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Generalizing the notion of right $\mathfrak{K}_0$-continuous regular rings (see [2], [3]) we define that of quasi-$\mathfrak{K}_0$- and $\mathfrak{K}_0$-continuous modules and mainly study the directly finiteness of nonsingular $\mathfrak{K}_0$-continuous modules over (von Neumann) regular rings.

Let $R$ be a regular ring. By $\mathcal{F}$ we denote the family of all essentially $\mathfrak{K}_0$-generated essential right ideals of $R$. It is shown that $\mathcal{F}$ becomes a right Gabriel topology on $R$ (Proposition 5). From this fact the divisible hull $E_{\mathcal{F}}(M)$ of a given right $R$-module $M$ is considered. Our main purpose of this note is to prove that a nonsingular $\mathfrak{K}_0$-continuous $R$-module $M$ is directly finite if and only if so is $E_{\mathcal{F}}(M)$. This is a generalization of a result due to Goodearl [3].

Throughout this paper $R$ is a ring with identity and all $R$-modules considered are unitary right $R$-modules.

For a given $R$-module $M$, we denote its injective hull by $E(M)$ and the family of all submodules of $M$ by $\mathcal{L}(M)$.

For $N \subseteq \mathcal{L}(M)$, $N \leq M$ means that $N$ is an essential submodule of $M$ and $(N : x)$, for $x \in M$, denotes the right ideal $\{r \in R | xr \in N\}$.

Let $M$ be an $R$-module. An $\mathcal{S}$-closed submodule of $M$ is a submodule $B$ such that $M/B$ is nonsingular. For any submodule $A$ of $M$ there exists the smallest $\mathcal{S}$-closed submodule $C$ of $M$ containing $A$, which is called the $\mathcal{S}$-closure of $A$ in $M$ (see [1]). We note that, when $M$ is nonsingular, the $\mathcal{S}$-closure $C$ of $A$ in $M$ is uniquely determined as a submodule $C$ such that $A \leq C$ and $C$ is $\mathcal{S}$-closed in $M$.

**Lemma 1.** Let $M$ be an $R$-module, and let $A$ and $B$ be submodules of $M$ such that $A \leq B$. Then $B$ is contained in the $\mathcal{S}$-closure of $A$ in $M$. In addition if $M$ is nonsingular and $B$ is a direct summand of $M$, then $B$ coincides with the $\mathcal{S}$-closure of $A$ in $M$.

Proof. Let $C$ be the $\mathcal{S}$-closure of $A$ in $M$. Since $(B+C)/C$ is an epimorphic image of a singular module $B/A$, we see that $(B+C)/C$ is singular. On the other hand, $(B+C)/C$ is a submodule of a nonsingular module $M/C$,
whence \((B+C)/C\) is nonsingular. As a result, we have that \((B+C)/C=0\), and so \(B\leq C\) as desired.

Let \(M\) be an \(R\)-module. We consider a subfamily \(\mathcal{A}\) of \(\mathcal{L}(M)\) which is closed under isomorphic images and essential extensions. For such an \(\mathcal{A}\) the following conditions are studied in [5]:

(C1) For any \(A\in\mathcal{A}\) there exists a direct summand \(A^*\) of \(M\) such that \(A\leq_s A^*\).

(C2) If \(A\in\mathcal{A}\) is a direct summand of \(M\), then any exact sequence \(0\rightarrow A\rightarrow M\) splits.

(C3) If \(A\in\mathcal{A}\), \(N\in\mathcal{L}(M)\) and both of them are direct summands of \(M\) with \(A\cap N=0\), then \(A\oplus N\) is also a direct summand of \(M\).

Note that if \(M\) is nonsingular and \(\mathcal{A}\) satisfies the condition (C1), then, for each \(A\in\mathcal{A}\), \(A^*\) coincides with the \(S\)-closure of \(A\) in \(M\).

Following [5] we call \(M\ \mathcal{A}\)-continuous (resp. \(\mathcal{A}\)-quasi-continuous) if \(M\) satisfies the conditions (C1) and (C2) (resp. (C1) and (C3)). Especially if \(M\) is \(\mathcal{L}(M)\)-continuous (resp. \(\mathcal{L}(M)\)-quasi-continuous) we simply call \(M\) continuous (resp. quasi-continuous). It follows from [5] that \(\mathcal{A}\)-continuous modules are \(\mathcal{A}\)-quasi-continuous, quasi-injective modules are continuous and that \(M\) is \(\mathcal{A}\)-quasi-continuous if and only if \(M\) satisfies (C1) and the condition:

(*) For any \(A\in\mathcal{A}\) and \(N\in\mathcal{L}(M)\) such that \(N\) is a direct summand of \(M\) and \(A\cap N=0\), every homomorphism from \(A\) to \(N\) can be extended to a homomorphism from \(M\) to \(N\).

We now introduce the notion of quasi-\(\aleph_0\)-continuous modules and \(\aleph_0\)-continuous modules. Let \(M\) be an \(R\)-module and consider the family \(\mathcal{A}(M)\) of all submodules \(A\) of \(M\) such that \(A\) contains a countably generated essential submodule. Then \(\mathcal{A}(M)\) is closed under isomorphic images and essential extensions. We say that \(M\) is \(\aleph_0\)-continuous (resp. quasi-\(\aleph_0\)-continuous) if \(M\) is \(\mathcal{A}(M)\)-continuous (resp. \(\mathcal{A}(M)\)-quasi-continuous).

An \(R\)-module \(M\) is directly finite provided that \(M\) is not isomorphic to any proper direct summand of itself. If \(M\) is not directly finite, then \(M\) is said to be directly infinite. It is well-known that \(M\) is directly finite if and only if for all \(f, g \in \text{End}_R(M)\), \(fg=1\) implies \(gf=1\).

**Theorem 2.** For a given nonsingular \(\aleph_0\)-continuous \(R\)-module \(M\), the following conditions are equivalent:

(a) \(M\) is directly finite.

(b) \(M\) contains no infinite direct sums of nonzero pairwise isomorphic submodules.

(c) Any submodule of \(M\) is directly finite.

Proof. (a)⇒(b): Assume that \(M\) is directly finite. It suffices to show
that if \( \{A_1, A_2, \cdots\} \) is an independent sequence of pairwise isomorphic cyclic submodules of \( M \), then \( A_i = 0 \). Set \( B_i = \bigoplus_{n=0}^{\infty} A_{3n+i} \) for \( i = 1, 2, 3 \). Then \( \bigoplus_{n=1}^{\infty} A_n = B_1 \oplus B_2 \oplus B_3 \) and \( B_2 \cong B_3 \), and \( B_1 \oplus B_2 \cong B_3 \). By the condition (C_1), there exists a direct summand \( B_i^* \) of \( M \) such that \( B_i \leq B_i^* \) for each \( i \). Using the condition (*) we have a homomorphism \( f: B_2^* \to B_3^* \) which is an extension of the isomorphism \( B_2 \cong B_3 \). Then \( f \) is a monomorphism, because \( B_2 \leq B_2^* \). Also, using the condition (C_2) we see that \( f(B_2^*) \) is a direct summand of \( M \) containing \( B_3^* \); hence by the uniqueness of the \( S \)-closure \( (B_2^*) = B_3^* \). Thus \( B_2^* \) is isomorphic to \( B_3^* \). Similarly, \( (B_1 \oplus B_2)^* \) is isomorphic to \( B_3^* \). By the condition (C_3), \( B_1^* \oplus B_2^* \) is the \( S \)-closure of \( B_1 \oplus B_2 \) in \( M \). Therefore \( (B_1 \oplus B_2)^* = B_1^* \oplus B_2^* \), and so \( B_1^* \oplus B_2^* \). As \( M \) is directly finite, \( B_1^* \oplus B_2^* \) is also directly finite, from which \( B_1^* = 0 \). Thus we see that \( A_i = 0 \) as desired.

(b)\(\Rightarrow\)(c): Let \( N \) be a submodule of \( M \), and let \( N = N_1 \oplus N_2 \) with an isomorphism \( f: N \cong N_1 \). Then,

\[
N = N_1 \oplus N_2 = f(N_1) \oplus f(N_2) \oplus N_2 = f^2(N_1) \oplus f^2(N_2) \oplus f(N_2) \oplus N_2 = \cdots
\]

It follows that \( \{N_2, f(N_2), f^2(N_2), \cdots\} \) is an independent sequence of pairwise isomorphic submodules of \( M \). By assumption \( N_2 = 0 \) and so \( N \) is directly finite.

(c)\(\Rightarrow\)(a) is clear.

The following lemma is well-known and, as is easily seen, the same conclusion is valid for sums of infinite many submodules.

**Lemma 3.** Let \( A, B, C \) and \( D \) be submodules of a nonsingular module \( M \) such that \( A \leq B \) and \( C \leq D \). Then \( (A + C) \leq (B + D) \).

**Lemma 4 ([2, Lemma 14.10]).** Let \( M \) be a projective module over a regular ring \( R \), and let \( \mathcal{L} \) denote the collection of all countably generated submodules of \( M \).

(a) If \( J, K \in \mathcal{L} \), then \( J \cap K \in \mathcal{L} \).

(b) If \( J, K \in \mathcal{L} \) and \( f \in \text{Hom}_R(J, M) \), then \( f^{-1}(K) = \{ x \in J \mid f(x) \in K \} \in \mathcal{L} \).

Now, let \( \mathcal{E} \) be the collection of all countably generated essential right ideals of a ring \( R \) and let \( \mathcal{F} \) be the collection of all right ideals which contain a member of \( \mathcal{E} \). Then we have the following proposition.

**Proposition 5.** If \( R \) is a regular ring, then \( \mathcal{F} \) is a right Gabriel topology, i.e., \( \mathcal{F} \) is not empty and satisfies the following conditions:
(T') If $I \in \mathcal{F}$ and $a \in R$, then $(I: a) \in \mathcal{F}$.

Proof. Suppose that $I \in \mathcal{F}$ and $a \in R$. Then there exists $J \in \mathcal{E}$ such that $J \subseteq I$. Noting that $(I: a) \supseteq (J: a)$, we may show that $(J: a) \in \mathcal{E}$. $J \subseteq R$ implies that $(J: a) \subseteq R$ and by Lemma 4 $(J: a)$ is countably generated. Therefore $(J: a) \in \mathcal{E}$. Now suppose that $I$ is a right ideal and that there exists $J \in \mathcal{F}$ such that $(I: a) \notin \mathcal{F}$ for every $a \in J$. Then there exists $K \in \mathcal{E}$ with $K \leq J$. Put $K = \sum a_n R$. Then by the assumption, there exists $I_n \in \mathcal{E}$ with $(I: a_n) \subseteq I_n$ for each $n$. The mapping $f: \oplus R \to K$ given by $f(a_n) = \sum a_n r_n$ is an epimorphism. Note that $f(\oplus I_n) \subseteq K \cap I$ and $f(\oplus I_n)$ is countably generated.

Since $f$ induces an epimorphism $\oplus R/\oplus I_n \to Kf(\oplus I_n)$ and $\oplus R/\oplus I_n$ is singular by Lemma 3, $Kf(\oplus I_n)$ is also singular. Hence $f(\oplus I_n) \subseteq K \subseteq R$ and so $f(\oplus I_n) \in \mathcal{E}$. Thus $I \in \mathcal{F}$ as desired.

For a given module $M$ over a regular ring $R$, we put

$$E_{\mathcal{F}}(M) = \{x \in E(M) | (M: x) \in \mathcal{F}\}.$$

$E_{\mathcal{F}}(M)$ is called the $\mathcal{F}$-injective hull or $\mathcal{F}$-divisible hull of $M$ (cf. [7, p. 30]).

**Lemma 6.** Let $R$ be a regular ring and let $M$ be an $R$-module.

(a) If $M = A \oplus B$ for some submodules $A$ and $B$, then $E_{\mathcal{F}}(M) = E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$.

(b) Any $R$-homomorphism from $M$ to an $R$-module $N$ can be extended to an $R$-homomorphism from $E_{\mathcal{F}}(M)$ to $E_{\mathcal{F}}(N)$.

Proof. (a) It is clear that $E(M) = E(A) \oplus E(B)$. Let $m \in E_{\mathcal{F}}(M)$. Then $m = a + b$ for some $a \in E(A)$ and $b \in E(B)$. Since $(M: m) \in \mathcal{F}$ and $(M: m) = (A: a) \cap (B: b)$, $(A: a) \in \mathcal{F}$ and $a \in E_{\mathcal{F}}(A)$. Likewise we have $b \in E_{\mathcal{F}}(B)$. Therefore $m \in E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$ and hence $E_{\mathcal{F}}(M) = E_{\mathcal{F}}(A) \oplus E_{\mathcal{F}}(B)$.

(b) Let $f: M \to N$ be a homomorphism. Then $f$ can be extended to an $R$-homomorphism $\tilde{f}: E(M) \to E(N)$. Let $m \in E_{\mathcal{F}}(M)$. Then $(M: m) \subseteq (N: \tilde{f}(m)) \in \mathcal{F}$. Hence $\tilde{f}(m)$ lies in $E_{\mathcal{F}}(N)$. Thus the restriction map $\tilde{f}|E_{\mathcal{F}}(M)$ of $\tilde{f}$ is the desired one.

**Proposition 7.** Let $R$ be a regular ring. If $M$ is a non-singular quasi-$\aleph_0$-continuous $R$-module, then so is $E_{\mathcal{F}}(M)$.

Proof. First we show that the condition $(C_1)$ holds for $\mathcal{A}(E_{\mathcal{F}}(M))$. Let
There exists a countably generated essential submodule $N$ of $L$; say $N = \sum_{n=1}^{\infty} x_n R$. We can take $I_n \in \mathcal{E}$ such that $x_n I_n \subseteq M$ for each $n$. Since $K = \sum_{n=1}^{\infty} x_n I_n$ is countably generated, there exists a direct summand $K^*$ of $M$ with $K \subseteq \Sigma K^*$. It follows that $K$ is an essential submodule of $E_{\mathcal{F}}(K^*)$ and $E_{\mathcal{F}}(K^*)$ is a direct summand of $E_{\mathcal{F}}(M)$ by Lemma 6 (a). On the other hand, $K$ is an essential submodule of $N$ by Lemma 3 and hence is that of $L$. Therefore from Lemma 1 we see that $L \subseteq E_{\mathcal{F}}(K^*)$.

Next, we prove that the condition (*) for $\mathcal{A}(E_{\mathcal{F}}(M))$ holds. Let $A \subseteq \mathcal{A}(E_{\mathcal{F}}(M))$ and $N \in \mathcal{L}(E_{\mathcal{F}}(M))$ such that $N$ is a direct summand of $E_{\mathcal{F}}(M)$ and $A \cap N = 0$, and let $f: A \to N$ be a homomorphism. Now there exists a countably generated essential submodule $B$ of $A$; say $B = \sum_{n=1}^{\infty} x_n R$. Then $f(B) = \sum_{n=1}^{\infty} f(x_n) R \subseteq f(A) \subseteq N$. Since both $x_n$ and $f(x_n)$ are in $E_{\mathcal{F}}(M)$ for each $n$, there exist $I_n'$ and $I_n''$ in $\mathcal{E}$ such that $x_n I_n' \subseteq M$ and $f(x_n) I_n'' \subseteq M$. Then $I_n = I_n' \cap I_n''$ lies in $\mathcal{E}$ by Lemma 4 (a), and $x_n I_n \subseteq M$ and $f(x_n) I_n \subseteq M$. Putting $C = \sum_{n=1}^{\infty} x_n I_n$ and $D = \sum_{n=1}^{\infty} f(x_n) I_n$, we see that $C$ and $D$ are countably generated submodules of $M$ with $C \subseteq B$ and $D \subseteq f(B)$. There exists a direct summand $D^*$ of $M$ such that $D \subseteq f(D^*)$. Using the condition (*) for $\mathcal{A}(M)$, the restriction $f | C: C \to D$ of $f$ can be extended to a homomorphism $M \to D^*$. This also can be extended to a homomorphism $h: E_{\mathcal{F}}(M) \to E_{\mathcal{F}}(D^*)$ by Lemma 6 (b). Since $E_{\mathcal{F}}(D^*)$ is the $S$-closure of $D$ in $E_{\mathcal{F}}(M)$, $E_{\mathcal{F}}(D^*)$ is contained in $N$. Therefore $h$ is a homomorphism from $E_{\mathcal{F}}(M)$ to $N$. Since $f | C = h | C$ and $C \subseteq B$, $h = f$.

At the end of this note we provide an example to show that the converse of Proposition 7 is not true in general.

Now we are in position to prove our main theorem.

**Theorem 8.** Let $R$ be a regular ring, and let $M$ be a non-singular $\mathfrak{a}_c$-continuous $R$-module. Then $M$ is directly finite if and only if so is $E_{\mathcal{F}}(M)$.

Proof. The “only if” part. Assume that $M$ is directly finite. If $E_{\mathcal{F}}(M)$ is directly infinite, then there exists an independent sequence of nonzero pairwise isomorphic cyclic submodules of $E_{\mathcal{F}}(M)$; say $\{x_n R\}_{n=1}^{\infty}$. Let $f_n: x_n R \to x_{n+1} R$ be an isomorphism with $f_n(x_n) = x_{n+1}$; $n=1, 2, \ldots$. For each $n$, there exists $I_n \in \mathcal{E}$ such that $x_n I_n \subseteq M$. Then $A_n = x_n (I_1 \cap \cdots \cap I_n)$ and $B_n = x_n (I_1 \cap \cdots \cap I_{n+1})$ are countably generated submodules of $M$ such that $B_n \subseteq A_n \subseteq x_n R$ and the restriction $f_n | B_n$ of $f_n$ to $B_n$ is an isomorphism between $B_n$ and $A_{n+1}$. By the assumption, there exist direct summands $A_n^*$ and $B_n^*$ of $M$ with $A_n \subseteq A_n^*$ and $B_n \subseteq B_n^*$ and further $f_n | B_n$ can be extended to an isomorphism $g_n$ between
Proposition 9. Assume that $M$ is a nonsingular $\mathfrak{R}$-continuous $R$-module with the following condition:

(\#) For any submodules $A$ and $B$ of $M$ with $A \cap B = 0$, any isomorphism from $A$ to $B$ can be extended to a homomorphism of $A$ to $B$, where $A$ and $B$ are the $\mathfrak{S}$-closures of $A$ and $B$ in $M$ respectively. Then, $M$ is directly finite if and only if so is $E(M)$.

Proof. Noting that $E(M)$ is nonsingular $\mathfrak{R}$-continuous, the "if" part is clear by Theorem 2.

The "only if" part. Assume that $M$ is directly finite. If $E(M)$ is directly infinite, there exists an infinite and independent sequence of nonzero pairwise isomorphic submodules of $E(M)$; say $\{A_n\}_{n=1}^\infty$. Let $f_n$ be the isomorphism between $A_n$ and $A_{n+1}$ for each $n$. Set $B_1 = A_1 \cap M$ and define inductively $B_{n+1} = f_n(B_n) \cap M$ and $C_n = f_n^{-1}(B_{n+1})$ for $n=1, 2, \ldots$. Then, for each $n$, $B_n$ and $C_n$ are submodules of $A_n \cap M$ with $C_n \subseteq B_n$ and the $\mathfrak{S}$-closure $\bar{B}_n$ of $B_n$ in $M$ coincides with $\bar{C}_n$, the $\mathfrak{S}$-closure of $C_n$ in $M$, by the similar way in the proof of Theorem 8. The restriction map $f_n|_{C_n}$ is an isomorphism from $C_n$ to $B_{n+1}$. Using the condition (\#), $f_n|_{C_n}$ can be extended to a monomorphism $\bar{f}_n$: $\bar{C}_n \rightarrow \bar{B}_{n+1} = \bar{C}_{n+1}$. Consequently $\{\bar{C}_n, \bar{f}_n(\bar{C}_n), \bar{f}_n(\bar{f}_n(\bar{C}_n)), \ldots\}$ is an independent sequence of nonzero pairwise isomorphic submodules of $M$, which is a contradiction by Theorem 2. Thus the proof is completed.

Corollary 10. Let $M$ be a nonsingular continuous $R$-module. Then $M$ is directly finite if and only if so is $E(M)$.

Finally we show the following result.

Theorem 11. Let $R$ be a regular ring and $M$ a finitely generated projective $\mathfrak{R}$-continuous $R$-module. If $A$ is a projective maximal submodule of $M$, then $A$ is a direct summand of $M$. 
Proof. $A$ can be written as a direct sum of cyclic submodules; say $A = \bigoplus_{a \in I} x_a R$ [4]. We claim that $I$ is a finite set. If $I$ is an infinite set, then we have a countable subset $J$ of $I$ such that $I \sim J$ is an infinite set and so $A = (\bigoplus_{a \in J} x_a R) \oplus (\bigoplus_{\beta \in I \setminus J} x_\beta R)$. Since $M$ is $\mathfrak{K}_0$-continuous, we have a direct summand $B^*$ of $M$ such that $\bigoplus_{a \in J} x_a R \leq B^*$ and $A \leq B^* \bigoplus_{\beta \in I \setminus J} x_\beta R \leq M$. If $A = B^* \bigoplus_{\beta \in I \setminus J} x_\beta R$, then $\bigoplus_{a \in J} x_a R$ coincides with $B^*$ and is finitely generated. If $B^* \bigoplus_{\beta \in I \setminus J} x_\beta R = M$, then $\bigoplus_{a \in J} x_a R$ is finitely generated. In any case we have a contradiction. Therefore $A$ is finitely generated and so it is a direct summand of $M$.

As a consequence of Theorem 11, we obtain the following which is a slight generalization of [6, Corollary].

**Corollary 12.** If $R$ is a right hereditary, right $\mathfrak{K}_0$-continuous, regular ring, then $R$ is a semi-simple artinian ring.

**Remark.** In general, the converse of Proposition 7 is not true. For example, take a field $F$ and set $R_n = M_2(F)$ for all $n=1, 2, \ldots$. Map each $R_n \rightarrow R_{n+1}$ along the diagonal, i.e., map $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, and set $R = \lim \rightarrow R_n$. Then $R$ is a simple, right hereditary, not artinian, regular ring with a unique dimension function (see [2]). Note that for a regular ring $R$, $R$ is quasi-$\mathfrak{K}_0$-continuous if and only if $R$ is $\mathfrak{K}_0$-continuous. Therefore we see that $E\mathfrak{K}(R) = E(\mathfrak{K}_0)(R) = E(R)$ is a nonsingular quasi-$\mathfrak{K}_0$-continuous $R$-module, but $R$ is not quasi-$\mathfrak{K}_0$-continuous by Corollary 12.

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**References**


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