Osaka Mathematical Journal
Vol. 5, No. 2, December, 1953

On the Structure of the Plane Translation of Brouwer

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The so-called plane translation theorem of Brouwer [1] had been a starting point to a series of investigations concerning the sense preserving topological transformation of the Euclidean plane onto itself without fixed point. The general behaviour of such a transformation, called by Scorza Dragoni [10] a generalized translation, however, is not so simple and but little is definitely known until now. It was first pointed out by Sperner[1] that the study of singularities arising from a generalized translation should have an essential meaning for further investigation but no attempt seems to have been actually made. In the present paper is given a full treatment of the subject along this line.

In the attempt to prove the converse to the structure theorem, however, we were forced to the conclusion that the mere consideration of an individual homeomorphism is insufficient and a procedure should be devised of obtaining from a given homeomorphism some other simplified one in order to make clear the whole mechanism of generalized translations. In this way alone we shall be able to obtain the adequate classification of generalized translations comparable with the beautiful results of Kaplan [3] in his classification of curve-families filling the plane.

In the preliminary I the theory of free domains is developed to be applied in II to the study of singular lines. The structure theorem and an attempt to obtain its converse are given in III. In the final IV known and unknown examples of generalized translations are collected along with some theorems leading to the construction of several examples.

I. Free Domains

1. Throughout this paper f denotes, unless otherwise stated, a sense preserving topological transformation of the Euclidean plane E^2 onto itself without fixed point. The n-th iteration of f will be denoted by f^n for any positive or negative integer n, inclusive zero, provided that f^0 and f^1 stand for the identical transformation and the original transformation f itself respectively.

1) [11] p. 24, foot-note.

2. Regularity and irregularity.

If M is a point set, the cluster set of $f^n(M)$ for all positive integers n, that is a set of points whose arbitrarily small neighbourhoods meet an infinite number of $f^n(M)$, will be called the *final limit* or the $(+)$-*limit* of M and denoted by $\lim^+ M$; likewise the cluster set of $f^n(M)$ for all negative integers n will be called the *initial limit* or the $(-)$-*limit* of M and denoted by $\lim^- M$.

If the $(+)$-*limit* of M is vacuous, then M is said to be $(+)$-*regular* and if it is non vacuous, then M is said to be $(+)$-*irregular*. Similarly for $(-)$-*regularity* and $(-)$-*irregularity*.

A single point cannot be irregular in the above sense, for

Lemma (Brouwer [1]). *If p is a point, then $f^n(p)$ diverges to ∞ when $n \to +\infty$ and when $n \to -\infty$.*

holds true. However, a point p will be said to be $(+)$-*irregular*, if every neighbourhood $U(p)$ of p, where we understand by a *neighbourhood* always a *Jordan domain* containing the point in question, is $(+)$-*irregular*.

Cf. Example 2[2], where a is $(+)$-irregular.

If further the meet $P = \bigwedge \lim^+ U(p)$ for all neighbourhoods $U(p)$ of p is non vacuous, then p is said to be *strongly* $(+)$-*irregular*. If p is $(+)$-irregular but P vanishes, then p is said to be *weakly* $(+)$-*irregular*. Similarly for the $(-)$-*irregularity*. All points of the segment a_0a_1 other than $a_{1/2^n}$ in Example 6 are weakly $(+)$- as well as $(-)$-irregular.

If p is a strongly $(+)$-irregular point, then

$$P = \bigwedge \lim^+ U(p)$$

for all neighbourhoods $U(p)$ of p will be called the $(+)$-*singularity polar to p* and p will be called a *pole* of P. (Cf. Example 1 and 2). Similarly for $(-)$-*singularity*.

It is sometimes convenient to replace the Jordan domain U by a closed Jordan domain $\bar{U}$. If namely $U_1, U_2, \dots, U_n, \dots$ are a sequence of Jordan domains such that

(i) $U \supset U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots$,
(ii) $\bigwedge_{n=1}^{\infty} \bar{U}_n$ is a point $p \in U$,
(iii) $\bigwedge_{n=1}^{\infty} \lim^+ U_n = P$ is non vacuous,

then P will be called the $(+)$-*singularity polar to p with respect to the decreasing sequence of domains* $U_1, U_2, \dots$. Cf. Example 2.

2) All example are collected in IV.

If a point p is neither $(+)$- nor $(-)$-irregular, i.e., if there exists a neighbourhood $U(p)$ of p such that both $\lim^{+} U(p)$ and $\lim^{-} U(p)$ vanish, then p is said to be *regular*.

3. Duality.

Duality Theorem. *If p is strongly $(+)$-irregular, then every point q of the $(+)$-irregularity P polar to p are strongly $(-)$-irregular, and the $(-)$-singularity Q polar to q passes through p. The same holds true if p is strongly $(-)$-irregular*[3].

Proof. Let U and V be arbitrary neighbourhoods of p and q respectively. Since $q \in P = \bigcap_{U(p)} \lim^{+} U(p) \subset \lim^{+} U$, we have $f^{n}(U) \cdot V \neq 0$ for infinitely many $n > 0$, thus $U \cdot f^{-n}(V) \neq 0$ for infinitely many $n > 0$, and consequently $\bar{U} \cdot \lim^{-} V \neq 0$. Since U was arbitrary, it follows $p \in \lim^{-} V$. Since V was arbitrary, we have finally $p \in \bigcap_{q \in V} \lim^{-} V = Q$, q.e.d.

Remark 1. The same duality theorem holds if we take instead of the singularity polar to p the singularity polar to p with respect to a decreasing sequence of domains U_n with $p = \bigcap \bar{U}_n$.

As an application of the Duality Theorem we have

Lemma 1. *If U is a $(+)$-irregular bounded domain, then $\bar{U}$ contains at least one strongly $(+)$-irregular point.*

Proof. Let $q \in \lim^{+} U$. Then, if $U(q)$ is an arbitrary neighbourhood of q, we have $f^{n}(U) \cdot U(q) \neq 0$ for infinitely many $n > 0$, thus $U \cdot f^{-n}(U(q)) \neq 0$ for infinitely many $n > 0$. It follows therefore $\bar{U} \cdot \lim^{-} U(q) \neq 0$. Then a point $p \in \bar{U} \cdot \lim^{-} U(q)$ is by Duality Theorem a strongly $(+)$-irregular point contained in $\bar{U}$.

4. A point set M is said to be *free*, if $M \cdot f(M) = 0$.

A domain U is said to *touch* another domain V, if $U \cdot V = 0$ but $\dot{U} \cdot \dot{V} \neq 0$.[4]

A domain is said to be *critical*, if U touches its own image $f(U)$.

An *open line* is a closed set which is homeomorphic to a straight line. By an *arc* is meant a simple (closed or open) arc.

An arc τ which has with its image an end point in common, the interior of the arc being free, is called a *translation arc* according to

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3) A proposition remains true, if we interchange $(+)$ and $(-)$, or "initial" with "final". We often omit the enunciation of proposition obtainable in this way. Likewise for difinitions.

4) $\dot{U}$ (a dot above U) denotes the boundary of a domain U.

Brouwer. Then $\bigcup_{n=-\infty}^{\infty} f^n(\tau)$ will be called a *stream-line* (Bahnkurve according to Brouwer). A stream-line is a topological image of a straight line but is not necessarily a closed set, i.e., not necessarily an open line (Brouwer [1]). Cf. Example 3. If a stream-line is an open line, it will be called *regular*.

It may happen that every stream-line through a point p is not regular. Such a point will be called *irregular in the large*.

Remark 2. *A regular point can be irregular in the large.*

See Example 4. All points of D_n are regular but irregular in the large.

Concerning stream-lines we have the following important lemma due to Brouwer [1], from which follows for instance the lemma in § 2 as well as the property above mentioned of stream-lines immediately:

Lemma 2. FUNDAMENTAL LEMMA ON STREAM-LINES (BROUWER). *If an arc α makes together with an arc β of a stream-line a simple closed curve and if β contains a translation arc as pure subset, then α has at least one point in common with its own image $f(\alpha)$.*

We have now

Through every point passes at least one stream-line (Brouwer [1]).

For a simple proof see Terasaka [12].

The following lemma is a simple consequence of the fundamental lemma on stream-lines and is a main instrument for subsequent discussion.

Lemma 3. LEMMA OF FREE DOMAIN. *If U is a free domain, i.e., if $U \cdot f(U) = 0$, then $U \cdot f^n(U) = 0$ for all integers n other than 0. If U is a free Jordan domain, then there is a positive number ε such that*

$$d(U, f^n(U)) > \varepsilon > 0$$

for all n other than $n = -1$, 0 and 1.

In the latter part of the lemma we cannot replace the Jordan domain merely by a bounded domain.

5. Translation-field and area of translation.

Our principal concern is the investigation of the structure and the distribution of singularities alluded to in § 2, and the theory of translation-field of Brouwer might be a useful instrument for this purpose. As a matter of fact, the singularities are all contained in the boundary of the area of translation generated by the translation field. But a main difficulty arises from the fact that the complexities of boundary come not only from the transformation f itself but also from the way how we choose a translation-field, as can be shown by examples (cf. Examples 11 and 12). From this reason we devised a more direct approach by considering only iterated images of free Jordan domains instead of translation-field, and therefore only what is referred to later on the subject should be given here.

A simply connected domain which is bounded by two open lines, one of which is the image of the other, is called a *translation-field* (Brouwer). It is the important so-called *plane translation theorem of Brouwer* that any point of the plane is contained in some translation-field[5]. The open line γ which bounds together with its image $f(\gamma)$ the translation-field T, and in general $f^n(\gamma)\,(-\infty < n < \infty)$ will be called *level curves*.

If T is a translation-field, then

$$A = \bigcup_{n=-\infty}^{\infty} f^n(T)$$

will be called the *area of translation* generated by T. A is evidently simply connected and bounded by $\lim^- T$ and $\lim^+ T$, which will be called the *initial* and the *final boundary* of A respectively. These are evidently closed and their components are, if non vacuous, unbounded continua, since they are the cluster sets of level curves which are all open lines.

6. Intermediate and consecutive domains.

Let U and V be free Jordan domains which meet. Let D be a component of $V - \overline{\bigcup_{n=-\infty}^{\infty} f^n(U)}$ such that U and D have some boundary points in common within V. Such a domain D will be called an *intermediate domain contiguous* to U (in V). We say also that U is *contiguous* to D.

Then we have the following important property of intermediate domains.

Lemma 4. LEMMA ON INTERMEDIATE DOMAIN. *If D is an intermediate domain contiguous to U in V, then D cannot be contiguous other than* (i) *either to U alone,* (ii) *or to U and $f(U)$,* (iii) *or to U and $f^{-1}(U)$.*

Proof. If supposing on the contrary the boundaries of $f(U)$ and $f^{-1}(U)$ take parts in the boundary of D at the same time, join a pair

5) Brouwer [1]. For simplified proofs see Kerékjártó [5], Scherrer [8], Scorza-Dragoni [9]. Sperner [11], Terasaka [12].

of their points a and b within D by a Jordan arc ab. Construct a Jordan domain W within D such that W contains ab and has an arc β in common with $f^{-1}(U)$. Then, since W is disjoint from all $f^n(U)$ except for $n=-1$, 0 and 1, and since $W \subset V$ is free, $f^{-1}(U)+\beta+W=W'$ is a free domain, while W' touches its second image $f^2(W')=f(U)+f^2(\beta)+f^2(W)$ in the point a, which is impossible by Lemma 3. Thus the boundaries of $f(U)$ and $f^{-1}(U)$ cannot take parts in the boundary of D at the same time.

From the same reason $f^n(U)$ has no boundary point in common with D if $|n| \geqq 2$.

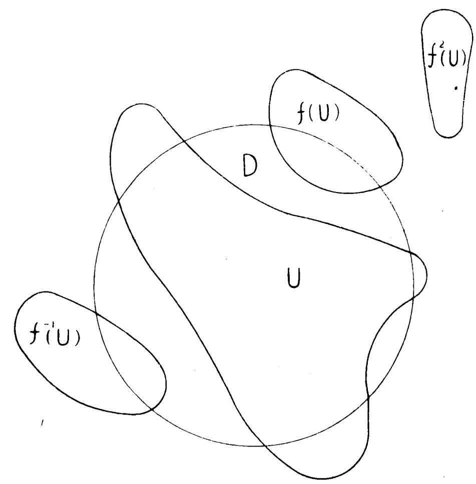

Fig. 1

Finally suppose there is a boundary point p of D which is a cluster point of an infinite number of $f^n(U)$. Then, since p cannot be a boundary point of $U, f(U)$ or of $f^{-1}(U)$, there is a neighbourhood $U(p)$ of p contained in V which is disjoint from $U, f(U)$ and from $f^{-1}(U)$. We can find therefore an accessible boundary point b of D within $U(p)$ which is a cluster point of an infinite number of $f^n(U)$. Construct as before a domain W within D having an arc β in common with U and having the point b on its boundary. Then $U+\beta+W$ must be a free Jordan domain whose arbitrarily small neighbourhood has points in common with an infinite number of its own images, in contradiction with Lemma 3.

Thus the proof of the lemma is complete.

If there is an intermediate domain contiguous to $f^n(U)$ and $f^{n+1}(U)$ in V, then $f^n(U)$ and $f^{n+1}(U)$ will be called *consecutive domains* with respect to V.

As a result of Lemma 4 we have

Lemma 5. *Let U and V be free Jordan domains. If an infinite number of $f^n(U)$ for positive n meet V, then almost all of them meet V.*

Proof. First suppose that $f^n(U)\cdot V \neq 0$ and $f^m(V)\cdot V \neq 0$ for some $n > m+1$. Join a point of $f^n(U)\cdot V$ to a point of $f^m(U)\cdot V$ within V by an arc and determine its subarc $a_n a_m$ such that it is disjoint from $f^n(\bar{U})$ and $f^m(\bar{U})$ except for its end points a_n and a_m which are on $f^n(\dot{U})$ and $f^m(\dot{U})$ respectively. If D is the intermediate domain contiguous to $f^n(U)$ at a_n, then D is contiguous either to $f^{n-1}(U)$ or to $f^{n+1}(U)$ other than $f^n(U)$. In the former case let a_{n-1} be the last point of the intersection of a_na_m with $f^{n-1}(\bar{U})$ when a point moves along a_na_m from a_n. If D' is the intermediate domain contiguous to $f^{n-1}(U)$ at a_{n-1}, then D' must be contiguous to $f^{n-2}(U)$, since it cannot be contiguous to $f^n(U)$ by the property of a_na_m. Proceeding in this way we see that $f^n(U), f^{n-1}(U), \dots, f^{m+1}(U), f^m(U)$ are a series of consecutive domains of V which meet a_na_m. If on the other hand $f^{n+1}(U)$ is contiguous to D, then we see by the same reasoning that $f^n(U), f^{n+1}(U), f^{n+2}(U), \dots$ ad inf. are a sequence of consecutive domains of V which meet a_na_m, and our lemma turns out to be true.

Now, if an infinite number of $f^n(U)$ for positive n meet V, then in either of the cases above considered we can conclude that $f^n(U)$'s meet V from a certain number n on, and our lemma is proved.

7. Let U be a free Jordan domain. Then, if T is a translation-field which contains U, we have $\lim^+ U \subset \lim^+ T$ and $\lim^- U \subset \lim^- T$. But since $\lim^+ T$ and $\lim^- T$ belong to the different components of E^2-T, they are disjoint. Hence

Lemma 6. *If U is a free Jordan domain, then* $\lim^+ U$ *and* $\lim^- U$ *are disjoint. U and all its images $f^n(U)$ $(-\infty < n < \infty)$ are contained in one and the same component of* $E^2-\lim^+ U-\lim^- U$.

If U is a free Jordan domain, then the components of $E^2-\lim^+ U$, of $E^2-\lim^- U$ and of $E^2-\lim^+ U-\lim^- U$ in which all $f^n(U)$ are contained will be called the *positive sides* of $\lim^+ U$, of $\lim^- U$ and of $\lim^+ U+\lim^- U$ respectively.

Now let U be a free Jordan domain and let V be another Jordan domain which meets $\lim^+ U$ and which is disjoint from $\lim^- U$. Suppose $f^n(U)$ meet V and draw from a point of $f^n(U)\cdot V$ within V an arc j such that it has no point in common with $\lim^+ U$ except for its end point a.

If a_n denotes the last point on j that lies on $f^n(U)$ when a point moves along j towards a, then the open arc (a_na) is disjoint from all $f^i(\bar{U})$ for $i \leqq n$. For first, since $V\cdot\lim^- U=0$, there is the least i with $f^i(\bar{U})\cdot a_na=0$. If a_i is the last point on a_na that lies on $f^i(\dot{U})$, and if D_i is the intermediate domain contiguous to $f^i(U)$ in V having a_i on its boundary, then $f^{i+1}(U)$ is contiguous to D_i and has evidently points in common with a_ia, and eo ipso with a_na. Continuing in this manner we see that all $f^m(U)$ have points in common with a_na if $m \geqq i$. If therefore $i<n$, then $f^n(U)$ must have points in common with $a_n a$, contrary to the property of a_n.

Such an arc $a_n a$ will be called a *bridge* between $f^n(U)$ aud $\lim^+ U$ in V, or briefly a bridge. The bridge $a_n a$ is by its definition a free arc such that a_n is on the boundary of $f^n(U)$ and a is on $\lim^+ U$ and such that it is disjoint from all $f^i(\bar{U})$ for $i \leqq n$ except for a_n.

8. Bordering cell.

Let the arcs $a_{n-1}a = j_{n-1}$ and $a_n a = j_n$ be bridges between $f^{n-1}(U)$ and $\lim^+ U$; $f^n(U)$ and $\lim^+ U$, respectively, j_n being a subarc of j_{n-1}, and let b_n be the first point of $f^n(\dot{U})$ on j_{n-1} proceeding from a_{n-1}. Then the arc $a_{n-1}b_n = k$ is an arc which connect $f^{n-1}(\bar{U})$ and $f^n(\bar{U})$ outside of all $f^i(U)$.

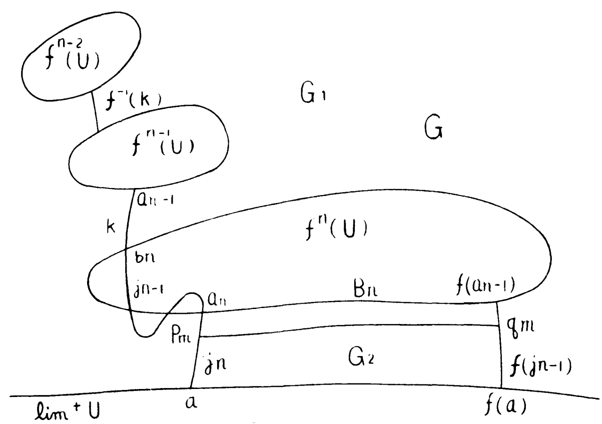

Fig. 2

Denote by β_n one of the two arcs of $f^n(\dot{U})$ divided by a_n and $f(a_{n-1})$ and which does not contain b_n. Since the component G of $E^2 - \lim^+ U - \lim^- U$ which contains all of $f^i(U)$ is simply connected, G is divided by $f(j_{n-1})+\beta_n + j_n$ into two domains G_1 and G_2. Suppose b_n belongs to G_1.

Now, since $f^i(U)$ for $i<n-1$ are all disjoint from j_{n-1} and hence $f^i(U)$ are for $i \leqq n-1$ also disjoint from $f(j_{n-1})$, each of $f^i(U)$ for $i \leqq n-1$ lies either in G_1 or in G_2. But since b_n, and hence $a_{n-1}b_n = k$ too belongs to G_1, $f^{n-1}(U)$ is contained in G_1; if we consider the inverse image $f^{-1}(k)$ of k, which joins the point $f^{-1}(b_n)$ of $f^{n-1}(\dot{U})$ to the point $f^{-1}(a_{n-1})$ of $f^{n-2}(\dot{U})$ and which is evidently disjoint from $f(j_{n-1})$ as well as from j_n and $f^n(\dot{U})$, we see at once that $f^{n-2}(U)$ lies likewise in G_1. Proceeding in this way we see that all $f^i(U)$ are contained in G_1 whenever $i \leqq n-1$.

We assert next that for each $m>n$ there is a subarc $p_m q_m$ of $f^m(\dot{U})$ which joins a point p_m of j_n to a point q_m of $f(j_{n-1})$ within G_2. Suppose on the contrary there is no such arc for some m. Then there must be an arc pq of $f^m(\dot{U})$ which joins a point p of j_n to a point q of $f(j_{n-1})$

within G_1, so that G_1 is divided by pq into two domains, one of which, say G', is a Jordan domain bounded by

$$pq+qf(a_{n-1})+\beta_n+a_n p = C.$$

Out of the two sides in the neighbourhood of β_n the one belongs to G_2, therefore outside of G', and hence the other one, i.e., the inside of $f^n(U)$, belongs to G'. Consequently b_n belongs to G', and the same argument as above leads to the conclusion that $f^i(U)$ for $i \leqq n-1$ are all confined to the Jordan domain G', which is absurd.

Thus for any $m > n$ there is a subarc $p_m q_m$ of $f^r(\dot{U})$ which joins a point p_m of j_n to a point q_m of $f(j_{n-1})$ within G_2.

The image $f(p_m q_m)$ lies outside of G_2; for first, if a point x moves along $p_m q_m$ from p_m, its image $f(x)$ in the neighbourhood of p_m lies outside of G_2 in the neighbourhood of the point $f(p_m)$ on $f(j_n)$ in consequence of the preservation of orientation, and since $f(x)$ cannot be a point of $f(j_{n-1})$ nor of $\beta_n (\subset f^n(\dot{U}))$, if it should happen that $f(x)$ is at a certain stage a point of G_2, there would be a point $x=p$ such that $f(p_m p)$ is an arc connecting the point $f(p_m)$ of $f(j_n)$ outside of G_2 to the point $f(p)$ of j_n, which is absurd as we have shown above. Thus $f(p_m q_m)$ lies outside of G_2.

If therefore H_m denotes one of the two domains into which G_2 is divided by $p_m q_m$ and whose boundary consists of

(i) the subarc $p_m a$ of j^n,
(ii) the subarc $q_m f(a)$ of $f(j_{n-1})$,
(iii) $p_m q_m$, and
(iv) a closed subset σ of $\lim^+ U$,

then the image $f(H_m)$ has no point in common with H_m, and thus H_m is a free domain. The image $f(p_m)$ of p_m lies on $q_m f(a)$, and the arc

$$p_m q_m + q_m f(p_m) = \tau$$

is a translation arc. We call H_m a *bordering cell*, the arc

$$ap_m + \tau + f(ap_m)$$

the *link* with the *translation arc* τ, and the point set σ the *singularity segment* bordering H_m.

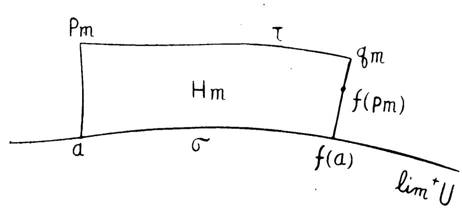

Fig. 3

The series of arcs ..., $p_m q_m$, $p_{m+1} q_{m+1}$, ... in the above construction can be so taken that $p_{m+1} q_{m+1}$ is *contained in* H_m except for its end points.

For, if there is no such arc, then a certain point c of $p_m q_m$ can be joined by an arc cd within H_m to a point d of some $p_k q_k$ with sufficiently large k that lies within H_m such that cd has no point in common with $f^{m+1}(\dot{U})$. Let c_m be the last point of $f^m(\dot{U})$ on cd and let $d_{m'}$ be the first point of $f^{m+2}(U)$, $f^{m+3}(U)$, ... that eventually encounter cd. Then $f^m(U)$ can be joined to $f^{m'}(U)$ with $m' \geqq m+2$ by the arc $c_m d_{m'}$ which is free since $c_m d_{m'} \subset H_m$, but this is impossible by Lemma 4.

The singularity segment σ bordering the bordering cell H_m is evidently the limit of $p_m q_m$, $p_{m+1} q_{m+1}$, ... If this sequence of arcs possesses the above property, then it will be called a *simple sequence* defining the singularity segment σ bordering H_m.

We will define and develop the theory of singular lines in the next section.

II. Properties of Singular Lines

9. Singular lines.

Since the singularity segment σ is the limit of a simple sequence of arcs, it is either a bounded continuum or consists of unbounded continua. σ may have points which are inaccessible from H_m (cf. Example 5 (1)), but has clearly a dense set of points which are accessible from H_m. The point set

$$S = \overline{\bigvee_{n=-\infty}^{\infty} f^n(\sigma)}$$

will be called the $(+)$-*singular line* generated by σ, the set

$$\bigvee_{n=-\infty}^{\infty} f^n(\sigma)$$

the *interior*, $\lim^+\sigma$ the *final end* or $(+)$-*end*, and $\lim^-\sigma$ the *initial end* or $(-)$-*end* of S. The component of E^2-S which contains the bordering

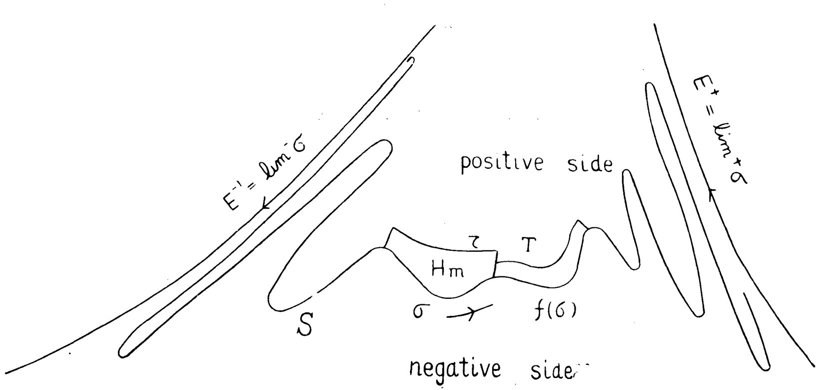

Fig. 4

What follows is almost evident:

S 2. *If two singular lines having no points in common can be joined by a free arc within the domain D bounded by them and coinciding with its own image, then they move in the opposite directions with respect to D.*

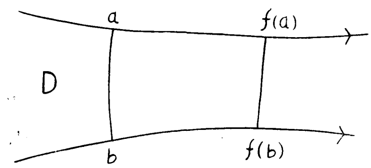

Fig. 5

A *singular arc* of a singular line S is a set of all prime ends lying between two distinct points (i.e. prime ends) of S. A singularity segment is a kind of singular arc if we consider its points as consisting of prime ends. It is to be noted that when we speak of the points or prime ends of S the *point at infinity* ∞ may happen to be counted.

10. To establish the first non trivial property of singular lines (S 3), we begin with the following

Lemma 7. *Every arc which joins a point of a free Jordan domain U to a point of* $\lim^+ U$ *meets almost all $f^n(U)$ for positive n.*

Proof. Suppose on the contrary that there is an arc j which joins a point a of U to a point b of $\lim^+ U$ and which meets only a finite number of $f^n(U)$. Choose a neighbourhood, i.e. a Jordan domain, V, containing b and not meeting U and let j' be an arc connecting b to a point c of some $f^n(U)\cdot V(\neq 0)$ within V such that j' has only the end point b in common with j and such that j' meets an infinite number of $f^n(U)$. Join the end point c of j' to a by an arc j'' meeting a finite number of $f^n(U)$ to make together with j and j' a closed Jordan curve C: $C = j+j'+j''$.

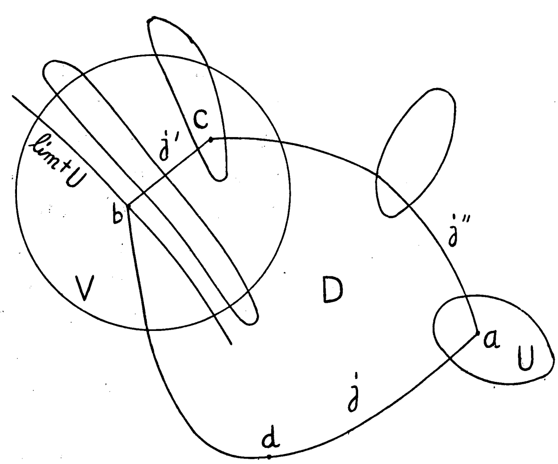

Fig. 6

Take a point d of j such that the arc bd of j does not meet any of $f^n(\bar{U})$ and let D be the component of

$$(\text{Interior of } C)-\overline{\bigvee_{n=-\infty}^{\infty} f^n(U)}$$

cells will be called the *positive side* of the singular line S or of the singularity segment σ, the other components, if any, being called the *negative side* (cf. Example 5(2)).

Similarly we can define $(-)$-*singular lines* etc. beginning from $\lim^- U$.

For the sake of simplicity of expression we say hereafter simply *singular lines* etc. instead of $(+)$-singular lines etc. unless otherwise stated.

The following is the first property of singular lines:

S 1. *A singular line is a closed set composed of unbounded continua, and bounds an unbonnded simply connected domain called its positive side.*

Taking for each m an inner point c'_m of $f^m(U)$ on j_n (cf. fig. 2), let c be a cluster point of the points $f^{-m}(c'_m) = c_m$ which belong all to U and consider a monotone decreasing sequence of domains $U_n \subset U : U_1 \supset U_2 \supset \cdots U_n \supset \cdots$ such that U_n converges to c and each U_n contains infinitely many c_m. Then $\bigcap_{n=1}^{\infty} \lim^+ U_n$ is evidently contained in $\lim^+ U$ and contains itself the singular line S. We say that S is *derived from the decreasing sequence of domains* U_n. We say also that S is *derived from the point* c.

Remark 3. It is not true that given a singular line there is a decreasing sequence of domains deriving this and only this line.

See Example 7.

According to Carathéodory [2] the boundary of a simply connected domain consists of prime ends which are cyclically ordered. Therefore, since the singular line S is the boundary of an unbounded simply connected domain, we can assign to it the positive sense in accordance with the fixed orientation of the plane, and speak for example of the positive side and the negative side of a point (i.e. a prime end) p on S. Then, if the image $f(p)$ of an interior point p of S falls into the positive side of p, it is clear that for any other interior point q of S the image $f(q)$ falls also into the positive side of q. In this case we say that the singular line S *moves in the positive direction.* Similarly for the motion in the negative.

The motion on the negative side of a singular line can also be defined, provided that there exists among the components of the negative side one, say D, such that D coincides with its own image: $D = f(D)$. In this case we can speak of the *motion in the positive* or *in the negative direction with respect to* D, just as we have done on the positive side of the singular line.

whose boundary contains bd. Then an infinite number of, and in reality almost all of, $f^n(\dot{U})$ appear on the boundary of D. For if not, suppose $f^{n_1}(\dot{U}), f^{n_2}(\dot{U}), \dots, f^{n_i}(\dot{U})$ are the only Jordan curves among $f^n(\dot{U})$ which take parts in the boundary of D. Let a point move along C from d towards b passing through a and c, and let p be the last intersection with $f^{n_1}(\dot{U}), f^{n_2}(\dot{U}), \dots, f^{n_i}(\dot{U})$. Then p cannot lie between a and c, since otherwise the first Jordan curve $f^n(\dot{U})$ which meets the moving point x after starting from p would be distinct from $f^{n_1}(U), \dots, f^{n_i}(U)$, which is absurd. But if on the other hand p is a point of j', suppose p is a point of $f^{n_i}(\dot{U})$. Then if we move the point x from p further, it will meet the consecutive domain $f^m(U)$ of $f^{n_i}(U)$ with respect to V, where $m=n_i+1$ or $m=n_i-1$, so that $f^m(\dot{U})$ also appears on the boundary of D, which again contradicts the supposition that only $f^{n_1}(\dot{U}), \dots, f^{n_i}(\dot{U})$ appear on the boundary.

Thus there must be an infinite number of $f^n(\dot{U})$ which appear on the boundary of D. By the argument of consecutive domains we see at once that in reality all $f^n(\dot{U})$ from a certain number on appear consecutively on D.

Now take for each n a point p_n of $f^n(\dot{U})$ which lies upon the boundary of D. By the theory of prime ends of Carathéodory there can be found a sequence of cuttings s_i of D such that s_i converges to a point q of $\dot{D}$ and such that if D_i denotes the subdomain cut from D by s_i and not containing a definite point o of D, then each D_i contains an infinite number of p_n on its boundary, the sequence D_i defining a prime end.

(i) q cannot be an accessible point of D. For otherwise let W be a free neighbourhood of q. Then almost all s_i belongs to W and consequently an infinite number of $f^n(U)$ are contiguous to a single intermediate domain with respect to W, contrary to Lemma 4.

(ii) q cannot be an inaccessible point of D. For otherwise there must be a cutting s_i joining a point of some $f^n(\dot{U})\cdot\dot{D}$ to a point of $\lim^+ U$ within a free neighbourhood W of q, contrary again to Lemma 4.

Thus our supposition leads to contradictions, and the proof of the lemma is complete.

A similar argument yields, if we use instead of Lemma 4, which results out of Lemma 3, the property of translation arc and the streamline generated by the arc:

Lemma 8. *Every arc which joins a point of translation arc* τ *to a point of* $\lim^+\tau$ *meets almost all* $f^n(\tau)$ *for positive* n.

To see that the same holds true for a singularity segment σ, we need the following

Lemma 9. *If σ is a singularity segment generating a singular line S, then there is no free arc joining a point of* $\lim^{+}\sigma$ *to a point of some $f^{n}(\sigma)$ within the positive side of S.*

Proof. Suppose on the contrary that there is a free arc pq joining a point p of $\lim^{+}\sigma$ to a point q of $f^{i}(\sigma)$ within the positive side of S. Retaining the earlier notation (§ 8), let the steam-line generated by the translation arc τ bordering H_m be denoted by T, and the domain bounded by T and S, by B. Then, if p lies on $\lim^{+}\tau$, pq has by Lemma 8 points in common with an infinite number of $f^{n}(\tau)$, and hence with an infinite number of $f^{n}(H_m)$; if on the other hand p is disjoint from $\lim^{+}\tau$, then either pq belongs entirely to B or there is a point q' on T such that $q'p$ is contained wholly in B, and if we proceed along qp from q or q' (which is a point of some $f^{n}(H_m)$), we encounter infinitely many $f^{n}(H_m)$. Thus in either case pq has points in common with an infinite number of $f^{n}(H_m)$ for $n>0$. Now take m so large that $f^{i-1}(H_m)$ and $f^{i+1}(H_m)$ become disjoint from pq, and let $f^{k}(H_m)$ be the first among $f^{n}(H_m)$ that pq intersect when a point moves along pq from q; let this intersection be $f^{k}(q_0)$. If we construct within H_m a domain H containing the points q and q_0' on its boundary, then $f^{i}(H)$ and $f^{k}(H)$ with $|i-k|\geqq 2$ are joined by a free subarc of pq outside of all $f^{n}(H)$, contrary to Lemma 4.

Finally we have

Lemma 10. *Every arc which joins a point of a singularity segment σ to a point of* $\lim^{+}\sigma$ *within the positive side of σ meet almost all $f^{n}(\sigma)$ for positive n.*

The proof of this lemma may be carried out quite analogously to that of Lemma 7 by virtue of Lemma 9 besides Lemma 7 and will be omitted. It should be added that in applying Lemma 4 a certain modification of H_m to H, as we have done in the above proof of Lemma 9, will be needed.

Lemma 10 may be stated thus:

S 3. *Every point of the final as well as the initial end of a singular line is inaccessible from its positive side.*

11. Relations between two or more singular lines. Initial and final ends.

S 4. *Two singular lines cannot cross, that is, either their positive sides* *have no point in common or one of them is contained wholly in the positive side of the other.*

Proof. Suppose the singular lines S and S' have a point o of their positive sides in common, and let D be the component of the intersection of the positive sides that contains o. Then clearly either $D\cdot f(D)=0$ or D coincides with $f(D)$.

(i) Suppose $D\cdot f(D)=0$. If α is an arc which joins two points of S on the boundary of D within D, then we have $\alpha\cdot f(\alpha)=0$, which shows that any pair of points of S on $\dot{D}$ cannot be separated by any other pair of points of S on $f^n(\dot{D})$ on the positive side of S, and it follow that there is a singular arc, even a singularity segment ab, of S which contains all those points of S appearing on D. Now take a point p of S' on D disjoint from S and construct a bordering cell H_m with the bordering singularity segment ab and not containing p and $f(p)$. Then if i is chosen sufficiently large, $f^i(\dot{U})$ runs sufficiently near to p and to $f(p)$ so that it cuts across H_m, and hence across $p_m q_m$ of $f^m(\dot{U})$ and across the singularity segment ab. It follows therefore that $f^i(U)$ meets $f^n(U)$ for all n greater than m, and if i has been chosen $< m$ beforehand, then $f^i(U)\cdot f(U')\neq 0$, whence we have $U\cdot U'\neq 0$.

By the consideration of decreasing sequence of domains, U' may be taken as small as we please and so, since $U\cdot U'\neq 0$, U' may be contained wholly within the positive side of S and consequently S' will be wholly contained in the positive side of S, contrary to the supposition that $D\cdot f(D)=0$.

(ii) Suppose D coincides with $f(D)$, but coincides neither with the positive side of S nor with that of S'. Then there is an accessible point a of S on $\dot{D}$ disjoint from S'. Join a and $f(a)$ within D by an arc α. Then there is within the domain bounded by α and by the singularity segment $af(a)=\sigma$ at least one point, say c, of S', since otherwise σ would be wholly contained in the boundary of D. Now if j is an arc within D ending in a and if H_m is the corresponding bordering cell with sufficiently large m so that the singularity segment σ' of S' beginning from c meets the translation arc τ of H_m, then σ' cuts across τ as well as σ, and the same reasoning as in (i) leads to the conclusion that $U\cdot U'\neq 0$, whence a contradictiou arises exactly as above.

The property S 4 is thus proved.

Now suppose the singular lines S and S' whose positive sides intersect have an interior point a in common. Since S and S' cannot cross by S 4, one of them, say S', is wholly contained in the positive side of S. Then almost all $f^n(U)$ intersect a singularity segment σ' of S' containing

a, hence U intersect almost all $f^{-n}(\sigma')$. Thus the initial end of S' is non vacuous and S will be derived from one of its points.

Conversely, if the initial end of S' is non vacuous, then the singular line S derived from its points has, if non vacuous, clearly interior points in common with S'. S can vanish if the singularity segment σ' deriving S' is unbounded. Thus we have

S 5. *If two distinct singular lines have some interior points in common, then one of them is derived from a point of the initial end of the other, provided that their positive sides have points in common. Conversely, the singular line derived from a point of the initial end of another singular line has, if non vacuous, interior points in common with the latter.*

If two singular lines have some interior points in common, we say, since they do not cross by S 4, that they *touch* each other. If a singular line S touches S' and if S' is on the positive side of S, we say *S' touches S from the positive side.*

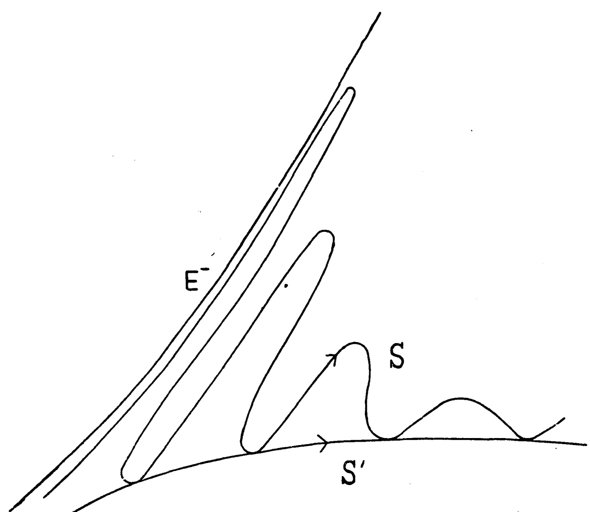

Fig. 7

If S' touches S from the positive side, then S is derived from a point of the initial end E' of S' by S 5, and hence E' cannot be the initial end of S. Thus

S 6. *If two distinct singular lines touch each other, then their initial ends are disjoint.*

S 7. *Two distinct singular lines whose positive sides have points in common cannot have a common initial end.*

For if they have a common initial end, their interiors are by S 5 and S 6 disjoint. But if we take a point p on their initial end, then the singular line derived from p must touch by S 5 both singular lines, which is evidently impossible.

The following is a direct consequence of the foregoing properties of singular lines:

S 8. *Three singular lines connot touch one another, if their positive sides have points in common.*

In connection with S 7 note the following

Remark 4. *Two distinct singular lines whose positive sides have points in common may have a common final end.*

See Example 8.
Further we have

S 9. *The final end of a singular line is disjoint from the initial end of another singular line, if their positive sides have points in common.*

Proof. Suppose on the contrary that the final end of the singular line S has a point p in common with the initial end of another S', and join S and S' within their common positive side by an arc aa'. If D denotes the domain bounded by S, S', aa' and by the ends of S and S' having p in common, then $f(a)$ is again a boundary point of D, while $f(a')$ lie outside of D, and consequently aa' intersect its image $f(aa')$. But aa' could have been taken in any free neighbourhood of p, and we have a contradiction.

The latter part of S 5 may be stated thus:

S* 5. *The initial end of a $(+)$-singular line is a $(-)$-singular line of an interior point of S, if the singularity segment deriving S is bounded.*

From a similar reasoning results at once:

S 10. *The final end of a singular line S is derived from an interior point of S, if the singularity segment deriving S is bounded.*

12. For an infinite collection of singular lines we have the following property.

S 11. *If $S_1, S_2, \dots, S_n, \dots$ are an infinite number of disjoint singular lines such that for each S_i all the other S_n belong to the same component of E^2-S_i, then they have no cluster set.*

Proof. Suppose on the contrary that they have a cluster set. Then let pq be a free arc meeting infinitely many S_n, of which p is a point of the cluster set in question, and a sole one upon pq. If S_i is an S_n which meets pq and which does not eventually pass through p, then p lies evidently on the same side of S_i on which all other S_n belong, and as a consequence, if a is the first intersection of pq with S_i proceeding from p, then ap is contained wholly within that side of S_i. Next choose another S_n, say S_j, meeting ap and not passing eventually through p and let b be the first intersection of ap with S_j proceeding this time from a. Then the arc ab of ap is a free arc which connect S_i and S_j in the domain D_1 bounded by S_i and S_j, and it follows from S 2 that S_i and S_j move in the opposite directions with respect to D_1.

In the same way let b' be the first intersection of pb with S_j proceeding from p and choose among S_n one, say S_k, which meets pb' but does not meet $b'a$ and let c be the first intersection of $b'p$ with S_k proceeding from b'. Then we see as above that S_j and S_k move in the opposite directions with respect to the domain D_2 bounded by S_j and S_k. But since the arc ac is again a free arc connecting S_i and S_k in the domain D_3 bounded by S_i and S_k, these must move in the opposite directions with respect to D_3, and we have a contradiction, since D_1 and D_3 overlap along S_i.

If the condition of this theorem is not fulfilled, singular lines may have cluster set. Indeed, let a be a weakly $(+)$-irregular point. If U_1 is a circular neighbourhood of a of radius 1/2, then there is by Lemma 1 a strongly $(+)$-irregular point p in $\bar{U}_1$. Let P_1 be the $(+)$-singularity passing through p_1 and determine a singular line S_1 through an accessible point (which exists) in $\bar{U}_1$. Next let U_2 be a circular neighbourhood of a disjoint from S and of radius smaller than 1/4 and determine likewise a singular line S_2 intersecting $\bar{U}_2$. Proceeding in this manner we obtain a sequence of distinct singular lines $S_1, S_2, \ldots$ whose cluster set passes evidently through a.

The cluster set of a sequence of singular lines distinct from any singular line will be called a *weak singular line*. We have thus

Lemma 11. *Through every weak $(+)$-irregular point passes a weak singular line.*

A singular line will sometimes be called a *strong singular line* in opposition to weak singular line.

Since we can assign to weak singular lines directions of motion, we have

S 11′. *The proposition* S 11 *remains true, if S_n are weak singular lines.*

As an application of S 11 we have

Lemma 12. *If U is a free Jordan domain, then* $\lim^+ U$ *consists of at most a countable number of singular lines whose interiors have points accessible from the positive side of* $\lim^+ U$.

Proof. If **S** denotes the family of all singular lines passing through points which are accessible from the positive side of $\lim^+ U$, then they satisfy the condition of S 11 and it follows at once that there are at most a countable number of distinct singular lines in **S**. But if a point p of $\lim^+ U$ fails to be a point of any one of **S**, then p must belong to some cluster set of **S**, which is absurd by S 11.

Lemma 13. *There exist at most a countable number of distinct* (*strong*) *singular lines.*

Proof. If **S** denotes the famity of all singular lines corresponding to all circular neighbourhoods with center at rational points and rational radii, then **S** consists by the preceding lemma of at most a countable number of singular lines. Since every singular line corresponds to a decreasing sequence of Jordan domains, it belongs to **S**, and the lemma follows.

13. As a relation between a $(+)$-singular line and a $(-)$-singular line we have first

Lemma 14. *If a $(+)$-singular line S^+ and a $(-)$-singular line S^- have an interior point a in common and if in any small neighbourhood of a there are points of S^- on the positive side of S^+, then the initial end of S^- is non vacuous and S^+ is derived from its points. Conversely, if the initial end E^- of a $(-)$-singular line S^- is non vacuous, then a $(+)$-singular line derived from a point of E^- has some interior points in common with S^- with the above property.*

Proof. Let S^+ be derived from U and let S^- be generated by a singularity segment σ containing a. Then by hypothesis $f^n(U)\cdot\sigma \neq 0$ and hence $U\cdot f^{-n}(\sigma)\neq 0$ for almost all n, therefore the initial end of S^- is non vacuous and S^+ is seen to be derived from a point of this initial end.

The converse is also clear, if we note that the point at infinity may happen to be an interior point of a singular line, since we consider prime ends instead of, or along with, the ordinary points on every singular line.

If a $(+)$-singular line S^+ and a $(-)$-singular line S^- have some interior points in common and if S^- is wholly contained in the positive side of S^+, we say that S^- *touches* S^+ from the positive side. Then, combining Lemma 14 with its dual, we obtain

S 12. *If a $(-)$-singular line S^- touches a $(+)$-singular line S^+ from* *the positive side, then the initial end* E^- *of* S^- *is non vacuous and* S^+ *is derived from its points. Conversely,* S^- *touches from the positive side a* $(+)$*-singular line* S *derived from a point of* E^-*, provided that the final end of* S *vanishes.*

S 13. *It is possible that a* $(+)$*-singular line* S^+ *and a* $(-)$*-singular line* S^- *touch each other from the positive side of the other, or that* S^+ *and* S^- *cross. In either case* S^+ *is derived from a point of the initial end of* S^-*, and* S^- *is derived from a point of the final end of* S^+*.*

For the first part of S 13 see Example 9. The second part is an immediate consequence of Lemma 14 and its dual.

III. Structure Theorems. More Properties of Singular Lines.

14. The set of all regular points makes evidently an open set. Each component of this set will be called a *maximal regular domain.* A maximal regular domain may be free whether bounded or not; but if it is bounded, it must evidently be free. If a maximal regular domain is not free, it coincides with its own images; then it will be called an *area of total regularity.* Mapping the area of total resulartiy on the whole plane E^2 and applying the theorem of Kerékjártó-Sperner ([4], [11]), we see at once that

Lemma 15. *An area of total regularity can be filled with a regular family of regular stream-lines.*

Thus we have the following

First Structure Theorem. *Let* f *be a generalized translation, i. e., a sense preserving topological transformation of the plane* E^2 *onto itself without fixed point. Then* E^2 *is divided into three kinds of disjoint sets:* $O_1, O_2, \dots; O_1', O_2', \dots;$ *and* F*. Each* O_n*, the area of total regularity, is an unbounded simply connected domain and can be filled with a regular family of regular stream-lines. Each* O_n' *is a simply connected free domain and its points are all regular.* F *is closed, consists of all irregular points of* f *and filled with at most a countable number of* $(+)$*- and* $(-)$*-singular lines and their cluster set, the singular lines having the properties* S 1-13 *and their duals.*

15. The converse of the first structure theorem seems to be too complicated to formulate. If we take only the $(+)$-singularities into consideration, the structure theorem undergoes a certain weakening, but instead we have the advantage of obtaining its converse to some measure.

In the following we understand by a singular line a strong as well as a weak $(+)$-singular line.

If two singular lines S and S' are disjoint and S' lies on the positive side of S, then the point p from which S is derived lies evidently either on S' or within the domain bounded by S and S'. Now the $(-)$-singular line S^- which passes through p and which is derived from a decreasing sequence of domains U_n contained on the positive side of S and converging to an interior point a of S lies wholly within the domain bounded by S and S' or at most within its closure. Thus:

S 14. *If the singular lines S and S' are disjoint and S' lies on the positive side of S, then a $(-)$-singular line S^- from whose points S is derived lies wholly within the domain bounded by S and S' or at most in its closure.*

If two singular lines S and S' touch each other, then each component D of the open set bounded by them is evidently a free domain. But there are several cases that occur according to the nature of S and S'.

If their positive sides have points in common, then by S 8 there is no singular line that enter D.

If their positive sides have no points in common, there can be two more singular lines which enter D. In this case they touch together S and S' at the very points where S and S' touch each other.

If S' is a weak singular line, then the positive sides of S and S' have no points in common, and there can be only one more singular line that enter D.

Finally if S and S' are both weak singular lines, no more singular line enter D.

In each case there is a least domain D_0 bounded by singular lines (weak or strong) disjoint from all singular lines, and thus the points of D_0 are all $(-)$-regular.

A single singular line may bound a free domain, but if D is a free domain bounded by several singular lines, then one of them must be touched by others, since otherwise D cannot be free, and we arrive at the same situation above considered. Thus

S 15. *If D is a free domain bounded by several singular lines, and disjoint from all singular lines, then D is either bounded by a single* *singular line or bounded by two singular lines touching each other. All points of D are* $(-)$*-regular.*

Suppose now that S_n is a sequence of disjoint singular lines such that each S_n passes through a point a_n which converges to an interior point of a singular lines S from its positive side. If U is a free neighbourhood of an interior point of S^- considered in S 14, then $\lim^+ U \supset S$ and hence for sufficiently large n some $f^m(U)$ has points in common with the opposite side to that of S_n in which S lies. But $f^m(U)$ can not be wholly contained in that side of S_n, since otherwise U would be wholly contained in that side of S_n, which is impossible by the property of S^- and by the hypothesis on U, and consequently $f^m(U)$ intersect S_n, hence U intersect $f^{-m}(S_n)=S_n$. Since U was arbitrary, this indicates that S^- belongs to a cluster set of S_n and S^- is seen to be also a $(+)$-singular line or at least a weak $(+)$-singular line. Thus

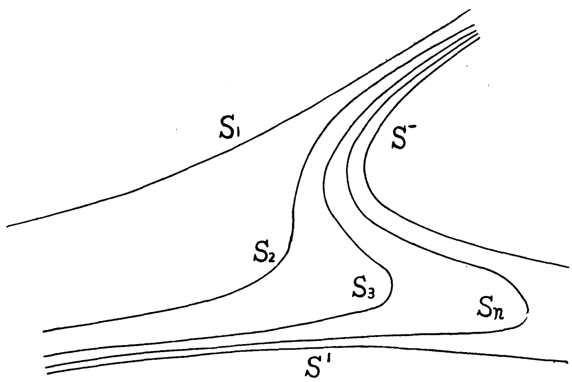

Fig. 8

S 16. *If a singular line S belongs to the cluster set of a sequence of singular lines, then a* $(-)$*-singular line from whose points S is derived belongs also to the cluster set of the sequence.*

If a domain D is bounded by one or more singular lines, disjoint from all singular lines and not free, then D coincides with its images $f^n(D)$. Since every point of D is by hypothesis $(-)$-regular, D will be called an *area of semi-regularity*. If U is a free Jordan domain contained in D, then $\lim^+ U$ is also by hypothesis disjoint from D. It follows therefore that if D is mapped topologically onto the whole plane by a mapping g, then gfg^{-1} is a regular (or singularity free) transformation in the sense of Kerékjártó-Sperner ([4], [11]) and is consequently equivalent to an ordinary translation. The inverse mapping shows that D can be filled with a regular family of regular stream-lines with respect to D.

By filling up an area of semi-regularity with a regular family of regular stream-lines with respect to it we see at once that

S 17. *The singular lines bounding an area of semi-regularity D either* *move all in the same direction or are divided into two groups of consecutive singular lines such that all singular lines belonging to the same group move in the same direction and any two singular lines belonging to different groups move in the opposite directions. In the former case D belongs to the negative side of some one of the singular lines bounding D.*

Cf. Example 2.

16. Now to the construction of generalized translations f.

Let us call a fixed point free topological mapping φ of the boundary of an open set O onto itself *natural*, if the boundary $\dot{D}$ of every component D of O is mapped onto $\dot{D}$ or onto the boundary of another component and if, whenever $\dot{D}$ is mapped onto itself, then φ satisfies the condition of S 17. We call an open set O *inwardly extendible*, if a natural mapping φ of $\dot{O}$ can be extended to a fixed point free topological mapping of the whole $\bar{O}$ onto itself coinciding withe φ on $\dot{O}$.

A closed set S is said to be *admissible* or called an admissible line, if S bounds an unbounded simply connected domain D and if it is *periodic*, i.e., if there exists a fixed point free topological transformation φ of S onto itself such that φ moves every point of S in the same direction with respect to D. Open lines are the simplest example of admissible lines. φ will be called a *periodicity associated with* S. The simplest example of an inwardly extendible domain is the one bounded by a set of open lines having no cluster set. Cf. Extension Theorem in IV (p. 146).

Periodically related are by definition:

1) Two admissible lines S and S', if there are periodicities φ and φ' on S and S' respectively such that the transformations φ and φ' coincides on their eventual intersection;

2) A finite number of admissible lines, if there are periodicities on each lines such that they coincide on their respective intersections;

3) An infinite number of admissible lines $\{S\}$, if (i) they constitute together a closed set $C = \bigcup S$, (ii) there is associated a periodicity on each line such that they coincide on their respective intersections, and (iii) these periodicities make a continuous family of transformation on C.

17. We are now in a positition to state the second structure theorem and its converse as follows:

Second Structure Theorem. *Let f be a generalized translation of the plane E^2. Then E^2 is divided into three kinds of disjoint sets: O_1, O_2,*

... ; O_1', O_2', ... ; *and* F. *Each* O_n, *the area of semi-regularity, is an unbounded simply connected domain and can be filled with a regular family of regular stream-lines with respect to itself. Each* O_n' *is a simply connected free domain and its points are all* $(-)$*-regular.* F *is closed, consists of all* $(-)$*-irregular points of* f *and filled with strong and weak* $(+)$*-singular lines, the singular lines having the properties* S 1-11 *and* S 14-17.

Conversely, given a closed family of periodically related admissible closed set $\{S\}$ *which, regarded as* $(+)$*-singular lines, have the properties* S 1-11, S 14-17 *such that* $E^2-\bigcup S$ *is inwardly extendible, then there exists a generalized translation* f *of* E^2 *having* $\{S\}$ *as the family of its* $(+)$*-singular lines.*

Proof. The first part of the theorem is clear from what we have shown above. To prove the converse, let $D_1, D_2, \dots, D_n, \dots$ be the totality of the components of $E^2-\bigcup S$ which should be the area of semi-regularity. We observe first that

S 18. *If a circle* C *is given, then there are at most a finite number of* D_n *such that at least three different singular lines on the boundary of* D_n *have points in common with* C.

To prove this, let us call such a domain D_n a *special domain*, the three (or more) singular lines on the boundary of D_n intersecting C *special lines*, that side of a singular line S bordering D_n which contains D_n the *inside*, the other side of S the *outside* and finally let us call a special line S a *separating line*, if there are infinitely many special domains outside of S.

Now suppose on the contrary that there are infinitely many special domains and suppose first that a system of a finite number of special lines $S_1, \dots, S_n$ have been already chosen such that each one of them lies inside of the others and such that an infinite number of special lines are contained in the domain G bounded by $S_1, \dots, S_n$. Consider then any special line for any special domain contained in G, and let m be the number of individuals out of $S_1, \dots, S_n$ contained outside of this special line. Let m_0 be the minimum of

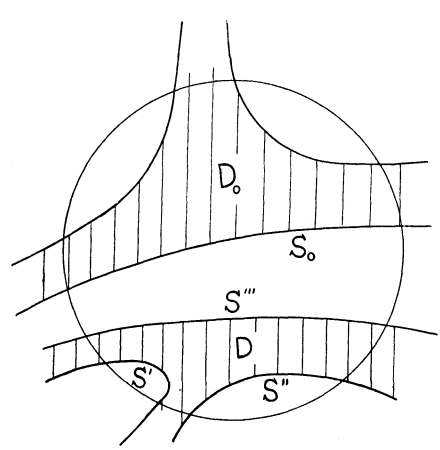

Fig. 9

such numbers m for all special lines, and let D_0 be a special domain with the special line S_0 outside of which there are m_0 individuals of $S_1, \dots, S_n$. Take next an arbitrary special domain D' outside of S_0 with three special lines S', S'' and S'''. Since there is one and only one singular line on the boundary of D' such that D_0 is contained outside of it, suppose D_0 lies inside of S' and S''. Now by the definition of m_0 there are m_0 individuals of $S_1, \dots, S_n$ outside of S' as well as of S'', and hence the number of individuals $S_1, \dots, S_n$ outside of S_0 amounts to at least $2m_0$, which is true when and only when $m_0 = 0$. It follows therefore that $S_1, \dots, S_n$ together with S' and S'' lie inside one another. Putting $S_{n+1} = S'$ if S' and S'' are both separating lines, and $S_{n+1} = S'$ or S'' according as S' or S'' respectively is not a separating line, we get a system of special lines $S_1, \dots, S_n, S_{n+1}$ having the same property as the system $S_1, \dots, S_n$.

Proceeding in this manner we get an infinite number of singular lines $S_1, \dots, S_n, \dots$ intersecting C such that each one of them lies wholly inside of the others, which is impossible by S 11, and the proof of S 18 is complete.

Now by the hypothesis of periodical relatedness a transformation φ is associated to each admissible closed set S such that $\{\varphi\}$ forms a continuous family of transformation on $\bigvee S$. But it is not known, and in fact not postulated, whether or not $\{\varphi\}$ satisfies the condition of S 17 concerning the direction of motion on the boundary of D_n. Our next step is to modify φ so that it satisfies the condition S 17.

To this end consider first D_1 and let $\varphi_1, \varphi_2, \dots$ be the transformations of $\{\varphi\}$ associated with the singular lines $S_1, S_2, \dots$ constituting the boundary of D_1. Substituting some of φ_i suitably with its inverse φ_i^{-1} we obtain a new system $\varphi_1^{n_1}, \varphi_2^{n_2}, \dots$ of transformations of $S_1, S_2, \dots$ such that they satisfy S 17 with respect to D_1, where n_i stand either for 1 or for -1. Describe a circle about a point of D_1 with radius >1 and large enough and let $D_{n_1}, D_{n_2}, \dots, D_{n_N}$ be all the special domains with respect to C_1, D_1 itself being counted as a special domain. Call two special domains D_{n_i} and D_{n_j} *consecutive*, it these are not separated by another D_{n_k} and call any domain lying between two consecutive domain *intermediate domain.* Further, the unique pair of singular lines S_i and S_j on the boundary of consecutive domains D_{n_i} and D_{n_j} respectively such that S_i and S_j can be connected by a series of intermediate domains will be called the *line of transmission* of the consecutive domains.

Suppose D_1 and D_{n_i} are consecutive and S_1 and S_i their lines of transmission, with the associated transformations φ_1 and φ_i respectively. Then substitute φ_i by $\varphi_i^{n_1}$ and for all singular lines intersecting C_1 on the boundaries of the intermediate domains between D_1 and D_{n_i} substitute the associated transformation φ by φ^{n_1}. On the basis of this substitution perform a similar substitution for all singular lines on the boundary of D_{n_i} in accordance with S 17. Perform such substitution for all pairs of consecutive domains D_1 and D_{n_j} independently. This can be carried out effectively, since for two D_{n_i} and D_{n_j} under consideration at most their eventually existing lines of transmission have to do with in the operation. If there are consecutive domains of D_{n_i} etc. other than D_1 or those taken up thus far, perform the substitution further, until all special domains with respect to C_1 are exhausted.

Next describe a circle C_2 concentric with C_1 and radius greater than that of C_1 and 2. If there are among the new special domains with respect to C_2 those which were intermediate domains with respect to C_1, effect first the substitution on them in conformity with S 17 and then perform for the rest the substitution just as above. Proceeding in this manner we obtain a system of homeomorphisms $\{\varphi_\lambda^{n_\lambda}\}$ satisfying S 17 in addition to the other conditions. Thus the conditions of inward extension are now satisfied, and we obtain the desired generalized translation f.

Remark 5. The converse of the structure theorem given above is unsatisfactory, since it is still undecided what is the necessary and sufficient condition that a domain should be inwardly inextendible. Let the infimum of the diameters of the arcs joining two accessible boundary points a and b of a domain D within D be called the *inner distance* of a and b. In order that a domain D should be inwardly extendible it is necessary that if (a_n, b_n) is a sequence of pairs of accessible boundary points of D converging to a point of $\dot{D}$ and if the inner distance of a_n, b_n tends to 0, then the inner distance of $\varphi(a_n)$ and $\varphi(b_n)$ should tend also to 0. *Is this condition sufficient?*

Remark 6. Examples 8 and 9 and their extension, Example 10, show that a slight modification of a generalized translation f, which takes place indeed only in a bounded domain, gives rise to considerable complications to singular lines. It is most desirable to divise a procedure of simplification so that we may reduce the generalized translation to some simplest forms.

IV. Examples

18. Several examples which may serve to facilitate the understanding of the theorems in the preceding sections will be collected in this

The domain bounded by A and B, the domain bounded by A and C are domains of total regularity, and the domain bounded by B and C is a domain of semi-regularity.

Example 3. In Example 1 join the points $a(0, 0)$, $b(0, \pi/4)$ and $c(1, 0)$ successively by segments ab and bc. Then $ab+bc$ is a translation arc, but the stream line generated by this arc has the straight line $y=\pi$ as cluster set and so is not an open line. This is essentially equivalent to the example given by Brouwer [1], p. 40, but simpler.

Example 4. Let f be defined as follows:

(i) $x' = x-1,\ y' = y$ for $y \geqq 0$.

(ii) Let C_n be the curve $y = \sin 2\pi x$ for $n-\frac{1}{2} < x < n$ and let $x' = x+1,\ y' = y$

for the points on C_n and for the points of the convex domains D_n bounded by C_n.

(iii) Fill up the rest of the plane with a system of stream-lines as indicated in Fig. 12, which will be effected as follows: let D be the domain bounded by all of C_n and by the x-axis. Map the stripe in Example 1 topologically onto D by a mapping h such that h remains continuous on the boundary of the stripe except for points $x=n$ (integer), $y=0$, and define mapping on D by hfh^{-1}.

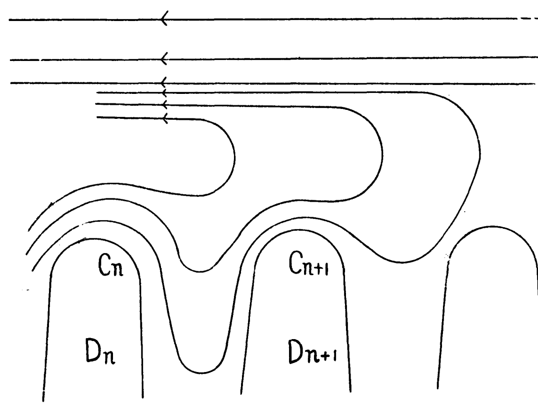

Fig. 12

The points of D_n are all regular, but are irregular in the large. $\bigcup_{n=-\infty}^{\infty} C_n$ is a $(-)$-singular line generated by unbounded singularity segment C_n.

Example 5. (1) In Example 6, C_n can be so modified that it has inaccessible points. Thus a singularity segment can have inaccessible points.

(2) If C_n are the half rays

$$x = n, \quad y \leqq -1$$

then the $(-)$-singular line $\bigcup C_n$ has no negative side.

section. Some theorems leading to the construction of examples are also given.

Example 1. The following is the (simplified) classical example of an f which is not topologically equivalent to the ordinary translation and is instructive for further discussion. Cf. Kerékjártó: Vorlesungen über Topologie, I (1923), p. 195.; cf. also Brouwer [1], p. 40. These examples turn out to be topologically equivalent each other.

f: $(x, y) \to (x', y')$ is defined as follows:

(i) $x' = x-1$, $y' = y$ for $y \geqq \pi$,
(ii) $x' = x+\cos y$, $y' = y+\sin y$ for $0 \leqq y \leqq \pi$,
(iii) $x' = x+1$, $y' = y$ for $y \leqq 0$.

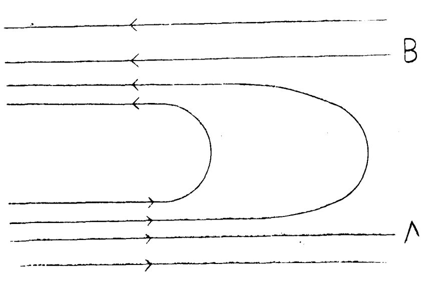

Fig. 10

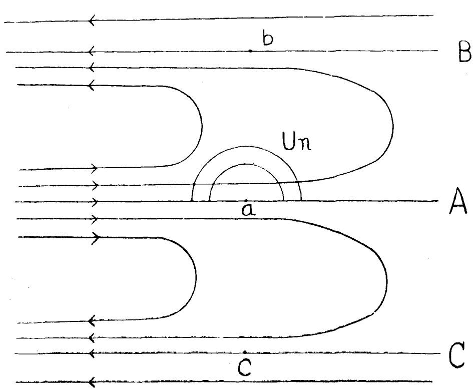

Fig. 11

Example 2. Changing the range of definition, let f be defined as follows:

(i) $x' = x-1$, $y' = y$ for $y \geqq \pi$,
(ii) $x' = x+\cos y$, $y' = y+\sin y$ for $-\pi \leqq y \leqq \pi$,
(iii) $x' = x-1$, $y' = y$ for $y \leqq -\pi$.

Every point a of A is strongly $(+)$-irregular and $B+C$ is the $(+)$-singularity polar to a. Every point b on B and every point c of C are strongly $(-)$-irregular and A is the $(-)$-singularity polar to b as well as the $(-)$-singularity polar to c. B and C are $(+)$-singular lines, while A is a $(-)$-singular line.

Let C_n be the circle with center a and radius $1/n$. If U_n is the upper semi-circle of C_n above the line A, then the singular line B is derived from the decreasing sequence of domains U_n.

The construction of Example 4 is capable of extension by using the following lemmas.

Lemma 16. Let g be a topological mapping of the circumference C of a circle onto itself such that the set of fixed points F is non vacuous and non dense on C. Moreover let the direction of motion under g be the same all over C except at fixed points of g. Then g can be extended to a topological mapping of the whole circle onto itself having no fixed point in the interior and leaving invariant every circle touching C at one of its fixed points.

The proof is easily done as follows. Let the direction of motion be positive. Then, if a is a fixed point and if we represent the point x of C by the arc length measured from a in the positive direction, the condition of the lemma is expressed by the inequality

$$x \leqq g(x).$$

If we set $g(2\pi)=2\pi$, the whole length of C being 2π, the curve expressed by the equation $y=g(x)$ in the (x, y)-plane lies on the upper side of the straight line $y=x$ and touches the latter from above. Let $G(x)$ be an upper-function of $g(x)$, i.e. a monotone function such that

$$g(x) < G(x)$$

except for $x=0$ and $x=2\pi$, where $G(0)=g(0)=0$, and $G(2\pi)=g(2\pi)=2\pi$. We have then $x < G(x)$ for $0 < x < 2\pi$. If we put

$$g_t(x)=(1-t)G(x)+tg(x),$$

then $g_t(x)$ is monotone, satisfies the relation

$$x < g_t(x) \quad \text{for} \quad 0 < x < 2\pi$$

and

$$g_t(0)=0 \quad \text{and} \quad g_t(2\pi)=2\pi,$$

and finally $g_t(x)$ converges monotonously to $g(x)$ when $t \to 1$.

Let C_t be the circle of radius t touching C at a from inside. Describe the circle orthogonal to C at a and let its intersections with C and C_t correspond by a function h_t. Then the mapping $h=h_t \cdot g_t \cdot h_t^{-1}$ is the desired one.

We have similarly

Lemma 17. Let g be a topological mapping of a circle C onto itself such that the set of fixed points F is non vacuous and non dense on C. Moreover let C be divided into two semi-circles with end points a and b

such that on one of them the direction of motion is positive and on the other, negative. Then g can be extended to a topological mapping of the whole circle onto itself having no fixed point in the interior and leaving invariant every circular arc passing through a and b.

The proof may be carried out as above.

The following is an immediate consequence of the preceding lemmas by virtue of the mapping theorem of Carathéodory.

Extension Theorem. Let D be a simply connected domain bounded by a discrete family of admissible closed sets S_n whose points are all accessible from D. Then D is inwardly extendible, and moreover D can be filled with a system of regular stream-lines with respect to D.

Example 6. The following is an example of an f where all points of the plane are both $(+)$- and $(-)$-irregular.

Let $T(p, q)$ denote in general the convex domain bounded by a segment pq parallel to the y-axis and by two rays $p\infty$ and $q\infty$ emerging from p and q respectively and running in the positive direction parallel to the x-axis. $T(p, q)$ will be called a *tube* with vertices p, q.

First let $T(a_0, a_1)$ be a tube and let $T(b, c)$ be another tube wholly contained in the former, called an *inner tube* of $T(a_0, a_1)$. Take points a_0' and a_1' on the rays $c\infty$ and $a_1\infty$ respectively such that the segment $a_0'a_1'$ is parallel to the y-axis. Then the domain

$$P(a_0, a_1) = T(a_0, a_1) - \overline{T}(b, c) - \overline{T}(a_0', a_1')$$

will be called a *pipe* with *base* a_0a_1 and *opening* $a_0'a_1'$. Fill up this pipe $P_1 = P(a_0, a_1)$ with a regular family of stream-line segments as in Fig. 13.

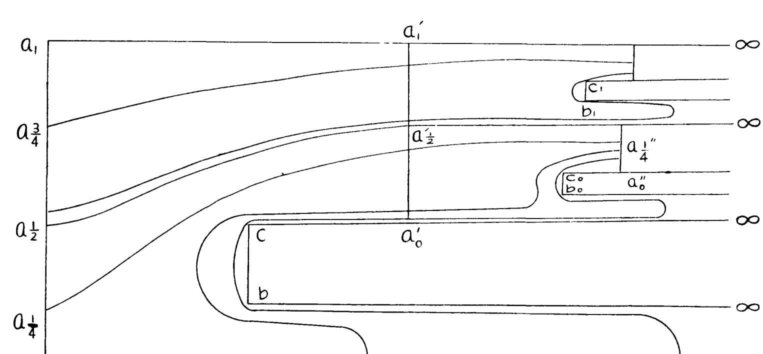

Fig. 13

Take the middle point $a_{\frac{1}{2}}$ of a_0a_1 and let $a_{\frac{1}{2}}'$ be the point where the stream-line segment through $a_{\frac{1}{2}}$ meet the segment $a_0'a_1'$. Next insert in the tube $T(a_0', a_1')$ inner tubes $T(b_0, c_0)$ and $T(b_1, c_1)$ respectively and fill up the corresponding pipes $P(a_0', a_{\frac{1}{2}}')$ and $P(a_{\frac{1}{2}}', a_1')$ with stream-lines. Take the middle point $a_{\frac{1}{4}}$ of the segment a_0a_1 and let $a_{\frac{1}{4}}''$ be the point where the streams-line segment through $a_{\frac{1}{2}}$ and its continuation in $P(a_0', a_{\frac{1}{2}}')$ meets the opening of the latter.

Continue in this manner indefinitely on the condition that the series of inner tubes which appear should not have cluster set. Then invert the configuration on the opposite side of a_0a_1. Thus we have a domain which is bounded by two parallel lines and by an infinite number of broken lines (the boundaries of inner tubes) and which is filled with a system of stream-lines.

Insert in each of inner tubes topologically the configuration thus obtained indefinitely and translate the whole configuration suitably parallel to the y-axis. Then we have finally a regular family of stream-lines which fill up the whole plane and every point of the plane becomes $(+)$- as well as $(-)$-irregular.

Example 7. In Fig. 14,

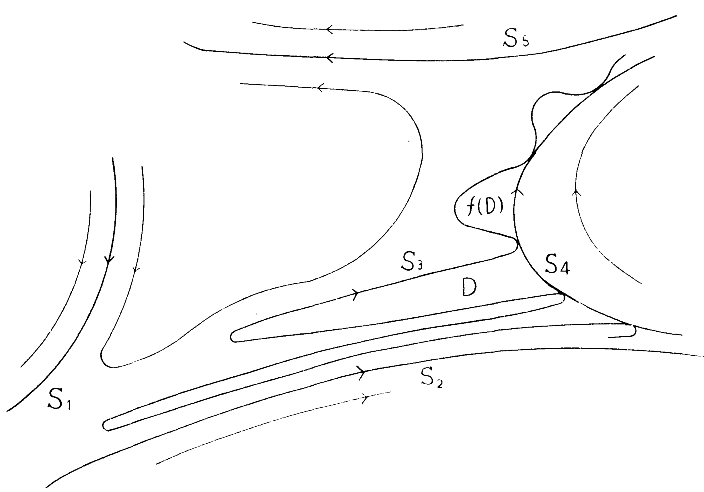

Fig. 14

S_1 is a $(-)$-singular line derived from a point of S_2.
S_2+S_1 is a $(-)$-singular line derived from a point of S_4.
$S_3+S_2+S_1$ is a $(-)$-singular line derived from a point of S_5.
$S_3+S_2+S_5$ is a $(+)$-singular line derived from a point of S_1.
S_4+S_5 is a $(+)$-singular line derived from a point of S_2.
S_5 is a $(+)$-singular line derived from a point of S_4.

Especially, S_4 can be derived only from the point p of S_2. But every decreasing sequence of domains converging to p derives not only S_4 but also S_5 at the same time.

Example 8. Let φ be a given generalized translation as indicated by stream-lines in Fig. 15, B and C being the $(+)$-singular lines derived from the point p and q of S_1 and S_2 respectively, and E being the $(+)$-singular line derived from a point of B. In the figure let a, b, c, d be transformed by φ and φ^2 into a', b', c', d' and a'', b'', c'', d'' respectively. Let $b'fb''$ and $c'ec''$ be arcs as indicated in the figure. Let f be a modification of φ such that f maps the domain $abb'a'$, $bcc'b'$, $cdd'c'$ onto domains $a'b'fb''a''$, $b'c'ec''b''f$, $c'd'd''c''e$ respectively as indicated in the figure, f coinciding with φ elsewhere. Then the $(+)$-singular line derived from p and q have the same final end E in common.

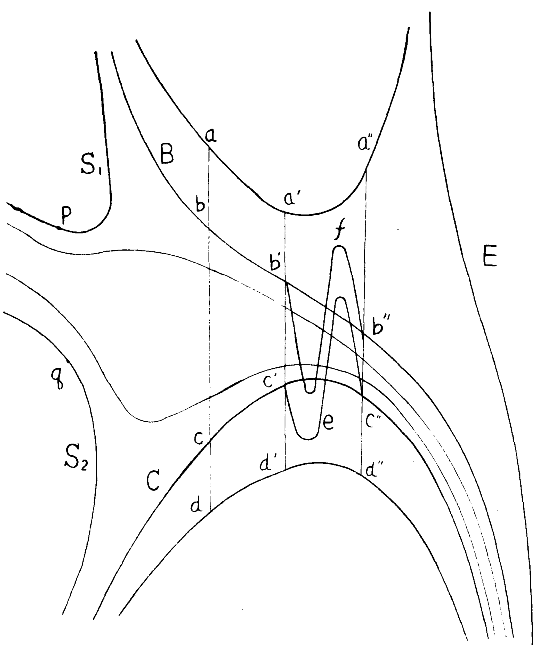

Fig. 15

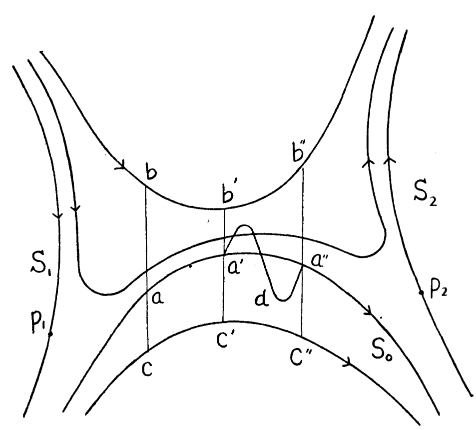

Fig. 16

Example 9. Let φ be a generalized translation as indicated in Fig. 16 by stream-lines, S_0+S_2 being the $(+)$-singular line derived from a point p_1 of S_1. Let a, b, c be transformed by φ and φ^2 into a', b', c' and a'', b'', c'' respectively, and let $a'da''$ be an arc as indicated in the figure. If f is a modification of φ such that f maps the domain $aa'b'b$ and $aa'c'c$ into domains $a'da''b''b'$ and $a'da''c''c'$ respectively, for the rest coinciding with φ, then the $(+)$-singular line α^+ derived from a point of S_1 and the

$(-)$-singular line α^- derived from a point of S_2 cross in $f^n(a)$ $(n=0, \pm 1, \pm 2, \ldots)$

Example 10. The above process of modification shows that simple singular lines give rise to a complicated final or initial end. In general let $S_1, S_2, \ldots, S_n, \ldots$ be a sequence of singular lines appearing successively in this order and in the positive sense on the boundary of a domain D of semi-regularity such that D is filled with a regular family of stream-lines. On every S_n make a modification of homeomorphism as in the above examples. Then we have an example of a generalized translation f such that the singular line derived from a point of S_n becomes the final end of the singular line derived from the point of S_{n-1}.

Example 11. Let f be the ordinary translation given by

$$x' = x+1, \quad y' = y.$$

Then the domain T bounded by two parallel lines

$$x = \frac{1}{\sin y} \quad (0 < y < \pi)$$

and

$$x = \frac{1}{\sin y} + 1 \quad (0 < y < \pi)$$

is a translation-field. The area of translaton generated by T is then the stripe bounded by two parallel lines $y=0$ and $y=\pi$.

In general we have

Example 12. Let f be the ordinary translation

$$x' = x+1, \quad y' = y.$$

If D is a simply connected domain and periodic, that is, $f(D)=D$, then D becomes an area of translation generated by some translation-field.

Proof (cf. Sperner [11], p. 19). Through a point of D we can find a stream-line j lying wholly within D. The boundary of D will then be separated by j into two parts: the upper boundary and the lower one. Then there is on the y-axis a segment ab such that a belongs to the upper and b belongs to the lower boundary, ab being admitted to be the whole y-axis or to be a half line. Choose a sequence of points a_n $(-\infty < n < +\infty)$ such that a_n lies between a_{n-1} and a_{n+1} and $\lim_{n\to+\infty} a_n = a$, $\lim_{n\to-\infty} a_n = b$. Join a_n and $f(a_{n+1})$ within the sub-domain D' of D bounded by ab and $f(ab)$ by an arc C_n such that C_n are disjoint and that C_n converges to the lower bundary of D' when $n \to -\infty$. If we put $\gamma = \bigcup_{n=-\infty}^{\infty} f^n(C_n)$, then the domain bounded by γ and $f(\gamma)$ is a translation-field which generates D.

(Received October 1, 1953)

Bibliography

[1] L. E. J. Brouwer: Beweis des ebenen Translationssatzes, Math. Ann. **72** (1912), 37–54.

[2] C. Carathéodory: Über die Begrenzung einfach zusammenhängender Gebiete, Math. Ann. **73** (1913), 323–370.

[3] W. Kaplan: Regular curve-families filling the plane. I, Duke Math. J. **7** (1940), 154–185. II, ibid. **8** (1941), 11–46.

[4] B. v. Kerékjártó: On a geometric theory of continuous groups, Ann. of Math. **26** (1925), 105–117.

[5] ——— : The plane translation theorem of Brouwer and the last geometric theorem of Poincaré, Acta litt. ac sci. **4** (1928), 87–102.

[6] ——— : Über die fixpunktfreien Abbildungen der Ebene, ibid. **6** (1934), 226–234.

[7] ——— : Topologische Charakterisierung der linearen Abbildungen, ibid. **6** (1934), 235–262.

[8] W. Scherrer: Translationen über einfach zusammenhängenden Gebiete, Viert. Jahrschr. Nat. Ges. **70** (1925), 77–84.

[9] G. Scorza-Dragoni: Über die fixpunktfreien Abbildungen der Ebene, Abh. Math. Sem. Hans. Univ. **14** (1941), 1–21.

[10] ——— : Sugli autoomeomorfismi del piano privi di punti uniti, Rend. Sem. Mat. Padova **18** (1949), 1–53.

[11] E. Sperner: Über die fixpunktfreien Abbildungen der Ebene, Abh. Math. Sem. Hamburg. **10** (1934), 1–47.

[12] H. Terasaka: Ein Beweis des Brouwerschen ebenen Translationssatzes, Japan. J. Math. **7** (1930), 61–69.