 1. *Let $\bar{M}$ be a complex projective algebraic manifold and D an* *effective divisor with only simple normal crossings. If $K_{\bar{M}}\otimes[D]$ is ample, then there is a unique (up to constant multiple) complete Kähler-Einstein metric with negative Ricci curvature on $M=\bar{M}-D$, where $K_{\bar{M}}$ and $[D]$ denote the canonical bundle of $\bar{M}$ and the line bundle associated to D, respectively.*

The ampleness of $K_{\bar{M}}\otimes[D]$ assures the existence of a volume form on M with the properties stated above. In the proof of the theorem, we use the method of deformation of a Kähler metric developed in [2] and [10]. The starting metric of the deformation is the Carlson-Griffiths Kähler metric constructed in [5], and the ending metric is the required complete Kähler-Einstein metric. In this case, a "model" of such metrics is the pictured disk with Poincaré metric:

$$(1)\qquad 2\sqrt{-1}\,dz\wedge d\bar{z}/|z|^2(\log|z|^2)^2 = -\sqrt{-1}\,\partial\bar{\partial}\log\{|z|^2(\log|z|^2)^2\}\ .$$

The complete Kähler-Einstein manifold obtained here has the following properties:

(i) with negative Ricci curvature,

(ii) with finite volume,

(iii) the curvature tensor and its covariant derivatives have bounded length.

The characterization of such manifolds will be an interesting problem in this field.

As an application of Theorem 1, we obtain the following theorem.

Theorem 2. *Let $\bar{M}$ and D be as in Theorem 1, and $n=\dim\bar{M}\geqq 2$. Let $D=\sum_{i=1}^k D_i$ be the decomposition into irreducible components. Then the following inequality holds.*

$$2(n+1)\,(-c_1+\delta)^{n-2}(c_2-c_1\delta+2^{-1}\delta^2+2\textstyle\sum_{i=1}^k\delta_i^2)\geqq n(-c_1+\delta)^n\,,$$

where c_i, δ and δ_i denote the i-th Chern class of $\bar{M}$, the cohomology class of D, and that of D_i, respectively.

If $D=\phi$, this inequality reduces to the Chen-Ogiue-Miyaoka-Yau inequality for compact Kähler manifolds with ample canonical bundle.

After finishing this work, the author learned that Yau obtained an existence theorem of a complete Kähler-Einstein metric for a broader class of manifolds containing those in Theorem 1.

Finally, the author would like to express his thanks to Professors M. Takeuchi, H. Ozeki and Y. Sakane for their valuable suggestions.

1. Singular volume form with negative Ricci curvature

Let $\bar{M}$ be a compact complex manifold with dim $\bar{M}=n$ and E a holomorphic line bundle on $\bar{M}$. Let $\bar{M}=\cup_{\alpha\in A}U_\alpha$ be a covering of $\bar{M}$ by local trivializing neighborhoods for E, and $\{g_{\alpha\beta}\}_{\alpha\,\beta\in A}$ transition functions for E. Let h be a Hermitian metric for E, i.e., the collection of positive functions $\{h_\alpha\}_{\alpha\in A}$ such that $h_\alpha=|g_{\alpha\beta}|^2h_\beta$ for any α, $\beta\in A$. Then $\sqrt{-1}\,\partial\bar\partial\log h_\alpha$ is well defined real closed (1, 1)-form on $\bar{M}$ and is called the *curvature form* of (E, h). The de Rham cohomology class of the curvature form multiplied by $1/2\pi$ is independent of the choice of h, and is the first Chern class of E. In particular, a volume form Ψ of $\bar{M}$ may by regarded as a Hermitian metric for the anti-canonical bundle of $\bar{M}$. The curvature form of Ψ is called the *Ricci form* of Ψ and denoted by Ric Ψ. In the following, we assume that $\bar{M}$ is compact. A holomorphic line bundle E is called *ample* iff its first Chern class contains a positive definite real closed (1, 1)-form. By the fact mentioned in the introduction, E is ample iff E has a Hermitian metric with positive definite curvature form. Let D be an effective divisor on $\bar{M}$ with only simple normal crossings. (i.e., If $D=\sum_{i=1}^k D_i$ is the decomposition into irreducible components, then each D_i is nonsingular and at any $x\in\bar{M}$, there is a coordinate polydisk where all of D_i's through x are coordinate hyperplanes.) Let $M=\bar{M}-D$. Throughout in this and the next sections we assume $K_{\bar{M}}\otimes[D]$ is ample. These spaces $\bar{M}$ and M are treated in equidimensional Nevanlinna theory [5]. Although the following lemma is proved in [5], we give a proof for later purposes.

Lemma 1. *Under the above situation, there is a singular volume form* Ψ *on* $\bar{M}$ *with the following properties*:

(i) $-Ric\ \Psi$ *is positive definite on* M, *and* $(M, -Ric\ \Psi)$ *is a complete Kähler manifold with finite volume*,

(ii) *there is a positive constant* C *such that*

$$C^{-1}<\Psi/(-Ric\ \Psi)^n<C \text{ on } M\,.$$

Proof. Let σ_i be a holomorphic section of $[D_i]$ such that $D_i=\{x\in\bar{M};\ \sigma_i(x)=0\}$. The norm with respect to a Hermitian metric for $[D_i]$ and also the product norm on $[D]=\otimes_{i=1}^k[D_i]$ are denoted by $||\cdot||$. By the previous remark, there is a volume form Ω on $\bar{M}$ such that

$$-\mathrm{Ric}\ \Omega-\textstyle\sum_{i=1}^k\sqrt{-1}\,\partial\bar\partial\log||\sigma_i||^2$$

is positive definite on $\bar{M}$, because $K_{\bar{M}}\otimes[D]$ is ample. We define a volume form Ψ on M by

$$\Psi = \Omega/\textstyle\prod_{i=1}^k||\sigma_i||^2(\log||\sigma_i||^2)^2\,.$$

Direct computation shows

$$(2)\qquad \begin{aligned} -\mathrm{Ric}\ \Psi &= -\mathrm{Ric}\ \Omega-\textstyle\sum_{i=1}^k\sqrt{-1}\,\partial\bar\partial\log||\sigma_i||^2 \\ &\quad -2\textstyle\sum_{i=1}^k(\sqrt{-1}\,\partial\bar\partial\log||\sigma_i||^2)/\log||\sigma_i||^2 \end{aligned}$$

$$+2\sum_{i=1}^{k}\sqrt{-1}(\partial\log\|\sigma_i\|^2\wedge\bar\partial\log\|\sigma_i\|^2)/(\log\|\sigma_i\|^2)^2 .$$

Taking an appropriate constant multiple of $\|\cdot\|$, we can sasume that

$$-\mathrm{Ric}\,\Omega-\sum_{i=1}^{k}\sqrt{-1}\,\partial\bar\partial\log\|\sigma_i\|^2-2\sum_{i=1}^{k}(\sqrt{-1}\,\partial\bar\partial\log\|\sigma_i\|^2)/\log\|\sigma_i\|^2$$

defined on M is bounded from below by

$$2^{-1}(\mathrm{Ric}\,\Omega-\sum_{i=1}^{k}\sqrt{-1}\,\partial\bar\partial\log\|\sigma_i\|^2) ,$$

therefore, $-\mathrm{Ric}\,\Psi$ is positive definite on M. Now we show that $-\mathrm{Ric}\,\Psi$ is a complete Kähler metric on M. Let $x\in D_1\cap\cdots\cap D_m-D_{m+1}\cup\cdots\cup D_k$. There exists an n-polydisk Δ^n centered at x such that $D\cap\Delta^n=\cup_{i=1}^{m}\{z\in\Delta^n;\ z^i=0\}$. Then $\Delta^n\cap M=(\Delta^*)^m\times\Delta^{n-m}$, where Δ^* denotes the punctured disk. In this polydisk, $\|\sigma_i\|^2=|z^i|^2/h_i$ where h_i is a smooth positive function on Δ^n. Around x, the last term in (2), which is positive definite, is

$$(3)\qquad 2\sum_{i=1}^{m}\sqrt{-1}(dz^i\wedge d\bar z^i+|z^i|^2\alpha_i)/|z^i|^2(\log|z^i|^2-\log h_i)^2$$
$$+\text{positive semidefinite smooth term,}$$

where

$$\alpha_i=-(dz^i\wedge\bar\partial\log h_i)/z^i-(\partial\log h_i\wedge d\bar z^i)/\bar z^i+\partial\log h_i\wedge\partial\log h_i .$$

Therefore, by comparing (3) with (1), and from the completeness of Poincaré metric of Δ^* at the origin, we know that the length of a curve approaching to D measured by $-\mathrm{Ric}\,\Psi$ is infinity, which means the completeness of $-\mathrm{Ric}\,\Psi$.

The finiteness of the volume of $(M,\ -\mathrm{Ric}\,\Psi)$ follows immediately from

$$\int_{0<|z|<c}\frac{\sqrt{-1}\,dz\wedge d\bar z}{|z|^2(\log|z|^2)^2}=-\pi/\log c<\infty\quad\text{if}\quad 0<c<1 .$$

The second assertion of Lemma 1 follows from (2), (3) and the definition of Ψ. Thus the proof is completed.

REMARK. In the above proof, it is essential that the Poincaré metric is a complete metric with negative constant Gaussian curvature.

2. Nice coordinate system on $(M,\ -\mathrm{Ric}\,\Psi)$

In this section, we introduce a nice coordinate system at infinity of M. Let $x\in D_1\cap\cdots\cap D_m-D_{m+1}\cup\cdots\cup D_k$. Let Δ^n be a coordinate polydisc centered at x such that

$$\Delta^n\cap D_i=\{z\in\Delta^n;\ z^i=0\}\quad(1\leqq i\leqq m) ,$$
$$\Delta^n\cap M=(\Delta^*)^m\times\Lambda^{n-m} .$$

Define the universal covering map $\Delta^m\times\Delta^{n-m}\to(\Delta^*)^m\times\Delta^{n-m}$, $(w^1,\cdots,w^m,w^{m+1},$

$\cdots, w^n) \mapsto (z^1, \cdots, z^m, z^{m+1}, \cdots, z^n)$ by

$$z^i = \exp(w^i+1)/(w^i-1) \quad \text{if} \quad 1 \leqq i \leqq m\,,$$
$$z^j = w^j \qquad\qquad\qquad \text{if} \quad m+1 \leqq j \leqq n\,.$$

A fundamental domain of the universal covering map $\Delta \to \Delta^*$, $w \mapsto z = \exp(w+1)/(w-1)$, is as the figure 1 (i.e., the domain bounded by two geodesics tending to 1).

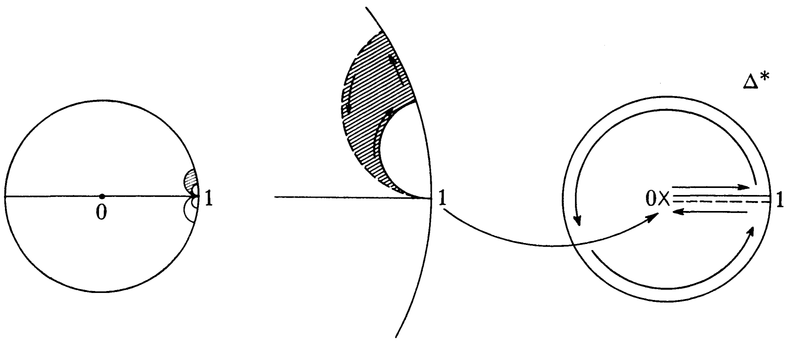

fig. 1.

By this map, each sequence in Δ tending to 1 is mapped to a sequence in Δ^* tending to 0. Now we can introduce a nice coordinate system similar to that of [2]. Firstly, we introduce a coordinate system on an open set in Δ close to 1 as follows. Let η be a real number close to 1 in Δ, and Φ_η a biholomorphism of Δ sending η to 0 defined by

$$\Phi_\eta(w) = (w-\eta)/(1-\eta w)\,.$$

Fix a positive number R with $2^{-1} < R < 1$. Around η, consider the open set $\Phi_\eta^{-1}(B(0, R))$, where $B(0, R) = \{z \in C;\ |z| < R\}$. On $\Phi_\eta^{-1}(B(0, R))$, define a coordinate function $\Phi_\eta^{-1}(B(0, R)) \to B(0, R)$, $w \mapsto v$, by

$$v = \Phi_\eta(w) = (w-\eta)/(1-\eta w)\,.$$

Secondly, let $z \in (\Delta^*)^m \times \Delta^{n-m}$ be a point close to D, so that z^i's $(1 \leqq i \leqq m)$ are close to 0. By the universal covering map defined above, we can find in Δ^m a point $(w^1, \cdots, w^m)$ that is projected on $(z^1, \cdots, z^m)$. Since w^i's $(1 \leqq i \leqq m)$ are close to 1, we can introduce the coordinates constructed above by

$$v^i = (w^i - \eta^i)/(1-\eta^i w^i)\,,$$

by choosing suitable real numbers η^i's in Δ close to 1. In fact, if η^i ranges real numbers close to 1 in Δ, the set $\cup_\eta \Phi_\eta^{-1}(B(0, R))$ covers the open subsets (shaded portion in figure 2) of fundamental domains of $\Delta \to \Delta^*$.

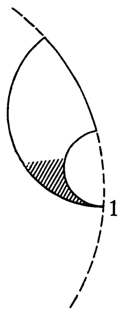

fig. 2.

This fact is proved as follows. Relations $v=(w-\eta)/(1-\eta w)$ and $v=Re^{\sqrt{-1}\mu}$ imply

$$Re(w) = \{\eta(1+R^2)+(1+\eta^2)R\cos\theta\}/\{1+\eta^2R^2+2\eta R\cos\theta\},$$
$$\mathrm{Im}(w) = (1-\eta^2)R\sin\theta/\{1+\eta^2R^2+2\eta R\cos\theta\},$$

and if $\theta=\frac{\pi}{2}$, then

$$\mathrm{Re}(w) = \eta(1+R^2)/(1+\eta^2R^2)\geqq\eta,$$
$$\mathrm{Im}(w) = R(1-\eta^2)/(1+\eta^2R^2)\geqq\frac{R}{2}(1-\eta^2).$$

Therefore, $\cup_\eta\Phi_\eta^{-1}(B(0, R))$ is as figure 3, and the assertion follows.

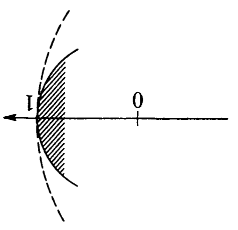

fig. 3.

We define a "coordinate function" v^i around a point of $(\Delta^*)^m\times\Delta^{n-m}$ close to D, by

$$v^i = (w^i-\eta^i)/(1-\eta^iw^i) \quad (1\leqq i\leqq m),$$

where $z^i=\exp(w^i+1)/(w^i-1)$, η^i is a real number close to 1,

$$v^j = w^j = z^j \quad (m+1\leqq j\leqq n).$$

Although this "coordinate function" is not a coordinate function in the usual sense, it has a meaning to take components with respect to v^i's of a tensor field on $(\Delta^*)^m\times\Delta^{n-m}$ by lifting it to a tensor field on Δ^n. To examine the behavior of a function defined on a neighborhood of D is equivalent to examine the behavior of the (locally) lifted function in a neighborhood of $(1, \cdots, 1, *)$ in Λ^n using the coordinates v^i's and the components of the lifted metric with respect to v^i's. So, we introduce the follgwing notion.

DEFINITION. Let V be an open set in C^n. A holomorphic map from V into a complex manifold M of dimension n is called a *quasi-coordinate map* iff it is of maximal rank everywhere in V. (V; Euclidean coordinates of C^n) is called a *local quasi-coordinate* of M.

Then our map $V=B(0, R)^m \times \Delta^{n-m} \to (\Delta^*)^m \times \Delta^{n-m} \hookrightarrow M$, defined by $(v^1, \cdots, v^m, \cdots, v^n) \mapsto (\cdots, \exp(\Phi_{\eta^i}{}^{-1}(v^i)+1)/(\Phi_{\eta^i}{}^{-1}(v^i)-1), \cdots, v^{m+1}, \cdots, v^n)$ where $1 \leqq i \leqq m$, is a quasi-coordinate map.

Lemma 2. *There exists a family of local quasi-coordinates* $\mathcal{V} = \{(V; v^1, \cdots, v^n\}$ *of* $M = \bar{M} - D$ *with the following properties.*

(i) *M is covered by the images of* $(V; v^1, \cdots, v^n)$*'s.*

(ii) *The complement of some open neighborhood of D is covered by a finite number of* $(V; v^1, \cdots, v^n)$*'s which are local coordinates in the usual sense.*

(iii) *Each V, as an open subset of the complex Euclidean space C^n, contains a ball of radius* $\frac{1}{2}$.

(iv) *There exist positive constants c and $\mathcal{A}_k$ $(k=0, 1, 2, \cdots)$ independent of V's such that at each* $(V; v^1, \cdots, v^n)$, *the inequalities*

$$\frac{1}{c}(\delta_{ij}) < (g_{i\bar{j}}) < c(\delta_{ij}),$$

$$(|\partial^{|p|+|q|}/\partial v^p \partial \bar{v}^q) g_{i\bar{j}}| < \mathcal{A}_{|p|+|q|}, \quad \textit{for any multiindices } p \textit{ and } q,$$

hold, where $g_{i\bar{j}}$ denote the components of $-Ric\ \Psi$ with respect to v^i's.

Proof. Cover an open neighborhood U of D by ourlocal quasi-coordinates $(V; v^1, \cdots, v^n)$ and then cover $M-U$ by a finite number of unit balls of C^n. Then assertion (i), (ii), (iii) are clear. Assertion (iv) is proved by the local expression of $-\text{Ric}\ \Psi$ with respect to v^i's as follows. From

$$z^i = \exp(w^i+1)/(w^i-1) = \exp((1+\eta^i)(v^i+1)/(1-\eta^i)(v^i-1)) \qquad (1 \leqq i \leqq m),$$

we have

$$\begin{aligned} & dz^i \wedge d\bar{z}^i / |z^i|^2 (\log|z^i|^2 - \log h_i)^2 \\ &= 4dv^i \wedge d\bar{v}^i / \{2(|v^i|^2-1) - (1-\eta^i)(\log h_i)|v^i-1|^2/1+\eta^i\}^2, \\ & (dz^i/z^i) \wedge \bar{\partial} \log h_i / |(\log|z^i|^2 - \log h_i)^2 \\ &= -2(1-\eta^i)(1+\eta^i) dv^i \wedge \bar{\partial} \log h_i / \{2(1+\eta^i)(|v^i|^2-1)/|v^i-1|^2 - (1-\eta^i) \end{aligned}$$

$$\begin{aligned}&\log h_i\}^2(v^i-1)\,,\quad \partial/\partial v^i\\&=-\{2(1+\eta^i)\exp((1+\eta^i)(v^i+1)/(1-\eta^i)(v^i-1))\partial/\partial z^i\}/(1-\eta^i)(v^i-1)^2\,,\\&\log|z^i|^2\\&=2(1-\eta^i)(|v^i|^2-1)/(1-\eta^i)(|v^i-1|^2)\,.\end{aligned}$$

Substituting these into (3), we obtain (iv), making use of the fact $\lim_{x\uparrow\infty} e^{-x}x^p=0$ for any real number p.

REMARK. In the above proof, the invariance of Poincaré metric under biholomorphisms is essential.

Now we define the Hölder space of $C^{k,\lambda}$-functions on $M=\bar{M}-D$ by using the quasi-coordinate system of Lemma 2. For a nonnegative integer k, $\lambda\in(0, 1)$, and $u\in C^k(M)$, we define

$$\begin{aligned}||u||_{k,\lambda}=\sup_{v\in\mathcal{V}}\{&\sup_{z\in V}(\sum_{|p|+|q|\leq k}|(\partial^{|p|+|q|}/\partial v^p\partial\bar{v}^q)u(z)|)\\&+\sup_{z,z'\in V}(\sum_{|p|+|q|=k}|z-z'|^{-\lambda}|(\partial^{|p|+|q|}/\partial v^p\partial\bar{v}^q)u(z)\\&\qquad-(\partial^{|p|+|q|}/\partial v^p\partial\bar{v}^q)u(z')|)\}\,.\end{aligned}$$

The function space $C^{k,\lambda}(M)$ is, by definition,

$$C^{k,\lambda}(M)=\{u\in C^k(M);\ ||u||_{k,\lambda}<\infty\}\,,$$

which is a Banach space with respect to the norm $||\cdot||_{k,\lambda}$.

The quasi coordinate system of Lemma 2 is useful in the Schauder estimate on M. In the interior Schauder estimate

$$||u||_{C^{k,\lambda}(V')}\leqq C(\sup_V|u|+||Lu||_{C^{k-2,\lambda}(V)}),\ \text{where } V'\subset\subset V\subset R^m\,,$$

(See Chapter 6 of [4].) for a linear elliptic operator L, the constant C is determined by m, k, ellipticity of L, $C^{k-2,\lambda}$-norms of coefficients of L, and the distance between V' and ∂V. Therefore, the interior Schauder estimate on M is reduced to that on a bounded domain in Euclidean space, because of (iii) and (iv) in Lemma 2.

3. The existence of a complete Kähler-Einstein metric on M

In this section, Theorem 1 is proved. The complete Kähler metric $-\mathrm{Ric}\ \Phi$ on M can be approximated by Poincaré metrics of punctured disks transversal to D. Therefore, it is suspected that $-\mathrm{Ric}\ \Psi$ should be deformed into a complete Kähler-Einstein metric.

Set $\omega=-\mathrm{Ric}\ \Psi$. The deformation of ω is defined by

$$\omega\mapsto\omega+\sqrt{-1}\,\partial\bar{\partial}u\,,$$

where u is a smooth function on M. We want to find u such that $\omega+\sqrt{-1}\,\partial\bar{\partial}u$ is a complete Kähler-Einstein metric on M. Suppose u satisfies the following equation.

$$(4)\qquad \begin{cases} (\omega+\sqrt{-1}\,\partial\bar{\partial}u)^n = (\exp u)\Psi \\ \omega+\sqrt{-1}\,\partial\bar{\partial}u \text{ is positive definite on } M. \end{cases}$$

The Ricci form of $\omega+\sqrt{-1}\,\partial\bar{\partial}u$ is Ric $\Psi-\sqrt{-1}\,\partial\bar{\partial}u=-(\omega+\sqrt{-1}\,\partial\bar{\partial}u)$, hence $u+\sqrt{-1}\,\partial\bar{\partial}u$ is a Kähler-Einstein metric with negative Ricci curvature. To get a complete one, we define an open subset U in $C^{k,\lambda}(M)$ by

$$U=\{u\in C^{k,\lambda}(M);\ \frac{1}{c}<\omega+\omega\sqrt{-1}\,\partial\bar{\partial}u<c\omega, \text{ for some positive constant } c\}.$$

If u satisfies (4) and belongs to U, $\omega+\sqrt{-1}\,\partial\bar{\partial}u$ is a complete Kähler-Einstein metric of M. The procedure to find a solution of (4) in U is the same as [2], [10]. Here, we give a brief review of it. We consider a C^0-map Φ: $C^{k,\lambda}(M)\to C^{k-2,\lambda}(M)$ defined by

$$\Phi\colon u\mapsto e^{-u}(\omega+\sqrt{-1}\,\partial\bar{\partial}u)^n/\omega^n\,.$$

We claim that for any $F\in C^{k-2,\lambda}(M)$ $(k\geqq 6)$, there is a solution of

$$(5)\qquad \Phi(u)=\exp(F),\ u\in U\,.$$

Define C, a subset of the interval [0, 1], by

$$C=\{t\in[0,1];\ \text{there is a solution } u\in U \text{ of } \Phi(u)=e^{tF}\}\,.$$

Since 0 belongs to C, to prove $1\in C$, it is sufficient to show that C is open and closed.

Openness follows from the inverse mapping theorem. The Fréchet derivative, $\Phi'(u)$: $C^{k,\lambda}(M)\to C^{k-2,\lambda}(M)$, of Φ at $u\in U$ is given by

$$h\mapsto e^F(\tilde{\Delta}h-h)\,,$$

where $\tilde{\Delta}$ denotes the Laplacian with respect to the Kähler metric $\omega+\sqrt{-1}\,\partial\bar{\partial}u$. The openness of C comes from the openness of Φ at u. It suffices to show that $\Phi'(u)$ has a C^0-inverse, i.e., for any $v\in C^{k-2,\lambda}(M)$, there is a unique solution $h\in C^{k,\lambda}(M)$ of the equation

$$\tilde{\Delta}h-h=v\,,$$

with an estimate $||h||_{k,\lambda}\leqq c||v||_{k-2,\lambda}$, where c is a constant independent of the choice of $v\in C^{k-2,\lambda}(M)$. The equation on a relatively compact domain Ω in M:

$$\begin{cases} \tilde{\Delta}h-h=v & \text{in } \Omega \\ h=0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution. See for example [4], Theorem 6.13. To prove the convergence of $\{h_i\}$ obtained in [2], p. 521 (h_i is a unique solution for the above Dirichlet problem, for $\Omega=\Omega_i$, where $\{\Omega_i\}$ is a exhustion of M), as well as the above estimate, we use the interior Schauder estimate with respect to our local quasi-coordinate $(V; v^1, \cdots, v^n)$. See for example [4], Corollary 6.3.

Main tools in the proof of closedness of C are the *a priori* estimate of the equation (5) and the interior Schauder estimate of the linealized equation of (5). The former estimate goes exactly as in [2] using our quasi-coordinate system. In the latter estimate, Lemma 2 plays an essential role. Here, we restrict ourselves only to give a proof simpler than [2] of the C^0-estimate of (5). Let $u\in U$ be a solution of (5). Then

$$\begin{aligned} u+F &= \log\{\det(g_{i\bar{j}}+u_{i\bar{j}})\} - \log\{\det(g_{i\bar{j}})\} \\ &= \int_0^1 \frac{d}{dt}\log\{\det(g_{i\bar{j}}+tu_{i\bar{j}})\}\,dt \\ &= \int_0^1 (g+tu)^{i\bar{j}}u_{i\bar{j}}dt\,, \end{aligned}$$

where $(g+tu)^{i\bar{j}}$ denotes the inverse matrics of $(g_{i\bar{j}}+tu_{i\bar{j}})$ and $\omega=\sqrt{-1}\sum g_{i\bar{j}}dz^i \wedge d\bar{z}^j$. At a point $x\in M$, we may assume $g_{i\bar{j}}=\delta_{ij}$ and $u_{i\bar{j}}=\delta_{ij}u_{i\bar{i}}$, hence

$$(g+tu)^{i\bar{j}}u_{i\bar{j}} = \sum \frac{u_{i\bar{i}}}{1+tu_{i\bar{i}}}$$
$$\begin{cases} = \sum\left(u_{i\bar{i}} - \dfrac{tu_{i\bar{i}}^2}{1+tu_{i\bar{i}}}\right) \leqq \sum u_{i\bar{i}} = \Delta u \\ = \sum\left(\dfrac{u_{i\bar{i}}}{1+u_{i\bar{i}}} + \dfrac{(1-t)u_{i\bar{i}}^2}{(1+u_{i\bar{i}})(1+tu_{i\bar{i}})}\right) \geqq \sum \dfrac{u_{i\bar{i}}}{1+u_{i\bar{i}}} = \tilde{\Delta} u \end{cases}$$

if $0\leqq t\leqq 1$. Hence $u+F\leqq\Delta u$ and $u+F\geqq\tilde{\Delta}u$. Here, Δ denotes the Laplacian with respect to ω. Since u belongs to U, u is a bounded C^2-function and both ω and $\omega+\sqrt{-1}\partial\bar{\partial}u$ define complete Riemannian metric with bounded curvature (in particular, with Ricci curvature bounded from below). Hence, applying Yau's maximum principle (Theorem 1 of [9]), it follows

$$\sup u\leqq \sup|F| \text{ and } \inf u\geqq -\sup|F|\,.$$

To complete the proof of Theorem 1, it suffices to show that $F_0=\log\{\Psi/\omega^n\}$ belongs to $C^{k-2,\lambda}(M)$, because the equation (4) can be written as

$$(4)' \qquad e^{-u}(\omega+\sqrt{-1}\partial\bar{\partial}u)^n/\omega^n = \exp(F_0)\,.$$

F_0 belongs to $C^{k-2,\lambda}(M)$ for any k, λ, because F_0 is a bounded smooth function on M (Lemma 1, (ii)), and

$$\partial/\partial v^i = -2(z^i\log|z^i|)\ \{(\bar{v}^i-1)/(|v^i|^2-1)\ (v^i-1)\}\partial/\partial z^i$$

implies the boundedness of the derivatives of F_0. See (2) and (3).

The uniqueness of a complete Kähler-Einstein metric with negative Ricci curvature follows from Yau's Schwarz lemma [9]. Thus the proof of Theorem 1 is completed.

Now we give some examples of complete Kähler-Einstein manifold obtained in Theorem 1.

(i) P^n—$(n+k)$ hyperplanes in general position $(2\leqq k)$,
P^n—a nonsingular hypersurface of degree $n+k$ $(2\leqq k)$,

(ii) an Abelian variety — an effective ample divisor with only simple normal crossings,

(iii) a compact quotient of B^2 — a nonsingular curve; or more generally, a compact Kähler surface of negative holomorphic bisectional curvature — a nonsingular curve.

We prove that examples in (iii) satisfy our condition. Let M be a compact surface with negative holomorphic bisectional curvature. Then from the decreasing property of holomorphic bisectional curvature, any nonsingular curve C on M admits a Kähler metric with negative Gaussian curvature, hence its genus $g(C)$ is greater than one. From the adjunction formula ([6], p. 471), $2g(C)-2=KC+C^2>0$, where K denotes the canonical divisor of M. Hence $(K+C)^2=K^2+2KC+C^2>K^2+KC>0$, using the ampleness of K_M. Therefore, $K_M\otimes[C]$ satisfies the condition of the criterion of Nakai, and $K_M\otimes[C]$ is ample for any nonsingular curve C.

REMARK 1. In example (iii), the genus of C is not smaller than $1+(KC+c_1^2-3c_2)/4$. This follows from Theorem 2.

REMARK 2. B. Wong [8] and Yau proved that for a bounded strongly pseudoconvex domain Ω in C^n with smooth boundary, the following conditions are equivalent.

(a) Ω is biholomorphic to B^n;

(b) Ω is homogeneous;

(c) $\mathrm{Aut}(\Omega)$, the group of biholomorphisms of Ω, is noncompact;

(d) There is a subgroup $Z\subset\mathrm{Aut}(\Omega)$ acting properly discontinuously on Ω such that the volume of the quotient $Z\backslash\Omega$ with respect to the canonical complete Kähler-Einstein metric is finite.

Our complete Kähler-Einstein manifold $\bar{M}-D$ has finite volume, but may admit an entire holomorphic curve. For example, P^2 minus four lines in general position (an example in (i)) contains C^*, the diagonal line minus two points, which is a holomorphic image of C.

REMARK 3. Let M be a complete Kähler manifold and ω ist Kähler form. (M, ω) is called of *bounded geometry* or *homogeneous regular* iff there is a quasi-coordinates $\mathcal{V}$ which satisfies the condition (i), (ii), (iv) of Lemma 2. Let (M, ω) be a noncompact complete Kähler manifold with bounded geometry. Then we can define the Banach space $C^{k,\lambda}(M)$ as in section 2. Under these definitions Theorem 1 is generalized in the following way.

Theorem 1′. *Let M be a noncompact complex manifold of complex dimension n. Suppose M admits a volume form Ψ such that $\omega=-Ric\ \Psi$ is a complete Kähler metric of M and (M, ω) is of bounded geometry. Then for any $f\in C^{k,\lambda}(M)$, $k\geqq 3$, the equation*

$$(\omega+\partial\bar{\partial}u)^n = e^u e^f \omega^n$$

has a solution u such that $\omega+\partial\bar{\partial}u$ is a complete Kahler metric of M which is equivalent to ω. In particular, if $\log(\Psi/\omega^n)$ is in $C^{k,\lambda}(M)$ for any k, λ, then M admits a unique complete Kähler-Einstein metric with negative Ricci curvature.

4. An inequality for Chern numbers

In this section, we prove Theorem 2 using our complete Kähler-Einstein metric with negative Ricci curvature on $M=\bar{M}-D$.

In [1], B.Y. Chen and K. Ogiue proved an inequality: $(-1)^n 2(n+1)c_1^{n-2}c_2 \geqq (-1)^n n c_1^n$ for a compact Kähler-Einstein manifold. By [10], this inequality holds for every projective algebraic manifold with ample canonical bundle. This inequality can be proved easily by computing the Chern forms using the curvature tensor of a Kähler Einstein metric. In this paper, we apply this method to our complete Kähler-Einstein metric. From the proof of Theorem 1, such manifolds have properties (i), (ii), (iii) stated in the introduction. So, the "Chern numbers" are computed from our complete Kähler-Einstein metric (which is unique up to constant multiple and they are determined only by the complex structure of M and the divisor D.

We fix the notations as follows.

$\bar{M}$, M, $D=\sum_{i=1}^{k} D_i$ are as in Theorem 1,

$\omega=-\mathrm{Ric}\ \Psi$: Carlson-Griffiths metric on $M=\bar{M}-D$,

$\tilde{\omega}=\omega+\sqrt{1-}\ \partial\bar{\partial}u$: our complete Kahler-Einstein metric on M,

γ_i (resp. $\tilde{\gamma}_i$): i-th Chern form computed from the Riemannian connection of ω (resp. $\tilde{\omega}$) on tangent bundle TM.

Lemma (Gaffney [3]). *Let X be an m-dimentional complete Riemannian manifold, η an $(m-1)$ form on X. Assume $\int_X \|\eta\| dv<\infty$, and $\int_X \|d\eta\| dv<\infty$, where $\|\cdot\|$ and dv denote Riemannian norm and Riemannian measure, respectively. Then*

$$\int_X d\eta = 0\,.$$

Proposition 1. *Assume $n \geqq 2$ and define two "Chern numbers" by*

$$\tilde{c}_1^n = \int_M \tilde{\gamma}_1^n , \quad \tilde{c}_1^{n-2}\tilde{c}_2 = \int_M \tilde{\gamma}_1^{n-2}\tilde{\gamma}_2 .$$

Then

(i) $(-1)^n \tilde{c}_2^n = (k+\delta)^n$

(ii) $(-1)^n \tilde{c}_1^{n-2}\tilde{c}_2 = (k+\delta)^{n-2}(c_2 + k\delta + 2^{-1}\delta^2 + 2^{-1}\sum_{i=1}^k \delta_i^2)$,

where k denotes the first Chern class of $K_{\bar{M}}$

Proof. We claim firstly that the following equalities hold.

$$(6) \qquad (-1)^n \tilde{c}_1^n = \int_M (\omega/2\pi)^n ,$$

$$(7) \qquad (-1)^n \tilde{c}_1^{n-2}\tilde{c}_2 = \int_M (\omega/2\pi)^{n-2} \wedge \gamma_2 .$$

Since both equalities can be proved similarly, we prove here only (6).

$$\tilde{\gamma}_1 = (1/2\pi)\operatorname{Ric}(\tilde{\omega}^n) = -(1/2\pi)\tilde{\omega} = -(1/2\pi)\omega - dd^c u ,$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$. Hence

$$\begin{aligned}(-1)^n \tilde{c}_1^n &= \int_M (-\tilde{\gamma}_1)^n \\ &= \int_M (\omega/2\pi)^n + \int_M d\left\{\sum_{r=1}^{n-1} \binom{n}{r} (\omega/2\pi)^r < d^c u \wedge (dd^c u)^{n-r-1}\right\} .\end{aligned}$$

Because u is in U and vol (M, ω) is finite, we can apply Lemma 3 to obtain (6). Assertion (i) can be proved as follows. We have

$$\omega/2\pi = (-(\operatorname{Ric}\Omega)/2\pi - dd^c \textstyle\sum_{i=1}^k \log\|\sigma_i\|^2) - dd^c \sum_{i=1}^k \log(\log\|\sigma_i\|^2)^2 ,$$

where $-(\operatorname{Ric}\Omega)/2\pi - dd^c \sum_{i=1}^k \log\|\sigma_i\|^2$ represents $k+\delta$. Hence $\int_M (\omega/2\pi)^n$ can be written as

$$\begin{aligned}(k+\delta)^n + \int_M d\Big\{\sum_{r=0}^{n-1} \binom{n}{r} & (\gamma_1(K_{\bar{M}}) + \gamma_1([D]))^r \\ & \wedge d^c(-\textstyle\sum_{i=1}^k \log(\log\|\sigma_i\|^2)^2) \\ & \wedge dd^c(-\textstyle\sum_{i=1}^k \log(\log\|\sigma_i\|^2)^2)^{n-r-1}\Big\} ,\end{aligned}$$

where $\gamma_i(E)$ denotes the i-th Chern form of a complex vector bundle E over $\bar{M}$. Here,

$$\begin{aligned}& d^c \log(\log\|\sigma_i\|^2)^2 \\ &= \sqrt{-1}(d\bar{z}^i/\bar{z}^i - \bar{\partial}\log h_i - dz^i/z^i + \log h_i)/4\pi \log\|\sigma_i\|^2 ,\end{aligned}$$

and

$$dd^c \log(\log\|\sigma_i\|^2)^2 = -2\gamma_1([D_i])/\log\|\sigma_i\|^2$$

$$-\sqrt{-1}(dz^i\wedge d\bar{z}^i+|z^i|^2\alpha_i)/\pi|z^i|^2(\log\|\sigma_i\|^2)^2$$

have bounded norms with respect to ω on M. By Lemma 3, $(-1)^n\tilde{c}_1^n$ equals $(k+\delta)^n$.

To prove the assertion (ii), we introduce a Hermitian metric on $\bar{M}$ and let $\mathring{\theta}$, $\mathring{\Theta}=\bar{\partial}\mathring{\theta}$ be the connection form and the curvature form of its Hermitian connection, respectively. Let θ and Θ be the connection form and the curvature form of the Kähler metric ω. By the formula giving the difference between the Chern forms defined by two connections ([6], pp. 400–406), we have

$$(\omega/2\pi)^{n-2}\wedge\gamma_2=(\gamma_1+\gamma_1([D]))^{r-2}\wedge\gamma_2(T\bar{M})+d(I+II+III)$$

where

$$I=\{\textstyle\sum_{r=0}^{n-3}\binom{n-2}{r}(\gamma_1(K_{\bar{M}})+\gamma_1([D])^r\wedge d^c(-\sum_{i=1}^k\log(\log\|\sigma_i\|^2)^2$$
$$(dd^c(-\textstyle\sum_{i=1}^k\log(\log\|\sigma_i\|^2)^2))^{n-3-r}\}\wedge\gamma_2(T\bar{M}),$$
$$II=(\gamma_1(K_{\bar{M}})+\gamma_1([D]))^{n-2}\wedge A,$$
$$III=\textstyle\sum_{r=0}^{n-3}\binom{n-2}{r}(\gamma_1(K_{\bar{M}})+\gamma_1([D]))^r\wedge$$
$$(dd^c(-\textstyle\sum_{i=1}^k\log(\log\|\sigma_i\|^2)^2))^{n-2-r}\wedge A,$$
$$A=\textstyle\sum_{\sharp I=2}\{\det\begin{bmatrix}\text{the first row of }\sqrt{(-1}\theta-\mathring{\theta})_I{}^I/2\pi\\ \text{the second row of }\sqrt{-1}(\Theta+\mathring{\Theta})_I{}^I/4\pi\end{bmatrix}$$
$$+\det\begin{bmatrix}\text{the first row of }\sqrt{-1}(\Theta+\mathring{\Theta})_I{}^I/4\pi\\ \text{the second row of }\sqrt{(-1}\theta-\mathring{\theta})_I{}^I/2\pi\end{bmatrix}\}.$$

Now, we take an exhaustion of M by relatively compact domains $\{\Omega_j\}_{j\in N}$. For example, $\Omega_j=\{z\in M;\ \Pi_{i=1}^k\|\sigma_i(z)\|>1/j\}$ $(j\in N)$. Then $\cup_j\Omega_j=M$, $\cap_j(\bar{M}-\Omega_j)=D$. By Stokes' theorem,

$$\int_M d(I+II+III)=-\lim_{j\uparrow\infty}\int_{\partial(\bar{M}-\Omega_j)}(I+II+III)$$
$$=\lim_{j\uparrow\infty}\int_{\partial\Omega_j}(I+II+III).$$

The argument in the proof of (i) shows $\lim_{j\uparrow\infty}\int_{\partial\Omega_j}I=0$. To compute $\lim_{j\uparrow\infty}\int_{\partial\Omega_j}(II+III)$, let $\Delta^n\subset\bar{M}$ be a coordinate polydisk such that $\Delta^n\cap D=\Delta^n\cap D_1=\{z\in\Delta^n;\ z^1=0\}$. Consider the boundary $\{z\in\Delta^n;\ \|\sigma_1\|^2=|z^1|^2/h_1=\varepsilon^2\}$ of the tube of radius $\varepsilon>0$ along D_1 in Δ^n, which will be abbreviated to $\{\|\sigma_1\|=\varepsilon\}$, and let $\varepsilon\to0$. If we set $z^1=\varepsilon\sqrt{h_1}e^{\sqrt{-1}\theta}$, then $dz^1=\sqrt{-1}z^1d\theta+\varepsilon e^{\sqrt{-1}\theta}d\sqrt{h_1}$, hence $dz^1/z^1=\sqrt{-1}d\theta+d\log\sqrt{h_1}$ on $\{\|\sigma_1\|=\varepsilon\}$. The order of $(g_{i\bar{j}})$ when ε tends to 0 is in the following.

$$(g_{i\bar{j}}) = \begin{array}{c} \left(\begin{array}{c|c} \mathcal{O}(\varepsilon^{-2}(\log\varepsilon)^{-2}) & \mathcal{O}(\varepsilon^{-1}(\log\varepsilon)^{-2}) \\ \hline \mathcal{O}(\varepsilon^{-1}(\log\varepsilon)^{-2}) & \mathcal{O}(1) \end{array}\right) \begin{array}{l} \}1 \\ \}n-1 \end{array} \\ \underbrace{\qquad\qquad}_{1} \quad \underbrace{\qquad\qquad}_{n-1} \end{array} .$$

Here, $g_{i\bar{j}}$ denotes the $(i, \bar{j})$-component of ω with respect to z^i's. Thus the inverse matrix $(g^{i\bar{j}})$ of $(g_{i\bar{j}})$ has the following order when ε tends to 0.

$$(g^{i\bar{j}}) = \begin{array}{c} \left(\begin{array}{c|c} \mathcal{O}(\varepsilon^{2}(\log\varepsilon)^{2}) & \mathcal{O}(\varepsilon) \\ \hline \mathcal{O}(\varepsilon) & \mathcal{O}(1) \end{array}\right) \begin{array}{l} \}1 \\ \}n-1 \end{array} \\ \underbrace{\qquad\qquad}_{1} \quad \underbrace{\qquad\qquad}_{n-1} \end{array}$$

Let the $\mathcal{O}(1)$ part of $(g_{i\bar{j}})$ be denoted by $(H_{i\bar{j}})_{i,j\geqq 2}$ and $g_{1\bar{p}}=H_{1\bar{p}}$. By direct computation,

$$\begin{aligned}
\theta_1^{\ 1} &= \textstyle\sum_{p,\alpha} g^{1\bar{p}}(\partial g_{1\bar{p}}/\partial z^\alpha)dz^\alpha = -(1+o(1))dz^1/z^1+\sum_{\alpha\geqq 2}o(1)dz^\alpha , \\
\theta_1^{\ j} &= \textstyle\sum_{p,\alpha} g^{j\bar{p}}(\partial g_{1\bar{p}}/\partial x^\alpha)dz^\alpha \\
&= (\textstyle\sum_p H^{j\bar{p}}H_{1\bar{p}}+o(1))dz^1/z^1+\sum_{\alpha\geqq 2}\mathcal{O}(\varepsilon^{-1}(\log\varepsilon)^{-2})dz^\alpha \quad (j\geqq 2) \\
\theta_i^{\ 1} &= o(1)dz^1+\textstyle\sum_{\alpha\geqq 2}\mathcal{O}(\varepsilon)dz^\alpha \qquad (i\geqq 2) \\
\theta_i^{\ j} &= o(1)dz^1/z^1+\textstyle\sum_{\alpha\geqq 2}\mathcal{O}(1)dz^\alpha \quad (i, j\geqq 2) .
\end{aligned}$$

Using $dz^1/z^1=\sqrt{-1}d\theta+d\log\sqrt{h_1}$, we estimate Θ on $\{\|\sigma_1\|=\varepsilon\}$ as follows.

$$\begin{aligned}
\Theta_1^{\ 1} &= \textstyle\sum_{\alpha\geqq 2}o(1)d\theta\wedge dz^\alpha+\sum_{\alpha\geqq 2}o(1)d\theta\wedge d\bar{z}^\alpha+\sum_{\alpha,\beta\geqq 2}o(1)dz^\alpha\wedge d\bar{z}^\beta , \\
\Theta_1^{\ j} &= -\sqrt{-1}d\theta\wedge\bar{\partial}(\textstyle\sum_{p\geqq 2}H^{i\bar{p}}H_{1\bar{p}})+\sum_{\alpha\geqq 22}o(1)d\theta\wedge dz^\alpha \\
&\quad +\textstyle\sum_{\alpha\geqq 1}\mathcal{O}(\varepsilon^{-1}(\log\varepsilon)^{-2})d\theta\wedge dz^\alpha \\
&\quad +\textstyle\sum_{\alpha,\beta\geqq 2}\mathcal{O}(\varepsilon^{-1}(\log\varepsilon)^{-2})dz^\alpha\wedge d\bar{z}^\beta \qquad (j\geqq 2) \\
\Theta_i^{\ 1} &= \textstyle\sum_{\alpha\geqq 2}\mathcal{O}(\varepsilon)d\theta\wedge dz^\alpha+\sum_{\alpha\geqq 2}\mathcal{O}(\varepsilon)d\theta\wedge d\bar{z}^\alpha \\
&\quad +\textstyle\sum_{\alpha,\beta\geqq 2}\mathcal{O}(\varepsilon)dz^\alpha\wedge d\bar{z}^\beta \qquad (i\geqq 2) \\
\Theta_i^{\ j} &= \textstyle\sum_{\alpha\geqq 2}o(1)d\theta\wedge dz^\alpha+\sum_{\alpha\geqq 2}o(1)d\theta\wedge d\bar{z}^\alpha \\
&\quad +\textstyle\sum_{\alpha,\beta\geqq 2}\mathcal{O}(1)dz^\alpha\wedge d\bar{z}^\beta \qquad (i, j\geqq 2) .
\end{aligned}$$

Therefore, on $\{\|\sigma_1\|=\varepsilon\}$,

$$\begin{aligned}
A &= \textstyle\sum_{j\geqq 2}\{\det\begin{pmatrix} \sqrt{-1}(\theta-\mathring{\theta})_1^{\ 1}/2\pi & \sqrt{-1}(\theta-\mathring{\theta})_1^{\ j}/2\pi \\ \sqrt{-1}(\Theta+\mathring{\Theta})_j^{\ 1}/4\pi & \sqrt{-1}(\Theta+\mathring{\Theta})_j^{\ j}/4\pi \end{pmatrix} \\
&\qquad +\det\begin{pmatrix} \sqrt{-1}(\Theta+\mathring{\Theta})_1^{\ 1}/4\pi & \sqrt{-1}(\Theta+\mathring{\Theta})_1^{\ j}/4\pi \\ \sqrt{-1}(\theta-\mathring{\theta})_j^{\ 1}/2\pi & \sqrt{-1}(\theta-\mathring{\theta})_j^{\ j}/2\pi \end{pmatrix}\} \\
&\quad +\textstyle\sum_{I=(i,j),j>i\geqq 2}
\end{aligned}$$

$$= (1+o(1))\,(d\theta/2\pi)\wedge\sqrt{-1}(\textstyle\sum_{j\geq 2}\Theta_j{}^j+\mathring{\Theta}_j{}^j)/4\pi$$
$$+(d\theta/2\pi)\wedge\textstyle\sum_{j\geq 2}\{(\sum_{p\geq 2}H^{j\bar{p}}H_{1\bar{p}})\sqrt{-1}\mathring{\Theta}_j{}^1/4\pi+o(1)\}$$
$$-(d\theta/4\pi)\wedge\{\textstyle\sum_{j\geq 2}(\sqrt{-1}\mathring{\theta}_j{}^1/2\pi)<\bar{\partial}(\sum_{p\geq 2}H^{j\bar{p}}H_{1\bar{p}})+o(1)\}$$
$$+o\left(\frac{1}{\varepsilon}\right)\text{ (terms which do not contain } d\theta).$$

Using the estimate of θ and Θ, it follows on $\{\|\sigma_1\|=\varepsilon\}$,

$$II = (\gamma_1(K_{\bar{M}})+\gamma_1([D]))^{n-2}\wedge A'+o(1)d\theta\wedge dz^2\wedge d\bar{z}^2\wedge\cdots\wedge dz^n\wedge d\bar{z}^n\,,$$
$$III = \textstyle\sum_{r=0}^{n-3}\binom{n-2}{r}(\gamma_1(K_{\bar{M}})+\gamma_1([D]))^r$$
$$\wedge(dd^c(-\textstyle\sum_{i=1}^{k}\log(\log\|\sigma_i\|^2)^2))^{n-2-r}\wedge\mathcal{A}'$$
$$+o(1)d\theta\wedge dz^2\wedge\cdots\wedge d\bar{z}^n\,,$$

where

$$A' = (d\theta/2\pi)\wedge[\sqrt{-1}(\textstyle\sum_{j\geq 1}\Theta_j{}^j)/4\pi$$
$$+\sqrt{-1}\bar{\partial}\{\textstyle\sum_{j\geq 2}(\mathring{\theta}_j{}^j+(\sum_{p\geq 2}H^{j\bar{p}}H_{1\bar{p}})\mathring{\theta}_j{}^1)\}/4\pi]\,.$$

Here, we need two lemmas.

Lemma 4. *$\tilde{\theta}=\sum_{j\geq 2}\{\mathring{\theta}_j{}^j+(\sum_{p\geq 2}H^{j\bar{p}}H_{1\bar{p}})\mathring{\theta}_j{}^1\}$ defines a connection of type* (1, 0) *on the anticanonical bundle of $D_1-(D_2\cup\cdots\cup D_k)$.*

Proof. Let $(U; z^1, z^2, \cdots, z^n)$ and $(V; w^1, w^2, \cdots, w^n)$ be holomorphic local coordinates such that D_1 is realized by the equations $z^1=0$ and $w^1=0$, respectively. Suppose $U\cap V\neq\phi$. Let ${}^t(\partial/\partial w^1, \cdots, \partial/\partial w^n)=G^t(\partial/\partial z^1, \cdots, \partial/\partial z^n)$, then

$$G=\left(\begin{array}{c|c}\partial z^1/\partial w^1 & z\partial^j/\partial w^1\\ \hline \partial z^1/\partial w^i & g\end{array}\right)\text{ where } g=(\partial z^j/\partial w^i)_{i,j\geq 2}\,.$$

On D_1, ${}^t(\partial/\partial w^2, \cdots, \partial/\partial w^n)=g^t(\partial/\partial z^2, \cdots, \partial/\partial z^n)$. The components with respect to w-coordinates will be denoted with prime "$'$". Then the direct computation shows that on $U\cap V\cap D_1$,

$$\tilde{\theta}' = \textstyle\sum_{j\geq 2}\mathring{\theta}'^j_j+(\sum_{p\geq 2}H'^{j\bar{p}}H'_{1\bar{p}})\mathring{\theta}'^1_j$$
$$=\tilde{\theta}+d\{\det(\partial z^j/\partial w^i)_{i,j\geq 2}\}/\det(\partial z^j/\partial w^i)_{i,j\geq 2}\,.$$

This completes the proof of Lemma 4.

Lemma 5 (Mumford [7]). *Let E be a holomorphic vector bundle on a compact complex manifold $\bar{M}$ and D a divisor on $\bar{M}$ with only simple normal crossings. Let θ and Θ denote the connection form and the curvature form of a connection for the restriction E_M on $M=\bar{M}-D$. If both θ and Θ are of Poincaré growth* *along D with respect to an open covering of D by polydisks, then the current defined by the k-th Chern form $\gamma_k(\Theta)$ represents the cohomology class of $c_k(E)$ in $H^{2k}(\bar{M}; C)$. In particular, the Chern numbers of E is the same as those of $\gamma_k(\Theta)$'s.*

A complex valued $C^\infty p$-form η is said to *have Poincaré growth along* $D=\bar{M}-M$ if there is a set of polydisks $U_\alpha\subset\bar{M}$ covering D such that in each U_α, an estimate

$$|\eta(X_1, \cdots, X_p)|^2\leqq C_\alpha P_\alpha(X_1) \cdots P_\alpha(X_p)$$

holds, where P_α is the metric on $U_\alpha\cap M=(\Delta^*)^m\times\Delta^{n-m}$ which is a product of the Poincaré metric on Δ^*'s and tue usual flat metric on Δ's.

We return to the proof of (ii). The connection form $\tilde{\theta}$ and the curvature form $\tilde{\Theta}$ of the anticanonical bundle of $D_1-(D_2\cup \cdots \cup D_k)$ satisfies the condition of Lemma 5. In fact, on Δ^n such that $\Delta^n\cap D=\cup_{i=1}^m \{z\in\Delta^n;\ z^i=0\}$,

$$\begin{aligned}
H_{i\bar{i}} &= 2(1+o(1))/|z^i|^2(\log\|\sigma_i\|^2)^2 && (2\leqq i\leqq m)\,,\\
H_{i\bar{j}} &= -2(\partial\log h_i/\partial\bar{z}^j+o(1))/z^i(\log\|\sigma_i\|^2)^2\\
&\quad -2(\partial\log h_j/\partial z^i+o(1))/\bar{z}^j(\log\|\sigma_j\|^2)^2 && (2\leqq i\neq j\leqq m)\,,\\
H_{i\bar{j}} &= -2(\partial\log h_i/\partial\bar{z}^j+o(1))/z^i(\log\|\sigma_i\|^2)^2 && (2\leqq i\leqq m,\ m+1\leqq j)\,,\\
H_{i\bar{j}} &= \mathcal{O}(1) && (m+1\leqq i, j)\,,\\
H^{i\bar{i}} &= 2^{-1}|z^i|^2(\log\|\sigma_i\|^2)^2(1+o(1)) && (2\leqq i\leqq m),\\
H^{i\bar{j}} &= z^i|z^j|^2\log\|\sigma_j\|^2\mathcal{O}(1)+\bar{z}^j|z^i|^2\log\|\sigma_i\|^2\mathcal{O}(1) && (2\leqq i\neq j\leqq m)\,,\\
H^{i\bar{j}} &= z^i\mathcal{O}(1) && (2\leqq i\leqq m,\ m+1\leqq j)\,,\\
H^{i\bar{j}} &= \mathcal{O}(1) && (m+1\leqq i, j)\,.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
H_{1\bar{p}} &= g_{1\bar{p}} = \mathcal{O}(1)-\sqrt{-1}\textstyle\sum_{i=1}^k(\partial\bar{\partial}\log(\log\|\sigma_i\|^2)^2)_{1\bar{p}} && (2\leqq p\leqq m)\\
H_{1\bar{p}} &= \mathcal{O}(1) && (m+1\leqq p)\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\textstyle\sum_{p\geqq 2}H^{j\bar{p}}H_{1\bar{p}} = \mathcal{O}(1)\\
&\bar{\partial}(\textstyle\sum_{p\geqq 2}H^{j\bar{p}}H_{1\bar{p}}) = \textstyle\sum_{2\geqq i\geqq m}\mathcal{O}(1/\bar{z}^i(\log\|\sigma_i\|^2)^2)d\bar{z}^i+\textstyle\sum_{m+1\geqq i}\mathcal{O}(1)d\bar{z}^i\,.
\end{aligned}$$

Therefore, $\tilde{\theta}$ and $\bar{\partial}\tilde{\theta}$ are of Poincaré growth along $D_1\cap(D_2\cup \cdots \cup D_k)$. Next, we consider $\sum_{j\geqq 1}\Theta_j{}^j$. Because it is the Ricci form of ω,

$$\textstyle\sum_{j\geqq 1}\Theta_j{}^j = \mathrm{Ric}(-\mathrm{Ric}\ \Psi)^n = \mathrm{Ric}(\Psi/\exp F_0) = -\omega+\sqrt{-1}\,\partial\bar{\partial}F_0\,.$$

From the proof of Theorem 1, $\partial\bar{\partial}F_0$ is of Poincaré growth along $D_1\cap(D_2\cup \cdots \cup D_k)$, when $\partial\bar{\partial}F_0$ is considered as a differential form on D_1. Since

$$\begin{aligned}\omega = &\ 2(1+o(1))dz^1\wedge d\bar{z}^1/|z^1|^2(\log\|\sigma_1\|^2)^2\\ &-2(\partial\log h_1/\partial\bar{z}^j+o(1))dz^1\wedge d\bar{z}^j/z^1(\log\|\sigma_1\|^2)^2\\ &-2(\partial\log h_1/\partial z^j+o(1))dz^j\wedge d\bar{z}^1/\bar{z}^1(\log\|\sigma_1\|^2)^2\\ &+\gamma_1(K_{\bar{M}})+\gamma_1([D])-\sqrt{-1}\textstyle\sum_{i\geqq 2}\partial\bar{\partial}\log(\log\|\sigma_i\|^2)^2\,,\end{aligned}$$

where the last term is of Poincaré growth along $D_1\cap(D_2\cup\cdots\cup D_k)$, it follows by Lemma 5 that

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_{\|\sigma_1\|=\varepsilon}(\gamma_1(K_{\bar{M}})+\gamma_1([D]))^{n-2}\wedge d\theta/2\pi\wedge\sqrt{-1}\textstyle\sum_{j\geqq 1}\Theta_j{}^j/4\pi\\ &=\int_{D_1}(\gamma_1(K_{\bar{M}})+\gamma_1([D]))^{n-2}\wedge(-2^{-1})\,(\gamma_1(K+\gamma_1([D]))\\ &=-2^{-1}(k+\delta)^{n-2}(k\delta_1+\delta\delta_1)\,,\end{aligned}$$

and

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_{\|\sigma_1\|=\varepsilon}&\textstyle\sum_{r=0}^{n-3}\binom{n-2}{r}\,(\gamma_1(K)+\gamma_1([D]))^r\\ &\wedge(dd^c(-\textstyle\sum_i\log(\log\|\sigma_i\|^2)^2))^{n-2-r}\\ &\wedge d\theta/2\pi\wedge\sqrt{-1}\textstyle\sum_{j\geqq 1}\Theta_j{}^j/4\pi\\ =0\,.&\end{aligned}$$

On the other hand, by Lemma 5

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_{\|\sigma_1\|=\varepsilon}(\gamma_1(K_{\bar{M}})+\gamma_1([D]))^{n-2}\wedge(d\theta/2\pi)\wedge\sqrt{-1}\,\bar{\partial}\tilde{\theta}/4\pi\\ &=\int_{D_1}(\gamma_1(K_{\bar{M}})+\gamma_1([D]))^{n-2}\wedge 2^{-1}\gamma_1(K_{D_1^*})\\ &=-\int_{D_1}(\gamma_1(K_{\bar{M}})+\gamma_1([D]))^{n-2}\wedge 2^{-1}(\gamma_1(K_{\bar{M}})+\gamma_1([D_1]))\\ &=-2^{-1}(k+\delta)^{n-2}(k\delta_1+\delta_1^2) \qquad\qquad \text{(by the adjunction formula)}\end{aligned}$$

and the remaining integral equals zero.

Summing up the above arguments, we get

$$\begin{aligned}&\lim_{j\uparrow\infty}\int_{\partial\Omega_j}(I+II+III)\\ &=-\textstyle\sum_{i=1}^k\{-2^{-1}(k+\delta)^{n-2}(k\delta_i+\delta\delta_i)-2^{-1}(k+\delta)^{n-2}(k\delta_i+\delta_i^2)\}\\ &=(k+\delta)^{n-2}(k\delta+2^{-1}\delta^2+2^{-1}\textstyle\sum_{i=1}^k\delta_i^2)\,,\end{aligned}$$

hence

$$(-1)^n\bar{c}_1{}^{n-2}\bar{c}_2=(k+\delta)^{n-2}(c_2+k\delta+2^{-1}\delta^2+2^{-1}\textstyle\sum_{i=1}^k\delta_i^2)\,.$$

We have finished the proof of Proposition 1.

Now we can prove Theorem 2. Let (N,ω) be an n-dimensional Kähler manifold. Let γ_1 and γ_2 denote the first and second Chern forms computed from the Riemannian connection of ω. Then

$$(8)\qquad \begin{aligned} \omega^{n-2}\wedge\gamma_1^2 &= \{(n-2)!(\tau^2-2\|\rho\|^2)/16\pi^2\}*1\,, \\ \omega^{n-2}\wedge\gamma_2 &= \{(n-2)!(\tau^2-4\|\rho\|^2+\|R\|^2)/32\pi^2\}*1\,, \end{aligned}$$

where R, ρ, τ, and $\|\cdot\|$ denote the curvature tensor, the Ricci tensor, the scalar curvature and the Riemannian norm of ω, respectively. If ω is Kähler-Einstein, then $\gamma_1=\lambda\omega$ for some $\lambda\in R$ and $\tau^2-4\|\rho\|^2=2(n-2)\|\rho\|^2$. Therefore, if $\lambda=-1/2\pi$, i.e., if $\rho=-$(metric), (8) can be written as

$$(9)\qquad \begin{aligned} (-1)^n\gamma_1^n &= \{(n-1)!\|\rho\|^2/2(2\pi)^n\}*1\,, \\ (-1)^n\gamma_1^{n-2}\gamma_2 &= \{(n-2)!(2(n-2)\|\rho\|^2+\|R\|^2)/(2\pi)^n\cdot 8\}*1\,. \end{aligned}$$

On the other hand, we have the following inequality for a general Kähler manifold:

$$(10)\qquad (n+1)\|R\|^2\geqq 4\|\rho\|^2\,.$$

Combining (9) and (10), for a Kähler-Einstein manifold with negative Ricci curvature, we get

$$2(n+1)\,(-1)^n\gamma_1^{n-2}\gamma_2\geqq n(-1)^n\gamma_1^n\,.$$

This pointwise inequality holds for our complete Kähler-Einstein manifold $(\bar{M}-D,\ \tilde{\omega})$. Integrating this inequality over $M=\bar{M}-D$, and applying Proposition 1, we get Theorem 2.

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Department of Mathematics
Faculty of Science
Tôhoku University
Aramaki, Sendai 980
Japan