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## $\aleph_0$ -CONTINUOUS MODULES

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Generalizing the notion of right  $\aleph_0$ -continuous regular rings (see [2], [3]) we define that of quasi- $\aleph_0$ - and  $\aleph_0$ -continuous modules and mainly study the directly finiteness of nonsingular  $\aleph_0$ -continuous modules over (von Neumann) regular rings.

Let  $R$  be a regular ring. By  $\mathcal{F}$  we denote the family of all essentially  $\aleph_0$ -generated essential right ideals of  $R$ . It is shown that  $\mathcal{F}$  becomes a right Gabriel topology on  $R$  (Proposition 5). From this fact the divisible hull  $E_{\mathcal{F}}(M)$  of a given right  $R$ -module  $M$  is considered. Our main purpose of this note is to prove that a nonsingular  $\aleph_0$ -continuous  $R$ -module  $M$  is directly finite if and only if so is  $E_{\mathcal{F}}(M)$ . This is a generalization of a result due to Goodearl [3].

Throughout this paper  $R$  is a ring with identity and all  $R$ -modules considered are unitary right  $R$ -modules.

For a given  $R$ -module  $M$ , we denote its injective hull by  $E(M)$  and the family of all submodules of  $M$  by  $\mathcal{L}(M)$ .

For  $N \in \mathcal{L}(M)$   $N \leq_e M$  means that  $N$  is an essential submodule of  $M$  and  $(N: x)$ , for  $x \in M$ , denotes the right ideal  $\{r \in R \mid xr \in N\}$ .

Let  $M$  be an  $R$ -module. An  $\mathcal{S}$ -closed submodule of  $M$  is a submodule  $B$  such that  $M/B$  is nonsingular. For any submodule  $A$  of  $M$  there exists the smallest  $\mathcal{S}$ -closed submodule  $C$  of  $M$  containing  $A$ , which is called the  $\mathcal{S}$ -closure of  $A$  in  $M$  (see [1]). We note that, when  $M$  is nonsingular, the  $\mathcal{S}$ -closure  $C$  of  $A$  in  $M$  is uniquely determined as a submodule  $C$  such that  $A \leq_e C$  and  $C$  is  $\mathcal{S}$ -closed in  $M$ .

**Lemma 1.** *Let  $M$  be an  $R$ -module, and let  $A$  and  $B$  be submodules of  $M$  such that  $A \leq_e B$ . Then  $B$  is contained in the  $\mathcal{S}$ -closure of  $A$  in  $M$ . In addition if  $M$  is nonsingular and  $B$  is a direct summand of  $M$ , then  $B$  coincides with the  $\mathcal{S}$ -closure of  $A$  in  $M$ .*

**Proof.** Let  $C$  be the  $\mathcal{S}$ -closure of  $A$  in  $M$ . Since  $(B+C)/C$  is an epimorphic image of a singular module  $B/A$ , we see that  $(B+C)/C$  is singular. On the other hand,  $(B+C)/C$  is a submodule of a nonsingular module  $M/C$ ,

whence  $(B+C)/C$  is nonsingular. As a result, we have that  $(B+C)/C=0$ , and so  $B \leq C$  as desired.

Let  $M$  be an  $R$ -module. We consider a subfamily  $\mathcal{A}$  of  $\mathcal{L}(M)$  which is closed under isomorphic images and essential extensions. For such an  $\mathcal{A}$  the following conditions are studied in [5]:

(C<sub>1</sub>) For any  $A \in \mathcal{A}$  there exists a direct summand  $A^*$  of  $M$  such that  $A \leq_e A^*$ .

(C<sub>2</sub>) If  $A \in \mathcal{A}$  is a direct summand of  $M$ , then any exact sequence  $0 \rightarrow A \rightarrow M$  splits.

(C<sub>3</sub>) If  $A \in \mathcal{A}$ ,  $N \in \mathcal{L}(M)$  and both of them are direct summands of  $M$  with  $A \cap N = 0$ , then  $A \oplus N$  is also a direct summand of  $M$ .

Note that if  $M$  is nonsingular and  $\mathcal{A}$  satisfies the condition (C<sub>1</sub>), then, for each  $A \in \mathcal{A}$ ,  $A^*$  coincides with the  $\mathcal{S}$ -closure of  $A$  in  $M$ .

Following [5] we call  $M$   $\mathcal{A}$ -continuous (resp.  $\mathcal{A}$ -quasi-continuous) if  $M$  satisfies the conditions (C<sub>1</sub>) and (C<sub>2</sub>) (resp. (C<sub>1</sub>) and (C<sub>3</sub>)). Especially if  $M$  is  $\mathcal{L}(M)$ -continuous (resp.  $\mathcal{L}(M)$ -quasi-continuous) we simply call  $M$  continuous (resp. quasi-continuous). It follows from [5] that  $\mathcal{A}$ -continuous modules are  $\mathcal{A}$ -quasi-continuous, quasi-injective modules are continuous and that  $M$  is  $\mathcal{A}$ -quasi-continuous if and only if  $M$  satisfies (C<sub>1</sub>) and the condition:

(\*) For any  $A \in \mathcal{A}$  and  $N \in \mathcal{L}(M)$  such that  $N$  is a direct summand of  $M$  and  $A \cap N = 0$ , every homomorphism from  $A$  to  $N$  can be extended to a homomorphism from  $M$  to  $N$ .

We now introduce the notion of quasi- $\aleph_0$ -continuous modules and  $\aleph_0$ -continuous modules. Let  $M$  be an  $R$ -module and consider the family  $\mathcal{A}(M)$  of all submodules  $A$  of  $M$  such that  $A$  contains a countably generated essential submodule. Then  $\mathcal{A}(M)$  is closed under isomorphic images and essential extensions. We say that  $M$  is  $\aleph_0$ -continuous (resp. quasi- $\aleph_0$ -continuous) if  $M$  is  $\mathcal{A}(M)$ -continuous (resp.  $\mathcal{A}(M)$ -quasi-continuous).

An  $R$ -module  $M$  is directly finite provided that  $M$  is not isomorphic to any proper direct summand of itself. If  $M$  is not directly finite, then  $M$  is said to be directly infinite. It is well-known that  $M$  is directly finite if and only if for all  $f, g \in \text{End}_R(M)$ ,  $fg=1$  implies  $gf=1$ .

**Theorem 2.** *For a given nonsingular  $\aleph_0$ -continuous  $R$ -module  $M$ , the following conditions are equivalent:*

- (a)  $M$  is directly finite.
- (b)  $M$  contains no infinite direct sums of nonzero pairwise isomorphic submodules.
- (c) Any submodule of  $M$  is directly finite.

**Proof.** (a)  $\Rightarrow$  (b): Assume that  $M$  is directly finite. It suffices to show

that if  $\{A_1, A_2, \dots\}$  is an independent sequence of pairwise isomorphic cyclic submodules of  $M$ , then  $A_1=0$ . Set  $B_i = \bigoplus_{n=0}^{\infty} A_{3n+i}$  for  $i=1, 2, 3$ . Then  $\bigoplus_{n=1}^{\infty} A_n = B_1 \oplus B_2 \oplus B_3$  and  $B_2 \cong B_3$ , and  $B_1 \oplus B_2 \cong B_3$ . By the condition  $(C_1)$ , there exists a direct summand  $B_i^*$  of  $M$  such that  $B_i \leq_e B_i^*$  for each  $i$ . Using the condition  $(*)$  we have a homomorphism  $f: B_2^* \rightarrow B_3^*$  which is an extension of the isomorphism  $B_2 \cong B_3$ . Then  $f$  is a monomorphism, because  $B_2 \leq_e B_2^*$ . Also, using the condition  $(C_2)$  we see that  $f(B_2^*)$  is a direct summand of  $M$  containing  $B_3$ ; hence by the uniqueness of the  $\mathcal{S}$ -closure  $f(B_2^*) = B_3^*$ . Thus  $B_2^*$  is isomorphic to  $B_3^*$ . Similarly,  $(B_1 \oplus B_2)^*$  is isomorphic to  $B_3^*$ . By the condition  $(C_3)$ ,  $B_1^* \oplus B_2^*$  is the  $\mathcal{S}$ -closure of  $B_1 \oplus B_2$  in  $M$ . Therefore  $(B_1 \oplus B_2)^* = B_1^* \oplus B_2^*$ , and so  $B_1^* \oplus B_2^* \cong B_2^*$ . As  $M$  is directly finite,  $B_1^* \oplus B_2^*$  is also directly finite, from which  $B_1^* = 0$ . Thus we see that  $A_1 = 0$  as desired.

(b)  $\Rightarrow$  (c): Let  $N$  be a submodule of  $M$ , and let  $N = N_1 \oplus N_2$  with an isomorphism  $f: N \cong N_1$ . Then,

$$\begin{aligned} N &= N_1 \oplus N_2 \\ &= f(N_1) \oplus f(N_2) \oplus N_2 \\ &= f^2(N_1) \oplus f^2(N_2) \oplus f(N_2) \oplus N_2 \\ &= \dots \end{aligned}$$

It follows that  $\{N_2, f(N_2), f^2(N_2), \dots\}$  is an independent sequence of pairwise isomorphic submodules of  $M$ . By assumption  $N_2 = 0$  and so  $N$  is directly finite.

(c)  $\Rightarrow$  (a) is clear.

The following lemma is well-known and, as is easily seen, the same conclusion is valid for sums of infinite many submodules.

**Lemma 3.** *Let  $A, B, C$  and  $D$  be submodules of a nonsingular module  $M$  such that  $A \leq_e B$  and  $C \leq_e D$ . Then  $(A+C) \leq_e (B+D)$ .*

**Lemma 4** ([2, Lemma 14.10]). *Let  $M$  be a projective module over a regular ring  $R$ , and let  $\mathcal{L}$  denote the collection of all countably generated submodules of  $M$ .*

(a) *If  $J, K \in \mathcal{L}$ , then  $J \cap K \in \mathcal{L}$ .*

(b) *If  $J, K \in \mathcal{L}$  and  $f \in \text{Hom}_R(J, M)$ , then  $f^{-1}(K) = \{x \in J \mid f(x) \in K\} \in \mathcal{L}$ .*

Now, let  $\mathcal{E}$  be the collection of all countably generated essential right ideals of a ring  $R$  and let  $\mathcal{F}$  be the collection of all right ideals which contain a member of  $\mathcal{E}$ . Then we have the following proposition.

**Proposition 5.** *If  $R$  is a regular ring, then  $\mathcal{F}$  is a right Gabriel topology, i.e.,  $\mathcal{F}$  is not empty and satisfies the following conditions:*

(T<sub>1</sub>) If  $I \in \mathcal{F}$  and  $a \in R$ , then  $(I: a) \in \mathcal{F}$ .

(T<sub>2</sub>) If  $I$  is a right ideal and there exists  $J \in \mathcal{F}$  such that  $(I: a) \in \mathcal{F}$  for every  $a \in J$ , then  $I \in \mathcal{F}$ .

Proof. Suppose that  $I \in \mathcal{F}$  and  $a \in R$ . Then there exists  $J \in \mathcal{E}$  such that  $J \leq I$ . Noting that  $(I: a) \geq (J: a)$ , we may show that  $(J: a) \in \mathcal{E}$ .  $J \leq_e R$  implies that  $(J: a) \leq_e R$  and by Lemma 4  $(J: a)$  is countably generated. Therefore  $(J: a) \in \mathcal{E}$ . Now suppose that  $I$  is a right ideal and that there exists  $J \in \mathcal{F}$  such that  $(I: a) \in \mathcal{F}$  for each  $a \in J$ . Then there exists  $K \in \mathcal{E}$  with  $K \leq J$ . Put  $K = \sum_{n=1}^{\infty} a_n R$ . Then by the assumption, there exists  $I_n \in \mathcal{E}$  with  $(I: a_n) \geq I_n$  for each  $n$ . The mapping  $f: \bigoplus_{n=1}^{\infty} R \rightarrow K$  given by  $f((r_n)) = \sum a_n r_n$  is an epimorphism. Note that  $f(\bigoplus_{n=1}^{\infty} I_n) \leq K \cap I$  and  $f(\bigoplus_{n=1}^{\infty} I_n)$  is countably generated. Since  $f$  induces an epimorphism  $\bigoplus_{n=1}^{\infty} R / \bigoplus_{n=1}^{\infty} I_n \rightarrow K / f(\bigoplus_{n=1}^{\infty} I_n)$  and  $\bigoplus_{n=1}^{\infty} R / \bigoplus_{n=1}^{\infty} I_n$  is singular by Lemma 3,  $K / f(\bigoplus_{n=1}^{\infty} I_n)$  is also singular. Hence  $f(\bigoplus_{n=1}^{\infty} I_n) \leq_e K \leq_e R$  and so  $f(\bigoplus_{n=1}^{\infty} I_n) \in \mathcal{E}$ . Thus  $I \in \mathcal{F}$  as desired.

For a given module  $M$  over a regular ring  $R$ , we put

$$\begin{aligned} E_{\mathcal{F}}(M) &= \{x \in E(M) \mid (M: x) \in \mathcal{F}\} \\ &= \{x \in E(M) \mid xI \leq M \text{ for some } I \in \mathcal{E}\}. \end{aligned}$$

$E_{\mathcal{F}}(M)$  is called the  $\mathcal{F}$ -injective hull or  $\mathcal{F}$ -divisible hull of  $M$  (cf. [7, p. 30]).

**Lemma 6.** Let  $R$  be a regular ring and let  $M$  be an  $R$ -module.

(a) If  $M = A \oplus B$  for some submodules  $A$  and  $B$ , then  $E_{\mathcal{F}}(M) = E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$ .

(b) Any  $R$ -homomorphism from  $M$  to an  $R$ -module  $N$  can be extended to an  $R$ -homomorphism from  $E_{\mathcal{F}}(M)$  to  $E_{\mathcal{F}}(N)$ .

Proof. (a) It is clear that  $E(M) = E(A) \oplus E(B)$ . Let  $m \in E_{\mathcal{F}}(M)$ . Then  $m = a + b$  for some  $a \in E(A)$  and  $b \in E(B)$ . Since  $(M: m) \in \mathcal{F}$  and  $(M: m) = (A: a) \cap (B: b)$ ,  $(A: a) \in \mathcal{F}$  and  $a \in E_{\mathcal{F}}(A)$ . Likewise we have  $b \in E_{\mathcal{F}}(B)$ . Therefore  $m \in E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$  and hence  $E_{\mathcal{F}}(M) = E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$ .

(b) Let  $f: M \rightarrow N$  be a homomorphism. Then  $f$  can be extended to an  $R$ -homomorphism  $\tilde{f}: E(M) \rightarrow E(N)$ . Let  $m \in E_{\mathcal{F}}(M)$ . Then  $(M: m) \leq (N: \tilde{f}(m)) \in \mathcal{F}$ . Hence  $\tilde{f}(m)$  lies in  $E_{\mathcal{F}}(N)$ . Thus the restriction map  $\tilde{f}|_{E_{\mathcal{F}}(M)}$  of  $\tilde{f}$  is the desired one.

**Proposition 7.** Let  $R$  be a regular ring. If  $M$  is a non-singular quasi- $\mathfrak{n}_0$ -continuous  $R$ -module, then so is  $E_{\mathcal{F}}(M)$ .

Proof. First we show that the condition (C<sub>1</sub>) holds for  $\mathcal{A}(E_{\mathcal{F}}(M))$ . Let

$L \in \mathcal{A}(E_{\mathcal{F}}(M))$ . There exists a countably generated essential submodule  $N$  of  $L$ ; say  $N = \sum_{n=1}^{\infty} x_n R$ . We can take  $I_n \in \mathcal{E}$  such that  $x_n I_n \leq M$  for each  $n$ . Since  $K = \sum_{n=1}^{\infty} x_n I_n$  is countably generated, there exists a direct summand  $K^*$  of  $M$  with  $K \leq_e K^*$ . It follows that  $K$  is an essential submodule of  $E_{\mathcal{F}}(K^*)$  and  $E_{\mathcal{F}}(K^*)$  is a direct summand of  $E_{\mathcal{F}}(M)$  by Lemma 6 (a). On the other hand,  $K$  is an essential submodule of  $N$  by Lemma 3 and hence is that of  $L$ . Therefore from Lemma 1 we see that  $L \leq_e E_{\mathcal{F}}(K^*)$ .

Next, we prove that the condition (\*) for  $\mathcal{A}(E_{\mathcal{F}}(M))$  holds. Let  $A \in \mathcal{A}(E_{\mathcal{F}}(M))$  and  $N \in \mathcal{L}(E_{\mathcal{F}}(M))$  such that  $N$  is a direct summand of  $E_{\mathcal{F}}(M)$  and  $A \cap N = 0$ , and let  $f: A \rightarrow N$  be a homomorphism. Now there exists a countably generated essential submodule  $B$  of  $A$ ; say  $B = \sum_{n=1}^{\infty} x_n R$ . Then  $f(B) = \sum_{n=1}^{\infty} f(x_n) R \leq_e f(A) \leq N$ . Since both  $x_n$  and  $f(x_n)$  are in  $E_{\mathcal{F}}(M)$  for each  $n$ , there exist  $I_n'$  and  $I_n''$  in  $\mathcal{E}$  such that  $x_n I_n' \leq M$  and  $f(x_n) I_n'' \leq M$ . Then  $I_n = I_n' \cap I_n''$  lies in  $\mathcal{E}$  by Lemma 4 (a), and  $x_n I_n \leq M$  and  $f(x_n) I_n \leq M$ . Putting  $C = \sum_{n=1}^{\infty} x_n I_n$  and  $D = \sum_{n=1}^{\infty} f(x_n) I_n$ , we see that  $C$  and  $D$  are countably generated submodules of  $M$  with  $C \leq_e B$  and  $D \leq_e f(B)$ . There exists a direct summand  $D^*$  of  $M$  such that  $D \leq_e D^*$ . Using the condition (\*) for  $\mathcal{A}(M)$ , the restriction  $f|_C: C \rightarrow D$  of  $f$  can be extended to a homomorphism  $M \rightarrow D^*$ . This also can be extended to a homomorphism  $h: E_{\mathcal{F}}(M) \rightarrow E_{\mathcal{F}}(D^*)$  by Lemma 6 (b). Since  $E_{\mathcal{F}}(D^*)$  is the  $\mathcal{S}$ -closure of  $D$  in  $E_{\mathcal{F}}(M)$ ,  $E_{\mathcal{F}}(D^*)$  is contained in  $N$ . Therefore  $h$  is a homomorphism from  $E_{\mathcal{F}}(M)$  to  $N$ . Since  $f|_C = h|_C$  and  $C \leq_e B$ ,  $h = f$ .

At the end of this note we provide an example to show that the converse of Proposition 7 is not true in general.

Now we are in position to prove our main theorem.

**Theorem 8.** *Let  $R$  be a regular ring, and let  $M$  be a non-singular  $\aleph_0$ -continuous  $R$ -module. Then  $M$  is directly finite if and only if so is  $E_{\mathcal{F}}(M)$ .*

*Proof.* The “only if” part. Assume that  $M$  is directly finite. If  $E_{\mathcal{F}}(M)$  is directly infinite, then there exists an independent sequence of nonzero pairwise isomorphic cyclic submodules of  $E_{\mathcal{F}}(M)$ ; say  $\{x_n R\}_{n=1}^{\infty}$ . Let  $f_n: x_n R \rightarrow x_{n+1} R$  be an isomorphism with  $f_n(x_n) = x_{n+1}$ ,  $n = 1, 2, \dots$ . For each  $n$ , there exists  $I_n \in \mathcal{E}$  such that  $x_n I_n \leq M$ . Then  $A_n = x_n (I_1 \cap \dots \cap I_n)$  and  $B_n = x_n (I_1 \cap \dots \cap I_{n+1})$  are countably generated submodules of  $M$  such that  $B_n \leq_e A_n \leq_e x_n R$  and the restriction  $f_n|_{B_n}$  of  $f_n$  to  $B_n$  is an isomorphism between  $B_n$  and  $A_{n+1}$ . By the assumption, there exist direct summands  $A_n^*$  and  $B_n^*$  of  $M$  with  $A_n \leq_e A_n^*$  and  $B_n \leq_e B_n^*$  and further  $f_n|_{B_n}$  can be extended to an isomorphism  $g_n$  between

$B_n^*$  and  $A_{n+1}^*$ . Hence we have  $\bigoplus_{n=1}^{\infty} B_n^* \cong \bigoplus_{n=1}^{\infty} A_{n+1}^*$ . On the other hand, by Lemma 1, for each  $n$   $A_n \leq B_n^*$ . So  $A_n \leq_e B_n^*$  which implies that  $B_n^* = A_n^*$ . Therefore  $\bigoplus_{n=1}^{\infty} B_n^* = B_1^* \oplus (\bigoplus_{n=1}^{\infty} A_{n+1}^*)$ . However, this is a contradiction, because  $\bigoplus_{n=1}^{\infty} B_n^*$  is directly finite by Theorem 2. Therefore  $E_{\mathcal{F}}(M)$  is directly finite.

The “if” part. Assume that  $E_{\mathcal{F}}(M)$  is directly finite, and consider  $f$  and  $g$  in  $End_R(M)$  such that  $fg = 1_{End_R(M)}$ . By Lemma 6 (b), there exist  $\bar{f}$  and  $\bar{g}$  in  $End_R(E_{\mathcal{F}}(M))$  such that  $\bar{f}$  and  $\bar{g}$  are extensions of  $f$  and  $g$  respectively. Noting that  $M \leq_e E_{\mathcal{F}}(M)$  and  $E_{\mathcal{F}}(M)$  is nonsingular, we obtain that  $\bar{f}\bar{g} = 1_{End_R(E_{\mathcal{F}}(M))}$ . By the assumption  $\bar{g}\bar{f} = 1_{End_R(E_{\mathcal{F}}(M))}$ , from which  $gf = 1_{End_R(M)}$ . Therefore  $M$  is directly finite.

**Proposition 9.** *Assume that  $M$  is a nonsingular  $\aleph_0$ -continuous  $R$ -module with the following condition:*

(#) *For any submodules  $A$  and  $B$  of  $M$  with  $A \cap B = 0$ , any isomorphism from  $A$  to  $B$  can be extended to a homomorphism of  $\bar{A}$  to  $\bar{B}$ , where  $\bar{A}$  and  $\bar{B}$  are the  $\mathcal{S}$ -closures of  $A$  and  $B$  in  $M$  respectively. Then,  $M$  is directly finite if and only if so is  $E(M)$ .*

Proof. Noting that  $E(M)$  is nonsingular  $\aleph_0$ -continuous, the “if” part is clear by Theorem 2.

The “only if” part. Assume that  $M$  is directly finite. If  $E(M)$  is directly infinite, there exists an infinite and independent sequence of nonzero pairwise isomorphic submodules of  $E(M)$ ; say  $\{A_n\}_{n=1}^{\infty}$ . Let  $f_n$  be the isomorphism between  $A_n$  and  $A_{n+1}$  for each  $n$ . Set  $B_1 = A_1 \cap M$  and define inductively  $B_{n+1} = f_n(B_n) \cap M$  and  $C_n = f_n^{-1}(B_{n+1})$  for  $n = 1, 2, \dots$ . Then, for each  $n$ ,  $B_n$  and  $C_n$  are submodules of  $A_n \cap M$  with  $C_n \leq_e B_n$  and the  $\mathcal{S}$ -closure  $\bar{B}_n$  of  $B_n$  in  $M$  coincides with  $\bar{C}_n$ , the  $\mathcal{S}$ -closure of  $C_n$  in  $M$ , by the similar way in the proof of Theorem 8. The restriction map  $f_n|_{C_n}$  is an isomorphism from  $C_n$  to  $B_{n+1}$ . Using the condition (#),  $f_n|_{C_n}$  can be extended to a monomorphism  $\bar{f}_n: \bar{C}_n \rightarrow \bar{B}_{n+1} = \bar{C}_{n+1}$ . Consequently  $\{\bar{C}_1, \bar{f}_1(\bar{C}_1), \bar{f}_2(\bar{f}_1(\bar{C}_1)), \dots\}$  is an independent sequence of nonzero pairwise isomorphic submodules of  $M$ , which is a contradiction by Theorem 2. Thus the proof is completed.

**Corollary 10.** *Let  $M$  be a nonsingular continuous  $R$ -module. Then  $M$  is directly finite if and only if so is  $E(M)$ .*

Finally we show the following result.

**Theorem 11.** *Let  $R$  be a regular ring and  $M$  a finitely generated projective  $\aleph_0$ -continuous  $R$ -module. If  $A$  is a projective maximal submodule of  $M$ , then  $A$  is a direct summand of  $M$ .*

Proof.  $A$  can be written as a direct sum of cyclic submodules; say  $A = \bigoplus_{\alpha \in I} x_\alpha R$  [4]. We claim that  $I$  is a finite set. If  $I$  is an infinite set, then we have a countable subset  $J$  of  $I$  such that  $I - J$  is an infinite set and so  $A = (\bigoplus_{\alpha \in J} x_\alpha R) \oplus (\bigoplus_{\beta \in I - J} x_\beta R)$ . Since  $M$  is  $\aleph_0$ -continuous, we have a direct summand  $B^*$  of  $M$  such that  $\bigoplus_{\alpha \in J} x_\alpha R \leq_e B^*$  and  $A \leq B^* \oplus (\bigoplus_{\beta \in I - J} x_\beta R) \leq M$ . If  $A = B^* \oplus (\bigoplus_{\beta \in I - J} x_\beta R)$ , then  $\bigoplus_{\alpha \in J} x_\alpha R$  coincides with  $B^*$  and is finitely generated. If  $B^* \oplus (\bigoplus_{\beta \in I - J} x_\beta R) = M$ , then  $\bigoplus_{\beta \in I - J} x_\beta R$  is finitely generated. In any case we have a contradiction. Therefore  $A$  is finitely generated and so it is a direct summand of  $M$ .

As a consequence of Theorem 11, we obtain the following which is a slight generalization of [6, Corollary].

**Corollary 12.** *If  $R$  is a right hereditary, right  $\aleph_0$ -continuous, regular ring, then  $R$  is a semi-simple artinian ring.*

REMARK. In general, the converse of Proposition 7 is not true. For example, take a field  $F$  and set  $R_n = M_{2^n}(F)$  for all  $n = 1, 2, \dots$ . Map each  $R_n \rightarrow R_{n+1}$  along the diagonal, i.e., map  $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ , and set  $R = \varinjlim R_n$ . Then  $R$  is a simple, right hereditary, not artinian, regular ring with a unique dimension function (see [2]). Note that for a regular ring  $R$ ,  $R_R$  is quasi- $\aleph_0$ -continuous if and only if  $R_R$  is  $\aleph_0$ -continuous. Therefore we see that  $E_{\mathcal{F}}(R) = E_{\mathcal{L}}(R)(R) = E(R)$  is a nonsingular quasi- $\aleph_0$ -continuous  $R$ -module, but  $R$  is not quasi- $\aleph_0$ -continuous by Corollary 12.

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