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UNIFORM ALGEBRA GENERATED BY $z_1, \dots, z_n, f_1(\mathbf{z}), \dots, f_s(\mathbf{z})$

Dedicated to Professor Yukinari Tōki on his 60th birthday

AKIRA SAKAI

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Introduction

Let C^n be the complex Euclidean space with complex coordinates $z = (z_1, \dots, z_n)$ and K a compact subset of C^n . For any complex valued C^∞ -functions f_1, \dots, f_s defined on an open subset U of C^n containing K , we shall consider the uniform algebra A consisting of uniform limits of polynomials of $z_1, \dots, z_n, f_1, \dots, f_s$ on K .

Hörmander-Wermer [1] proved that, if $s=n$ and if each f_j is 'close' to \bar{z}_j in some sense, then A coincides with $C(K)$, the algebra of all complex valued continuous functions on K . In this paper, we shall deal with the case where $0 < s < n$ and each f_j is holomorphic in z_{s+1}, \dots, z_n near K . In Section 3, an approximation theorem on the graph of f_1, \dots, f_s will be proved. In Section 4, we shall give a sufficient condition on f_j and K assuring that every function holomorphic in z_{s+1}, \dots, z_n near K belongs to A .

1. The graph of f_1, \dots, f_s

Let f_1, \dots, f_s be C^∞ -functions defined on an open subset U of C^n . The graph of f_1, \dots, f_s

$$M = \{(z_1, \dots, z_n, f_1(z), \dots, f_s(z)) \in C^{n+s}; z = (z_1, \dots, z_n) \in U\}$$

is a real $2n$ -dimensional submanifold of C^{n+s} . If g is a C^∞ -function on M , then the function g_0 defined by

$$(1.1) \quad g_0(z_1, \dots, z_n) = g(z_1, \dots, z_n, f_1(z_1, \dots, z_n), \dots, f_s(z_1, \dots, z_n))$$

is a C^∞ -function on U .

We denote by $H_r(U)$, $r=n-s$, the class of functions of $C^\infty(U)$ which are holomorphic in z_{s+1}, \dots, z_n .

We shall now consider the following assumptions on f_1, \dots, f_s :

$$(1.2) \quad f_1, \dots, f_s \text{ belong to } H_r(U), \text{ and}$$

$$(1.3) \quad \det \left(\frac{\partial f_j}{\partial \bar{z}_k} \right)_{j,k=1,\dots,s} \text{ has no zeros on } U.$$

These conditions imply that, for every point p of M , the dimension of maximal complex submanifold of \mathbf{C}^{n+s} through p contained in M is just r . It follows from the following lemma, which is easily proved by linear algebra.

Lemma 1. *The complex tangent space of M at every point is of r -dimension if and only if*

$$\text{rank} \left(\frac{\partial f_j}{\partial \bar{z}_k} \right)_{j=1,\dots,s; k=1,\dots,n} = n-r$$

holds at every point of U .

A C^∞ -function on M which is holomorphic in complex coordinates of M is called a *CR-function*. (1.1) gives an isomorphism of $H_r(U)$ and the algebra of *CR-functions* on M .

2. Holomorphic convexity of M

By a *region of holomorphy* we mean a disjoint sum of domains of holomorphy. We define

$$\phi(z) = \sum_{j=1}^s |f_j(z_1, \dots, z_n) - z_{n+j}|^2, \quad z \in U \times \mathbf{C}^s,$$

and

$$G_\varepsilon(V) = \{z \in V; \phi(z) < \varepsilon\},$$

for any open subset V of $U \times \mathbf{C}^s$ and for any positive number ε .

Lemma 2. *Suppose that f_j satisfy (1.2) and (1.3). Let V be a region of holomorphy in \mathbf{C}^{n+s} such that $\bar{V} \subset U \times \mathbf{C}^s$. Then there exists a positive number ε_0 such that $G_\varepsilon(V)$ is a region of holomorphy in \mathbf{C}^{n+s} for any ε , $0 < \varepsilon < \varepsilon_0$.*

Proof. We consider the complex Hessian form

$$H(\xi, \bar{\xi})_z = \sum_{\nu, \mu=1}^{n+s} \frac{\partial^2 \phi}{\partial z_\nu \partial \bar{z}_\mu}(z) \xi_\nu \bar{\xi}_\mu, \quad z \in U \times \mathbf{C}^s.$$

Let $z=(z_1, \dots, z_{n+s})$ be any point of M and $z_0=(z_1, \dots, z_n)$ the corresponding point in U . Then we have

$$H(\xi, \bar{\xi})_z = \sum_{j=1}^s \left\{ \left| \sum_{\nu=1}^n \frac{\partial f_j}{\partial z_\nu}(z_0) \xi_\nu - \xi_{n+j} \right|^2 + \left| \sum_{\nu=1}^s \frac{\partial f_j}{\partial \bar{z}_\nu}(z_0) \bar{\xi}_\nu \right|^2 \right\}.$$

The right member can vanish only if

$$\sum_{\nu=1}^s \frac{\partial f_j}{\partial \bar{z}_\nu}(z_0) \bar{\xi}_\nu = 0 \quad \text{and} \quad \xi_{n+j} = \sum_{\nu=1}^n \frac{\partial f_j}{\partial z_\nu}(z_0) \xi_\nu, \quad j = 1, \dots, s.$$

By (1.3), $H(\xi, \bar{\xi})_z$ can vanish only when ξ is a complex tangent vector of M at z :

$$\xi = \left(0, \dots, 0, \xi_{s+1}, \dots, \xi_n, \sum_{\nu=s+1}^n \frac{\partial f_1}{\partial z_\nu}(z_0) \xi_\nu, \dots, \sum_{\nu=s+1}^n \frac{\partial f_s}{\partial z_\nu}(z_0) \xi_\nu \right).$$

Therefore the matrix $H_z = \left(\frac{\partial^2 \phi}{\partial z_\nu \partial \bar{z}_\mu}(z) \right)$ has $n+s-r$ non-zero eigenvalues for every point z of M . Let V_1 be an open set such that $\bar{V} \subset V_1 \subset \bar{V}_1 \subset U \times \mathbb{C}^s$. By continuity of H_z , there exists a positive number ε_0 such that H_z has at least $n+s-r$ non zero eigen values for every z in $G_{\varepsilon_0}(V_1)$.

Let S_ε be the hypersurface $\{z \in V_1; \phi(z) = \varepsilon\}$. Fix an arbitrary point $\alpha = (\alpha_1, \dots, \alpha_{n+s})$ on S_ε . We define a non-singular holomorphic map $z = \Phi(\zeta)$ of $U(\alpha) = \{\zeta \in \mathbb{C}^r; (\alpha_1, \dots, \alpha_s, \zeta_1, \dots, \zeta_r) \in U\}$ into \mathbb{C}^{n+s} by

$$\Phi_j(\zeta) = \begin{cases} \alpha_j & j = 1, \dots, s, \\ \zeta_{j-s} & j = s+1, \dots, n, \\ f_{j-n}(\alpha_1, \dots, \alpha_s, \zeta_1, \dots, \zeta_r) - f_{j-n}(\alpha_1, \dots, \alpha_n) + \alpha_j, & j = n+1, \dots, n+s. \end{cases}$$

The Φ -image of $U(\alpha)$ is an r -dimensional complex submanifold of \mathbb{C}^{n+s} containing α . Since $\sum_{j=1}^s |f_j(\alpha_1, \dots, \alpha_n) - \alpha_{n+j}|^2 = \varepsilon$, it is contained in S_ε . Hence the complex Hessian H_α evaluated at α has at least r zero eigenvalues with complex eigenvectors tangent to S_ε (see Wells [2], Lemma 2.5'). Thus, $H(\xi, \bar{\xi})_\alpha$ is non-negative for any tangent vector ξ to S_ε . Since V is a region of holomorphy in \mathbb{C}^{n+s} , so is $G_\varepsilon(V)$. The lemma is proved.

A compact set F of \mathbb{C}^n (or of \mathbb{C}^{n+s}) is called an H -convex set in \mathbb{C}^n (or in \mathbb{C}^{n+s} resp.), if F is the intersection of regions of holomorphy containing F in \mathbb{C}^n (or in \mathbb{C}^{n+s} resp.). If U_1 is a region of holomorphy in U , then $U_1 \times \mathbb{C}^s$ is a region of holomorphy in \mathbb{C}^{n+s} . Therefore, we have

Corollary. *If K is an H -convex compact subset of U , then $K^* = \{(z_1, \dots, z_{n+s}) \in M; (z_1, \dots, z_n) \in K\}$ is H -convex in \mathbb{C}^{n+s} .*

3. Holomorphic approximation on M

In this section, we suppose that f_1, \dots, f_s satisfy (1.2) and (1.3).

Lemma 3. *Suppose g is a CR-function on M . Then for every positive integer N and for every relatively compact open subset U_0 of U , there exist a function $\tilde{g} \in C^\infty(U \times \mathbb{C}^s)$ and a positive constant γ such that*

- (i) $\bar{g}|_M = g$, and
(ii) $\left| \frac{\partial g}{\partial \bar{z}_\nu}(z) \right| \leq \gamma \cdot d(z, M)^N$, $z \in U_0 \times \mathbf{C}^s$, $\nu = 1, \dots, n+s$,

where $d(z, M)$ is the Euclidean distance in \mathbf{C}^{n+s} between z and M .

Proof. We consider the system of linear equations at every point of U

$$(3.1) \quad \sum_{j=1}^s h_j \frac{\partial f_j}{\partial \bar{z}_\nu} = \frac{\partial g_0}{\partial \bar{z}_\nu}, \quad \nu = 1, \dots, s,$$

where g_0 is the function defined by (1.1). By (1.3), there exist the uniquely determined solutions $h_j(z_1, \dots, z_n)$, $j=1, \dots, s$. Since f_j and g_0 are of $H_r(U)$, so are h_j . We shall define the function h_J inductively for every multi-index $J=(j_1, \dots, j_k)$, $1 \leq i \leq s$. Suppose h_J is given in $H_r(U)$. Then h_{Jj} , $j=1, \dots, s$, will be defined as the solutions of the equations

$$(3.2) \quad \sum_{j=1}^s h_{Jj} \frac{\partial f_j}{\partial \bar{z}_\nu} = \frac{\partial h_J}{\partial \bar{z}_\nu}, \quad \nu = 1, \dots, s.$$

The condition (1.3) guarantees the existence of the solutions h_{Jj} in $H_r(U)$.

We shall prove that h_J are symmetric with respect to J . By differentiating each equation of (3.1) by \bar{z}_μ , we have

$$\sum_{j=1}^s \frac{\partial f_j}{\partial \bar{z}_\nu} \cdot \frac{\partial h_j}{\partial \bar{z}_\mu} = \frac{\partial^2 g_0}{\partial \bar{z}_\nu \partial \bar{z}_\mu} - \sum_{j=1}^s \frac{\partial^2 f_j}{\partial \bar{z}_\nu \partial \bar{z}_\mu} h_j.$$

Since the right member is symmetric in ν and μ , we have

$$(3.3) \quad \sum_{j=1}^s \frac{\partial f_j}{\partial \bar{z}_\nu} \cdot \frac{\partial h_j}{\partial \bar{z}_\mu} = \sum_{j=1}^s \frac{\partial f_j}{\partial \bar{z}_\mu} \cdot \frac{\partial h_j}{\partial \bar{z}_\nu}, \quad \nu, \mu = 1, \dots, s.$$

Substituting (3.2) for $k=1$ to (3.3), we obtain

$$\sum_{j=1}^s \left(\sum_{i=1}^s h_{ji} \frac{\partial f_i}{\partial \bar{z}_\mu} \right) \frac{\partial f_j}{\partial \bar{z}_\nu} = \sum_{j=1}^s \left(\sum_{i=1}^s h_{ji} \frac{\partial f_i}{\partial \bar{z}_\nu} \right) \frac{\partial f_j}{\partial \bar{z}_\mu}, \quad \nu, \mu = 1, \dots, s,$$

or equivalently

$$\sum_{i,j} (h_{ji} - h_{ij}) \frac{\partial f_j}{\partial \bar{z}_\mu} \cdot \frac{\partial f_i}{\partial \bar{z}_\nu} = 0, \quad \nu, \mu = 1, \dots, s.$$

By using (1.3), we can find that $h_{ji} = h_{ij}$ for every i and j .

General cases will be proved by induction. For simplicity, we write $J=(j_1, \dots, j_k)$, $I=(j_1, \dots, j_{i-1}, j_{i+1}, \dots, j_k)$, $i=j_i$ and $J'=(j_1, \dots, j_{i-1}, j, j_{i+1}, \dots, j_k)$. Since $h_{J'} = h_{Ij}$ and $h_J = h_{Ii}$ by assumption of induction, we have

$$\sum_{i=1}^s h_J \frac{\partial f_i}{\partial \bar{z}_\nu} = \frac{\partial h_J}{\partial \bar{z}_\nu} \quad \text{and} \quad \sum_{j=1}^s h_{J'} \frac{\partial f_j}{\partial \bar{z}_\mu} = \frac{\partial h_{J'}}{\partial \bar{z}_\mu}.$$

By differentiating the first identity by \bar{z}_μ and the second by \bar{z}_ν , we obtain

$$(3.4) \quad \sum_{i=1}^s \frac{\partial h_J}{\partial \bar{z}_\mu} \cdot \frac{\partial f_i}{\partial \bar{z}_\nu} = \sum_{j=1}^s \frac{\partial h_{J'}}{\partial \bar{z}_\nu} \cdot \frac{\partial f_j}{\partial \bar{z}_\mu}, \quad \nu, \mu = 1, \dots, s.$$

Substituting the equalities

$$\sum_{j=1}^s h_{J_j} \frac{\partial f_j}{\partial \bar{z}_\mu} = \frac{\partial h_J}{\partial \bar{z}_\mu} \quad \text{and} \quad \sum_{i=1}^s h_{J'_i} \frac{\partial f_i}{\partial \bar{z}_\nu} = \frac{\partial h_{J'}}{\partial \bar{z}_\nu}$$

to (3.4), we have

$$\sum_{i,j} (h_{J_j} - h_{J'_i}) \frac{\partial f_i}{\partial \bar{z}_\nu} \frac{\partial f_j}{\partial \bar{z}_\mu} = 0, \quad \nu, \mu = 1, \dots, s.$$

By (1,3), we find that $h_{J_j} = h_{J'_i}$, which implies the symmetry of h_J for all J .

Now we define \tilde{g} by

$$\begin{aligned} \tilde{g}(z_1, \dots, z_{n+s}) &= g_0(z_1, \dots, z_n) \\ &+ \sum_{k=1}^N \frac{1}{k!} \sum_{(j_1, \dots, j_k)} h_{j_1 \dots j_k}(z_1, \dots, z_n) (z_{n+j_1} - f_{j_1}(z_1, \dots, z_n)) \cdots (z_{n+j_k} - f_{j_k}(z_1, \dots, z_n)). \end{aligned}$$

If $\nu = s+1, \dots, n+s$, we have $\frac{\partial \tilde{g}}{\partial \bar{z}_\nu} \equiv 0$. For $\nu = 1, \dots, s$, we have

$$\frac{\partial \tilde{g}}{\partial \bar{z}_\nu} = \frac{1}{N!} \sum_{(j_1, \dots, j_N)} \frac{\partial h_{j_1 \dots j_N}}{\partial \bar{z}_\nu} (z_{n+j_1} - f_{j_1}(z_1, \dots, z_n)) \cdots (z_{n+j_N} - f_{j_N}(z_1, \dots, z_n)),$$

which proves the lemma.

We consider two uniform algebras on a compact subset K^* of M . $H(K^*)$ is the algebra of uniform limits on K^* of functions each holomorphic in a neighborhood (in \mathbf{C}^{n+s}) of K^* . $CR(K^*)$ is the algebra of uniform limits on K^* of functions each of which is a CR -function on a neighborhood (in M) of K^* .

Suppose K^* is H -convex in \mathbf{C}^{n+s} . Let g be a CR -function on a neighborhood M_1 (in M) of K^* . We can find a region of holomorphy V such that $K^* \subset V$ and $\bar{V} \cap M \subset M_1$. We denote by K and U_0 the projections of K^* and $M_0 = V \cap M$ respectively by the map $(z_1, \dots, z_{n+s}) \rightarrow (z_1, \dots, z_n)$. Let d denote the distance between K and ∂U_0 . By the way of construction of $G_\varepsilon(V)$ in Lemma 2, we can find a positive constant η such that, for every point z^0 of K^* , the ball $B_{\varepsilon\eta}(z^0) = \{z \in \mathbf{C}^{n+s}; |z - z^0| \leq \varepsilon\eta\}$ is contained in $G_\varepsilon(V)$, whenever $\varepsilon < d$. Therefore, by using Lemma 2 and Lemma 3 for $N = n+1$, and applying the same technique as one developed in [1], we obtain the following

Theorem 1. *If K^* is a compact subset of M which is H -convex in \mathbf{C}^{n+s} , then we have $H(K^*)=CR(K^*)$.*

4. Polynomial approximation.

We consider the following conditions for a compact subset K of \mathbf{C}^n and for functions f_j of $C^\infty(\mathbf{C}^n)$;

- (a) f_1, \dots, f_s are of $H_r(U)$ for some open set U containing K ,
 (b) there exists a constant k , $0 < k < 1$, such that

$$\sum_{j=1}^s |f_j(z+\xi) - f_j(z) - \bar{\xi}_j| \leq k \sum_{j=1}^s |\xi_j|^2$$

holds for any z and $\xi = (\xi_1, \dots, \xi_s, 0, \dots, 0)$ in \mathbf{C}^n , and

- (c) for any vector $\alpha' = (\alpha_1, \dots, \alpha_s)$, $K \cap E_{\alpha'}$ is polynomially convex in $E_{\alpha'}$, where $E_{\alpha'}$ is the subspace $\{z \in \mathbf{C}^n; z_j = \alpha_j, j=1, \dots, s\}$ of \mathbf{C}^n .

The condition (b) implies (1.3). In fact, we can find a constant k_1 , $0 < k_1 < 1$, such that

$$\sum_{j=1}^s \left| \sum_{v=1}^s \frac{\partial f_j}{\partial \bar{z}_v} \bar{\xi}_v + \bar{\xi}_j \right|^2 \leq k_1 \sum_{j=1}^s |\xi_j|^2,$$

and hence the system of linear equations

$$\sum_{v=1}^s \frac{\partial f_j}{\partial \bar{z}_v} \bar{\xi}_v = 0, \quad j = 1, \dots, s$$

has only trivial solution.

We consider two uniform algebras on K . A is the algebra of uniform limits on K of polynomials of $z_1, \dots, z_n, f_1(z), \dots, f_s(z)$. $H_r(K)$ is the algebra of uniform limits of functions each of which is holomorphic in z_{s+1}, \dots, z_n , in a neighborhood of K .

Theorem 2. *Suppose the conditions (a), (b) and (c) are satisfied. Then we have $A=H_r(K)$.*

Proof. We shall first prove that K^* is polynomially convex in \mathbf{C}^{n+s} . To do this, it is sufficient to show that the maximal ideal space of $P(K^*)$, the algebra of uniform limits of polynomials in z_1, \dots, z_{n+s} on K^* , coincides with K^* , or equivalently that every complex homomorphism of A is a point evaluation for some point of K . Let φ be any complex homomorphism on A . Set $\alpha_j = \varphi(z_j)$, $j=1, \dots, n$, and $\alpha = (\alpha_1, \dots, \alpha_n)$. We consider the function

$$f(z) = \sum_{j=1}^s (z_j - \alpha_j) (f_j(z) - f_j(\alpha)).$$

Then $f(z)$ is in A . By the condition (b), we have

$$\operatorname{Re} f(z) > 0 \quad \text{for } z \notin E_{\alpha'}, \alpha' = (\alpha_1, \dots, \alpha_s).$$

Let m be a representing measure for φ of A supported on K . Then we have

$$0 = \operatorname{Re} \varphi(f) = \int \operatorname{Re} f \, dm.$$

Therefore, the support of m must be contained in $K \cap E_{\alpha'}$ and, in particular, $K \cap E_{\alpha'}$ is not empty.

Let $h(z)$ be any polynomial of $z_1, \dots, z_n, f_1(z), \dots, f_s(z)$. For simplicity, we write $h_1(z_{s+1}, \dots, z_n) = h(\alpha_1, \dots, \alpha_s, z_{s+1}, \dots, z_n)$. Then we have

$$\begin{aligned} \varphi(h) &= \int h(z) \, dm(z) \\ &= \int h_1(z_{s+1}, \dots, z_n) \, dm(z_{s+1}, \dots, z_n). \end{aligned}$$

By the condition (a), h_1 is holomorphic in $U \cap E_{\alpha'}$. Since $K \cap E_{\alpha'}$ is polynomially convex, by Oka-Weil's theorem, h_1 is approximated uniformly on $K \cap E_{\alpha'}$ by polynomials of z_{s+1}, \dots, z_n . Since every polynomial of z_{s+1}, \dots, z_n is considered as a polynomial of z_1, \dots, z_n , φ can be considered as a complex homomorphism ψ of $P_0(K \cap E_{\alpha'})$, the algebra of uniform limits on $K \cap E_{\alpha'}$ of polynomials of z_{s+1}, \dots, z_n . Polynomial convexity of $K \cap E_{\alpha'}$ implies that ψ is a point evaluation at α . Therefore we have

$$\varphi(h) = \psi(h_1) = h_1(\alpha_{s+1}, \dots, \alpha_n) = h(\alpha),$$

which proves the polynomial convexity of K^* .

By Oka-Weil's theorem, $H(K^*)$ coincides with $P(K^*)$. Since K^* is the intersection of polynomial polyhedra containing K^* , it is H -convex, and therefore we have $H(K^*) = CR(K^*)$ by Theorem 1. A is isomorphic to $P(K^*)$ and $H_r(K)$ to $CR(K^*)$. Since $A \subset H_r(K)$, we obtain $A = H_r(K)$.

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