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ON THE GROUPS $J_{Z_{m,q}}(*)$

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1. Introduction

Let G be a compact topological group. If V is an orthogonal representation space of G , we denote by $S(V)$ its unit sphere with respect to some G -invariant inner product. Two orthogonal representation spaces V and W of G are called *J-equivalent* if there exists an orthogonal representation space U such that $S(V \oplus U)$ and $S(W \oplus U)$ are G -homotopy equivalent. Let $RO(G)$ denote the real representation ring of G , and let $T_G(*) \subset RO(G)$ denote an additive subgroup consisting of all elements $V - W$ such that V and W are J -equivalent.

In [6] and [7], Kawakubo considered the quotient group $J_G(*) = RO(G)/T_G(*)$ and the natural epimorphism $J_G: RO(G) \rightarrow J_G(*)$, and determined the structure of $J_G(*)$ for compact abelian topological groups G .

The purpose of this paper is to determine $J_G(*)$ in case G is the metacyclic group

$$Z_{m,q} = \{a, b \mid a^m = b^q = e, bab^{-1} = a^r\},$$

where m is a positive odd integer, q is an odd prime integer, $(r-1, m)=1$ and r is a primitive q -th root of 1 mod m . Our main results are Theorem 7.3 and Corollary 7.4.

The author wishes to express his hearty thanks to Professor K. Kawakubo for many invaluable advices.

2. The metacyclic group $Z_{m,q}$

In this section we recall some well-known results about the metacyclic group $Z_{m,q}$. The metacyclic group $Z_{m,q}$ is a non-abelian group of order mq and every element of $Z_{m,q}$ is written in the form

$$g = a^i b^j, \quad 0 \leq i \leq m-1, \quad 0 \leq j \leq q-1.$$

Let $m = p_1^{r(1)} p_2^{r(2)} \cdots p_t^{r(t)}$ be a prime decomposition of m . We can check easily from the definition of $Z_{m,q}$ the following:

- (2.1) $(m, r) = 1$,
 (2.2) $q \mid (p_i - 1)$ for $1 \leq i \leq t$ and $q \mid (m - 1)$,
 (2.3) $(m, q) = 1$.

The metacyclic group $Z_{m,q}$ has the following two subgroups:

- (2.4) $Z_m = \langle a \rangle$,
 (2.5) $K_q = \langle b \rangle$.

The groups Z_m and K_q are cyclic groups of order m and q respectively and we have

Lemma 2.6. *The group Z_m is a normal subgroup of $Z_{m,q}$ and K_q is a subgroup satisfying $N(K_q) = K_q$ where $N(K_q)$ denotes the normalizer of K_q in $Z_{m,q}$.*

Proof. Obviously Z_m is a normal subgroup of $Z_{m,q}$. Let $g = a^i b^j$ be an arbitrary element of $N(K_q)$. Then we have $g^{-1} b g = b^{-1} a^{i(r-1)} b^{j+1} \in K_q$. Hence $a^{i(r-1)} \in Z_m \cap K_q = \{e\}$. Therefore we obtain $m \mid i$ and $g = a^i b^j = b^j \in K_q$. Namely $N(K_q) \subset K_q$. q.e.d.

Lemma 2.7. *Let $H (\neq \{e\})$ be a subgroup of $Z_{m,q}$. If H satisfies $H \cap Z_m = \{e\}$, then H and K_q are conjugate.*

Proof. By assumption, there exists an element $a^i b^j \in H$ which satisfies $j \not\equiv 0 \pmod{q}$. Hence we obtain $Z_m H = Z_{m,q}$. Thus there exists a canonical isomorphism

$$Z_{m,q}/Z_m \cong H/H \cap Z_m.$$

Therefore $q = |Z_{m,q}/Z_m| = |H/H \cap Z_m| = |H|$. Since K_q is a Sylow q -subgroup of $Z_{m,q}$, H and K_q are conjugate. q.e.d.

REMARK 2.8. Let H be an arbitrary subgroup of $Z_{m,q}$. By Lemma 2.7, H satisfies one of the following:

- (i) $H = \{e\}$,
 (ii) H is conjugate to K_q ,
 (iii) $H \cap Z_m \neq \{e\}$.

REMARK 2.9. In general the metacyclic group $Z_{m,q}$ depends on not only the integers m, q but also the integer r . But the group $J_{Z_{m,q}}(*)$ depends only on the integers m, q (see Theorem 7.3).

3. The real representation ring $RO(Z_{m,q})$

In this section we determine the additive generators of the real representation ring $RO(Z_{m,q})$. First we recall the results, due to Curtis and Reiner [2;

§47], about the additive generators of the complex representation ring $R(Z_{m,q})$.

The metacyclic group $Z_{m,q}$ has the following unitary representations:

- (3.1) the trivial one-dimensional representation $1_{\mathcal{C}^1}$,
 (3.2) the complex q -dimensional representations T_h ($h \in \mathbf{Z}$) defined by

$$T_h(a) = \begin{pmatrix} L_0 & & & & \\ & L_1 & & & \\ & & L_2 & & \\ & & & \ddots & \\ & 0 & & & L_{q-1} \end{pmatrix} \in U(q)$$

and

$$T_h(b) = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & 0 & & 0 \\ & 1 & 0 & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 & 0 \end{pmatrix} \in U(q),$$

where $L_j = \exp(2\pi h r^j \sqrt{-1}/m)$ for $0 \leq j \leq q-1$,

- (3.3) the complex one-dimensional representations ρ_d ($d \in \mathbf{Z}$) defined by

$$\rho_d(a) = 1 \in U(1)$$

and

$$\rho_d(b) = \exp(2\pi d \sqrt{-1}/q) \in U(1).$$

The representations T_h ($h \in \mathbf{Z}$) satisfy the following (see [2; §47]):

- (3.4) If $(h, m) = 1$, then T_h is irreducible.
 (3.5) When T_h and T_k are irreducible, T_h and T_k are inequivalent if and only if $r^j h \not\equiv k \pmod{m}$ for $0 \leq j \leq q-1$.

Denote by $FR(Z_{m,q})$ the subgroup of $R(Z_{m,q})$ generated by $\{T_h | (h, m) = 1, h \in \mathbf{Z}\}$. When n is an integer such that $n|m$ and $n > 1$, we obtain the metacyclic group $Z_{n,q} = \{c, d | c^n = d^q = e, dcd^{-1} = c^r\}$ and define the natural epimorphism $\pi_n: Z_{m,q} \rightarrow Z_{n,q}$ by $\pi_n(a^i b^j) = c^i d^j$.

Theorem 3.6. *There is an isomorphism (additively)*

$$R(Z_{m,q}) \cong A' \oplus B' \oplus \bigoplus_{n|m, n>1} FR(Z_{n,q}),$$

where A' is the subgroup of $R(Z_{m,q})$ generated by $1_{\mathcal{C}^1}$ and B' is the subgroup of $R(Z_{m,q})$ generated by $\{\rho_d | (d, q) = 1, d \in \mathbf{Z}\}$.

Proof. It follows that

$$R(Z_{m,q}) = A' \oplus B' \oplus \bigoplus_{n|m, n>1} \pi_n^*(FR(Z_{n,q}))$$

(see [2; §47]). Since $\pi_n^*|FR(Z_{n,q}):FR(Z_{n,q})\rightarrow R(Z_{m,q})$ is injective, we obtain the result. q.e.d.

If \mathcal{X} is a complex representation, then the real representation $r(\mathcal{X})$ is defined to be the underlying real representation of \mathcal{X} , and $\bar{\mathcal{X}}$ denotes the complex conjugate representation of \mathcal{X} .

Lemma 3.7. *If $(h, m)=1$, then T_h and \bar{T}_h are inequivalent.*

Proof. Suppose that T_h is equivalent to $\bar{T}_h \cong T_{-h}$. It follows from (3.5) that there exists an integer j ($0 \leq j \leq q-1$) such that $r^j h \equiv -h \pmod{m}$. Since $(h, m)=1$, we have $r^j \equiv -1 \pmod{m}$. Thus we obtain $1 \equiv (r^j)^q \equiv (-1)^q \equiv -1 \pmod{m}$. This is a contradiction. Therefore T_h is inequivalent to \bar{T}_h . q.e.d.

Denote by $FRO(Z_{m,q})$ the subgroup of $RO(Z_{m,q})$ generated by $\{r(T_h) | (h, m)=1, h \in \mathbb{Z}\}$. Now we have

Theorem 3.8. *There is an isomorphism (additively)*

$$RO(Z_{m,q}) \cong A \oplus B \oplus \bigoplus_{n|m, n>1} FRO(Z_{n,q}),$$

where A is the subgroup of $RO(Z_{m,q})$ generated by the trivial one-dimensional representation 1_{R^1} and B is the subgroup of $RO(Z_{m,q})$ generated by $\{r(\rho_d) | (d, q)=1, d \in \mathbb{Z}\}$.

Proof. The result follows easily from Theorem 3.6 and Adams [1; Theorem 3.57].

In the following we write T_h and ρ_d instead of $r(T_h)$ and $r(\rho_d)$ respectively. We use the same symbol as a representation for its representation space.

REMARK 3.9. The representation T_h is identified with the following unitary representation space:

$$\begin{cases} T_h(a) \circ (z_0, z_1, \dots, z_{q-1}) = (\exp(2\pi h \sqrt{-1}/m) z_0, \exp(2\pi h r \sqrt{-1}/m) z_1, \dots, \\ \quad \exp(2\pi h r^{q-1} \sqrt{-1}/m) z_{q-1}), \\ T_h(b) \circ (z_0, z_1, \dots, z_{q-1}) = (z_{q-1}, z_0, z_1, \dots, z_{q-2}), \end{cases}$$

where $(z_0, z_1, \dots, z_{q-1}) \in \mathbb{C}^q$. Moreover we regard R^1 as 1_{R^1} .

4. G -homotopy equivalences of spheres of G -representation spaces

We begin by fixing some notations. Let G be a finite group and X be a

G -space. We denote the isotropy group at $x \in X$ by G_x . For a subgroup H of G , (H) denotes the conjugacy class of H in G and we set

$$X^H = \{x \in X \mid G_x \supset H\}.$$

For a G -map $f: X_1 \rightarrow X_2$, we denote by f^H the restriction $f|X_1^H: X_1^H \rightarrow X_2^H$. If V is a unitary G -representation space, then for a subgroup H of G , $S(V)^H$ has a canonical orientation defined by the complex structure of V^H . Let V, W be unitary G -representation spaces and $f: S(V) \rightarrow S(W)$ be a G -map. Then for a subgroup H of G satisfying $\dim S(V)^H = \dim S(W)^H$, we have the degree of the map $f^H: S(V)^H \rightarrow S(W)^H$. When $S(V)^H = S(W)^H = \phi$, we define $\deg f^H = 1$. Since G is a finite group, there are only finite conjugacy classes of subgroups of G , say

$$\{(H_1), (H_2), \dots, (H_n)\}.$$

By Theorem 1.1 of James-Segal [5], we have

Theorem 4.1. *Let V, W be unitary G -representation spaces which satisfy the condition $\dim S(V)^{H_i} = \dim S(W)^{H_i}$ for $1 \leq i \leq n$. If there exists a G -map $f: S(V) \rightarrow S(W)$ such that*

$$|\deg f^{H_i}| = 1 \quad \text{for } 1 \leq i \leq n,$$

then $S(V)$ and $S(W)$ are G -homotopy equivalent.

5. The group $J_{Z_{m,q}}(B)$

Let $\rho_{a_i} (1 \leq i \leq n)$ and $\rho_{b_j} (1 \leq j \leq n)$ be non-trivial $Z_{m,q}$ -representation spaces defined by (3.3). We set

$$M = \rho_{a_1} \oplus \rho_{a_2} \oplus \dots \oplus \rho_{a_n}, \quad M' = \rho_{b_1} \oplus \rho_{b_2} \oplus \dots \oplus \rho_{b_n}.$$

Theorem 5.1. *The following three conditions are equivalent:*

- (i) $S(M)$ and $S(M')$ are $Z_{m,q}$ -homotopy equivalent,
- (ii) M and M' are J -equivalent,
- (iii) $\prod_{i=1}^n a_i \equiv \pm \prod_{j=1}^n b_j \pmod{q}$.

Proof. From the definition of $\rho_d (d \in \mathbb{Z})$, it suffices to consider the K_q -actions instead of the $Z_{m,q}$ -actions. The K_q -representation $\rho_d|K_q$ is defined by $(\rho_d|K_q)(b) = \exp(2\pi d\sqrt{-1}/q)$. Since $(a_i, q) = (b_j, q) = 1$ for $1 \leq i, j \leq n$, it follows from Kawakubo [7; Theorem 2.6] that (i), (ii) and (iii) are equivalent. q.e.d.

Corollary 5.2. *There is an isomorphism*

$$J_{Z_{m,q}}(B) \cong \mathbb{Z} \oplus \mathbb{Z}_{(q-1)/2}.$$

Proof. See Kawakubo [7; §2 and §3].

6. The group $\text{Ker}(J_{Z_{m,q}} | FRO(Z_{m,q}))$

In this section we determine the group $\text{Ker}(J_{Z_{m,q}} | FRO(Z_{m,q}))$. Let T_h and T_k be $Z_{m,q}$ -representation spaces defined by (3.9). If T_h is contained in $FRO(Z_{m,q})$, then the integer h satisfies $(h, m)=1$. Thus there exists some integer \bar{h} such that $\bar{h}h \equiv 1 \pmod{m}$. We define a $Z_{m,q}$ -map $f_{\bar{h}k}: S(T_h) \rightarrow S(T_k)$ by

$$f_{\bar{h}k}(z_0, z_1, \dots, z_{q-1}) = \frac{(z_0^{\bar{h}k}, z_1^{\bar{h}k}, \dots, z_{q-1}^{\bar{h}k})}{\|(z_0^{\bar{h}k}, z_1^{\bar{h}k}, \dots, z_{q-1}^{\bar{h}k})\|}.$$

It is obvious that $f_{\bar{h}k}$ is a well-defined $Z_{m,q}$ -map.

Let T_{h_i} ($1 \leq i \leq n$) and T_{k_j} ($1 \leq j \leq n$) be $Z_{m,q}$ -representation spaces contained in $FRO(Z_{m,q})$. We set

$$N = T_{h_1} \oplus T_{h_2} \oplus \dots \oplus T_{h_n}, \quad N' = T_{k_1} \oplus T_{k_2} \oplus \dots \oplus T_{k_n}.$$

Let x_0 (resp. y_0) be the point $(0, 1)$ of $S(\mathbf{R}^2) \subset S(N \oplus \mathbf{R}^2)$ (resp. $S(\mathbf{R}^2) \subset S(N' \oplus \mathbf{R}^2)$). Since \mathbf{C}^1 is a complex vector space, the underlying real vector space \mathbf{R}^2 has a canonical orientation.

Lemma 6.1. *There exists a $Z_{m,q}$ -map $F: S(N \oplus \mathbf{R}^2) \rightarrow S(N' \oplus \mathbf{R}^2)$ such that*

(i) $F(x_0) = y_0$,

(ii) $\deg F = \prod_{i=1}^n (\bar{h}_i k_i)^q$, $\deg F^{K_q} = \prod_{i=1}^n \bar{h}_i k_i$ and $\deg F^H = 1$,

where H is an arbitrary subgroup of $Z_{m,q}$ satisfying $H \cap Z_m \neq \{e\}$.

Proof. First we study the $Z_{m,q}$ -map $f_{\bar{h}k}: S(T_h) \rightarrow S(T_k)$, where T_h and T_k are contained in $FRO(Z_{m,q})$. It follows from the definition of $f_{\bar{h}k}$ that $\deg f_{\bar{h}k} = (\bar{h}k)^q$. For the subgroup K_q , we have

$$\begin{aligned} S(T_h)^{K_q} &= \{(z_0, z_1, \dots, z_{q-1}) \in S(T_h) \mid z_0 = z_1 = \dots = z_{q-1}\}, \\ S(T_k)^{K_q} &= \{(w_0, w_1, \dots, w_{q-1}) \in S(T_k) \mid w_0 = w_1 = \dots = w_{q-1}\}. \end{aligned}$$

Hence $\deg (f_{\bar{h}k})^{K_q} = \bar{h}k$. Since $S(T_h)^H = S(T_k)^H = \phi$, we obtain $\deg (f_{\bar{h}k})^H = 1$. Then we put

$$F = f_{\bar{h}_1 k_1} * f_{\bar{h}_2 k_2} * \dots * f_{\bar{h}_n k_n} * id_{S(\mathbf{R}^2)},$$

where $*$ denotes the join. Now F is a $Z_{m,q}$ -map from $S(N \oplus \mathbf{R}^2)$ to $S(N' \oplus \mathbf{R}^2)$ which satisfies the conditions (i) and (ii). q.e.d.

The following lemma is due to Petrie [10].

Lemma 6.2. *Let G be a finite group and V, W be unitary G -representation spaces. Let H be a subgroup of G whose conjugacy class is contained in $\text{Iso}(V) =$*

$\{(G_v) | v \in V\}$. Suppose that $f: S(V) \rightarrow S(W)$ is an H -map, then there exists a G -map $\tau(G, H; f): S(V \oplus \mathbf{R}^2) \rightarrow S(W \oplus \mathbf{R}^2)$ which satisfies the following conditions:

- (i) $\tau(G, H; f)(x_0) = y_0$ where x_0, y_0 are those in Lemma 6.1.
- (ii) Let K be a subgroup of G such that $\dim V^K = \dim W^K$. If there exists some element g_0 of G such that $g_0^{-1}Kg_0 \subset H$, we have

$$\deg \tau(G, H; f)^K = |(G/H)^K| \deg f^{g_0^{-1}Kg_0}.$$

On the other hand, if $g^{-1}Kg \not\subset H$ for any element g of G , we have

$$\deg \tau(G, H; f)^K = 0.$$

Proof. By Meyerhoff-Petrie [9; Theorem 2.2] and Petrie [10; Lemma 2.3], there exists a G -map $\tilde{f}: S(V \oplus \mathbf{R}^1) \rightarrow S(W \oplus \mathbf{R}^1)$ which satisfies the condition (ii). Then we obtain a G -map $\tau(G, H; f) = \tilde{f} \ast id_{S(\mathbf{R}^1)}: S(V \oplus \mathbf{R}^2) \rightarrow S(W \oplus \mathbf{R}^2)$. It is obvious that the G -map $\tau(G, H; f)$ satisfies the conditions (i) and (ii).

q.e.d.

Lemma 6.3. *There exist two $Z_{m,q}$ -maps $\theta, \psi: S(N \oplus \mathbf{R}^2) \rightarrow S(N' \oplus \mathbf{R}^2)$ which satisfy the following two conditions:*

- (i) $\theta(x_0) = \psi(x_0) = y_0$,
 - (ii) $\deg \theta = mq$, $\deg \theta^{K_q} = \deg \theta^H = 0$, $\deg \psi = m$, $\deg \psi^{K_q} = 1$ and $\deg \psi^H = 0$,
- where H is an arbitrary subgroup of $Z_{m,q}$ satisfying $H \cap Z_m \neq \{e\}$.

Proof. We recall that N, N' are unitary $Z_{m,q}$ -representation spaces and remark that $\text{Iso}(N) = \{(e), (K_q), (Z_{m,q})\}$. Apply Lemma 6.2 to the identity map $id: S(N) \rightarrow S(N')$ which is an $\{e\}$ -map, then we have a $Z_{m,q}$ -map $\theta = \tau(Z_{m,q}, \{e\}; id): S(N \oplus \mathbf{R}^2) \rightarrow S(N' \oplus \mathbf{R}^2)$ such that $\theta(x_0) = y_0$, $\deg \theta = |Z_{m,q}| = mq$ and $\deg \theta^{K_q} = \deg \theta^H = 0$. Moreover the identity map is not only an $\{e\}$ -map but also a K_q -map. We also have a $Z_{m,q}$ -map $\psi = \tau(Z_{m,q}, K_q; id): S(N \oplus \mathbf{R}^2) \rightarrow S(N' \oplus \mathbf{R}^2)$ such that $\psi(x_0) = y_0$, $\deg \psi = |Z_{m,q}/K_q| = m$, $\deg \psi^{K_q} = |(Z_{m,q}/K_q)^{K_q}| = |N(K_q)/K_q| = 1$ and $\deg \psi^H = 0$.

q.e.d.

Now we have

Theorem 6.4. *The following three conditions are equivalent:*

- (i) $S(N \oplus \mathbf{R}^2)$ and $S(N' \oplus \mathbf{R}^2)$ are $Z_{m,q}$ -homotopy equivalent,
- (ii) N and N' are J -equivalent,
- (iii) $\prod_{i=1}^n h_i^q \equiv \pm \prod_{j=1}^n k_j^q \pmod{m}$.

Proof. Obviously (i) implies (ii).

First we show that (ii) implies (iii). By assumption, there exists an orthogonal $Z_{m,q}$ -representation space U such that $S(N \oplus U)$ and $S(N' \oplus U)$

are $Z_{m,q}$ -homotopy equivalent. Obviously $S(N \oplus U)$ and $S(N' \oplus U)$ are also Z_m -homotopy equivalent. Let $\mu_d (d \in \mathbf{Z})$ be the complex one-dimensional Z_m -representations defined by $\mu_d(a) = \exp(2\pi d \sqrt{-1}/m)$. Then we have $T_h|_{Z_m} \cong \mu_h \oplus \mu_{hr} \oplus \mu_{hr^2} \oplus \cdots \oplus \mu_{hr^{q-1}}$ as Z_m -representations. The integers h, r^s, k, r^s satisfy $(h, r^s, m) = (k, r^s, m) = 1$ for $1 \leq i, j \leq n$ and $0 \leq s \leq q-1$. It follows from Kawakubo [7; Theorem 2.6] that $r^{q(q-1)n/2} \prod_{i=1}^n h_i^q \equiv \pm r^{q(q-1)n/2} \prod_{j=1}^n k_j^q \pmod{m}$. Since $r^q \equiv 1 \pmod{m}$, we obtain the condition (iii).

Next we show that (iii) implies (i). By Lemma 6.1, there exists a $Z_{m,q}$ -map $F: S(N \oplus \mathbf{R}^2) \rightarrow S(N' \oplus \mathbf{R}^2)$ such that

$$(6.4.1) \quad \begin{cases} F(x_0) = y_0, \\ \deg F = \prod_{i=1}^n (\bar{h}_i k_i)^q, \deg F^{K_q} = \prod_{i=1}^n \bar{h}_i k_i \\ \text{and } \deg F^H = 1 \text{ where } H \cap Z_m \neq \{e\}. \end{cases}$$

On the other hand, by Lemma 6.3, there exists a $Z_{m,q}$ -map $\psi: S(N \oplus \mathbf{R}^2) \rightarrow S(N' \oplus \mathbf{R}^2)$ such that

$$(6.4.2) \quad \begin{cases} \psi(x_0) = y_0, \\ \deg \psi = m, \deg \psi^{K_q} = 1 \text{ and } \deg \psi^H = 0 \text{ where } H \cap Z_m \neq \{e\}. \end{cases}$$

We define $\varepsilon (= \pm 1)$ by $\prod_{i=1}^n h_i^q \equiv \varepsilon \prod_{j=1}^n k_j^q \pmod{m}$. The $Z_{m,q}$ -homotopy classes of $Z_{m,q}$ -maps from $S(N \oplus \mathbf{R}^2)$ to $S(N' \oplus \mathbf{R}^2)$ sending x_0 to y_0 form a group. Therefore by (6.4.1) and (6.4.2), we obtain a $Z_{m,q}$ -map $F_2 = F - (\prod_{i=1}^n \bar{h}_i k_i - \varepsilon) \psi$ which satisfies the following condition:

$$(6.4.3) \quad \begin{cases} F_2(x_0) = y_0, \\ \deg F_2 = \prod_{i=1}^n (\bar{h}_i k_i)^q - (\prod_{i=1}^n \bar{h}_i k_i - \varepsilon)m, \deg F_2^{K_q} = \varepsilon \\ \text{and } \deg F_2^H = 1 \text{ where } H \cap Z_m \neq \{e\}. \end{cases}$$

By Lemma 6.3, there exists a $Z_{m,q}$ -map $\theta: S(N \oplus \mathbf{R}^2) \rightarrow S(N' \oplus \mathbf{R}^2)$ such that

$$(6.4.4) \quad \begin{cases} \theta(x_0) = y_0, \\ \deg \theta = mq \text{ and } \deg \theta^{K_q} = \deg \theta^H = 0 \text{ where } H \cap Z_m \neq \{e\}. \end{cases}$$

On the other hand, by the assumption (iii), we have

$$\prod_{i=1}^n (\bar{h}_i k_i)^q \equiv \varepsilon \pmod{m}.$$

Then we obtain

$$(6.4.5) \quad \prod_{i=1}^n (\bar{h}_i k_i)^q - \left(\prod_{i=1}^n \bar{h}_i k_i - \varepsilon \right) m - \varepsilon \equiv 0 \pmod{m}.$$

Moreover it is well-known that

$$\prod_{i=1}^n (\bar{h}_i k_i)^q \equiv \prod_{i=1}^n \bar{h}_i k_i \pmod{q}.$$

Hence we obtain (see (2.2))

$$(6.4.6) \quad \prod_{i=1}^n (\bar{h}_i k_i)^q - \left(\prod_{i=1}^n \bar{h}_i k_i - \varepsilon \right) m - \varepsilon \equiv (1-m) \prod_{i=1}^n \bar{h}_i k_i + \varepsilon(m-1) \pmod{q} \\ \equiv 0 \pmod{q}.$$

Since m and q are relatively prime integers, by (6.4.5) and (6.4.6), we obtain

$$\prod_{i=1}^n (\bar{h}_i k_i)^q - \left(\prod_{i=1}^n \bar{h}_i k_i - \varepsilon \right) m - \varepsilon \equiv 0 \pmod{mq}.$$

Let n_0 be an integer such that

$$(6.4.7) \quad \prod_{i=1}^n (\bar{h}_i k_i)^q - \left(\prod_{i=1}^n \bar{h}_i k_i - \varepsilon \right) m - \varepsilon = n_0 m q.$$

By (6.4.3), (6.4.4) and (6.4.7), we obtain a $Z_{m,q}$ -map $F_3 = F_2 - n_0 \theta$ such that

$$\deg F_3 = \deg F_3^{K^q} = \varepsilon \quad \text{and} \quad \deg F_3^H = 1 \quad \text{where} \quad H \cap Z_m \neq \{e\}.$$

Therefore it follows from Remark 2.8 and Theorem 4.1 that $S(N \oplus \mathbf{R}^2)$ and $S(N' \oplus \mathbf{R}^2)$ are $Z_{m,q}$ -homotopy equivalent. q.e.d.

7. The group $J_{Z_{m,q}}(*)$

In this section we determine the group $J_{Z_{m,q}}(*)$. For this purpose we follow the procedure due to Kawakubo [7; §3 and §4]. To determine the group

$$C_m = J_{Z_{m,q}}(FRO(Z_{m,q})),$$

we define another group C'_m as follows. Let $m = p_1^{r(1)} p_2^{r(2)} \dots p_t^{r(t)}$ be a prime decomposition of m . We set

$$C'_m = \mathbf{Z} \oplus \left\{ \bigoplus_{i=1}^t \mathbf{Z}_{(p_i^{r(i)} - p_i^{r(i)-1})/q} \right\} / \mathbf{Z}_2,$$

where the inclusion of \mathbf{Z}_2 into $\bigoplus_{i=1}^t \mathbf{Z}_{(p_i^{r(i)} - p_i^{r(i)-1})/q}$ is given by $1 \rightarrow \bigoplus_{i=1}^t (p_i^{r(i)} - p_i^{r(i)-1}) /$

$2q$. Remark that $2q \mid (p_i - 1)$ for $1 \leq i \leq t$ (see (2.2)).

We also define a homomorphism

$$J'_m: FRO(Z_{m,q}) \rightarrow C'_m$$

as follows. As is well-known, there exist integers $\alpha(i)$ for $1 \leq i \leq t$ such that $\alpha(i)$ is a primitive root mod $p_i^{r(i)}$ and $\alpha(i) \equiv 1 \pmod{p_j^{r(j)}}$ for every $j \neq i$. For every integer h with $(h, m) = 1$ and for $1 \leq i \leq t$, there exists a unique $\mu(h, i) \in \mathbf{Z}_{p_i^{r(i)} p_i^{r(i)-1}}$ such that

$$h \equiv \prod_{i=1}^t \alpha(i)^{\mu(h, i)} \pmod{m}.$$

Let

$$\omega: \bigoplus_{i=1}^t \mathbf{Z}_{p_i^{r(i)} - p_i^{r(i)-1}} \rightarrow \left\{ \bigoplus_{i=1}^t \mathbf{Z}_{(p_i^{r(i)} - p_i^{r(i)-1})/q} \right\} / \mathbf{Z}_2$$

denote the natural projection. Let $\sum_{j=1}^u a(h_j) T_{h_j}$ be an arbitrary element of $FRO(Z_{m,q})$, that is, $a(h_j) \in \mathbf{Z}$. We define

$$J'_m \left(\sum_{j=1}^u a(h_j) T_{h_j} \right) = \sum_{j=1}^u a(h_j) \oplus \omega \left(\bigoplus_{i=1}^t \sum_{j=1}^u a(h_j) \mu(h_j, i) \right).$$

Denote by J_m the restricted homomorphism $J_{Z_{m,q}}|FRO(Z_{m,q})$. We have

Lemma 7.1. *J'_m is an epimorphism and $\text{Ker } J_m = \text{Ker } J'_m$. Hence there is an isomorphism*

$$C_m \cong C'_m$$

Proof. Let a, a_i ($1 \leq i \leq t$) be arbitrary integers. Then we have

$$\begin{aligned} J'_m \left((a - \sum_{i=1}^t a_i) T_1 + \sum_{i=1}^t a_i T_{\alpha(i)} \right) &= a \oplus \omega \left(\bigoplus_{i=1}^t a_i \right) \\ &\in C'_m = \mathbf{Z} \oplus \left\{ \bigoplus_{i=1}^t \mathbf{Z}_{(p_i^{r(i)} - p_i^{r(i)-1})/q} \right\} / \mathbf{Z}_2. \end{aligned}$$

This shows that J'_m is surjective.

Next we show that $\text{Ker } J_m = \text{Ker } J'_m$. Let $x = \sum_{\lambda=1}^u a(h_\lambda) T_{h_\lambda} - \sum_{\nu=1}^v b(k_\nu) T_{k_\nu}$ be an arbitrary element of $FRO(Z_{m,q})$, where $a(h_\lambda)$ ($1 \leq \lambda \leq u$) and $b(k_\nu)$ ($1 \leq \nu \leq v$) are non-negative integers. The element x is contained in $\text{Ker } J'_m$ if and only if the following condition (7.1.1) is satisfied.

$$(7.1.1) \quad \left\{ \begin{array}{l} \sum_{\lambda=1}^u a(h_\lambda) = \sum_{\nu=1}^v b(k_\nu), \\ \omega \left(\bigoplus_{i=1}^t \sum_{\lambda=1}^u a(h_\lambda) \mu(h_\lambda, i) \right) = \omega \left(\bigoplus_{i=1}^t \sum_{\nu=1}^v b(k_\nu) \mu(k_\nu, i) \right). \end{array} \right.$$

It is easy to see that the condition (7.1.1) is equivalent to the following condition (7.1.2):

$$(7.1.2) \quad \begin{cases} \sum_{\lambda=1}^u a(h_\lambda) = \sum_{\nu=1}^v b(k_\nu), \\ \prod_{\lambda=1}^u h_\lambda^{a(h_\lambda)q} \equiv \pm \prod_{\nu=1}^v k_\nu^{b(k_\nu)q} \pmod{m}. \end{cases}$$

By Theorem 6.4, x satisfies the condition (7.1.2) if and only if x is contained in $\text{Ker } J_m$. Therefore we have $\text{Ker } J_m = \text{Ker } J'_m$. q.e.d.

We recall that there is an isomorphism (see Theorem 3.8)

$$RO(Z_{m,q}) \cong A \oplus B \bigoplus_{n|m, n>1} FRO(Z_{n,q}).$$

Lemma 7.2. *There is an isomorphism*

$$T_{Z_{m,q}}(*) \cong \{0\} \oplus \text{Ker } (J_{Z_{m,q}}|B) \oplus \bigoplus_{n|m, n>1} \text{Ker } J_n.$$

Proof. The result is easily seen from tom Dieck [3; Proposition 4.1].

It follows from Corollary 5.2 and Lemma 7.2 that

$$J_{Z_{m,q}}(*) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_{(q-1)/2} \oplus \bigoplus_{n|m, n>1} C_n.$$

Therefore we obtain, by Lemma 7.1, the following main theorem.

Theorem 7.3. *There is an isomorphism*

$$J_{Z_{m,q}}(*) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_{(q-1)/2} \oplus \bigoplus_{m|n, n>1} C'_n.$$

Corollary 7.4. *Let V, W be orthogonal $Z_{m,q}$ -representation spaces. If V and W are J -equivalent, then $S(V \oplus \mathbf{R}^2)$ and $S(W \oplus \mathbf{R}^2)$ are $Z_{m,q}$ -homotopy equivalent.*

Proof. The result follows easily from Theorems 5.1, 6.5 and Lemma 7.2.

REMARK 7.5. M. Morimoto has succeeded to omit \mathbf{R}^2 in Corollary 7.4.

8. Appendix

In this section G will be a finite group. Denote by $RO_0(G)$ the additive subgroup of $RO(G)$

$$\{V - W \mid \dim V^H = \dim W^H \text{ for every subgroup } H \text{ of } G\}.$$

In [3] and [4], tom Dieck and Petrie defined the group $jO(G)$ to be $RO_0(G)/$

$T_G(*)$. Since $T_G(*) \subset RO_0(G) \subset RO(G)$, there exists a short exact sequence

$$0 \rightarrow jO(G) \rightarrow J_G(*) \rightarrow RO(G)/RO_0(G) \rightarrow 0.$$

Since $RO(G)/RO_0(G)$ is a free abelian group (see Lee-Wasserman [8; §3]), the above short exact sequence is split. Thus we have

Proposition 8.1. *There is an isomorphism*

$$J_G(*) \cong jO(G) \oplus RO(G)/RO_0(G).$$

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