Kakutani, S.
Osaka J. Math.
**17** (1980), 513–524

# ON THE GROUPS $J_{Z_{m,q}}(*)$

SHIN-ICHIRO KAKUTANI

(Received August 10, 1979)

## 1. Introduction

Let $G$ be a compact topological group. If $V$ is an orthogonal representation space of $G$, we denote by $S(V)$ its unit sphere with respect to some $G$-invariant inner product. Two orthogonal representation spaces $V$ and $W$ of $G$ are called *J-equivalent* if there exists an orthogonal representation space $U$ such that $S(V\oplus U)$ and $S(W\oplus U)$ are $G$-homotopy equivalent. Let $RO(G)$ denote the real representation ring of $G$, and let $T_G(*)\subset RO(G)$ denote an additive subgroup consisting of all elements $V-W$ such that $V$ and $W$ are $J$-equivalent.

In [6] and [7], Kawakubo considered the quotient group $J_G(*)=RO(G)/T_G(*)$ and the natural epimorphism $J_G\colon RO(G)\to J_G(*)$, and determined the structure of $J_G(*)$ for compact abelian topological groups $G$.

The purpose of this paper is to determine $J_G(*)$ in case $G$ is the metacyclic group

$$Z_{m,q}=\{a, b\,|\,a^m=b^q=e,\ bab^{-1}=a^r\}\ ,$$

where $m$ is a positive odd integer, $q$ is an odd prime integer, $(r-1, m)=1$ and $r$ is a primitive $q$-th root of 1 mod $m$. Our main results are Theorem 7.3 and Corollary 7.4.

The author wishes to express his hearty thanks to Professor K. Kawakubo for many invaluable advices.

## 2. The metacyclic group $Z_{m,q}$

In this section we recall some well-known results about the metacyclic group $Z_{m,q}$. The metacyclic group $Z_{m,q}$ is a non-abelian group of order $mq$ and every element of $Z_{m,q}$ is written in the form

$$g=a^ib^j\ ,\qquad 0\leqq i\leqq m-1,\quad 0\leqq j\leqq q-1\ .$$

Let $m=p_1^{r(1)}p_2^{r(2)}\cdots p_t^{r(t)}$ be a prime decomposition of $m$. We can check easily from the definition of $Z_{m,q}$ the following:

(2.1) $(m, r)=1$,
(2.2) $q|(p_i-1)$ for $1\leqq i\leqq t$ and $q|(m-1)$,
(2.3) $(m, q)=1$.

The metacyclic group $Z_{m,q}$ has the following two subgroups:

(2.4) $\boldsymbol{Z}_m=\langle a\rangle$,
(2.5) $K_q=\langle b\rangle$.

The groups $\boldsymbol{Z}_m$ and $K_q$ are cyclic groups of order $m$ and $q$ respectively and we have

**Lemma 2.6.** *The group $\boldsymbol{Z}_m$ is a normal subgroup of $Z_{m,q}$ and $K_q$ is a subgroup satisfying $N(K_q)=K_q$ where $N(K_q)$ denotes the normalizer of $K_q$ in $Z_{m,q}$.*

Proof. Obviously $\boldsymbol{Z}_m$ is a normal subgroup of $Z_{m,q}$. Let $g=a^ib^j$ be an arbitrary element of $N(K_q)$. Then we have $g^{-1}bg=b^{-1}a^{i(r-1)}b^{j+1}\in K_q$. Hence $a^{i(r-1)}\in\boldsymbol{Z}_m\cap K_q=\{e\}$. Therefore we obtain $m|i$ and $g=a^ib^j=b^j\in K_q$. Namely $N(K_q)\subset K_q$. q.e.d.

**Lemma 2.7.** *Let $H$ $(\neq\{e\})$ be a subgroup of $Z_{m,q}$. If $H$ satisfies $H\cap\boldsymbol{Z}_m=\{e\}$, then $H$ and $K_q$ are conjugate.*

Proof. By assumption, there exists an element $a^ib^j\in H$ which satisfies $j\not\equiv 0 \bmod q$. Hence we obtain $\boldsymbol{Z}_mH=Z_{m,q}$. Thus there exists a canonical isomorphism

$$Z_{m,q}/\boldsymbol{Z}_m\cong H/H\cap\boldsymbol{Z}_m\,.$$

Therefore $q=|Z_{m,q}\colon\boldsymbol{Z}_m|=|H\colon H\cap\boldsymbol{Z}_m|=|H|$. Since $K_q$ is a Sylow $q$-subgroup of $Z_{m,q}$, $H$ and $K_q$ are conjugate. q.e.d.

REMARK 2.8. Let $H$ be an arbitrary subgroup of $Z_{m,q}$. By Lemma 2.7, $H$ satisfies one of the following:

(i) $H=\{e\}$,
(ii) $H$ is conjugate to $K_q$,
(iii) $H\cap\boldsymbol{Z}_m\neq\{e\}$.

REMARK 2.9. In general the metacyclic group $Z_{m,q}$ depends on not only the integers $m$, $q$ but also the integer $r$. But the group $J_{Z_{m,q}}(*)$ depends only on the integers $m$, $q$ (see Theorem 7.3).

## 3. The real representation ring $RO(\boldsymbol{Z}_{m,q})$

In this section we determine the additive generators of the real representation ring $RO(Z_{m,q})$. First we recall the results, due to Curtis and Reiner [2;

§47], about the additive generators of the complex representation ring $R(Z_{m,q})$.

The metacyclic group $Z_{m,q}$ has the following unitary representations:

(3.1) the trivial one-dimensional representation $1_{C^1}$,

(3.2) the complex $q$-dimensional representations $T_h$ ($h\in\boldsymbol{Z}$) defined by

$$T_h(a)=\begin{pmatrix} L_0 & & & & \\ & L_1 & & 0 & \\ & & L_2 & & \\ & 0 & & \ddots & \\ & & & & L_{q-1} \end{pmatrix}\in U(q)$$

and

$$T_h(b)=\begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 & 1 \\ 1 & 0 & & & & 0 \\ & 1 & 0 & 0 & & \vdots \\ & & 1 & & & \vdots \\ & 0 & & \ddots & & \vdots \\ & & & & 1 & 0 \end{pmatrix}\in U(q)\,,$$

where $L_j=\exp(2\pi hr^j\sqrt{-1}/m)$ for $0\leqq j\leqq q-1$,

(3.3) the complex one-dimensional representations $\rho_d$ ($d\in\boldsymbol{Z}$) defined by

$$\rho_d(a)=1\in U(1)$$

and

$$\rho_d(b)=\exp(2\pi d\sqrt{-1}/q)\in U(1)\,.$$

The representations $T_h$ ($h\in\boldsymbol{Z}$) satisfy the following (see [2; §47]):

(3.4) *If* $(h, m)=1$, *then* $T_h$ *is irreducible.*

(3.5) *When* $T_h$ *and* $T_k$ *are irreducible,* $T_h$ *and* $T_k$ *are inequivalent if and only if* $r^jh\not\equiv k \mod m$ *for* $0\leqq j\leqq q-1$.

Denote by $FR(Z_{m,q})$ the subgroup of $R(Z_{m,q})$ generated by $\{T_h\,|\,(h,m)=1,\ h\in\boldsymbol{Z}\}$. When $n$ is an integer such that $n\,|\,m$ and $n>1$, we obtain the metacyclic group $Z_{n,q}=\{c, d\,|\,c^n=d^q=e,\ dcd^{-1}=c^r\}$ and define the natural epimorphism $\pi_n\colon Z_{m,q}\to Z_{n,q}$ by $\pi_n(a^ib^j)=c^id^j$.

**Theorem 3.6.** *There is an isomorphism (additively)*

$$R(Z_{m,q})\cong A'\oplus B'\oplus \bigoplus_{n|m,n>1} FR(Z_{n,q})\,,$$

*where* $A'$ *is the subgroup of* $R(Z_{m,q})$ *generated by* $1_{C^1}$ *and* $B'$ *is the subgroup of* $R(Z_{m,q})$ *generated by* $\{\rho_d\,|\,(d,q)=1,\ d\in\boldsymbol{Z}\}$.

Proof. It follows that

$$R(Z_{m,q})=A'\oplus B'\oplus \bigoplus_{n|m,n>1} \pi_n^*(FR(Z_{n,q}))$$

(see [2; §47]). Since $\pi_n^* | FR(Z_{n,q})\colon FR(Z_{n,q}) \to R(Z_{m,q})$ is injective, we obtain the result. q.e.d.

If $\chi$ is a complex representation, then the real representation $r(\chi)$ is defined to be the underlying real representation of $\chi$, and $\bar{\chi}$ denotes the complex conjugate representation of $\chi$.

**Lemma 3.7.** *If* $(h, m)=1$, *then* $T_h$ *and* $\bar{T}_h$ *are inequivalent.*

Proof. Suppose that $T_h$ is equivalent to $\bar{T}_h \simeq T_{-h}$. It follows from (3.5) that there exists an integer $j$ $(0 \leqq j \leqq q-1)$ such that $r^j h \equiv -h \bmod m$. Since $(h, m)=1$, we have $r^j \equiv -1 \bmod m$. Thus we obtain $1 \equiv (r^j)^q \equiv (-1)^q \equiv -1 \bmod m$. This is a contradiction. Therefore $T_h$ is inequivalent to $\bar{T}_h$. q.e.d.

Denote by $FRO(Z_{m,q})$ the subgroup of $RO(Z_{m,q})$ generated by $\{r(T_h) | (h, m)=1, h \in \boldsymbol{Z}\}$. Now we have

**Theorem 3.8.** *There is an isomorphism* (*additively*)

$$RO(Z_{m,q}) \simeq A \oplus B \oplus \bigoplus_{n|m, n>1} FRO(Z_{n,q}),$$

*where* $A$ *is the subgroup of* $RO(Z_{m,q})$ *generated by the trivial one-dimensional representation* $1_{\boldsymbol{R}^1}$ *and* $B$ *is the subgroup of* $RO(Z_{m,q})$ *generated by* $\{r(\rho_d) | (d, q)=1, d \in \boldsymbol{Z}\}$.

Proof. The result follows easily from Theorem 3.6 and Adams [1; Theorem 3.57].

In the following we write $T_h$ and $\rho_d$ instead of $r(T_h)$ and $r(\rho_d)$ respectively. We use the same symbol as a representation for its representation space.

Remark 3.9. The representation $T_h$ is identified with the following unitary representation space:

$$\begin{cases} T_h(a)\circ(z_0, z_1, \cdots, z_{q-1}) = (\exp(2\pi h\sqrt{-1}/m)z_0, \exp(2\pi hr\sqrt{-1}/m)z_1, \cdots, \\ \qquad\qquad\qquad\qquad\qquad \exp(2\pi hr^{q-1}\sqrt{-1}/m)z_{q-1}), \\ T_h(b)\circ(z_0, z_1, \cdots, z_{q-1}) = (z_{q-1}, z_0, z_1, \cdots, z_{q-2}), \end{cases}$$

where $(z_0, z_1, \cdots, z_{q-1}) \in \boldsymbol{C}^q$. Moreover we regard $\boldsymbol{R}^1$ as $1_{\boldsymbol{R}^1}$.

## 4. *G*-homotopy equivalences of spheres of *G*-representation spaces

We begin by fixing some notations. Let $G$ be a finite group and $X$ be a

$G$-space. We denote the isotropy group at $x\in X$ by $G_x$. For a subgroup $H$ of $G$, $(H)$ denotes the conjugacy class of $H$ in $G$ and we set

$$X^H = \{x\in X \mid G_x \supset H\} .$$

For a $G$-map $f\colon X_1\to X_2$, we denote by $f^H$ the restriction $f\mid X_1^H\colon X_1^H\to X_2^H$. If $V$ is a unitary $G$-representation space, then for a subgroup $H$ of $G$, $S(V)^H$ has a canonical orientation defined by the complex structure of $V^H$. Let $V$, $W$ be unitary $G$-representation spaces and $f\colon S(V)\to S(W)$ be a $G$-map. Then for a subgroup $H$ of $G$ satisfying $\dim S(V)^H=\dim S(W)^H$, we have the degree of the map $f^H\colon S(V)^H\to S(W)^H$. When $S(V)^H=S(W)^H=\phi$, we define $\deg f^H=1$. Since $G$ is a finite group, there are only finite conjugacy classes of subgroups of $G$, say

$$\{(H_1), (H_2), \cdots, (H_n)\} .$$

By Theorem 1.1 of James-Segal [5], we have

**Theorem 4.1.** *Let $V$, $W$ be unitary $G$-representation spaces which satisfy the condition* $\dim S(V)^{H_i}=\dim S(W)^{H_i}$ *for* $1\leqq i\leqq n$. *If there exists a $G$-map $f\colon S(V)\to S(W)$ such that*

$$|\deg f^{H_i}|=1 \qquad \textit{for} \quad 1\leqq i\leqq n ,$$

*then $S(V)$ and $S(W)$ are $G$-homotopy equivalent.*

## 5. The group $J_{Z_{m,q}}(B)$

Let $\rho_{a_i}$ $(1\leqq i\leqq n)$ and $\rho_{b_j}$ $(1\leqq j\leqq n)$ be non-trivial $Z_{m,q}$-representation spaces defined by (3.3). We set

$$M = \rho_{a_1}\oplus\rho_{a_2}\oplus\cdots\oplus\rho_{a_n} , \quad M'=\rho_{b_1}\oplus\rho_{b_2}\oplus\cdots\oplus\rho_{b_n} .$$

**Theorem 5.1.** *The following three conditions are equivalent:*

(i) *$S(M)$ and $S(M')$ are $Z_{m,q}$-homotopy equivalent,*

(ii) *$M$ and $M'$ are $J$-equivalent,*

(iii) $\prod_{i=1}^n a_i\equiv\pm\prod_{j=1}^n b_j \bmod q$.

Proof. From the definition of $\rho_d$ $(d\in \boldsymbol{Z})$, it suffices to consider the $K_q$-actions instead of the $Z_{m,q}$-actions. The $K_q$-representation $\rho_d|K_q$ is defined by $(\rho_d|K_q)(b)=\exp(2\pi d\sqrt{-1}/q)$. Since $(a_i, q)=(b_j, q)=1$ for $1\leqq i$, $j\leqq n$, it follows from Kawakubo [7; Theorem 2.6] that (i), (ii) and (iii) are equivalent. q.e.d.

**Corollary 5.2.** *There is an isomorphism*

$$J_{Z_{m,q}}(B)\cong \boldsymbol{Z}\oplus\boldsymbol{Z}_{(q-1)/2} .$$

Proof. See Kawakubo [7; §2 and §3].

## 6. The group Ker $(J_{Z_{m,q}}|FRO(Z_{m,q}))$

In this section we determine the group Ker $(J_{Z_{m,q}}|FRO(Z_{m,q}))$. Let $T_h$ and $T_k$ be $Z_{m,q}$-representation spaces defined by (3.9). If $T_h$ is contained in $FRO(Z_{m,q})$, then the integer $h$ satisfies $(h, m)=1$. Thus there exists some integer $\bar{h}$ such that $\bar{h}h\equiv 1 \bmod m$. We define a $Z_{m,q}$-map $f_{\bar{h}k}\colon S(T_h)\to S(T_k)$ by

$$f_{\bar{h}k}(z_0, z_1, \cdots, z_{q-1}) = \frac{(z_0^{\bar{h}k}, z_1^{\bar{h}k}, \cdots, z_{q-1}^{\bar{h}k})}{||(z_0^{\bar{h}k}, z_1^{\bar{h}k}, \cdots, z_{q-1}^{\bar{h}k})||}\,.$$

It is obvious that $f_{\bar{h}k}$ is a well-defined $Z_{m,q}$-map.

Let $T_{h_i}$ $(1\leqq i\leqq n)$ and $T_{k_j}$ $(1\leqq j\leqq n)$ be $Z_{m,q}$-representation spaces contained in $FRO(Z_{m,q})$. We set

$$N = T_{h_1}\oplus T_{h_2}\oplus\cdots\oplus T_{h_n},\quad N' = T_{k_1}\oplus T_{k_2}\oplus\cdots\oplus T_{k_n}\,.$$

Let $x_0$ (resp. $y_0$) be the point (0, 1) of $S(\boldsymbol{R}^2)\subset S(N\oplus\boldsymbol{R}^2)$ (resp. $S(\boldsymbol{R}^2)\subset S(N'\oplus\boldsymbol{R}^2)$). Since $\boldsymbol{C}^1$ is a complex vector space, the underlying real vector space $\boldsymbol{R}^2$ has a canonical orientation.

**Lemma 6.1.** *There exists a $Z_{m,q}$-map $F\colon S(N\oplus\boldsymbol{R}^2)\to S(N'\oplus\boldsymbol{R}^2)$ such that*

(i) $F(x_0)=y_0$,

(ii) $\deg F = \prod_{i=1}^{n} (\bar{h}_i k_i)^q$, $\deg F^{K_q} = \prod_{i=1}^{n} \bar{h}_i k_i$ *and* $\deg F^H=1$,

*where $H$ is an arbitrary subgroup of $Z_{m,q}$ satisfying $H\cap \boldsymbol{Z}_m \neq \{e\}$.*

Proof. First we study the $Z_{m,q}$-map $f_{\bar{h}k}\colon S(T_h)\to S(T_k)$, where $T_h$ and $T_k$ are contained in $FRO(Z_{m,q})$. It follows from the definition of $f_{\bar{h}k}$ that $\deg f_{\bar{h}k}= (\bar{h}k)^q$. For the subgroup $K_q$, we have

$$S(T_h)^{K_q} = \{(z_0, z_1, \cdots, z_{q-1})\in S(T_h)\,|\, z_0 = z_1 = \cdots = z_{q-1}\}\,,$$
$$S(T_k)^{K_q} = \{(w_0, w_1, \cdots, w_{q-1})\in S(T_k)\,|\, w_0 = w_1 = \cdots = w_{q-1}\}\,.$$

Hence $\deg (f_{\bar{h}k})^{K_q}=\bar{h}k$. Since $S(T_h)^H=S(T_k)^H=\phi$, we obtain $\deg (f_{\bar{h}k})^H=1$. Then we put

$$F = f_{\bar{h}_1k_1} * f_{\bar{h}_2k_2} *\cdots* f_{\bar{h}_nk_n} * id_{S(\boldsymbol{R}^2)}\,,$$

where $*$ denotes the join. Now $F$ is a $Z_{m,q}$-map from $S(N\oplus\boldsymbol{R}^2)$ to $S(N'\oplus\boldsymbol{R}^2)$ which satisfies the conditions (i) and (ii). q.e.d.

The following lemma is due to Petrie [10].

**Lemma 6.2.** *Let $G$ be a finite group and $V$, $W$ be unitary $G$-representation spaces. Let $H$ be a subgroup of $G$ whose conjugacy class is contained in* Iso$(V)=$

$\{(G_v) \mid v \in V\}$. *Suppose that* $f: S(V) \to S(W)$ *is an* $H$*-map, then there exists a* $G$*-map* $\tau(G, H; f)$: $S(V \oplus \boldsymbol{R}^2) \to S(W \oplus \boldsymbol{R}^2)$ *which satisfies the following conditions*:

(i) $\tau(G, H; f)(x_0) = y_0$ *where* $x_0$, $y_0$ *are those in Lemma* 6.1.

(ii) *Let* $K$ *be a subgroup of* $G$ *such that* $\dim V^K = \dim W^K$. *If there exists some element* $g_0$ *of* $G$ *such that* $g_0^{-1} K g_0 \subset H$, *we have*

$$\deg \tau(G, H; f)^K = |(G/H)^K| \deg f^{g_0^{-1} K g_0}.$$

*On the other hand, if* $g^{-1} K g \not\subset H$ *for any element* $g$ *of* $G$, *we have*

$$\deg \tau(G, H; f)^K = 0.$$

Proof. By Meyerhoff-Petrie [9; Theorem 2.2] and Petrie [10; Lemma 2.3], there exists a $G$-map $\tilde{f}$: $S(V \oplus \boldsymbol{R}^1) \to S(W \oplus \boldsymbol{R}^1)$ which satisfies the condition (ii). Then we obtain a $G$-map $\tau(G, H; f) = \tilde{f} * id_{S(\boldsymbol{R}^1)}$: $S(V \oplus \boldsymbol{R}^2) \to S(W \oplus \boldsymbol{R}^2)$. It is obvious that the $G$-map $\tau(G, H; f)$ satisfies the conditions (i) and (ii).

q.e.d.

**Lemma 6.3.** *There exist two* $Z_{m,q}$*-maps* $\theta$, $\psi$: $S(N \oplus \boldsymbol{R}^2) \to S(N' \oplus \boldsymbol{R}^2)$ *which satisfy the following two conditions*:

(i) $\theta(x_0) = \psi(x_0) = y_0$,

(ii) $\deg \theta = mq$, $\deg \theta^{K_q} = \deg \theta^H = 0$, $\deg \psi = m$, $\deg \psi^{K_q} = 1$ *and* $\deg \psi^H = 0$, *where* $H$ *is an arbitrary subgroup of* $Z_{m,q}$ *satisfying* $H \cap \boldsymbol{Z}_m \neq \{e\}$.

Proof. We recall that $N$, $N'$ are unitary $Z_{m,q}$-representation spaces and remark that $\mathrm{Iso}(N) = \{(e), (K_q), (Z_{m,q})\}$. Apply Lemma 6.2 to the identity map $id$: $S(N) \to S(N')$ which is an $\{e\}$-map, then we have a $Z_{m,q}$-map $\theta = \tau(Z_{m,q}, \{e\}; id)$: $S(N \oplus \boldsymbol{R}^2) \to S(N' \oplus \boldsymbol{R}^2)$ such that $\theta(x_0) = y_0$, $\deg \theta = |z_{m,q}| = mq$ and $\deg \theta^{K_q} = \deg \theta^H = 0$. Moreover the identity map is not only an $\{e\}$-map but also a $K_q$-map. We also have a $Z_{m,q}$-map $\psi = \tau(Z_{m,q}, K_q; id)$: $S(N \oplus \boldsymbol{R}^2) \to S(N' \oplus \boldsymbol{R}^2)$ such that $\psi(x_0) = y_0$, $\deg \psi = |Z_{m,q}/K_q| = m$, $\deg \psi^{K_q} = |(Z_{m,q}/K_q)^{K_q}| = |N(K_q)/K_q| = 1$ and $\deg \psi^H = 0$. q.e.d.

Now we have

**Theorem 6.4.** *The following three conditions are equivalent*:

(i) $S(N \oplus \boldsymbol{R}^2)$ *and* $S(N' \oplus \boldsymbol{R}^2)$ *are* $Z_{m,q}$*-homotopy equivalent,*

(ii) $N$ *and* $N'$ *are* $J$*-equivalent,*

(iii) $\prod_{i=1}^{n} h_i^q \equiv \pm \prod_{j=1}^{n} k_j^q \mod m$.

Proof. Obviously (i) implies (ii).

First we show that (ii) implies (iii). By assumption, there exists an orthogonal $Z_{m,q}$-representation space $U$ such that $S(N \oplus U)$ and $S(N' \oplus U)$

are $Z_{m,q}$-homotopy equivalent. Obviously $S(N\oplus U)$ and $S(N'\oplus U)$ are also $\boldsymbol{Z}_m$-homotopy equivalent. Let $\mu_d$ $(d\in \boldsymbol{Z})$ be the complex one-dimensional $\boldsymbol{Z}_m$-representations defined by $\mu_d(a)=\exp(2\pi d\sqrt{-1}/m)$. Then we have $T_h|\boldsymbol{Z}_m \cong \mu_h \oplus \mu_{hr} \oplus \mu_{hr^2} \oplus \cdots \oplus \mu_{hr^{q-1}}$ as $\boldsymbol{Z}_m$-representations. The integers $h_i r^s$, $k_j r^s$ satisfy $(h_i r^s, m)=(k_j r^s, m)=1$ for $1\leqq i,\ j\leqq n$ and $0\leqq s\leqq q-1$. It follows from Kawakubo [7; Theorem 2.6] that $r^{q(q-1)n/2}\prod_{i=1}^{n} h_i^q \equiv \pm r^{q(q-1)n/2}\prod_{j=1}^{n} k_j^q \bmod m$. Since $r^q\equiv 1 \bmod m$, we obtain the condition (iii).

Next we show that (iii) implies (i). By Lemma 6.1, there exists a $Z_{m,q}$-map $F\colon S(N\oplus \boldsymbol{R}^2)\to S(N'\oplus \boldsymbol{R}^2)$ such that

$$
(6.4.1)\quad \begin{cases} F(x_0)=y_0\,, \\ \deg F=\prod_{i=1}^{n}(\bar{h}_i k_i)^q,\ \deg F^{K_q}=\prod_{i=1}^{n}\bar{h}_i k_i \\ \text{and}\quad \deg F^H=1 \quad \text{where} \quad H\cap \boldsymbol{Z}_m\neq\{e\}. \end{cases}
$$

On the other hand, by Lemma 6.3, there exists a $Z_{m,q}$-map $\psi\colon S(N\oplus \boldsymbol{R}^2)\to S(N'\oplus \boldsymbol{R}^2)$ such that

$$
(6.4.2)\quad \begin{cases} \psi(x_0)=y_0\,, \\ \deg\psi=m,\ \deg\psi^{K_q}=1 \text{ and } \deg\psi^H=0 \text{ where } H\cap \boldsymbol{Z}_m\neq\{e\}\,. \end{cases}
$$

We define $\varepsilon$ $(=\pm 1)$ by $\prod_{i=1}^{n} h_i^q\equiv \varepsilon \prod_{j=1}^{n} k_j^q \bmod m$. The $Z_{m,q}$-homotopy classes of $Z_{m,q}$-maps from $S(N\oplus \boldsymbol{R}^2)$ to $S(N'\oplus \boldsymbol{R}^2)$ sending $x_0$ to $y_0$ form a group. Therefore by (6.4.1) and (6.4.2), we obtain a $Z_{m,q}$-map $F_2=F-(\prod_{i=1}^{n}\bar{h}_i k_i-\varepsilon)\psi$ which satisfies the following condition:

$$
(6.4.3)\quad \begin{cases} F_2(x_0)=y_0\,, \\ \deg F_2=\prod_{i=1}^{n}(\bar{h}_i k_i)^q-(\prod_{i=1}^{n}\bar{h}_i k_i-\varepsilon)m\,,\quad \deg F_2^{K_q}=\varepsilon \\ \text{and}\quad \deg F_2^H=1 \quad \text{where} \quad H\cap \boldsymbol{Z}_m\neq\{e\}\,. \end{cases}
$$

By Lemma 6.3, there exists a $Z_{m,q}$-map $\theta\colon S(N\oplus \boldsymbol{R}^2)\to S(N'\oplus \boldsymbol{R}^2)$ such that

$$
(6.4.4)\quad \begin{cases} \theta(x_0)=y_0\,, \\ \deg\theta=mq \quad \text{and} \quad \deg\theta^{K_q}=\deg\theta^H=0 \quad \text{where} \quad H\cap \boldsymbol{Z}_m\neq\{e\}. \end{cases}
$$

On the other hand, by the assumption (iii), we have

$$
\prod_{i=1}^{n}(\bar{h}_i k_i)^q\equiv\varepsilon \qquad \bmod m.
$$

Then we obtain

$$\prod_{i=1}^{n}(\bar{h}_i k_i)^q - (\prod_{i=1}^{n}\bar{h}_i k_i - \varepsilon)m - \varepsilon \equiv 0 \quad \mod m. \tag{6.4.5}$$

Moreover it is well-known that

$$\prod_{i=1}^{n}(\bar{h}_i k_i)^q \equiv \prod_{i=1}^{n}\bar{h}_i k_i \quad \mod q.$$

Hence we obtain (see (2.2))

$$\begin{aligned}\prod_{i=1}^{n}(\bar{h}_i k_i)^q - (\prod_{i=1}^{n}\bar{h}_i k_i - \varepsilon)m - \varepsilon &\equiv (1-m)\prod_{i=1}^{n}\bar{h}_i k_i + \varepsilon(m-1)\\ &\equiv 0 \quad \mod q.\end{aligned} \tag{6.4.6}$$

Since $m$ and $q$ are relatively prime integers, by (6.4.5) and (6.4.6), we obtain

$$\prod_{i=1}^{n}(\bar{h}_i k_i)^q - (\prod_{i=1}^{n}\bar{h}_i k_i - \varepsilon)m - \varepsilon \equiv 0 \quad \mod mq.$$

Let $n_0$ be an integer such that

$$\prod_{i=1}^{n}(\bar{h}_i k_i)^q - (\prod_{i=1}^{n}\bar{h}_i k_i - \varepsilon)m - \varepsilon = n_0 mq\,. \tag{6.4.7}$$

By (6.4.3), (6.4.4) and (6.4.7), we obtain a $Z_{m,q}$-map $F_3 = F_2 - n_0\theta$ such that

$$\deg F_3 = \deg F_3^{K_q} = \varepsilon \quad \text{and} \quad \deg F_3^H = 1 \quad \text{where} \quad H \cap \boldsymbol{Z}_m \neq \{e\}\,.$$

Therefore it follows from Remark 2.8 and Theorem 4.1 that $S(N \oplus \boldsymbol{R}^2)$ and $S(N' \oplus \boldsymbol{R}^2)$ are $Z_{m,q}$-homotopy equivalent. q.e.d.

## 7. The group $J_{Z_{m,q}}(*)$

In this section we determine the group $J_{Z_{m,q}}(*)$. For this purpose we follow the procedure due to Kawakubo [7; §3 and §4]. To determine the group

$$C_m = J_{Z_{m,q}}(FRO(Z_{m,q}))\,,$$

we define another group $C'_m$ as follows. Let $m = p_1^{r(1)} p_2^{r(2)} \cdots p_t^{r(t)}$ be a prime decomposition of $m$. We set

$$C'_m = \boldsymbol{Z} \oplus \{\bigoplus_{i=1}^{t} \boldsymbol{Z}_{(p_i^{r(i)} - p_i^{r(i)-1})/q}\} / \boldsymbol{Z}_2\,,$$

where the inclusion of $\boldsymbol{Z}_2$ into $\bigoplus_{i=1}^{t} \boldsymbol{Z}_{(p_i^{r(i)} - p_i^{r(i)-1})/q}$ is given by $1 \to \bigoplus_{i=1}^{t} (p_i^{r(i)} - p_i^{r(i)-1})/2q$. Remark that $2q \mid (p_i - 1)$ for $1 \leqq i \leqq t$ (see (2.2)).

We also define a homomorphism

$$J'_m\colon FRO(Z_{m,q}) \to C'_m$$

as follows. As is well-known, there exist integers $\alpha(i)$ for $1 \leqq i \leqq t$ such that $\alpha(i)$ is a primitive root mod $p_i^{r(i)}$ and $\alpha(i) \equiv 1 \bmod p_j^{r(j)}$ for every $j \neq i$. For every integer $h$ with $(h, m)=1$ and for $1 \leqq i \leqq t$, there exists a unique $\mu(h, i) \in \boldsymbol{Z}_{p^{r(i)}p^{r(i)-1}}$ such that

$$h \equiv \prod_{i=1}^{t} \alpha(i)^{\mu(h,i)} \qquad \mathrm{mod}\ m\,.$$

Let

$$\omega\colon \bigoplus_{i=1}^{t} \boldsymbol{Z}_{p_i^{r(i)}-p_i^{r(i)-1}} \to \{\bigoplus_{i=1}^{t} \boldsymbol{Z}_{(p_i^{r(i)}-p_i^{r(i)-1})/q}\}/\boldsymbol{Z}_2$$

denote the natural projection. Let $\sum_{j=1}^{u} a(h_j)T_{h_j}$ be an arbitrary element of $FRO(Z_{m,q})$, that is, $a(h_j) \in \boldsymbol{Z}$. We define

$$J'_m(\sum_{j=1}^{u} a(h_j)T_{h_j}) = \sum_{j=1}^{u} a(h_j) \oplus \omega(\bigoplus_{i=1}^{t} \sum_{j=1}^{u} a(h_j)\mu(h_j, i))\,.$$

Denote by $J_m$ the restricted homomorphism $J_{Z_{m,q}} | FRO(Z_{m,q})$. We have

**Lemma 7.1.** *$J'_m$ is an epimorphism and* $\mathrm{Ker}\, J_m = \mathrm{Ker}\, J'_m$. *Hence there is an isomorphism*

$$C_m \cong C'_m$$

Proof. Let $a$, $a_i$ $(1 \leqq i \leqq t)$ be arbitrary integers. Then we have

$$J'_m((a - \sum_{i=1}^{t} a_i)T_1 + \sum_{i=1}^{t} a_i T_{\alpha(i)}) = a \oplus \omega(\bigoplus_{i=1}^{t} a_i)$$

$$\in C'_m = \boldsymbol{Z} \oplus \{\bigoplus_{i=1}^{t} \boldsymbol{Z}_{(p_i^{r(i)}-p_i^{r(i)-1})/q}\}/\boldsymbol{Z}_2\,.$$

This shows that $J'_m$ is surjective.

Next we show that $\mathrm{Ker}\, J_m = \mathrm{Ker}\, J'_m$. Let $x = \sum_{\lambda=1}^{u} a(h_\lambda)T_{h_\lambda} - \sum_{\nu=1}^{v} b(k_\nu)T_{k_\nu}$ be an arbitrary element of $FRO(Z_{m,q})$, where $a(h_\lambda)$ $(1 \leqq \lambda \leqq u)$ and $b(k_\nu)$ $(1 \leqq \nu \leqq v)$ are non-negative integers. The element $x$ is contained in $\mathrm{Ker}\, J'_m$ if and only if the following condition (7.1.1) is satisfied.

$$\text{(7.1.1)} \qquad \begin{cases} \sum_{\lambda=1}^{u} a(h_\lambda) = \sum_{\nu=1}^{v} b(k_\nu)\,, \\ \omega(\bigoplus_{i=1}^{t} \sum_{\lambda=1}^{u} a(h_\lambda)\mu(h_\lambda, i)) = \omega(\bigoplus_{i=1}^{t} \sum_{\nu=1}^{v} b(k_\nu)\mu(k_\nu, i))\,. \end{cases}$$

It is easy to see that the condition (7.1.1) is equivalent to the following condition (7.1.2):

$$(7.1.2) \qquad \begin{cases} \sum_{\lambda=1}^{u} a(h_\lambda) = \sum_{\nu=1}^{v} b(k_\nu)\,, \\ \prod_{\lambda=1}^{u} h_\lambda^{a(h_\lambda)q} \equiv \pm \prod_{\nu=1}^{v} k_\nu^{b(k_\nu)q} \qquad \bmod m\,. \end{cases}$$

By Theorem 6.4, $x$ satisfies the condition (7.1.2) if and only if $x$ is contained in $\mathrm{Ker}\, J_m$. Therefore we have $\mathrm{Ker}\, J_m = \mathrm{Ker}\, J'_m$. q.e.d.

We recall that there is an isomorphism (see Theorem 3.8)

$$RO(Z_{m,q}) \cong A \oplus B \bigoplus_{n|m, n>1} FRO(Z_{n,q})\,.$$

**Lemma 7.2.** *There is an isomorphism*

$$T_{Z_{m,q}}(*) \cong \{0\} \oplus \mathrm{Ker}\,(J_{Z_{m,q}} | B) \oplus \bigoplus_{n|m, n>1} \mathrm{Ker}\, J_n\,.$$

Proof. The result is easily seen from tom Dieck [3; Proposition 4.1].

It follows from Corollary 5.2 and Lemma 7.2 that

$$J_{Z_{m,q}}(*) \cong \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus \boldsymbol{Z}_{(q-1)/2} \oplus \bigoplus_{n|m, n>1} C_n\,.$$

Therefore we obtain, by Lemma 7.1, the following main theorem.

**Theorem 7.3.** *There is an isomorphism*

$$J_{Z_{m,q}}(*) \cong \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus \boldsymbol{Z}_{(q-1)/2} \oplus \bigoplus_{m|n, n>1} C'_n\,.$$

**Corollary 7.4.** *Let $V$, $W$ be orthogonal $Z_{m,q}$-representation spaces. If $V$ and $W$ are $J$-equivalent, then $S(V \oplus \boldsymbol{R}^2)$ and $S(W \oplus \boldsymbol{R}^2)$ are $Z_{m,q}$-homotopy equivalent.*

Proof. The result follows easily from Theorems 5.1, 6.5 and Lemma 7.2.

REMARK 7.5. M. Morimoto has succeeded to omit $\boldsymbol{R}^2$ in Corollary 7.4.

## 8. Appendix

In this section $G$ will be a finite group. Denote by $RO_0(G)$ the additive subgroup of $RO(G)$

$$\{V - W \mid \dim V^H = \dim W^H \quad \text{for every subgroup } H \text{ of } G\}\,.$$

In [3] and [4], tom Dieck and Petrie defined the group $jO(G)$ to be $RO_0(G)/$

$T_G(*)$. Since $T_G(*)\subset RO_0(G)\subset RO(G)$, there exists a short exact sequence

$$0 \to jO(G) \to J_G(*) \to RO(G)/RO_0(G) \to 0\,.$$

Since $RO(G)/RO_0(G)$ is a free abelian group (see Lee-Wasserman [8; §3]), the above short exact sequence is split. Thus we have

**Proposition 8.1.** *There is an isomorphism*

$$J_G(*)\cong jO(G)\oplus RO(G)/RO_0(G)\,.$$

---

## References

[1] J.F. Adams: Lectures on Lie groups, Benjamin Inc., New York, 1969.

[2] C.R. Curtis and I. Reiner: Representation theory of finite groups and associative algebras, Interscience, 1962.

[3] T. tom Dieck: *Homotopy-equivalent group representations*, J. Reine Angew. Math. **298** (1978), 182–195.

[4] T. tom Dieck and T. Petrie: *Geometric modules over the Burnside ring*, Invent. Math. **47** (1978), 273–287.

[5] I.M. James and G.B. Segal: *On equivariant homotopy type*, Topology **17** (1978), 267–272.

[6] K. Kawakubo: *The groups $J_G(*)$ for compact abelian topological groups $G$*, Proc. Japan Acad. **54** (1978), 76–78.

[7] K. Kawakubo: *Equivariant homotopy equivalence of group representations*, J. Math. Soc. Japan **32** (1980), 105–118.

[8] C.N. Lee and A.G. Wasserman: *On the groups $JO(G)$*, Memoirs of A.M.S. **159** (1975).

[9] A. Meyerhoff and T. Petrie: *Quasi equivalence of G modules*, Topology **15** (1976), 69–75.

[10] T. Petrie: Geometric modules over the Burnside ring, Aarhus Univ. preprint No. 26, (1976).

Department of Mathematics
Osaka University
Toyonaka, Osaka
560 Japan