Kutami, M.
Osaka J. Math.
21 (1984), 31-38

$\aleph_0$-CONTINUOUS MODULES

MAMORU KUTAMI

(Received June 8, 1982)

Generalizing the notion of right $\aleph_0$-continuous regular rings (see [2], [3]) we define that of quasi-$\aleph_0$- and $\aleph_0$-continuous modules and mainly study the directly finiteness of nonsingular $\aleph_0$-continuous modules over (von Neumann) regular rings.

Let R be a regular ring. By $\mathcal{F}$ we denote the family of all essentially $\aleph_0$-generated essential right ideals of R. It is shown that $\mathcal{F}$ becomes a right Gabriel topology on R (Proposition 5). From this fact the divisible hull $E_{\mathcal{F}}(M)$ of a given right R-module M is considered. Our main purpose of this note is to prove that a nonsingular $\aleph_0$-continuous R-module M is directly finite if and only if so is $E_{\mathcal{F}}(M)$. This is a generalization of a result due to Goodearl [3].

Throughout this paper R is a ring with identity and all R-modules considered are unitary right R-modules.

For a given R-module M, we denote its injective hull by $E(M)$ and the family of all submodules of M by $\mathcal{L}(M)$.

For $N\in\mathcal{L}(M)$ $N\leqq_e M$ means that N is an essential submodule of M and $(N: x)$, for $x\in M$, denotes the right ideal $\{r\in R\,|\,xr\in N\}$.

Let M be an R-module. An $\mathcal{S}$-closed submodule of M is a submodule B such that M/B is nonsingular. For any submodule A of M there exists the smallest $\mathcal{S}$-closed submodule C of M containing A, which is called the $\mathcal{S}$-closure of A in M (see [1]). We note that, when M is nonsingular, the $\mathcal{S}$-closure C of A in M is uniquely determined as a submodule C such that $A\leqq_e C$ and C is $\mathcal{S}$-closed in M.

Lemma 1. *Let M be an R-module, and let A and B be submodules of M such that $A\leqq_e B$. Then B is contained in the $\mathcal{S}$-closure of A in M. In addition if M is nonsingular and B is a direct summand of M, then B coincides with the $\mathcal{S}$-closure of A in M.*

Proof. Let C be the $\mathcal{S}$-closure of A in M. Since $(B+C)/C$ is an epimorphic image of a singular module B/A, we see that $(B+C)/C$ is singular. On the other hand, $(B+C)/C$ is a submodule of a nonsingular module M/C, whence $(B+C)/C$ is nonsingular. As a result, we have that $(B+C)/C=0$, and so $B\leqq C$ as desired.

Let M be an R-module. We consider a subfamily $\mathcal{A}$ of $\mathcal{L}(M)$ which is closed under isomorphic images and essential extensions. For such an $\mathcal{A}$ the following conditions are studied in [5]:

(C_1) For any $A\in\mathcal{A}$ there exists a direct summand A^* of M such that $A\leqq_e A^*$.

(C_2) If $A\in\mathcal{A}$ is a direct summand of M, then any exact sequence $0\to A\to M$ splits.

(C_3) If $A\in\mathcal{A}$, $N\in\mathcal{L}(M)$ and both of them are direct summands of M with $A\cap N=0$, then $A\oplus N$ is also a direct summand of M.

Note that if M is nonsingular and $\mathcal{A}$ satisfies the condition (C_1), then, for each $A\in\mathcal{A}$, A^* coincides with the $\mathcal{S}$-closure of A in M.

Following [5] we call M $\mathcal{A}$-continuous (resp. $\mathcal{A}$-quasi-continuous) if M satisfies the conditions (C_1) and (C_2) (resp. (C_1) and (C_3)). Especially if M is $\mathcal{L}(M)$-continuous (resp. $\mathcal{L}(M)$-quasi-continuous) we simply call M continuous (resp. quasi-continuous). It follows from [5] that $\mathcal{A}$-continuous modules are $\mathcal{A}$-quasi-continuous, quasi-injective modules are continuous and that M is $\mathcal{A}$-quasi-continuous if and only if M satisfies (C_1) and the condition:

(*) For any $A\in\mathcal{A}$ and $N\in\mathcal{L}(M)$ such that N is a direct summand of M and $A\cap N=0$, every homomorphism from A to N can be extended to a homomorphism from M to N.

We now introduce the notion of quasi-$\aleph_0$-continuous modules and $\aleph_0$-continuous modules. Let M be an R-module and consider the family $\mathcal{A}(M)$ of all submodules A of M such that A contains a countably generated essential submodule. Then $\mathcal{A}(M)$ is closed under isomorphic images and essential extensions. We say that M is $\aleph_0$-continuous (resp. quasi-$\aleph_0$-continuous) if M is $\mathcal{A}(M)$-continuous (resp. $\mathcal{A}(M)$-quasi-continuous).

An R-module M is directly finite provided that M is not isomorphic to any proper direct summand of itself. If M is not directly finite, then M is said to be directly infinite. It is well-known that M is directly finite if and only if for all $f, g\in End_R(M)$, $fg=1$ implies $gf=1$.

Theorem 2. *For a given nonsingular $\aleph_0$-continuous R-module M, the following conditions are equivalent*:

(a) *M is directly finite.*

(b) *M contains no infinite direct sums of nonzero pairwise isomorphic submodules.*

(c) *Any submodule of M is directly finite.*

Proof. (a)$\Rightarrow$(b): Assume that M is directly finite. It suffices to show that if $\{A_1, A_2, \cdots\}$ is an independent sequence of pairwise isomorphic cyclic submodules of M, then $A_1=0$. Set $B_i=\bigoplus_{n=0}^{\infty} A_{3n+i}$ for $i=1, 2, 3$. Then $\bigoplus_{n=1}^{\infty} A_n = B_1 \oplus B_2 \oplus B_3$ and $B_2 \cong B_3$, and $B_1 \oplus B_2 \cong B_3$. By the condition (C_1), there exists a direct summand B_i^* of M such that $B_i \leqq_e B_i^*$ for each i. Using the condition (*) we have a homomorphism f: $B_2^* \to B_3^*$ which is an extension of the isomorphism $B_2 \cong B_3$. Then f is a monomorphism, because $B_2 \leqq_e B_2^*$. Also, using the condition (C_2) we see that $f(B_2^*)$ is a direct summand of M containing B_3; hence by the uniqueness of the $\mathcal{S}$-closure $f(B_2^*)=B_3^*$. Thus B_2^* is isomorphic to B_3^*. Similarly, $(B_1 \oplus B_2)^*$ is isomorphic to B_3^*. By the condition (C_3), $B_1^* \oplus B_2^*$ is the $\mathcal{S}$-closure of $B_1 \oplus B_2$ in M. Therefore $(B_1 \oplus B_2)^* = B_1^* \oplus B_2^*$, and so $B_1^* \oplus B_2^* \cong B_2^*$. As M is directly finite, $B_1^* \oplus B_2^*$ is also directly finite, from which $B_1^*=0$. Thus we see that $A_1=0$ as desired.

(b)$\Rightarrow$(c): Let N be a submodule of M, and let $N=N_1 \oplus N_2$ with an isomorphism f: $N \cong N_1$. Then,

$$\begin{aligned} N &= N_1 \oplus N_2 \\ &= f(N_1) \oplus f(N_2) \oplus N_2 \\ &= f^2(N_1) \oplus f^2(N_2) \oplus f(N_2) \oplus N_2 \\ &= \cdots . \end{aligned}$$

It follows that $\{N_2, f(N_2), f^2(N_2), \cdots\}$ is an independent sequence of pairwise isomorphic submodules of M. By assumption $N_2=0$ and so N is directly finite.

(c)$\Rightarrow$(a) is clear.

The following lemma is well-known and, as is easily seen, the same conclusion is valid for sums of infinite many submodules.

Lemma 3. *Let A, B, C and D be submodules of a nonsingular module M such that $A \leqq_e B$ and $C \leqq_e D$. Then $(A+C) \leqq_e (B+D)$.*

Lemma 4 ([2, Lemma 14.10]). *Let M be a projective module over a regular ring R, and let $\mathcal{L}$ denote the collection of all countably generated submodules of M.*

(a) *If $J, K \in \mathcal{L}$, then $J \cap K \in \mathcal{L}$.*

(b) *If $J, K \in \mathcal{L}$ and $f \in Hom_R(J, M)$, then $f^{-1}(K) = \{x \in J \mid f(x) \in K\} \in \mathcal{L}$.*

Now, let $\mathcal{E}$ be the collection of all countably generated essential right ideals of a ring R and let $\mathcal{F}$ be the collection of all right ideals which contain a member of $\mathcal{E}$. Then we have the following proposition.

Proposition 5. *If R is a regular ring, then $\mathcal{F}$ is a right Gabriel topology, i.e., $\mathcal{F}$ is not empty and satisfies the following conditions:*

(T_1) *If* $I\in\mathcal{F}$ *and* $a\in R$, *then* $(I\colon a)\in\mathcal{F}$.

(T_2) *If* I *is a right ideal and there exists* $J\in\mathcal{F}$ *such that* $(I\colon a)\in\mathcal{F}$ *for every* $a\in J$, *then* $I\in\mathcal{F}$.

Proof. Suppose that $I\in\mathcal{F}$ and $a\in R$. Then there exists $J\in\mathcal{E}$ such that $J\leqq I$. Noting that $(I\colon a)\geqq(J\colon a)$, we may show that $(J\colon a)\in\mathcal{E}$. $J\leqq_e R$ implies that $(J\colon a)\leqq_e R$ and by Lemma 4 $(J\colon a)$ is countably generated. Therefore $(J\colon a)\in\mathcal{E}$. Now suppose that I is a right ideal and that there exists $J\in\mathcal{F}$ such that $(I\colon \mathrm{a})\in\mathcal{F}$ for each $a\in J$. Then there exists $K\in\mathcal{E}$ with $K\leqq J$. Put $K=\sum_{n=1}^{\infty}a_nR$. Then by the assumption, there exists $I_n\in\mathcal{E}$ with $(I\colon a_n)\geqq I_n$ for each n. The mapping $f\colon \bigoplus_{n=1}^{\infty}R\to K$ given by $f((r_n))=\sum a_nr_n$ is an epimorphism. Note that $f(\bigoplus_{n=1}^{\infty}I_n)\leqq K\cap I$ and $f(\bigoplus_{n=1}^{\infty}I_n)$ is countably generated. Since f induces an epimorphism $\bigoplus_{n=1}^{\infty}R/\bigoplus_{n=1}^{\infty}I_n\to K/f(\bigoplus_{n=1}^{\infty}I_n)$ and $\bigoplus_{n=1}^{\infty}R/\bigoplus_{n=1}^{\infty}I_n$ is singular by Lemma 3, $K/f(\bigoplus_{n=1}^{\infty}I_n)$ is also singular. Hence $f(\bigoplus_{n=1}^{\infty}I_n)\leqq_e K\leqq_e R$ and so $f(\bigoplus_{n=1}^{\infty}I_n)\in\mathcal{E}$. Thus $I\in\mathcal{F}$ as desired.

For a given module M over a regular ring R, we put

$$\begin{aligned} E_{\mathcal{F}}(M) &= \{x\in E(M)\mid (M\colon x)\in\mathcal{F}\} \\ &= \{x\in E(M)\mid xI\leqq M \text{ for some } I\in\mathcal{E}\}\ . \end{aligned}$$

$E_{\mathcal{F}}(M)$ is called the $\mathcal{F}$-injective hull or $\mathcal{F}$-divisible hull of M (cf. [7, p. 30]).

Lemma 6. *Let* R *be a regular ring and let* M *be an* R*-module.*

(a) *If* $M=A\oplus B$ *for some submodules* A *and* B, *then* $E_{\mathcal{F}}(M)=E_{\mathcal{F}}(A)\oplus E_{\mathcal{F}}(B)$.

(b) *Any* R*-homomorphism from* M *to an* R*-module* N *can be extended to an* R*-homomorphism from* $E_{\mathcal{F}}(M)$ *to* $E_{\mathcal{F}}(N)$.

Proof. (a) It is clear that $E(M)=E(A)\oplus E(B)$. Let $m\in E_{\mathcal{F}}(M)$. Then $m=a+b$ for some $a\in E(A)$ and $b\in E(B)$. Since $(M\colon m)\in\mathcal{F}$ and $(M\colon m)=(A\colon a)\cap(B\colon b)$, $(A\colon a)\in\mathcal{F}$ and $a\in E_{\mathcal{F}}(A)$. Likewise we have $b\in E_{\mathcal{F}}(B)$. Therefore $m\in E_{\mathcal{F}}(A)\oplus E_{\mathcal{F}}(B)$ and hence $E_{\mathcal{F}}(M)=E_{\mathcal{F}}(A)\oplus E_{\mathcal{F}}(B)$.

(b) Let $f\colon M\to N$ be a homomorphism. Then f can be extended to an R-homomorphism $\bar{f}\colon E(M)\to E(N)$. Let $m\in E_{\mathcal{F}}(M)$. Then $(M\colon m)\leqq(N\colon \bar{f}(m))\in\mathcal{F}$. Hence $\bar{f}(m)$ lies in $E_{\mathcal{F}}(N)$. Thus the restriction map $\bar{f}|_{E_{\mathcal{F}}(M)}$ of $\bar{f}$ is the desired one.

Proposition 7. *Let* R *be a regular ring. If* M *is a non-singular quasi-*$\aleph_0$*-continuous* R*-module, then so is* $E_{\mathcal{F}}(M)$.

Proof. First we show that the condition (C_1) holds for $\mathcal{A}(E_{\mathcal{F}}(M))$. Let

$L \in \mathcal{A}(E_{\mathcal{F}}(M))$. There exists a countably generated essential submodule N of L; say $N = \sum_{n=1}^{\infty} x_n R$. We can take $I_n \in \mathcal{E}$ such that $x_n I_n \leqq M$ for each n. Since $K = \sum_{n=1}^{\infty} x_n I_n$ is countably generated, there exists a direct summand K^* of M with $K \leqq_e K^*$. It follows that K is an essential submodule of $E_{\mathcal{F}}(K^*)$ and $E_{\mathcal{F}}(K^*)$ is a direct summand of $E_{\mathcal{F}}(M)$ by Lemma 6 (a). On the other hand, K is an essential submodule of N by Lemma 3 and hence is that of L. Therefore from Lemma 1 we see that $L \leqq_e E_{\mathcal{F}}(K^*)$.

Next, we prove that the condition (*) for $\mathcal{A}(E_{\mathcal{F}}(M))$ holds. Let $A \in \mathcal{A}(E_{\mathcal{F}}(M))$ and $N \in \mathcal{L}(E_{\mathcal{F}}(M))$ such that N is a direct summand of $E_{\mathcal{F}}(M)$ and $A \cap N = 0$, and let f: $A \to N$ be a homomorphism. Now there exists a countably generated essential submodule B of A; say $B = \sum_{n=1}^{\infty} x_n R$. Then $f(B) = \sum_{n=1}^{\infty} f(x_n) R \leqq_e f(A) \leqq N$. Since both x_n and $f(x_n)$ are in $E_{\mathcal{F}}(M)$ for each n, there exist I_n' and I_n'' in $\mathcal{E}$ such that $x_n I_n' \leqq M$ and $f(x_n) I_n'' \leqq M$. Then $I_n = I_n' \cap I_n''$ lies in $\mathcal{E}$ by Lemma 4 (a), and $x_n I_n \leqq M$ and $f(x_n) I_n \leqq M$. Putting $C = \sum_{n=1}^{\infty} x_n I_n$ and $D = \sum_{n=1}^{\infty} f(x_n) I_n$, we see that C and D are countably generated submodules of M with $C \leqq_e B$ and $D \leqq_e f(B)$. There exists a direct summand D^* of M such that $D \leqq_e D^*$. Using the condition (*) for $\mathcal{A}(M)$, the restriction $f|_C$: $C \to D$ of f can be extended to a homomorphism $M \to D^*$. This also can be extended to a homomorphism h: $E_{\mathcal{F}}(M) \to E_{\mathcal{F}}(D^*)$ by Lemma 6 (b). Since $E_{\mathcal{F}}(D^*)$ is the $\mathcal{S}$-closure of D in $E_{\mathcal{F}}(M)$, $E_{\mathcal{F}}(D^*)$ is contained in N. Therefore h is a homomorphism from $E_{\mathcal{F}}(M)$ to N. Since $f|_C = h|_C$ and $C \leqq_e B$, $h = f$.

At the end of this note we provide an example to show that the converse of Proposition 7 is not true in general.

Now we are in position to prove our main theorem.

Theorem 8. *Let R be a regular ring, and let M be a non-singular $\aleph_0$-continuous R-module. Then M is directly finite if and only if so is $E_{\mathcal{F}}(M)$.*

Proof. The "only if" part. Assume that M is directly finite. If $E_{\mathcal{F}}(M)$ is directly infinite, then there exists an independent sequence of nonzero pairwise isomorphic cyclic submodules of $E_{\mathcal{F}}(M)$; say $\{x_n R\}_{n=1}^{\infty}$. Let f_n: $x_n R \to x_{n+1} R$ be an isomorphism with $f_n(x_n) = x_{n+1}$, $n = 1, 2, \cdots$. For each n, there exists $I_n \in \mathcal{E}$ such that $x_n I_n \leqq M$. Then $A_n = x_n(I_1 \cap \cdots \cap I_n)$ and $B_n = x_n(I_1 \cap \cdots \cap I_{n+1})$ are countably generated submodules of M such that $B_n \leqq_e A_n \leqq_e x_n R$ and the restriction $f_n|_{B_n}$ of f_n to B_n is an isomorphism between B_n and A_{n+1}. By the assumption, there exist direct summands A_n^* and B_n^* of M with $A_n \leqq_e A_n^*$ and $B_n \leqq_e B_n^*$ and further $f_n|_{B_n}$ can be extended to an isomorphism g_n between

B_n^* and A_{n+1}^*. Hence we have $\bigoplus_{n=1}^{\infty} B_n^* \simeq \bigoplus_{n=1}^{\infty} A_{n+1}^*$. On the other hand, by Lemma 1, for each n $A_n \leqq B_n^*$. So $A_n \leqq_e B_n^*$ which implies that $B_n^* = A_n^*$. Therefore $\bigoplus_{n=1}^{\infty} B_n^* = B_1^* \oplus (\bigoplus_{n=1}^{\infty} A_{n+1}^*)$. However, this is a contradiction, because $\bigoplus_{n=1}^{\infty} B_n^*$ is directly finite by Theorem 2. Therefore $E\mathcal{F}(M)$ is directly finite.

The "if" part. Assume that $E\mathcal{F}(M)$ is directly finite, and consider f and g in $End_R(M)$ such that $fg = 1_{End_R(M)}$. By Lemma 6 (b), there exist $\bar{f}$ and $\bar{g}$ in $End_R(E\mathcal{F}(M))$ such that $\bar{f}$ and $\bar{g}$ are extensions of f and g respectively. Noting that $M \leqq_e E\mathcal{F}(M)$ and $E\mathcal{F}(M)$ is nonsingular, we obtain that $\bar{f}\bar{g} = 1_{End_R(E\mathcal{F}(M))}$. By the assumption $\bar{g}\bar{f} = 1_{End_R(E\mathcal{F}(M))}$, from which $gf = 1_{End_R(M)}$. Therefore M is directly finite.

Proposition 9. *Assume that M is a nonsingular $\aleph_0$-continuous R-module with the following condition:*

($\sharp$) *For any submodules A and B of M with $A \cap B = 0$, any isomorphism from A to B can be extended to a homomorphism of $\bar{A}$ to $\bar{B}$, where $\bar{A}$ and $\bar{B}$ are the $\mathcal{S}$-closures of A and B in M respectively. Then, M is directly finite if and only if so is $E(M)$.*

Proof. Noting that $E(M)$ is nonsingular $\aleph_0$-continuous, the "if" part is clear by Theorem 2.

The "only if" part. Assume that M is directly finite. If $E(M)$ is directly infinite, there exists an infinite and independent sequence of nonzero pairwise isomorphic submodules of $E(M)$; say $\{A_n\}_{n=1}^{\infty}$. Let f_n be the isomorphism between A_n and A_{n+1} for each n. Set $B_1 = A_1 \cap M$ and define inductively $B_{n+1} = f_n(B_n) \cap M$ and $C_n = f_n^{-1}(B_{n+1})$ for $n = 1, 2, \cdots$. Then, for each n, B_n and C_n are submodules of $A_n \cap M$ with $C_n \leqq_e B_n$ and the $\mathcal{S}$-closure $\bar{B}_n$ of B_n in M coincides with $\bar{C}_n$, the $\mathcal{S}$-closure of C_n in M, by the similar way in the proof of Theorem 8. The restriction map $f_n|_{C_n}$ is an isomorphism from C_n to B_{n+1}. Using the condition ($\sharp$), $f_n|_{C_n}$ can be extended to a monomorphism $\bar{f}_n$: $\bar{C}_n \rightarrow \bar{B}_{n+1} = \bar{C}_{n+1}$. Consequently $\{\bar{C}_1, \bar{f}_1(\bar{C}_1), \bar{f}_2(\bar{f}_1(\bar{C}_1)), \cdots\}$ is an independent sequence of nonzero pairwise isomorphic submodules of M, which is a contradiction by Theorem 2. Thus the proof is completed.

Corollary 10. *Let M be a nonsingular continuous R-module. Then M is directly finite if and only if so is $E(M)$.*

Finally we show the following result.

Theorem 11. *Let R be a regular ring and M a finitely generated projective $\aleph_0$-continuous R-module. If A is a projective maximal submodule of M, then A is a direct summand of M.*

Proof. A can be written as a direct sum of cyclic submodules; say $A=\bigoplus_{\alpha\in I} x_\alpha R$ [4]. We claim that I is a finite set. If I is an infinite set, then we have a countable subset J of I such that $I-J$ is an infinite set and so $A=(\bigoplus_{\alpha\in J} x_\alpha R)\oplus(\bigoplus_{\beta\in I-J} x_\beta R)$. Since M is $\aleph_0$-continuous, we have a direct summand B^* of M such that $\bigoplus_{\alpha\in J} x_\alpha R\leqq_e B^*$ and $A\leqq B^*\oplus(\bigoplus_{\beta\in I-J} x_\beta R)\leqq M$. If $A=B^*\oplus(\bigoplus_{\beta\in J-J} x_\beta R)$, then $\bigoplus_{\alpha\in J} x_\alpha R$ coincides with B^* and is finitely generated. If $B^*\oplus(\bigoplus_{\beta\in I-J} x_\beta R)=M$, then $\bigoplus_{\beta\in I-J} x_\beta R$ is finitely generated. In any case we have a contradiction. Therefore A is finitely generated and so it is a direct summand of M.

As a consequence of Theorem 11, we obtain the following which is a slight generalization of [6, Corollary].

Corollary 12. *If R is a right hereditary, right $\aleph_0$-continuous, regular ring, then R is a semi-simple artinian ring.*

REMARK. In general, the converse of Proposition 7 is not true. For example, take a field F and set $R_n=M_{2^n}(F)$ for all $n=1, 2, \cdots$. Map each $R_n\to R_{n+1}$ along the diagonal, i.e., map $x\mapsto\begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix}$, and set $R=\varinjlim R_n$. Then R is a simple, right hereditary, not artinian, regular ring with a unique dimension function (see [2]). Note that for a regular ring R, R_R is quasi-$\aleph_0$-continuous if and only if R_R is $\aleph_0$-continuous. Therefore we see that $E_{\mathcal{F}}(R)=E_{\mathcal{L}(R)}(R)=E(R)$ is a nonsingular quasi-$\aleph_0$-continuous R-module, but R is not quasi-$\aleph_0$-continuous by Corollary 12.

Acknowledgement. The author would like to express his thanks to Professor Y. Kurata and Professor K. Oshiro for their advice in preparing this paper.

References

[1] K.R. Goodearl and A.K. Boyle: Dimension theory for nonsingular injective modules, Mem. Amer. Math. Soc. **177** (1976).

[2] K.R. Goodearl: Von Neumann regular rings, Pitman, London-San Francisco-Melbourne, 1979.

[3] K.R. Goodearl: *Directly finite aleph-nought-continuous regular rings*, to appear.

[4] I. Kaplansky: *Projective modules*, Ann. of Math. **68** (1958), 372–377.

[5] K. Oshiro: *Continuous modules and quasi-continuous modules*, Osaka J. Math. **20** (1983), 681–694.

[6] B.L. Osofsky: *Rings all of whose finitely generated modules are injective*, Pacific J. Math. **14** (1964), 645–650.

[7] B. Stenström: Rings and modules of quotients, Lecture Notes in Math. 237, Springer-Verlag, 1971.

Department of Mathematics
Yamaguchi University
Yoshida, Yamaguchi 753
Japan