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\aleph_0 -CONTINUOUS MODULES

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Generalizing the notion of right \aleph_0 -continuous regular rings (see [2], [3]) we define that of quasi- \aleph_0 - and \aleph_0 -continuous modules and mainly study the directly finiteness of nonsingular \aleph_0 -continuous modules over (von Neumann) regular rings.

Let R be a regular ring. By \mathcal{F} we denote the family of all essentially \aleph_0 -generated essential right ideals of R . It is shown that \mathcal{F} becomes a right Gabriel topology on R (Proposition 5). From this fact the divisible hull $E_{\mathcal{F}}(M)$ of a given right R -module M is considered. Our main purpose of this note is to prove that a nonsingular \aleph_0 -continuous R -module M is directly finite if and only if so is $E_{\mathcal{F}}(M)$. This is a generalization of a result due to Goodearl [3].

Throughout this paper R is a ring with identity and all R -modules considered are unitary right R -modules.

For a given R -module M , we denote its injective hull by $E(M)$ and the family of all submodules of M by $\mathcal{L}(M)$.

For $N \in \mathcal{L}(M)$ $N \leq_e M$ means that N is an essential submodule of M and $(N: x)$, for $x \in M$, denotes the right ideal $\{r \in R \mid xr \in N\}$.

Let M be an R -module. An \mathcal{S} -closed submodule of M is a submodule B such that M/B is nonsingular. For any submodule A of M there exists the smallest \mathcal{S} -closed submodule C of M containing A , which is called the \mathcal{S} -closure of A in M (see [1]). We note that, when M is nonsingular, the \mathcal{S} -closure C of A in M is uniquely determined as a submodule C such that $A \leq_e C$ and C is \mathcal{S} -closed in M .

Lemma 1. *Let M be an R -module, and let A and B be submodules of M such that $A \leq_e B$. Then B is contained in the \mathcal{S} -closure of A in M . In addition if M is nonsingular and B is a direct summand of M , then B coincides with the \mathcal{S} -closure of A in M .*

Proof. Let C be the \mathcal{S} -closure of A in M . Since $(B+C)/C$ is an epimorphic image of a singular module B/A , we see that $(B+C)/C$ is singular. On the other hand, $(B+C)/C$ is a submodule of a nonsingular module M/C ,

whence $(B+C)/C$ is nonsingular. As a result, we have that $(B+C)/C=0$, and so $B \leq C$ as desired.

Let M be an R -module. We consider a subfamily \mathcal{A} of $\mathcal{L}(M)$ which is closed under isomorphic images and essential extensions. For such an \mathcal{A} the following conditions are studied in [5]:

(C₁) For any $A \in \mathcal{A}$ there exists a direct summand A^* of M such that $A \leq_e A^*$.

(C₂) If $A \in \mathcal{A}$ is a direct summand of M , then any exact sequence $0 \rightarrow A \rightarrow M$ splits.

(C₃) If $A \in \mathcal{A}$, $N \in \mathcal{L}(M)$ and both of them are direct summands of M with $A \cap N = 0$, then $A \oplus N$ is also a direct summand of M .

Note that if M is nonsingular and \mathcal{A} satisfies the condition (C₁), then, for each $A \in \mathcal{A}$, A^* coincides with the \mathcal{S} -closure of A in M .

Following [5] we call M \mathcal{A} -continuous (resp. \mathcal{A} -quasi-continuous) if M satisfies the conditions (C₁) and (C₂) (resp. (C₁) and (C₃)). Especially if M is $\mathcal{L}(M)$ -continuous (resp. $\mathcal{L}(M)$ -quasi-continuous) we simply call M continuous (resp. quasi-continuous). It follows from [5] that \mathcal{A} -continuous modules are \mathcal{A} -quasi-continuous, quasi-injective modules are continuous and that M is \mathcal{A} -quasi-continuous if and only if M satisfies (C₁) and the condition:

(*) For any $A \in \mathcal{A}$ and $N \in \mathcal{L}(M)$ such that N is a direct summand of M and $A \cap N = 0$, every homomorphism from A to N can be extended to a homomorphism from M to N .

We now introduce the notion of quasi- \aleph_0 -continuous modules and \aleph_0 -continuous modules. Let M be an R -module and consider the family $\mathcal{A}(M)$ of all submodules A of M such that A contains a countably generated essential submodule. Then $\mathcal{A}(M)$ is closed under isomorphic images and essential extensions. We say that M is \aleph_0 -continuous (resp. quasi- \aleph_0 -continuous) if M is $\mathcal{A}(M)$ -continuous (resp. $\mathcal{A}(M)$ -quasi-continuous).

An R -module M is directly finite provided that M is not isomorphic to any proper direct summand of itself. If M is not directly finite, then M is said to be directly infinite. It is well-known that M is directly finite if and only if for all $f, g \in \text{End}_R(M)$, $fg=1$ implies $gf=1$.

Theorem 2. *For a given nonsingular \aleph_0 -continuous R -module M , the following conditions are equivalent:*

- (a) M is directly finite.
- (b) M contains no infinite direct sums of nonzero pairwise isomorphic submodules.
- (c) Any submodule of M is directly finite.

Proof. (a) \Rightarrow (b): Assume that M is directly finite. It suffices to show

that if $\{A_1, A_2, \dots\}$ is an independent sequence of pairwise isomorphic cyclic submodules of M , then $A_1=0$. Set $B_i = \bigoplus_{n=0}^{\infty} A_{3n+i}$ for $i=1, 2, 3$. Then $\bigoplus_{n=1}^{\infty} A_n = B_1 \oplus B_2 \oplus B_3$ and $B_2 \cong B_3$, and $B_1 \oplus B_2 \cong B_3$. By the condition (C₁), there exists a direct summand B_i^* of M such that $B_i \leq_e B_i^*$ for each i . Using the condition (*) we have a homomorphism $f: B_2^* \rightarrow B_3^*$ which is an extension of the isomorphism $B_2 \cong B_3$. Then f is a monomorphism, because $B_2 \leq_e B_2^*$. Also, using the condition (C₂) we see that $f(B_2^*)$ is a direct summand of M containing B_3 ; hence by the uniqueness of the \mathcal{S} -closure $f(B_2^*) = B_3^*$. Thus B_2^* is isomorphic to B_3^* . Similarly, $(B_1 \oplus B_2)^*$ is isomorphic to B_3^* . By the condition (C₃), $B_1^* \oplus B_2^*$ is the \mathcal{S} -closure of $B_1 \oplus B_2$ in M . Therefore $(B_1 \oplus B_2)^* = B_1^* \oplus B_2^*$, and so $B_1^* \oplus B_2^* \cong B_2^*$. As M is directly finite, $B_1^* \oplus B_2^*$ is also directly finite, from which $B_1^* = 0$. Thus we see that $A_1 = 0$ as desired.

(b) \Rightarrow (c): Let N be a submodule of M , and let $N = N_1 \oplus N_2$ with an isomorphism $f: N \cong N_1$. Then,

$$\begin{aligned} N &= N_1 \oplus N_2 \\ &= f(N_1) \oplus f(N_2) \oplus N_2 \\ &= f^2(N_1) \oplus f^2(N_2) \oplus f(N_2) \oplus N_2 \\ &= \dots \end{aligned}$$

It follows that $\{N_2, f(N_2), f^2(N_2), \dots\}$ is an independent sequence of pairwise isomorphic submodules of M . By assumption $N_2 = 0$ and so N is directly finite.

(c) \Rightarrow (a) is clear.

The following lemma is well-known and, as is easily seen, the same conclusion is valid for sums of infinite many submodules.

Lemma 3. *Let A, B, C and D be submodules of a nonsingular module M such that $A \leq_e B$ and $C \leq_e D$. Then $(A+C) \leq_e (B+D)$.*

Lemma 4 ([2, Lemma 14.10]). *Let M be a projective module over a regular ring R , and let \mathcal{L} denote the collection of all countably generated submodules of M .*

(a) *If $J, K \in \mathcal{L}$, then $J \cap K \in \mathcal{L}$.*

(b) *If $J, K \in \mathcal{L}$ and $f \in \text{Hom}_R(J, M)$, then $f^{-1}(K) = \{x \in J \mid f(x) \in K\} \in \mathcal{L}$.*

Now, let \mathcal{E} be the collection of all countably generated essential right ideals of a ring R and let \mathcal{F} be the collection of all right ideals which contain a member of \mathcal{E} . Then we have the following proposition.

Proposition 5. *If R is a regular ring, then \mathcal{F} is a right Gabriel topology, i.e., \mathcal{F} is not empty and satisfies the following conditions:*

(T₁) If $I \in \mathcal{F}$ and $a \in R$, then $(I: a) \in \mathcal{F}$.

(T₂) If I is a right ideal and there exists $J \in \mathcal{F}$ such that $(I: a) \in \mathcal{F}$ for every $a \in J$, then $I \in \mathcal{F}$.

Proof. Suppose that $I \in \mathcal{F}$ and $a \in R$. Then there exists $J \in \mathcal{E}$ such that $J \leq I$. Noting that $(I: a) \geq (J: a)$, we may show that $(J: a) \in \mathcal{E}$. $J \leq_e R$ implies that $(J: a) \leq_e R$ and by Lemma 4 $(J: a)$ is countably generated. Therefore $(J: a) \in \mathcal{E}$. Now suppose that I is a right ideal and that there exists $J \in \mathcal{F}$ such that $(I: a) \in \mathcal{F}$ for each $a \in J$. Then there exists $K \in \mathcal{E}$ with $K \leq J$. Put $K = \sum_{n=1}^{\infty} a_n R$. Then by the assumption, there exists $I_n \in \mathcal{E}$ with $(I: a_n) \geq I_n$ for each n . The mapping $f: \bigoplus_{n=1}^{\infty} R \rightarrow K$ given by $f((r_n)) = \sum a_n r_n$ is an epimorphism. Note that $f(\bigoplus_{n=1}^{\infty} I_n) \leq K \cap I$ and $f(\bigoplus_{n=1}^{\infty} I_n)$ is countably generated. Since f induces an epimorphism $\bigoplus_{n=1}^{\infty} R / \bigoplus_{n=1}^{\infty} I_n \rightarrow K / f(\bigoplus_{n=1}^{\infty} I_n)$ and $\bigoplus_{n=1}^{\infty} R / \bigoplus_{n=1}^{\infty} I_n$ is singular by Lemma 3, $K / f(\bigoplus_{n=1}^{\infty} I_n)$ is also singular. Hence $f(\bigoplus_{n=1}^{\infty} I_n) \leq_e K \leq_e R$ and so $f(\bigoplus_{n=1}^{\infty} I_n) \in \mathcal{E}$. Thus $I \in \mathcal{F}$ as desired.

For a given module M over a regular ring R , we put

$$\begin{aligned} E_{\mathcal{F}}(M) &= \{x \in E(M) \mid (M: x) \in \mathcal{F}\} \\ &= \{x \in E(M) \mid xI \leq M \text{ for some } I \in \mathcal{E}\}. \end{aligned}$$

$E_{\mathcal{F}}(M)$ is called the \mathcal{F} -injective hull or \mathcal{F} -divisible hull of M (cf. [7, p. 30]).

Lemma 6. Let R be a regular ring and let M be an R -module.

(a) If $M = A \oplus B$ for some submodules A and B , then $E_{\mathcal{F}}(M) = E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$.

(b) Any R -homomorphism from M to an R -module N can be extended to an R -homomorphism from $E_{\mathcal{F}}(M)$ to $E_{\mathcal{F}}(N)$.

Proof. (a) It is clear that $E(M) = E(A) \oplus E(B)$. Let $m \in E_{\mathcal{F}}(M)$. Then $m = a + b$ for some $a \in E(A)$ and $b \in E(B)$. Since $(M: m) \in \mathcal{F}$ and $(M: m) = (A: a) \cap (B: b)$, $(A: a) \in \mathcal{F}$ and $a \in E_{\mathcal{F}}(A)$. Likewise we have $b \in E_{\mathcal{F}}(B)$. Therefore $m \in E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$ and hence $E_{\mathcal{F}}(M) = E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$.

(b) Let $f: M \rightarrow N$ be a homomorphism. Then f can be extended to an R -homomorphism $\tilde{f}: E(M) \rightarrow E(N)$. Let $m \in E_{\mathcal{F}}(M)$. Then $(M: m) \leq (N: \tilde{f}(m)) \in \mathcal{F}$. Hence $\tilde{f}(m)$ lies in $E_{\mathcal{F}}(N)$. Thus the restriction map $\tilde{f}|_{E_{\mathcal{F}}(M)}$ of \tilde{f} is the desired one.

Proposition 7. Let R be a regular ring. If M is a non-singular quasi- \mathfrak{n}_0 -continuous R -module, then so is $E_{\mathcal{F}}(M)$.

Proof. First we show that the condition (C₁) holds for $\mathcal{A}(E_{\mathcal{F}}(M))$. Let

$L \in \mathcal{A}(E_{\mathcal{F}}(M))$. There exists a countably generated essential submodule N of L ; say $N = \sum_{n=1}^{\infty} x_n R$. We can take $I_n \in \mathcal{E}$ such that $x_n I_n \leq M$ for each n . Since $K = \sum_{n=1}^{\infty} x_n I_n$ is countably generated, there exists a direct summand K^* of M with $K \leq_e K^*$. It follows that K is an essential submodule of $E_{\mathcal{F}}(K^*)$ and $E_{\mathcal{F}}(K^*)$ is a direct summand of $E_{\mathcal{F}}(M)$ by Lemma 6 (a). On the other hand, K is an essential submodule of N by Lemma 3 and hence is that of L . Therefore from Lemma 1 we see that $L \leq_e E_{\mathcal{F}}(K^*)$.

Next, we prove that the condition (*) for $\mathcal{A}(E_{\mathcal{F}}(M))$ holds. Let $A \in \mathcal{A}(E_{\mathcal{F}}(M))$ and $N \in \mathcal{L}(E_{\mathcal{F}}(M))$ such that N is a direct summand of $E_{\mathcal{F}}(M)$ and $A \cap N = 0$, and let $f: A \rightarrow N$ be a homomorphism. Now there exists a countably generated essential submodule B of A ; say $B = \sum_{n=1}^{\infty} x_n R$. Then $f(B) = \sum_{n=1}^{\infty} f(x_n)R \leq_e f(A) \leq N$. Since both x_n and $f(x_n)$ are in $E_{\mathcal{F}}(M)$ for each n , there exist I_n' and I_n'' in \mathcal{E} such that $x_n I_n' \leq M$ and $f(x_n) I_n'' \leq M$. Then $I_n = I_n' \cap I_n''$ lies in \mathcal{E} by Lemma 4 (a), and $x_n I_n \leq M$ and $f(x_n) I_n \leq M$. Putting $C = \sum_{n=1}^{\infty} x_n I_n$ and $D = \sum_{n=1}^{\infty} f(x_n) I_n$, we see that C and D are countably generated submodules of M with $C \leq_e B$ and $D \leq_e f(B)$. There exists a direct summand D^* of M such that $D \leq_e D^*$. Using the condition (*) for $\mathcal{A}(M)$, the restriction $f|_C: C \rightarrow D$ of f can be extended to a homomorphism $M \rightarrow D^*$. This also can be extended to a homomorphism $h: E_{\mathcal{F}}(M) \rightarrow E_{\mathcal{F}}(D^*)$ by Lemma 6 (b). Since $E_{\mathcal{F}}(D^*)$ is the \mathcal{S} -closure of D in $E_{\mathcal{F}}(M)$, $E_{\mathcal{F}}(D^*)$ is contained in N . Therefore h is a homomorphism from $E_{\mathcal{F}}(M)$ to N . Since $f|_C = h|_C$ and $C \leq_e B$, $h = f$.

At the end of this note we provide an example to show that the converse of Proposition 7 is not true in general.

Now we are in position to prove our main theorem.

Theorem 8. *Let R be a regular ring, and let M be a non-singular \aleph_0 -continuous R -module. Then M is directly finite if and only if so is $E_{\mathcal{F}}(M)$.*

Proof. The “only if” part. Assume that M is directly finite. If $E_{\mathcal{F}}(M)$ is directly infinite, then there exists an independent sequence of nonzero pairwise isomorphic cyclic submodules of $E_{\mathcal{F}}(M)$; say $\{x_n R\}_{n=1}^{\infty}$. Let $f_n: x_n R \rightarrow x_{n+1} R$ be an isomorphism with $f_n(x_n) = x_{n+1}$, $n = 1, 2, \dots$. For each n , there exists $I_n \in \mathcal{E}$ such that $x_n I_n \leq M$. Then $A_n = x_n(I_1 \cap \dots \cap I_n)$ and $B_n = x_n(I_1 \cap \dots \cap I_{n+1})$ are countably generated submodules of M such that $B_n \leq_e A_n \leq_e x_n R$ and the restriction $f_n|_{B_n}$ of f_n to B_n is an isomorphism between B_n and A_{n+1} . By the assumption, there exist direct summands A_n^* and B_n^* of M with $A_n \leq_e A_n^*$ and $B_n \leq_e B_n^*$ and further $f_n|_{B_n}$ can be extended to an isomorphism g_n between

B_n^* and A_{n+1}^* . Hence we have $\bigoplus_{n=1}^{\infty} B_n^* \cong \bigoplus_{n=1}^{\infty} A_{n+1}^*$. On the other hand, by Lemma 1, for each n $A_n \leq B_n^*$. So $A_n \leq_e B_n^*$ which implies that $B_n^* = A_n^*$. Therefore $\bigoplus_{n=1}^{\infty} B_n^* = B_1^* \oplus (\bigoplus_{n=1}^{\infty} A_{n+1}^*)$. However, this is a contradiction, because $\bigoplus_{n=1}^{\infty} B_n^*$ is directly finite by Theorem 2. Therefore $E_{\mathcal{F}}(M)$ is directly finite.

The “if” part. Assume that $E_{\mathcal{F}}(M)$ is directly finite, and consider f and g in $End_R(M)$ such that $fg = 1_{End_R(M)}$. By Lemma 6 (b), there exist \bar{f} and \bar{g} in $End_R(E_{\mathcal{F}}(M))$ such that \bar{f} and \bar{g} are extensions of f and g respectively. Noting that $M \leq_e E_{\mathcal{F}}(M)$ and $E_{\mathcal{F}}(M)$ is nonsingular, we obtain that $\bar{f}\bar{g} = 1_{End_R(E_{\mathcal{F}}(M))}$. By the assumption $\bar{g}\bar{f} = 1_{End_R(E_{\mathcal{F}}(M))}$, from which $gf = 1_{End_R(M)}$. Therefore M is directly finite.

Proposition 9. *Assume that M is a nonsingular \aleph_0 -continuous R -module with the following condition:*

(#) *For any submodules A and B of M with $A \cap B = 0$, any isomorphism from A to B can be extended to a homomorphism of \bar{A} to \bar{B} , where \bar{A} and \bar{B} are the \mathcal{S} -closures of A and B in M respectively. Then, M is directly finite if and only if so is $E(M)$.*

Proof. Noting that $E(M)$ is nonsingular \aleph_0 -continuous, the “if” part is clear by Theorem 2.

The “only if” part. Assume that M is directly finite. If $E(M)$ is directly infinite, there exists an infinite and independent sequence of nonzero pairwise isomorphic submodules of $E(M)$; say $\{A_n\}_{n=1}^{\infty}$. Let f_n be the isomorphism between A_n and A_{n+1} for each n . Set $B_1 = A_1 \cap M$ and define inductively $B_{n+1} = f_n(B_n) \cap M$ and $C_n = f_n^{-1}(B_{n+1})$ for $n = 1, 2, \dots$. Then, for each n , B_n and C_n are submodules of $A_n \cap M$ with $C_n \leq_e B_n$ and the \mathcal{S} -closure \bar{B}_n of B_n in M coincides with \bar{C}_n , the \mathcal{S} -closure of C_n in M , by the similar way in the proof of Theorem 8. The restriction map $f_n|_{C_n}$ is an isomorphism from C_n to B_{n+1} . Using the condition (#), $f_n|_{C_n}$ can be extended to a monomorphism $\bar{f}_n: \bar{C}_n \rightarrow \bar{B}_{n+1} = \bar{C}_{n+1}$. Consequently $\{\bar{C}_1, \bar{f}_1(\bar{C}_1), \bar{f}_2(\bar{f}_1(\bar{C}_1)), \dots\}$ is an independent sequence of nonzero pairwise isomorphic submodules of M , which is a contradiction by Theorem 2. Thus the proof is completed.

Corollary 10. *Let M be a nonsingular continuous R -module. Then M is directly finite if and only if so is $E(M)$.*

Finally we show the following result.

Theorem 11. *Let R be a regular ring and M a finitely generated projective \aleph_0 -continuous R -module. If A is a projective maximal submodule of M , then A is a direct summand of M .*

Proof. A can be written as a direct sum of cyclic submodules; say $A = \bigoplus_{\alpha \in I} x_\alpha R$ [4]. We claim that I is a finite set. If I is an infinite set, then we have a countable subset J of I such that $I - J$ is an infinite set and so $A = (\bigoplus_{\alpha \in J} x_\alpha R) \oplus (\bigoplus_{\beta \in I - J} x_\beta R)$. Since M is \aleph_0 -continuous, we have a direct summand B^* of M such that $\bigoplus_{\alpha \in J} x_\alpha R \leq_e B^*$ and $A \leq B^* \oplus (\bigoplus_{\beta \in I - J} x_\beta R) \leq M$. If $A = B^* \oplus (\bigoplus_{\beta \in I - J} x_\beta R)$, then $\bigoplus_{\alpha \in J} x_\alpha R$ coincides with B^* and is finitely generated. If $B^* \oplus (\bigoplus_{\beta \in I - J} x_\beta R) = M$, then $\bigoplus_{\beta \in I - J} x_\beta R$ is finitely generated. In any case we have a contradiction. Therefore A is finitely generated and so it is a direct summand of M .

As a consequence of Theorem 11, we obtain the following which is a slight generalization of [6, Corollary].

Corollary 12. *If R is a right hereditary, right \aleph_0 -continuous, regular ring, then R is a semi-simple artinian ring.*

REMARK. In general, the converse of Proposition 7 is not true. For example, take a field F and set $R_n = M_{2^n}(F)$ for all $n = 1, 2, \dots$. Map each $R_n \rightarrow R_{n+1}$ along the diagonal, i.e., map $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, and set $R = \varinjlim R_n$. Then R is a simple, right hereditary, not artinian, regular ring with a unique dimension function (see [2]). Note that for a regular ring R , R_R is quasi- \aleph_0 -continuous if and only if R_R is \aleph_0 -continuous. Therefore we see that $E_{\mathcal{F}}(R) = E_{\mathcal{L}}(R)(R) = E(R)$ is a nonsingular quasi- \aleph_0 -continuous R -module, but R is not quasi- \aleph_0 -continuous by Corollary 12.

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