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UNIFORM ALGEBRA GENERATED BY $z_1, \cdots, z_n, f_1(z), \cdots, f_s(z)$

Dedicated to Professor Yukinari Tôki on his 60th birthday

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Introduction

Let C^n be the complex Euclidean space with complex coordinates $z=(z_1, \cdots, z_n)$ and K a compact subset of C^n. For any complex valued C^∞-functions $f_1, \cdots, f_s$ defined on an open subset U of C^n containing K, we shall consider the uniform algebra A consisting of uniform limits of polynomials of $z_1, \cdots, z_n, f_1, \cdots, f_s$ on K.

Hörmander-Wermer [1] proved that, if $s=n$ and if each f_j is 'close' to $\bar{z}_j$ in some sense, then A coincides with $C(K)$, the algebra of all complex valued continuous functions on K. In this paper, we shall deal with the case where $0<s<n$ and each f_j is holomorphic in $z_{s+1}, \cdots, z_n$ near K. In Section 3, an approximation theorem on the graph of $f_1, \cdots, f_s$ will be proved. In Section 4, we shall give a sufficient condition on f_j and K assuring that every function holomorphic in $z_{s+1}, \cdots, z_n$ near K belongs to A.

1. The graph of $f_1, \cdots, f_s$

Let $f_1, \cdots, f_s$ be C^∞-functions defined on an open subset U of C^n. The graph of $f_1, \cdots, f_s$

$$M = \{(z_1, \cdots, z_n, f_1(z), \cdots, f_s(z)) \in C^{n+s};\ z = (z_1, \cdots, z_n) \in U\}$$

is a real $2n$-dimensional submanifold of C^{n+s}. If g is a C^∞-function on M, then the function g_0 defined by

$$g_0(z_1, \cdots, z_n) = g(z_1, \cdots, z_n, f_1(z_1, \cdots, z_n), \cdots, f_s(z_1, \cdots, z_n)) \tag{1.1}$$

is a C^∞-function on U.

We denote by $H_r(U)$, $r=n-s$, the class of functions of $C^\infty(U)$ which are holomorphic in $z_{s+1}, \cdots, z_n$.

We shall now consider the following assumptions on $f_1, \cdots, f_s$:

$$f_1, \cdots, f_s \text{ belong to } H_r(U), \text{ and} \tag{1.2}$$

$$\det\left(\frac{\partial f_j}{\partial \bar{z}_k}\right)_{j,k=1,\cdots,s} \text{ has no zeros on } U. \tag{1.3}$$

These conditions imply that, for every point p of M, the dimension of maximal complex submanifold of $\boldsymbol{C}^{n+s}$ through p contained in M is just r. It follows from the following lemma, which is easily proved by linear algebra.

Lemma 1. *The complex tangent space of M at every point is of r-dimension if and only if*

$$\operatorname{rank}\left(\frac{\partial f_j}{\partial \bar{z}_k}\right)_{j=1,\cdots,s\,;\,k=1,\cdots,n} = n-r$$

holds at every point of U.

A C^∞-function on M which is holomorphic in complex coordinates of M is called a *CR-function*. (1.1) gives an isomorphism of $H_r(U)$ and the algebra of CR-functions on M.

2. Holomorphic convexity of M

By a *region of holomorphy* we mean a disjoint sum of domains of holomorphy. We define

$$\phi(z) = \sum_{j=1}^{s} |f_j(z_1, \cdots, z_n) - z_{n+j}|^2, \qquad z \in U \times \boldsymbol{C}^s,$$

and

$$G_\varepsilon(V) = \{z \in V;\ \phi(z) < \varepsilon\},$$

for any open subset V of $U \times \boldsymbol{C}^s$ and for any positive number ε.

Lemma 2. *Suppose that f_j satisfy* (1.2) *and* (1.3). *Let V be a region of holomorphy in $\boldsymbol{C}^{n+s}$ such that $\bar{V} \subset U \times \boldsymbol{C}^s$. Then there exists a positive number ε_0 such that $G_\varepsilon(V)$ is a region of holomorphy in $\boldsymbol{C}^{n+s}$ for any ε, $0 < \varepsilon < \varepsilon_0$.*

Proof. We consider the complex Hessian form

$$H(\xi, \bar{\xi})_z = \sum_{\nu,\mu=1}^{n+s} \frac{\partial^2 \phi}{\partial z_\nu \partial \bar{z}_\mu}(z) \xi_\nu \bar{\xi}_\mu, \quad z \in U \times \boldsymbol{C}^s.$$

Let $z = (z_1, \cdots, z_{n+s})$ be any point of M and $z_0 = (z_1, \cdots, z_n)$ the corresponding point in U. Then we have

$$H(\xi, \bar{\xi})_z = \sum_{j=1}^{s} \left\{ \left| \sum_{\nu=1}^{n} \frac{\partial f_j}{\partial z_\nu}(z_0)\xi_\nu - \xi_{n+j} \right|^2 + \left| \sum_{\nu=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu}(z_0)\bar{\xi}_\nu \right|^2 \right\}.$$

The right member can vanish only if

$$\sum_{\nu=1}^{s}\frac{\partial f_j}{\partial \bar z_\nu}(z_0)\bar\xi_\nu = 0 \quad \text{and} \quad \xi_{n+j} = \sum_{\nu=1}^{n}\frac{\partial f_j}{\partial z_\nu}(z_0)\xi_\nu, \quad j = 1, \cdots, s\,.$$

By (1.3), $H(\xi, \bar\xi)_z$ can vanish only when ξ is a complex tangent vector of M at z:

$$\xi = \left(0, \cdots, 0, \xi_{s+1}, \cdots, \xi_n, \sum_{\nu=s+1}^{n}\frac{\partial f_1}{\partial z_\nu}(z_0)\xi_\nu, \cdots, \sum_{\nu=s+1}^{n}\frac{\partial f_s}{\partial z_\nu}(z_0)\xi_\nu\right).$$

Therefore the matrix $H_z = \left(\frac{\partial^2\phi}{\partial z_\nu \partial \bar z_\mu}(z)\right)$ has $n+s-r$ non-zero eigenvalues for every point z of M. Let V_1 be an open set such that $\bar V \subset V_1 \subset \bar V_1 \subset U \times C^s$. By continuity of H_z, there exists a positive number ε_0 such that H_z has at least $n+s-r$ non zero eigen values for every z in $G_{\varepsilon_0}(V_1)$.

Let S_ε be the hypersurface $\{z \in V_1;\ \phi(z) = \varepsilon\}$. Fix an arbitrary point $\alpha = (\alpha_1, \cdots, \alpha_{n+s})$ on S_ε. We define a non-singular holomorphic map $z = \Phi(\zeta)$ of $U(\alpha) = \{\zeta \in C^r;\ (\alpha_1, \cdots, \alpha_s, \zeta_1, \cdots, \zeta_r) \in U\}$ into C^{n+s} by

$$\Phi_j(\zeta) = \begin{cases} \alpha_j & j = 1, \cdots, s\,, \\ \zeta_{j-s} & j = s+1, \cdots, n\,, \\ f_{j-n}(\alpha_1, \cdots, \alpha_s, \zeta_1, \cdots, \zeta_r) - f_{j-n}(\alpha_1, \cdots, \alpha_n) + \alpha_j\,, & \\ & j = n+1, \cdots, n+s\,. \end{cases}$$

The Φ-image of $U(\alpha)$ is an r-dimensional complex submanifold of C^{n+s} containing α. Since $\sum_{j=1}^{s}|f_j(\alpha_1, \cdots, \alpha_n) - \alpha_{n+j}|^2 = \varepsilon$, it is contained in S_ε. Hence the complex Hessian H_α evaluated at α has at least r zero eigenvalues with complex eigenvectors tangent to S_ε (see Wells [2], Lemma 2.5′). Thus, $H(\xi, \bar\xi)_\alpha$ is non-negative for any tangent vector ξ to S_ε. Since V is a region of holomorphy in C^{n+s}, so is $G_\varepsilon(V)$. The lemma is proved.

A compact set F of C^n (or of C^{n+s}) is called an *H-convex set* in C^n (or in C^{n+s} resp.), if F is the intersection of regions of holomorphy containing F in C^n (or in C^{n+s} resp.). If U_1 is a region of holomorphy in U, then $U_1 \times C^s$ is a region of holomorphy in C^{n+s}. Therefore, we have

Corollary. *If K is an H-convex compact seubset of U, then $K^* = \{(z_1, \cdots, z_{n+s}) \in M;\ (z_1, \cdots, z_n) \in K\}$ is H-convex in C^{n+s}.*

3. Holomorphic approximation on M

In this section, we suppose that $f_1, \cdots, f_s$ satisfy (1.2) and (1.3).

Lemma 3. *Suppose g is a CR-function on M. Then for every positive integer N and for every relatively compact open subset U_0 of U, there exist a function $\tilde g \in C^\infty(U \times C^s)$ and a positive constant γ such that*

(i) $\tilde{g}|_M = g$, *and*

(ii) $\left|\dfrac{\partial g}{\partial \bar{z}_\nu}(z)\right| \leq \gamma \cdot d(z, M)^N, \quad z \in U_0 \times C^s, \quad \nu = 1, \cdots, n+s,$

where $d(z, M)$ *is the Euclidean distance in* C^{n+s} *between* z *and* M.

Proof. We consider the system of linear equations at every point of U

$$\sum_{j=1}^{s} h_j \frac{\partial f_j}{\partial \bar{z}_\nu} = \frac{\partial g_0}{\partial \bar{z}_\nu}, \quad \nu = 1, \cdots, s, \tag{3.1}$$

where g_0 is the function defined by (1.1). By (1.3), there exist the uniquely determined solutions $h_j(z_1, \cdots, z_n)$, $j=1, \cdots, s$. Since f_j and g_0 are of $H_r(U)$, so are h_j. We shall define the function h_J inductively for every multi-index $J=(j_1, \cdots, j_k)$, $1 \leq i_i \leq s$. Suppose h_J is given in $H_r(U)$. Then h_{Jj}, $j=1, \cdots, s$, will be defined as the solutions of the equations

$$\sum_{j=1}^{s} h_{Jj} \frac{\partial f_j}{\partial \bar{z}_\nu} = \frac{\partial h_J}{\partial \bar{z}_\nu}, \quad \nu = 1, \cdots, s. \tag{3.2}$$

The condition (1.3) guarantees the existence of the solutions h_{Jj} in $H_r(U)$.

We shall prove that h_J are symmetric with respect to J. By differentiating each equation of (3.1) by $\bar{z}_\mu$, we have

$$\sum_{j=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu} \cdot \frac{\partial h_j}{\partial \bar{z}_\mu} = \frac{\partial^2 g_0}{\partial \bar{z}_\nu \partial \bar{z}_\mu} - \sum_{j=1}^{s} \frac{\partial^2 f_j}{\partial \bar{z}_\nu \partial \bar{z}_\mu} h_j .$$

Since the right member is symmetric in ν and μ, we have

$$\sum_{j=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu} \cdot \frac{\partial h_j}{\partial \bar{z}_\mu} = \sum_{j=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\mu} \cdot \frac{\partial h_j}{\partial \bar{z}_\nu}, \quad \nu, \mu = 1, \cdots, s. \tag{3.3}$$

Substituting (3.2) for $k=1$ to (3.3), we obtain

$$\sum_{j=1}^{s} \left(\sum_{i=1}^{s} h_{ji} \frac{\partial f_i}{\partial \bar{z}_\mu} \right) \frac{\partial f_j}{\partial \bar{z}_\nu} = \sum_{j=1}^{s} \left(\sum_{i=1}^{s} h_{ji} \frac{\partial f_i}{\partial \bar{z}_\nu} \right) \frac{\partial f_j}{\partial \bar{z}_\mu}, \quad \nu, \mu = 1, \cdots, s,$$

or equivalently

$$\sum_{i,j} (h_{ji} - h_{ij}) \frac{\partial f_j}{\partial \bar{z}_\mu} \cdot \frac{\partial f_i}{\partial \bar{z}_\nu} = 0, \quad \nu, \mu = 1, \cdots, s.$$

By using (1.3), we can find that $h_{ji} = h_{ij}$ for every i and j.

General cases will be proved by induction. For simplicity, we write $J=(j_1, \cdots, j_k)$, $I=(j_1, \cdots, j_{i-1}, j_{i+1}, \cdots, j_k)$, $i=j_i$ and $J'=(j_1, \cdots, j_{i-1}, j, j_{i+1}, \cdots, j_k)$. Since $h_{J'} = h_{Ij}$ and $h_J = h_{Ii}$ by assumption of induction, we have

$$\sum_{i=1}^{s} h_J \frac{\partial f_i}{\partial \bar{z}_\nu} = \frac{\partial h_I}{\partial \bar{z}_\nu} \quad \text{and} \quad \sum_{j=1}^{s} h_{J'} \frac{\partial f_j}{\partial \bar{z}_\mu} = \frac{\partial h_I}{\partial \bar{z}_\mu}.$$

By differentiating the first identity by $\bar{z}_\mu$ and the second by $\bar{z}_\nu$, we obtain

$$(3.4) \qquad \sum_{i=1}^{s} \frac{\partial h_J}{\partial \bar{z}_\mu} \cdot \frac{\partial f_i}{\partial \bar{z}_\nu} = \sum_{j=1}^{s} \frac{\partial h_{J'}}{\partial \bar{z}_\nu} \cdot \frac{\partial f_j}{\partial \bar{z}_\mu}, \quad \nu, \mu = 1, \cdots, s.$$

Substituting the equalities

$$\sum_{j=1}^{s} h_{Jj} \frac{\partial f_j}{\partial \bar{z}_\mu} = \frac{\partial h_J}{\partial \bar{z}_\mu} \quad \text{and} \quad \sum_{i=1}^{s} h_{J'i} \frac{\partial f_i}{\partial \bar{z}_\nu} = \frac{\partial h_{J'}}{\partial \bar{z}_\nu}$$

to (3.4), we have

$$\sum_{i,j} (h_{Jj} - h_{J'i}) \frac{\partial f_i}{\partial \bar{z}_\nu} \frac{\partial f_j}{\partial \bar{z}_\mu} = 0, \quad \nu, \mu = 1, \cdots, s.$$

By (1,3), we find that $h_{Jj} = h_{J'i}$, which implies the symmetry of h_J for all J.

Now we define $\tilde{g}$ by

$$\tilde{g}(z_1, \cdots, z_{n+s}) = g_0(z_1, \cdots, z_n)$$
$$+ \sum_{k=1}^{N} \frac{1}{k!} \sum_{(j \cdots j_k)} h_{j_1 \cdots j_k}(z_1, \cdots, z_n) (z_{n+j_1} - f_{j_1}(z_1, \cdots, z_n)) \cdots (z_{n+j_k} - f_{j_k}(z_1, \cdots, z_n)).$$

If $\nu = s+1, \cdots, n+s$, we have $\frac{\partial \tilde{g}}{\partial \bar{z}_\nu} \equiv 0$. For $\nu = 1, \cdots, s$, we have

$$\frac{\partial \tilde{g}}{\partial \bar{z}_\nu} = \frac{1}{N!} \sum_{(j_1 \cdots j_N)} \frac{\partial h_{j_1 \cdots j_N}}{\partial \bar{z}_\nu} (z_{n+j_1} - f_{j_1}(z_1, \cdots, z_n)) \cdots (z_{n+j_N} - f_{j_N}(z_1, \cdots, z_n)),$$

which proves the lemma.

We consider two uniform algebras on a compact subset K^* of M. $H(K^*)$ is the algebra of uniform limits on K^* of functions each holomorphic in a neighborhood (in C^{n+s}) of K^*. $CR(K^*)$ is the algebra of uniform limits on K^* of functions each of which is a CR-function on a neighborhood (in M) of K^*.

Suppose K^* is H-convex in C^{n+s}. Let g be a CR-function on a neighborhood M_1(in M) of K^*. We can find a region of holomorphy V such that $K^* \subset V$ and $\bar{V} \cap M \subset M_1$. We denote by K and U_0 the projections of K^* and $M_0 = V \cap M$ respectively by the map $(z_1, \cdots, z_{n+s}) \to (z_1, \cdots, z_n)$. Let d denote the distance between K and ∂U_0. By the way of construction of $G_\varepsilon(V)$ in Lemma 2, we can find a positive constant η such that, for every point z^0 of K^*, the ball $B_{\varepsilon\eta}(z^0) = \{z \in C^{n+s};\ |z - z^0| \leq \varepsilon\eta\}$ is contained in $G_\varepsilon(V)$, whenever $\varepsilon < d$. Therefore, by using Lemma 2 and Lemma 3 for $N = n+1$, and applying the same technique as one developed in [1], we obtain the following

Theorem 1. *If K^* is a compact subset of M which is H-convex in $\boldsymbol{C}^{n+s}$, then we have $H(K^*)=CR(K^*)$.*

4. Polynomial approximation.

We consider the following conditions for a compact subset K of $\boldsymbol{C}^n$ and for functions f_j of $C^\infty(\boldsymbol{C}^n)$;

(a) $f_1, \cdots, f_s$ are of $H_r(U)$ for some open set U containing K,

(b) there exists a constant k, $0<k<1$, such that

$$\sum_{j=1}^{s} |f_j(z+\xi)-f_j(z)-\bar{\xi}_j|^2 \leq k \sum_{j=1}^{s} |\xi_j|^2$$

holds for any z and $\xi=(\xi_1, \cdots, \xi_s, 0, \cdots, 0)$ in $\boldsymbol{C}^n$, and

(c) for any vector $\alpha'=(\alpha_1, \cdots, \alpha_s)$, $K \cap E_{\alpha'}$ is polynomially convex in $E_{\alpha'}$, where $E_{\alpha'}$ is the subspace $\{z\in \boldsymbol{C}^n;\ z_j=\alpha_j, j=1, \cdots, s\}$ of $\boldsymbol{C}^n$.

The condition (b) implies (1.3). In fact, we can find a constant k_1, $0<k_1<1$, such that

$$\sum_{j=1}^{s} \left| \sum_{\nu=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu} \bar{\xi}_\nu + \bar{\xi}_j \right|^2 \leq k_1 \sum_{j=1}^{s} |\xi_j|^2,$$

and hence the system of linear equations

$$\sum_{\nu=1}^{s} \frac{\partial f_j}{\partial \bar{z}_\nu} \bar{\xi}_\nu = 0, \quad j = 1, \cdots, s$$

has only trivial solution.

We consider two uniform algebras on K. A is the algebra of uniform limits on K of polynomials of $z_1, \cdots, z_n, f_1(z), \cdots, f_s(z)$. $H_r(K)$ is the algebra of uniform limits of functions each of which is holomorphic in $z_{s+1}, \cdots, z_n$, in a neighborhood of K.

Theorem 2. *Suppose the conditions* (*a*), (*b*) *and* (*c*) *are satisfied. Then we have $A=H_r(K)$.*

Proof. We shall first prove that K^* is polynomially convex in $\boldsymbol{C}^{n+s}$. To do this, it is sufficient to show that the maximal ideal space of $P(K^*)$, the algebra of uniform limits of polynomials in $z_1, \cdots, z_{n+s}$ on K^*, ocincides with K^*, or equivalently that every complex homomorphism of A is a point evaluation for some point of K. Let φ be any complex homomorphism on A. Set $\alpha_j=\varphi(z_j)$, $j=1, \cdots, n$, and $\alpha=(\alpha_1, \cdots, \alpha_n)$. We consider the function

$$f(z) = \sum_{j=1}^{s} (z_j-\alpha_j)(f_j(z)-f_j(\alpha)).$$

Then $f(z)$ is in A. By the condition (*b*), we have

$$\operatorname{Re} f(z)>0 \qquad \text{for } z \notin E_{\alpha'},\ \alpha'=(\alpha_1, \cdots, \alpha_s)\,.$$

Let m be a representing measure for φ of A supported on K. Then we have

$$0 = \operatorname{Re} \varphi(f) = \int \operatorname{Re} f\, dm\,.$$

Therefore, the support of m must be contained in $K \cap E_{\alpha'}$ and, in particular, $K \cap E_{\alpha'}$ is not empty.

Let $h(z)$ be any polynomial of $z_1, \cdots, z_n, f_1(z), \cdots, f_s(z)$. For simplicity, we write $h_1(z_{s+1}, \cdots, z_n)=h(\alpha_1, \cdots, \alpha_s, z_{s+1}, \cdots, z_n)$. Then we have

$$\begin{aligned}\varphi(h) &= \int h(z)\, dm(z)\\ &= \int h_1(z_{s+1}, \cdots, z_n)\, dm(z_{s+1}, \cdots, z_n)\,.\end{aligned}$$

By the condition (a), h_1 is holomorphic in $U \cap E_{\alpha'}$. Since $K \cap E_{\alpha'}$ is polynomially convex, by Oka-Weil's theorem, h_1 is approximated uniformly on $K \cap E_{\alpha'}$ by polynomials of $z_{s+1}, \cdots, z_n$. Since every polynomial of $z_{s+1}, \cdots, z_n$ is considered as a polynomials of $z_1, \cdots, z_n$, φ can be considered as a complex homomorphism ψ of $P_0(K \cap E_{\alpha'})$, the algebra of uniform limits on $K \cap E_{\alpha'}$ of polynomials of $z_{s+1}, \cdots, z_n$. Polynomial convexity of $K \cap E_{\alpha'}$ implies that ψ is a point evaluation at α. Therefore we have

$$\varphi(h) = \psi(h_1) = h_1(\alpha_{s+1}, \cdots, \alpha_n) = h(\alpha)\,,$$

which proves the polynomial convexity of K^*.

By Oka-Weil's theorem, $H(K^*)$ coincides with $P(K^*)$. Since K^* is the intersection of polynomial polyhedra containing K^*, it is H-convex, and therefore we have $H(K^*)=CR(K^*)$ by Theorem 1. A is isomorphic to $P(K^*)$ and $H_r(K)$ to $CR(K^*)$. Since $A \subset H_r(K)$, we obtain $A=H_r(K)$.

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