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Author(s)	Kishi, Masanori
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Kishi, Masanori
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*Unicity Principles in the Potential Theory**)

By Masanori KISHI

Introduction

The theory of potentials in a locally compact space has been mainly concerned with potential theoretical principles and the characterization of the class of kernels which satisfy those principles. The principles are numerous and some of them are complicated, but typical ones are the balayage, equilibrium, energy principles and a few others. Roughly speaking, a kernel is said to satisfy the balayage (resp. equilibrium) principles when balayaged (resp. equilibrium) measures exist. The principle states only the existence of balayaged (resp. equilibrium) measures. Naturally the uniqueness of balayaged (resp. equilibrium) measures comes into question and the principle should be split into two principles, the one on the existence and the other on the uniqueness.

The present paper is devoted to those unicity principles, which are formulated as follows:

(i) if potentials of positive measure coincide with each other in the whole space except for at most a potential theoretical null set, then the measures are identical (U-*principle*),

(ii) the balayaged measure is uniquely determined by a given positive measure and a compact set (BU-*principle*),

(iii) the equilibrium measure is uniquely determined by a given compact set (EU-*principle*).

Needless to say, the BU- (resp. EU-) principle is considered as to kernels which satisfy the balayage (resp. equilibrium) principle.

By the classical theory it is known that the kernels of order α $(0<\alpha<n)$ in the n-dimensional Euclidean space R^n satisfy the U-principle and if $n-2\leq\alpha<n$, they satisfy both the BU- and EU-principles, and that the Green kernels on open Riemann surfaces satisfy the three principles. Recently, concerning symmetric kernels which satisfy the balayage principle and a regularity condition, Ninomiya [11] has given a necessary and sufficient condition for them to satisfy the U-principle. From his argument it follows that his condition is sufficient also for

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the BU-principle. He proposed a question which may be expressed as follows: Is it true that his condition is sufficient for the EU-principle? It will be shown that the answer is negative. A characterization of the U-principle was obtained by Anger [1], too. His result states that a kernel satisfies the U-principle if and only if every continuous function with compact support is approximated uniformly by a sequence of potentials of signed measure.

In the present paper we consider symmetric kernels and we aim to obtain simple necessary and/or sufficient conditions for the three unicity principles in terms of a kernel itself. In Chapter I, we give necessary conditions for the principles, and we get the relations among the conditions. Each of the following conditions is necessary in order that a symmetric kernel K satisfies one of the unicity principles:

(1) K is non-degenerate (§ 1),

(2) K-potentials separate points (§ 1),

(3) K satisfies Ninomiya's condition mentioned above (§ 2).

We remark that the necessity of (3) was proved by Ninomiya only for the U-principle. Condition (3) is stronger than (1), and (1) is stronger than (2). In § 3, we show that for symmetric composition kernels (1) and (2) are equivalent, and if they satisfy the balayage or equilibrium principle, then the three conditions are equivalent. We add to them two other necessary conditions for the unicity principles, one of which was obtained by Ninomiya.

To obtain sufficient conditions we prepare two approximation theorems in Chapter II. The first is due to Bourbaki [2] and the second is new so far as the author knows.

In Chapter III we give sufficient conditions. First we introduce three lower envelope principles; the strong, compact and ordinary lower envelope principles, and we obtain the relations among these principles, the domination (or balayage) and equilibrium principles. Deny [6] proved the equivalence between the strong lower envelope principle and the domination principle for his distribution-kernels of positive type, but for our kernels the equivalence fails to hold. If a kernel satisfies the domination principle, then it satisfies the compact lower envelope principle, but not necessarily the strong lower envelope principle, and if a kernel of positive type satisfies the compact lower envelope principle, then it satisfies the domination principle. A sufficient but not necessary condition will be obtained for balayable kernels to satisfy the strong lower envelope principle. Other relations between the lower envelope principle and the equilibrium principle will be added.

Next we define "regularity" of kernels. This is an auxiliary notion by which the approximation theorems mentioned above are available.

In the following section we prove that in order that a regular balayable kernel satisfies the U- and BU-principles, "non-degeneracy" is sufficient. From this follows immediately the sufficiency of Ninomiya's condition. The condition "potentials separate points" is not sufficient in general, but for kernels satisfying both the balayage and equilibrium principles it is sufficient. As to regular composition kernels satisfying the balayage principle it is shown that the conditions considered in Chapter I are sufficient. In the last section we show that the condition "potentials separate points" is sufficient for the EU-principle as to kernels satisfying the lower envelope principle.

Definitions and known results

We shall be concerned with a locally compact space X and positive measures μ on X with *compact support* S_μ. A real-valued continuous function $K(x, y)$ defined on the product space $X \times X$ is called a *kernel* when it satisfies the following conditions: (i) $0 < K(x, y) \leq +\infty$ and (ii) $K(x, y)$ is finite except for at most the diagonal set of $X \times X$. The kernel $\check{K}$ defined by $\check{K}(x, y) = K(y, x)$ is called the *adjoint kernel*. When $K \equiv \check{K}$, K is called the *symmetric kernel*. Given a positive measure μ, we define the $(K\text{-})$*potential* $K\mu(x)$ by the integral

$$K\mu(x) = \int K(x, y)\, d\mu(y)$$

and the adjoint potential $\check{K}\mu(x)$ by

$$\check{K}\mu(x) = \int \check{K}(x, y)\, d\mu(y) = \int K(y, x)\, d\mu(y)\,.$$

By Fubini's theorem it holds that

$$\int K\mu(x)\, d\nu(x) = \int \check{K}\nu(x)\, d\mu(x) \quad \text{for any positive measures } \mu \text{ and } \nu.$$

The integral $\int K\mu(x)\, d\mu(x)$ is called the *energy* of μ.

A Borel set $B \subset X$ is, by definition, *of positive capacity*, if it contains a compact set F which supports at least a positive measure $\mu \neq 0$ with finite energy; otherwise B is *of capacity zero*. We say that a property holds *nearly everywhere* on a Borel set B when the set of points of B at which the property fails to hold is a Borel set of capacity zero. It is easily verified that if a property holds nearly everywhere on Borel sets B_n $(n = 1, 2, 3, \cdots)$, then it holds nearly everywhere on $\bigcup_{n=1}^{\infty} B_n$.

Now we define potential theoretical principles which will be considered in this paper.

THE CONTINUITY PRINCIPLE. If $K\mu(x)$ is finite and continuous on the support S_μ, then it is continuous in the whole space.

THE EQUILIBRIUM PRINCIPLE. For any compact set F, there exists a positive measure μ supported by F such that $K\mu(x)=1$ nearly everywhere on F, and $K\mu(x)\leq 1$ everywhere in X. The measure μ is called the *equilibrium measure* of F and $K\mu(x)$ the *equilibrium potential*.

THE BALAYAGE (SWEEPING-OUT) PRINCIPLE. For any compact set F and for any positive measure μ, there exists a positive measure ν supported by F such that $K\nu(x)=K\mu(x)$ nearly everywhere on F and $K\nu(x)\leq K\mu(x)$ everywhere in X. The measure ν is called the *balayaged measure* of μ to F and $K\nu(x)$ the *balayaged potential*. A kernel satisfying the balayage principle is called *balayable*.

FROSTMAN'S MAXIMUM PRINCIPLE. If $K\mu(x)\leq 1$ on the support S_μ, then the same inequality holds everywhere in X.

THE DOMINATION PRINCIPLE (CARTAN'S MAXIMUM PRINCIPLE). If a potential $K\mu(x)$ of a positive measure with finite energy is dominated by a potential $K\nu(x)$ on the support S_μ, then $K\mu(x)$ is also dominated by a potential $K\nu(x)$ in the whole space.

THE ENERGY PRINCIPLE. This principle is considered only for symmetric kernels K. For any distinct positive measures μ and ν with finite energy the expression

$$(1) \qquad \int K\mu(x)\,d\mu(x)-2\int K\mu(x)\,d\nu(x)+\int K\nu(x)\,d\nu(x)$$

is finitely determined and >0. When the expression (1) is non-negative, the symmetric kernel K is said to be *of positive type*.

THE U-PRINCIPLE. If potentials of positive measure coincide with each other nearly everywhere in the whole space, then the measures are identical.

THE EU-PRINCIPLE. The equilibrium measure is uniquely determined by a given compact set.

THE BU-PRINCIPLE. The balayaged measure is uniquely determined by a given positive measure and a compact set.

Throughout this paper we assume that every open set is of positive capacity, that is, for any open set G, there exists a positive measure $\mu\neq 0$ such that the support S_μ is contained in G and the energy $\int K\mu\,d\mu$ is finite.

Concerning symmetric kernels Ninomiya [11] obtained the following theorems which we use throughout this paper.

Theorem A. *If K satisfies Frostman's maximum principle or the domination principle, then it is of positive type.*

Theorem B. *A kernel K satisfies Frostman's maximum principle if and only if it satisfies the equilibrium principle.*

Theorem B′. *If K satisfies the equilibrium principle, then the equilibrium potentials of a compact set coincide with each other nearly everywhere in X.*

Theorem C. *A kernel K satisfies the domination principle if and only if it satisfies the balayage principle.*

Theorem C′. *If K satisfies the balayage principle, then the balayaged potentials of a positive measure to a compact set coincide with each other nearly everywhere in X.*

Theorem D. *In order that K satisfies Frostman's maximum principle, it is necessary and sufficient that it has the property*

[P_1]. *Let x_0 be an arbitrarily fixed point and let λ be a positive measure with compact support S_λ which does not contain x_0. If $K\lambda(x) \leq K(x_0, x)$ on S_λ, then the total measure of λ is ≤ 1.*

Therem E. *In order that K satisfies the domination principle, it is necessary and sufficient that it has the property*

[P_2]. *Let x_0 be an arbitrarily fixed point and let λ be a positive measure with compact support S_λ which does not contain x_0. If $K\lambda(x) \leq K(x_0, x)$ on S_λ, then the same inequality holds in X.*

The following is also known.

Theorem F. *If K satisfies Frostman's maximum principle on the domination principle, then it satisfies the continuity principle.*

VAGUE TOPOLOGY. Let $\mathfrak{C}_0(X)$ be the totality of continuous real-valued functions with compact support in X. The vague topology is defined on the space of positive measures by the semi-norm

$\mu - \nu \rightarrow \left| \int f\,d\mu - \int f\,d\nu \right|$, $f \in \mathfrak{C}_0(X)$.

The following theorems are very important in the potential theory.

Theorem G (BOURBAKI [3]). *Let H be a subset of the space of positive measures. If, for every compact set $F \subset X$,* $\sup_{\mu \in H} \mu(F) < +\infty$, *then H is relatively compact with respect to the vague topology.*

Theorem H (OHTSUKA [12]). *Let a symmetric kernel K satisfy the continuity principle. If a subnet $T = \{\mu_\omega\,;\, \omega \in D, D$ is a directed set$\}$ of a sequence of positive measures, supported by a fixed compact set, converges vaguely to a positive measure μ_0, then*

$$\varliminf_{\omega} K\mu_\omega(x) \geq K\mu_0(x) \quad \textit{everywhere in } X \textit{ and}$$

$$\varliminf_{\omega} K\mu_\omega(x) = K\mu_0(x) \quad \textit{nearly everywhere in } X.$$

CHAPTER I. NECESSARY CONDITIONS

§ 1. Non-degnerate kernels

Throughout this paper we are concerned with positive symmetric kernels and we assume that every open set is of positive capacity. By Theorems B' and C' we can easily verify that *if a kernel K satisfies the* U-*principle and the equilibrium* (resp. *balayage*) *principle, then it satisfies the* EU- (resp. BU-) *principle.*

We shall say that a kernel $K(x, y)$ is *degenerate* if there exist distinct points x_1 and x_2 such that

$$\frac{K(x_1, x)}{K(x_2, x)} \equiv \text{constant in } X.$$

When K is not degenerate, it is called *non-degenerate.* With this terminology we give a necessary condition for the unicity principles.

Theorem 1.1. *In order that a kernel K satisfies one of the* U-, BU- *and* EU-*principles, it is necessary that it is non-degenerate.*

Proof. Suppose that K is degenerate, i.e., there exist distinct points x_1 and x_2 and a positive constant α such that

$$K(x_1, x) = \alpha \cdot K(x_2, x) \qquad \text{in } X. \tag{2}$$

Then K does not satisfy the U-principle, since the potentials of distinct measures ε_{x_1} and $\alpha \cdot \varepsilon_{x_2}$ are identical in X by (2), where ε_x denotes a point measure at x with total measure 1. We shall show that K does not satisfy the BU- nor EU-principles.

Let K be a balayable kernel. We balayage ε_{x_1} onto the compact set $F = \{x_1, x_2\}$. Evidently ε_{x_1} itself is a balayaged measure. By (2) it is seen that a measure $\alpha \cdot \varepsilon_{x_2}$ is also a balayaged measure. Hence K does not satisfy the BU-principle.

Next let K satisfy the equilibrium principle. Then $K(x, x_i) = K(x_i, x) \leq K(x_i, x_i)$ in X $(i=1, 2)$ and by (2) $K(x_1, x) = K(x_2, x)$ in X. Therefore $\mu_i = \dfrac{1}{K(x_i, x_i)} \cdot \varepsilon_{x_i}$ $(i=1, 2)$ are distinct equilibrium measures of F and hence K does not satisfy the EU-principle.

We shall say that *K-potentials separate points* when for any distinct points x_1 and x_2 in X, there exists a potentials $K\mu$ of a positive measure μ with finite energy such that $K\mu(x_1) \neq K\mu(x_2)$.

Theorem 1.2. *In order that K-potentials separate points it is necessary and sufficient that for any distinct points x_1 and x_2, $K(x_1, x) \not\equiv K(x_2, x)$ in X.*

Proof. Necessity is obvious. We shall show sufficiency. Suppose that K-potentials do not separate distinct points x_1 and x_2. Let G be an open set in X. Then the continuous function $K(x_1, x)-K(x_2, x)$ must vanish at some point x' in G, otherwise it contradicts our assumption $K\mu(x_1)=K\mu(x_2)$ for any positive measure μ with finite energy. Hence there exists a point x' in G such that $K(x_1, x')=K(x_2, x')$. Consequently, taking the base of fundamental neighborhoods of an arbitrarily fixed point x_0, we obtain

$$K(x_1, x_0) = K(x_2, x_0).$$

This completes the proof.

Corollary 1. *If K is non-degenerate, then K-potentials separate points.*

Corollary 2. *In order that a kernel K satisfies one of the* U-, BU- *and* EU-*principles, it is necessary that K-potentials separate points.*

REMARK. The converse of Corollary 1 is not valid in general. In § 3, it will be shown that the converse is true for composition kernels.

§ 2. Ninomiya's necessary condition

Consider the following condition [S], which was obtained by Ninomiya in his thesis [11].

[S] Let x_0 be an arbitrarily fixed point and let λ be a positive measure with compact support S_λ which does not contain x_0. If $K\lambda(x) \leq K(x_0, x)$ on S_λ, then $K\lambda(x) < K(x_0, x)$ in some neighborhood of x_0.

Ninomiya proved that Condition [S] is necessary for the U-principle, and it is sufficient under additional conditions. In this section we shall show that it is necessary for the BU- and EU-principles.

First we remark that *if K satisfies Condition* [S], *then it is non-degenerate.* In fact, suppose that K is degenerate, that is, there exist distinct points x_1 and x_2 and a positive constant α such that

$$K(x_1, x) = \alpha \cdot K(x_2, x) \qquad \text{in } X.$$

Put $\lambda=\alpha\cdot\varepsilon_{x_2}$. Then, on the support of λ, $K\lambda(x_2)=K(x_1, x_2)$, and at $x=x_1$, $K\lambda(x_1)=K(x_1, x_1)$. Therefore Condition [$S$] is not satisfied.

The converse of the above assertion is not valid, in general. For example, let X be a compact space consisting of three points, x_1, x_2 and x_3, and let a kernel K be given by a matrix

$$\begin{pmatrix} 1, & 1, & 1 \\ 1, & \frac{3}{2}, & \frac{1}{2} \\ 1, & \frac{1}{2}, & \frac{3}{2} \end{pmatrix},$$

that is, let $K(x_i, x_j)$ $(i, j=1, 2, 3)$ be given by the element which is in the i-th row and j-th column. Then K is a symmetric non-degenerate kernel, but Condition [S] is not satisfied. It is easily verified that K satisfies the equilibrium principle. In § 3, it will be remarked that if a composition kernel K is non-degenerate and satisfies the equilibrium principle or the balayage principle, then it satisfies Condition [S].

Now we shall prove the necessity of [S].

Theorem 1.3. *In order that a kernel K satisfies the* U- *or* BU-*principle, it is necessary that it satisfies Condition* [S].

Proof. What we have to show is that [S] is necessary for the BU-principle, since the necessity of [S] for the U-principle was proved in Ninomiya's thesis [11]. Suppose that K satisfies the BU-principle and Condition [S] is not satisfied. Then there exist a point x_0 and a positive measure λ with compact support S_λ which does not contain x_0 such that

$$K\lambda(x) \leq K(x_0, x) \qquad \text{on } S_\lambda \text{ and}$$
$$K\lambda(x_0) \geq K(x_0, x_0).$$

Applying Theorem E to the first inequality, we have $K\lambda(x) \leq K(x_0, x)$ in X and hence

$$K\lambda(x_0) = K(x_0, x_0).$$

We show that $K\lambda(x) = K(x_0, x)$ nearly everywhere on S_λ. On the contrary suppose that a compact set

$$F = \{x \in S_\lambda;\ K\lambda(x) \leq K(x_0, x) - \delta_0,\ \delta_0 > 0\}$$

is of positive capacity. Then there exists a positive measure ν supported by F such that $\int d\nu = 1$ and $\int K\nu d\nu$ is finite. We balayage this measure ν to the compact set $\{x_0\}$, which is of positive capacity since $K(x_0, x_0)$ is finite, and we obtain a balayaged measure $\alpha \cdot \varepsilon_{x_0}$ $(\alpha > 0)$ such that

$$\alpha \cdot K\varepsilon_{x_0}(x) \leq K\nu(x) \qquad \text{in } X \text{ and}$$
$$\alpha \cdot K\varepsilon_{x_0}(x_0) = K\nu(x_0).$$

Then

$$\begin{aligned}\alpha\cdot K(x_0, x_0) &= \alpha\cdot K\lambda(x_0) = \alpha\cdot\int K(x_0, x)d\lambda(x)\\ &= \alpha\cdot\int K\varepsilon_{x_0}(x)d\lambda(x)\leq\int K\nu(x)d\lambda(x)\\ &= \int K\lambda(x)d\nu(x)\leq\int\{K(x_0, x)-\delta_0\}d\nu(x)\\ &= \int K(x_0, x)d\nu(x)-\delta_0 = K\nu(x_0)-\delta_0\\ &= \alpha\cdot K\varepsilon_{x_0}(x_0)-\delta_0 = \alpha\cdot K(x_0, x_0)-\delta_0\,,\end{aligned}$$

which is impossible. Thus we have seen that $K\lambda(x)=K(x_0, x)$ nearly everywhere on S_λ. Consequently λ is a balayaged measure of ε_{x_0} to the compact set $F'=S_\lambda\cup\{x_0\}$. This contradicts our assumption that K satisfies the BU-principle, since ε_{x_0} itself is a balayaged measure to F'.

Theorem 1.4. *In order that a kernel K satisfies the* EU-*principle, it is necessary that Condition* [*S*] *is satisfied.*

Proof. Suppose that K satisfies the equilibrium principle and that Condition [S] is not satisfied, that is, there exist a point x_0 and a positive measure λ with compact support $S_\lambda\not\ni x_0$ such that

$$K\lambda(x)\leq K(x_0, x)\quad\text{on } S_\lambda \text{ and }\quad K\lambda(x_0)\geq K(x_0, x_0)\,.$$

Since K satisfies the equilibrium principle, $K(x_0, x)\leq K(x_0, x_0)$ at every point x in X, and from Theorem D it follows that $\int d\lambda\leq 1$. Therefore

$$K(x_0, x_0)\leq K\lambda(x_0) = \int K(x_0, x)d\lambda(x)\leq K(x_0, x_0)\cdot\int d\lambda\leq K(x_0, x_0)$$

and hence

$$K(x_0, x) = K(x_0, x_0)\quad\text{on } S_\lambda \text{ and }\quad \int d\lambda = 1\,.$$

Consequently the measure $\dfrac{1}{K(x_0, x_0)}\cdot\varepsilon_{x_0}$ is an equilibrium measure of a compact set $F'=S_\lambda\cup\{x_0\}$.

On the other hand we minimize $\int K\mu(x)d\mu(x)$ among positive measures μ which are supported by S_λ and of total measure 1. Since S_λ supports λ with finite energy, there exists a minimizing measure μ_0. It is well known (see, for example, Frostman [7]) that the potential $K\mu_0$ of μ_0 has the following properties:

$K\mu_0(x)\geq$ a positive constant α nearly everywhere on S_λ and
$K\mu_0(x)\leq\alpha$ on S_{μ_0}.

Then by Frostman's maximum principle (K satisfies the maximum principle by Theorem B),

$$K\mu_0(x) = \alpha \quad \text{nearly everywhere on } S_\lambda \text{ and}$$
$$K\mu_0(x) \leq \alpha \quad \text{in } X.$$

We put $\nu_0 = \frac{1}{\alpha}\mu_0$. Then ν_0 is an equilibrium measure of S_λ and $\int d\nu_0 = \frac{1}{\alpha}$. Since λ is a competing measure, we have

$$\alpha = \int K\mu_0(x)\, d\mu_0(x) \leq \int K\lambda(x)\, d\lambda(x) \leq \int K(x_0, x)\, d\lambda(x)$$
$$= K\lambda(x_0) = K(x_0, x_0),$$

and hence $1 \leq \frac{1}{\alpha} \cdot K(x_0, x_0)$. Therefore we have

$$K\nu_0(x_0) = \int K(x_0, x)\, d\nu_0(x) = K(x_0, x_0) \cdot \int d\nu_0 = \frac{1}{\alpha} K(x_0, x_0) \geq 1\,.$$

Consequently $K\nu_0(x_0) = 1$ and ν_0 is an equilibrium measure of F'. Thus K does not satisfy the EU-principle.

§ 3. Composition kernels

In this section we deal with composition kernels K and investigate equivalent expressions of "non-degeneracy". Let X be a locally compact abelian group and let $k(x)$ be a positive symmetric function defined on X which is finite and continuous at every point $x \neq 0$ and $k(0) = \lim k(x) \leq +\infty$. We set $K(x, y) = k(x-y)$. The kernels K thus defined are called *composition kernels*. Note that if $k(x)$ is locally summable with respect to Haar measure μ in X, then every set is of positive capacity. In fact, let G be an open neighborhood of the origin 0, then there exists an open neighborhood G_1 of 0 such that $\bar{G}_1 \subset G$, $\bar{G}_1$ is compact and $\int_{\bar{G}_1} k(x)\, d\mu(x)$ is finite. We can take an open neighborhood G_2 of 0 such that for any x, y in $\bar{G}_2$, $x-y$ is contained in G_1. Then the potential $K\mu'(x)$ of the restriction of μ to $\bar{G}_2$ is bounded on $\bar{G}_2$, since

$$K\mu'(x) = \int_{\bar{G}_2} k(x-y)\, d\mu(y) \leq \int_{\bar{G}_1} k(z)\, d\mu(z) \qquad \text{for any } x \in \bar{G}_2\,.$$

Therefore $\bar{G}_2$ supports a positive measure with finite energy and hence G is of positive capacity.

Theorem 1.5. *For a composition kernel K, the following three statements are equivalent*:

(1) *K is non-degenerate,*

(2) *K-potentials separate points,*

(3) *$k(x)$ is not periodic.*

Proof. The implication (1)$\Rightarrow$(2) follows from Theorem 1.2. The implication (2)$\Rightarrow$(3) is immediate. We shall prove the implication (3)$\Rightarrow$(1), that is, if a composition kernel K is degenerate, then there exists a point $x_0 \neq 0$ such that $k(x)=k(x-x_0)$ in X. Suppose that there exist distinct points x_1 and x_2 and a positive constant α such that $K(x_1, x)=\alpha \cdot K(x_2, x)$, namely, $k(x_1-x)=\alpha \cdot k(x_2-x)$ in X. Then

$$\begin{aligned} k(x) = k(x_1-(x_1-x)) &= \alpha \cdot k(x_2-(x_1-x)) \\ &= \alpha \cdot k(x-(x_1-x_2)) . \end{aligned}$$

Putting $x_0=x_1-x_2 \neq 0$, we have $k(x)=\alpha \cdot k(x-x_0)$ in X and particularly $k(0)=\alpha \cdot k(x_0)$ and $k(x_0)=\alpha \cdot k(0)$, and hence $\alpha=1$. Therefore

$$k(x) = k(x-x_0) \qquad \text{in } X .$$

Corollary. *For a composition kernel K, each of the three statements in Theorem 1.5 is a necessary condition in order that K satisfies one of the* U-, BU- *and* EU-*principles.*

Now we state another equivalent expression of "non-degeneracy" for a composition kernel satisfying the balayage or equilibrium principle.

Theorem 1.6. *Let K be a balayable composition kernel. It is non-degenerate if and only if $k(0)>k(x)$ at every point $x \neq 0$.*

Proof. First we remark that $k(0) \geq k(x)$ everywhere in X. In fact, let x_0 be an arbitrarily fixed point. We may assume that $k(0)$ is finite. Then the compact set $F=\{0\}$ is of positive capacity. Since K satisfies the balayage principle, we can balayage ε_{x_0} onto F and we obtain a balayaged measure $\alpha \cdot \varepsilon_0$ and

$$\alpha \cdot K\varepsilon_0(0) = K\varepsilon_{x_0}(0) \quad \text{and} \quad \alpha \cdot K\varepsilon_0(x) \leq K\varepsilon_{x_0}(x) \quad \text{everywhere in } X ,$$

namely,

$$\alpha \cdot k(0) = k(x_0) \quad \text{and} \quad \alpha \cdot k(x) \leq k(x-x_0) \quad \text{everywhere in } X .$$

From these inequalities it is immediately seen that $\alpha \leq 1$ and $k(0) \geq k(x_0)$. Consequently $k(0) \geq k(x)$ everywhere in X.

Now we prove our theorem. Suppose that $k(0)>k(x)$ at every point $x \neq 0$. Then $k(x)$ is not periodic and hence it is non-degenerate by Theorem 1.5.

Next, suppose that $k(0)=k(x_0)$ at some point $x_0 \neq 0$. We shall prove that $k(x)=k(x-x_0)$ and $k(x)$ is periodic. By the same argument as in the above remark we obtain

$$k(x) \leq k(x-x_0) \quad \text{and} \quad k(x) \leq k(x+x_0) \quad \text{everywhere in } X.$$

Into the second inequality we insert $x=y-x_0$ to obtain $k(y-x_0) \leq k(y)$. Consequently $k(x)=k(x-x_0)$ everywhere in X.

Corollary. *If a balayable composition kernel K satisfies one of the* U-, BU- *and* EU-*principles, then $k(0)>k(x)$ at every point $x \neq 0$.*

Theorem 1.7. *Let K be a composition kernel satisfying the equilibrium principle. It is non-degenerate if and only if $k(0)>k(x)$ at every point $x \neq 0$.*

Proof. It is evident that $k(0) \geq k(x)$ in X, since K satisfies the equilibrium principle. Same as in the proof of the preceding theorem it is sufficient to prove that if K is non-degenerate, then $k(0)>k(x)$ at every point $x \neq 0$. Suppose that there exists a point $x_1 \neq 0$ such that $k(0) \leq k(x_1)$, hence $k(0)=k(x_1)$ by the above remark. We shall show that $k(x)$ is periodic, $k(x)=k(x+x_1)$. Without loss of generality we may suppose that $k(0)=k(x_1)=1$ and that at some point x_2, $k(x_2)=a<1$. Consider an equilibrium measure $\mu=\frac{1}{1+a}(\varepsilon_0+\varepsilon_{x_2})$ of a compact set $F=\{0, x_2\}$. Then at every point x, $K\mu(x) \leq 1$ and hence

$$k(x)+k(x-x_2) \leq 1+a\,.$$

Here we put $x=x_1+x_2$ and $x=-x_1+x_2$, and we obtain

$$k(x_1+x_2) \leq k(x_2) \quad \text{and} \quad k(-x_1+x_2) \leq k(x_2)\,.$$

Hence $k(x_1+x_2)<1$ and we can repeat the above argument for x_1+x_2 instead of x_2 and we obtain

$$k(x_2) = k(-x_1+(x_1+x_2)) \leq k(x_1+x_2)\,.$$

Consequently we have $k(x_2)=k(x_1+x_2)$ for any point x_2 such that $k(x_2)<1$. This equality holds even if $k(x_2)=1$. In fact, if $k(x_1+x_2)<1$, then

$$1 > k(x_1+x_2) = k(-x_1-x_2) = k(x_1-x_1-x_2) = k(-x_2) = k(x_2) = 1\,,$$

which is impossible. Thus we have shown that $k(x)=k(x+x_1)$ in X, that is, $k(x)$ is periodic and hence K is degenerate by Theorem 1.5.

Corollary. *If a composition kernel K satisfies the equilibrium principle* *and one of the* U-, BU- *and* EU-*principles, then* $k(0)>k(x)$ *at every point* $x\neq 0$.

Closing this chapter, we state two remarks.

REMARK 1. *If a composition kernel* K *is non-degenerate and satisfies the equilibrium principle, then it satisfies Condition* [S]. In fact, if K is non-degenerate, then $k(0)>k(x)$ at every point $x\neq 0$ by Theorem 1.7. Suppose that λ is a positive measure with compact support S_λ which does not contain the origin, and that $K\lambda(x)\leq k(x)$ on S_λ. Then by Theorem D, the total measure of λ is ≤ 1 and

$$K\lambda(0)=\int k(x)\,d\lambda(x)<k(0)\cdot\int d\lambda\leq k(0)\,.$$

REMARK 2. Analogous statement as above is valid if a composition kernel satisfies the balayage principle. The proof will be given later.

CHAPTER II. APPROXIMATION THEOREMS

In this chapter we prepare approximation theorems of continuous functions in order to obtain sufficient conditions for the U-, BU- and EU-principles.

Let F be a compact space and $\mathfrak{C}(F)$ be a family consisting of all real-valued finite continuous functions on F. Let $\mathfrak{D}$ be a subfamily of $\mathfrak{C}(F)$. We inquire sufficient conditions in order that $\mathfrak{D}$ is dense in $\mathfrak{C}(F)$ with respect to the uniform convergence topology on F. Here we require, of course, that those sufficient conditions imposed on $\mathfrak{D}$ should be applicable to a family of potentials.

The theorem of Weierstrass and Stone is useful in case that kernels satisfy the domination principle.

Theorem 2.1. (BOURBAKI [2], p. 53) *Suppose that* $\mathfrak{D}$ *has the following properties*:

(a) *for any functions* f_1 *and* f_2 *of* $\mathfrak{D}$, *the functions* $(f_1\vee f_2)(x)=\max(f_1(x),f_2(x))$ *and* $(f_1\wedge f_2)(x)=\min(f_1(x),f_2(x))$ *belong to* $\mathfrak{D}$,

(b) *for any distinct points* x_1 *and* x_2 *in* F *and for any pair of real numbers* a_1 *and* a_2, *there exists a function* f *in* $\mathfrak{D}$ *such that* $f(x_i)=a_i$ $(i=1,2)$.

Then $\mathfrak{D}$ *is dense in* $\mathfrak{C}(F)$ *with respect to the uniform convergence topology on* F.

In the next chapter we apply this theorem to deduce sufficient conditions for the U- and BU-principles.

In case that kernels satisfy the equilibrium principle, we are permitted to assume that $\mathfrak{D}$ contains constant-functions and we obtain, from the above theorem,

Corollary 1. *Suppose that $\mathfrak{D}$ is a vector subspace of $\mathfrak{C}(F)$ such that*

(1) *$\mathfrak{D}$ contains constant-functions,*

(2) *for any functions f_1 and f_2 of $\mathfrak{D}$, the functions $f_1 \vee f_2$ and $f_1 \wedge f_2$ belong to $\mathfrak{D}$,*

(3) *for any distinct points x_1 and x_2 in F, there exists a function f in $\mathfrak{D}$ such that $f(x_1) \neq f(x_2)$.*

Then $\mathfrak{D}$ is dense in $\mathfrak{C}(F)$ with respect to the uniform convergence topology on F.

Proof. What we have to show is that our vector subspace $\mathfrak{D}$ has the property (b) in Theorem 2.1. Let x_1 and x_2 be distinct points in F and let a_1 and a_2 be real numbers. By the property (3) there exists a function $f \in \mathfrak{D}$ such that $f(x_1) \neq f(x_2)$. Since constant-functions belong to $\mathfrak{D}$, the function

$$g(x) = a_1 + (a_2 - a_1)\frac{f(x) - f(x_1)}{f(x_2) - f(x_1)}$$

belongs to $\mathfrak{D}$, and $g(x_i) = a_i$ $(i = 1, 2)$.

Now we denote by $\mathfrak{C}^+(F)$ the family consisting of all non-negative finite continuous functions of F. Then $\mathfrak{C}^+(F)$ is a half vector space, that is, if $f_i \in \mathfrak{C}^+(F)$ $(i=1,2)$, then $f_1 + f_2 \in \mathfrak{C}^+(F)$ and if $f \in \mathfrak{C}^+(F)$ and $a \geq 0$, then $a \cdot f \in \mathfrak{C}^+(F)$. Let $\mathfrak{D}^+$ be a subfamily of $\mathfrak{C}^+(F)$. Then Corollary 1 can be expressed as follows:

Corollary 2. *Suppose that $\mathfrak{D}^+$ is a half vector subspace of $\mathfrak{C}^+(F)$ such that*

(1′) *$\mathfrak{D}^+$ contains positive constant-functions,*

(2′) *for any functions f_1 and f_2 of $\mathfrak{D}^+$, the function $f_1 \wedge f_2$ belongs to $\mathfrak{D}^+$,*

(3′) *for any distinct points x_1 and x_2 in F, there exists a function f in $\mathfrak{D}^+$ such that $f(x_1) \neq f(x_2)$.*

Put $\mathfrak{D} = \{f\,;\, f = f_1 - f_2$ with $f_i \in \mathfrak{D}^+$ $(i=1,2)\}$. Then $\mathfrak{D}$ is dense in $\mathfrak{C}(F)$ with respect to the uniform convergence topology on F.

Proof. Evidently the vector subspace $\mathfrak{D}$ has the properties (1) and (3) of Corollary 1. We shall verify the property (2). Let

$$f_i = g_i - h_i \quad \text{with } g_i \text{ and } h_i \in \mathfrak{D}^+ \ (i=1,2).$$

Then

$$f_1 \wedge f_2 = (g_1 - h_1) \wedge (g_2 - h_2) = (g_1 + h_2) \wedge (g_2 + h_1) - (h_1 + h_2)$$

and $f_1 \wedge f_2$ belongs to $\mathfrak{D}$, since $(g_1+h_2) \wedge (g_2+h_1)$ belongs to $\mathfrak{D}^+$ by assumption (2'). By the identity $f_1 \vee f_2 = -\{(-f_1) \wedge (-f_2)\}$, the upper envelope $f_1 \vee f_2$ belongs to $\mathfrak{D}$, too. Consequently, by Corollary 1, $\mathfrak{D}$ is dense in $\mathfrak{C}(F)$ with respect to the uniform convergence topology.

Now we replace the conditions in Corollary 2 by weaker ones.

Theorem 2.2. *Suppose that $\mathfrak{D}^+$ is a half vector space of $\mathfrak{C}^+(F)$ such that*

(i) *if f belongs to $\mathfrak{D}^+$ and $0 \leq f \leq 1$, then $1-f$ belongs to $\mathfrak{D}^+$,*

(ii) *for any function f of $\mathfrak{D}^+$, the function $f \wedge 1$ belongs to $\mathfrak{D}^+$,*

(iii) *for any distinct points x_1 and x_2 in F, there exists a function f in $\mathfrak{D}^+$ such that $f(x_1) \neq f(x_2)$.*

Then $\mathfrak{D}^+$ is dense in $\mathfrak{C}^+(F)$ with respect to the uniform convergence topology on F.

This theorem follows from the following three lemmas.

Lemma 2.1. *Under the same assumptions in Theorem 2.2, there exists a function f in $\mathfrak{D}^+$, for any given distinct points x_1 and x_2 in F, such that $f(x_1)=0$, $f(x_2)=1$ and $0 \leq f \leq 1$.*

Proof. By the condition (iii), there exists an $f_1 \in \mathfrak{D}^+$ which takes distinct values at x_1 and x_2. In case that $f_1(x_1) > f_1(x_2)$, we multiply f_1 by a suitable positive number α so that we obtain $f_2 = \alpha \cdot f_1$ of $\mathfrak{D}^+$ such that $f_2(x_1) = 1 > f_2(x_2)$. We put $f_3 = 1-(f_2 \wedge 1)$. Then by conditions (ii) and (i), f_3 belongs to $\mathfrak{D}^+$ and $0 \leq f_3 \leq 1$, $f_3(x_1)=0$ and $1 \geq f_3(x_2) > 0$. Multiplying f_3 by a suitable positive number β, we have a function $f_4 = \beta \cdot f_3$ of $\mathfrak{D}^+$ such that $f_4(x_1)=0$ and $f_4(x_2)=1$. Then $f=f_4 \wedge 1$ is a function of $\mathfrak{D}^+$ and $0 \leq f \leq 1$, $f(x_1)=0$ and $f(x_2)=1$.

In case that $f_1(x_1) < f_1(x_2)$, we obtain, by the above argument, a function g of $\mathfrak{D}^+$ such that $0 \leq g \leq 1$, $g(x_1)=1$ and $g(x_2)=0$. Consequently $f=1-g$ is a required function.

Lemma 2.2. *Suppose that a half vector space $\mathfrak{D}^+$ has the properties in Theorem 2.2. Then for any closed subset A of F and for any point x_0 in the complement of A, there exists a function $g \in \mathfrak{D}^+$ such that $0 \leq g \leq 1$, $g(x_0)=1$ and $g(x)=0$ everywhere in A.*

Proof. Let y be an arbitrarily fixed point of A. Then by Lemma 2.1, there exists a function f_y of $\mathfrak{D}^+$ such that $0 \leq f_y \leq 1$, $f_y(x_0)=0$ and $f_y(y)=1$. Since f_y is continuous, there is a neighborhood $U(y)$ of y such that $f_y > 1/2$ everywhere in $U(y)$. Then, A being compact, a finite family $\{U(y_i)\}$, $1 \leq i \leq n$, covers A. We put

$$h = 2\left(\sum_{i=1}^{n} f_{y_i}\right) \wedge 1 .$$

By the condition (ii), h belongs to $\mathfrak{D}^+$ and $0 \leq h \leq 1$, $h(x_0)=0$ and $h(x)=1$ everywhere in A. Consequently $g=1-h$ is a required function of $\mathfrak{D}^+$.

Lemma 2.3. *Suppose that a half vector space $\mathfrak{D}^+$ has the properties in Theorem 2.2. Then for any disjoint closed subsets A and B, there exists an $h \in \mathfrak{D}^+$ such that $0 \leq h \leq 1$, $h=1$ on A and $h=0$ on B.*

Proof. Let y be an arbitrarily fixed point in A. Then by Lemma 2.2, there exists $g_y \in \mathfrak{D}^+$ such that $0 \leq g_y \leq 1$, $g_y(y)=1$ and $g_y=0$ on B. By the continuity of g_y, there is a neighborhood $U(y)$ of y, at every point of which $g_y > 1/2$. We can cover A by a finite family of these neighborhoods: $\bigcup_{i=1}^{m} U(y_i) \supset A$. Put $g=\sum_{i=1}^{m} g_{y_i}$. Then g belongs to $\mathfrak{D}^+$ and it vanishes on B and it is greater than $1/2$ on A. Consequently $h=2g \wedge 1$ satisfies the conditions required.

Now we prove Theorem 2.2.

Let φ be a positive finite continuous function on F. We prove that a sequence $\{f_n\}$ of functions of $\mathfrak{D}^+$ converges uniformly to φ. Without loss of generality we may suppose that $0 \leq \varphi \leq 1$. We put for each k, $1 \leq k \leq n$,

$$A_k = \{x\,;\ \varphi(x) \geq k/n\}\,,$$
$$B_k = \{x\,;\ \varphi(x) \leq (k-1)/n\}\,.$$

Then A_k and B_k are disjoint closed sets. By Lemma 2.3, there exists a function $g_k \in \mathfrak{D}^+$ such that $0 \leq g_k \leq 1$, $g_k=1$ on A_k and $g_k=0$ on B_k. Putting

$$f_n = \frac{1}{n} \cdot \sum_{k=1}^{n} g_k\,.$$

we have $|\varphi - f_n| \leq 1/n$ everywhere on F. Consequently $\{f_n\}$ converges uniformly on F to φ.

CHAPTER III. SUFFICIENT CONDITIONS

§ 1. Lower envelope principles

DEFINITION. We say that a kernel K satisfies the *strong lower envelope principle*, when, for any positive measures μ and ν, one of which has finite energy, there exists a positive measure λ such that

$$K\lambda = K\mu \wedge K\nu \quad \text{nearly everywhere in } X.$$

DEFINITION. We say that a kernel K satisfies the *compact lower envelope principle*, when, for any compact set $F \subset X$ and for any positive measures μ and ν, one of which has finite energy, the lower envelope $K\mu \wedge K\nu$ coincides nearly everywhere on F with a potential $K\lambda$ of a positive measure λ supported by F.

DEFINITION. We say that a kernel K satisfies the *lower envelope principle*, when, for any compact set F and for any signed measure $\sigma = \mu_1 - \mu_2$ with μ_i $(i=1,2)$ of positive measures with finite energy, the lower envelope $K\sigma \wedge 1$ coincides nearly everywhere on F with a potential $K\tau$ of a signed measure $\tau = \nu_1 - \nu_2$, where ν_i $(i=1,2)$ are positive measures with finite energy.

The notion of the lower envelope principles was first introduced by Deny [6]. He proved the equivalence of the strong lower envelope principle and the domination principle for his distribution-kernels of positive type. The principle was also investigated by Choquet-Deny [4] for the kernels defined on a finite space under the name of the lower envelope principle.

For later use we investigate in this section the relations among the above principles and the domination principle.

First we state

Lemma 3.1. *Let K be a kernel of positive type, and let μ_1 and μ_2 be positive measures whose energy are finite. If the energy of $\mu_1 - \mu_2$ vanishes, i.e., $\|\mu_1 - \mu_2\|^2 = \int K\mu_1 d\mu_1 - 2\int K\mu_1 d\mu_2 + \int K\mu_2 d\mu_2 = 0$, then $K\mu_1 = K\mu_2$ nearly everywhere in X.*

Proof.[1] On the contrary suppose that there exists a compact set F of positive capacity on which $K\mu_1(x) > K\mu_2(x)$ and $\|\mu_1 - \mu_2\|^2 = 0$. The compact set F supports a positive measure λ with finite energy, by which we integrate $K\mu_1 - K\mu_2$ and, noting that K is of positive type, we obtain

$$0 < \int (K\mu_1 - K\mu_2) d\lambda \leq \|\mu_1 - \mu_2\| \cdot \|\lambda\| = 0\,,$$

which is a contradiction.

By this lemma we have

Theorem 3.1. *If a kernel of positive type satisfies the compact lower envelope principle, then it satisfies the domination principle.*

Proof. Suppose that K is of positive type and satisfies the compact lower envelope principle. We shall show that K satisfies the domination principle. Let x_0 be an arbitrarily fixed point and μ be a positive measure

1) This proof is due to Ohtsuka.

with compact support S_μ which does not contain x_0 and let $K\mu(x)\leq K(x_0, x)$ on S_μ. By Theorem E, it is sufficient to show that the same inequality holds everywhere in X. On the contrary suppose that there exists a point x_1 such that $K\mu(x_1)>K(x_0, x_1)$. Then by the lower semi-continuity of $K\mu$, there exists a neighborhood U of x_1 with compact closure such that $S_\mu \cap \bar{U}=\phi$ and

$$(3) \qquad K\mu(x)>K(x_0, x) \quad \text{in } U.$$

By the compact lower envelope principle, $K\mu(x) \wedge K\varepsilon_{x_0}(x)$ coincides nearly everywhere on $F=S_\mu \cup \bar{U}$ with a potential $K\nu$ of a positive measure ν supported by F.

$$(4) \qquad K\nu(x) = K\mu(x) \wedge K(x_0, x) \quad \text{nearly everywhere on } F.$$

Then, noting that K is of positive type and that μ and ν have finite energy, we obtain

$$\begin{aligned} 0 &\leq \int (K\mu - K\nu) d(\mu-\nu) = \int_{S_\mu} (K\mu - K\nu) d\mu - \int_F (K\mu - K\nu) d\nu \\ &= -\int_F (K\mu - K\nu) d\nu \leq 0, \end{aligned}$$

namely the energy of the signed measure $\mu-\nu$ vanishes. Consequently, by Lemma 3.1,

$$K\mu(x) = K\nu(x) \quad \text{nearly everywhere in } X.$$

Therefore by (4), $K\mu(x)\leq K(x_0, x)$ nearly everywhere in U. This contradicts (3), since every open set is of positive capacity. This completes the proof.

REMARK. There exists a kernel K which satisfies the strong or compact lower envelope principle but not the domination principle. This kernel is necessarily not of positive type. We quote the following example from [4], p. 131: let X be a finite space consisting of three points x_1, x_2 and x_3 and let K be given by the matrix

$$\begin{pmatrix} 1, & 2, & 4 \\ 2, & 4, & 1 \\ 4, & 1, & 2 \end{pmatrix}.$$

This kernel satisfies the strong lower envelope principle but not the domination principle.

Now suppose that K satisfies the domination principle. We shall examine whether K satisfies the strong lower envelope principle. The following is rather obvious.

Lemma 3.2. *If $K\mu_1$ and $K\mu_2$ are finite-valued continuous potentials, then for any compact F, the lower envelope $K\mu_1 \wedge K\mu_2$ coincides nearly everywhere on F with a potential $K\lambda$ of positive measure λ supported by F.*

Proof. We note that K satisfies the continuity principle, since it satisfies the domination principle (Theorem F). Put

$$f(x) = K\mu_1(x) \wedge K\mu_2(x) .$$

Then f is continuous in X. By Gauss' (or Ninomiya's) variational method, the existence of the following positive measure λ is shown:

(i) λ is supported by F.
(ii) $K\lambda(x) \geq f(x)$ nearly everywhere on F,
(iii) $K\lambda(x) \leq f(x)$ on S_λ.

We shall show that $K\lambda(x) = f(x)$ nearly everywhere on F. By (ii) $K\lambda(x) \leq K\mu_i(x)$ on S_λ $(i=1,2)$. Hence λ has finite energy and, by the domination principle, $K\lambda(x) \leq K\mu_i(x)$ in X $(i=1,2)$, and hence $K\lambda(x) \leq f(x)$ in X. Together with (ii), we obtain $K\lambda(x) = f(x)$ nearly everywhere on F. Thus the proof is completed.

By the same method we can prove

Lemma 3.3. *Let $K\mu_1$ be a continuous potential. Then for any potential $K\mu_2$ of a positive measure μ_2 and for any compact set F, the lower envelope $K\mu_1 \wedge K\mu_2$ coincides nearly everywhere on F with potential $K\lambda$ of a positive measure λ supported by F.*

Proof. Put $f = K\mu_1 \wedge K\mu_2$. For an arbitrary positive number N, put

$$K_N(x, y) = K(x, y) \wedge N$$

$$K_N\mu_2(x) = \int K_N(x, y)\, d\mu_2(y)$$

$$f_N(x) = K\mu_1(x) \wedge K_N\mu_2(x) .$$

Then $K_N\mu_2$ is continuous in X, and $K_N\mu_2(x) \uparrow K\mu_2(x)$ and $f_N(x) \uparrow f(x)$ (as $N \uparrow \infty$) at every point $x \in X$. By Gauss' (or Ninomiya's) variational method there exists a positive measure λ_N such that

(i) λ_N is supported by F,
(ii) $K\lambda_N \geq f_N$ nearly everywhere on F,
(iii) $K\lambda_N \leq f_N$ on S_{λ_N}.

By (iii) and the domination principle we have

(iii)′ $K\lambda_N \leq f$ in X.

First we note that the total measures $\int d\lambda_N$ $(N=1, 2, \cdots)$ are bounded, since

$$\int d\lambda_N \leq \frac{1}{\alpha} K\lambda_N(x) \leq \frac{1}{\alpha} f_N(x) \leq \frac{M}{\alpha} \quad (x \in S_{\lambda_N}),$$

where $\alpha = \inf_{F \times F} K(x, y) > 0$ and $M = \max_F K\mu_1(x)$. Therefore by Theorem G a subnet $T = \{\lambda_\omega ;\ \omega \in D\}$ of the sequence $\{\lambda_N\}$ converges vaguely to a positive measure λ supported by F. Then by Theorem H

$$K\lambda(x) \leq \underline{\lim}_{\omega}\, K\lambda_\omega(x) \quad \text{everywhere in } X \text{ and}$$
$$K\lambda(x) = \underline{\lim}_{\omega}\, K\lambda_\omega(x) \quad \text{nearly everywhere in } X.$$

Hence by (iii)′, $K\lambda(x) \leq f(x)$ everywhere in X and by (ii), $K\lambda(x) \geq f(x)$ nearly everywhere on F. Consequently $K\lambda(x) = f(x)$ nearly everywhere on F. This completes the proof.

The following lemma was proved in [9].

Lemma 3.4. *If K satisfies the continuity principle, then for any positive measure μ with finite energy, there exists a sequence $\{\mu_n\}$ of positive measures with the following properties*: 1° *$\{\mu_n\}$ converges vaguely to μ,* 2° *the potentials $K\mu_n$ are all finite-valued continous in X, and* 3° *$K\mu_n(x) \uparrow K\mu(x)$ at every point x of X.*

We call each of the sequence $\{\mu_n\}$ a *smoothed measure* of μ.

Now we prove

Theorem 3.2. *If K satisfies the domination principle, then it satisfies the compact lower envelope principle.*

Proof. Let F be a compact set, and let μ be a positive measure with finite energy and ν be a positive measure. We shall show that the lower envelope $f = K\mu \wedge K\mu$ coincides nearly everywhere on F with a potential $K\lambda$ of a positive measure λ supported by F. By Lemma 3.4, a sequence $\{\mu_n\}$ of smoothed measures of μ converges vaguely to μ. Put

$$f_n = K\mu_n \wedge K\nu .$$

Then by Lemma 3.3, each f_n coincides nearly everywhere on F with a potential $K\lambda_n$ of a positive measure λ_n supported by F;

$$f_n = K\lambda_n \quad \text{nearly everywhere on } F.$$

The total measures $\int d\lambda_n$ $(n=1, 2, \cdots)$ are bounded, since

$$\alpha\left(\int d\lambda_n\right)^2 \leq \iint K(x, y)\, d\lambda_n d\lambda_n = \int K\lambda_n d\lambda_n$$
$$\leq \int K\mu_n d\lambda_n = \int K\lambda_n d\mu_n \leq \int K\mu_n d\mu_n \leq \int K\mu d\mu < \infty ,$$

where $\alpha = \inf_{F\times F} K(x, y) > 0$. Hence by Theorem G, a subnet $T = \{\lambda_\omega ; \omega \in D\}$ converges vaguely to λ. Then $K\lambda \leq \varliminf_\omega K\lambda_\omega$ in X and $K\lambda = \varliminf_\omega K\lambda_\omega$ nearly everywhere in X. Therefore

$$K\lambda = \varliminf f_n = f \quad \text{nearly everywhere on } F.$$

Consequently K satisfies the compact lower envelope principle.

Here we give an example of a kernel which satisfies the compact lower envelope principle, but not the strong lower envelope principle. Let X be a space consisting of a countably infinite number of points, $\{x_1, x_2, \cdots\}$, and let a kernel K be given by

$$K(x_i, x_i) = 2 \qquad (i = 1, 2, \cdots)$$
$$K(x_i, x_j) = 1 \qquad (i \neq j, \; i, j = 1, 2, \cdots) .$$

This symmetric kernel K satisfies the domination principle and hence the local lower envelope principle by Theorem 3. 2. In fact, let μ be a positive measure supported by $F = \{x_{n_1}, x_{n_2}, \cdots, x_{n_k}\}$, $\mu = \sum_{i=1}^{k} m_i \varepsilon_{x_{n_i}}$ $(m_i > 0)$, and let x' be a point $\notin F$ and $K\mu(x_{n_i}) \leq K(x_{n_i}, x')$ for every i. Then

$$\sum_{j=1}^{k} m_j < K\mu(x_{n_i}) \leq K(x_{n_i}, x') = 1 .$$

Therefore for any $x \notin F$,

$$K\mu(x) = \sum m_j < 1 \leq K(x, x') .$$

Consequently by Theorem F, K satisfies the domination principle.

Now let $\mu_i = \varepsilon_{x_i}$ $(i = 1, 2)$. Then the lower envelope $f = K\mu_1 \wedge K\mu_2$ is constantly 1 in X and does not coincide in X with a potential of a positive measure supported by a compact set, since the function 1 is not a potential. Thus K does not satisfy the strong lower envelope principle.

By this example it is also shown that a kernel does not necessarily satisfy the strong lower envelope principle even if it satisfies the domination principle or the balayage principle. Thus the following problem arises: Find a sufficient condition in order that a balayable kernel satisfies the strong lower envelope principle.

Here we give a sufficient condition. which is not necessary as shown later.

Let K be of positive type and denote by $\mathfrak{E}^+$ the totality of positive measures with finite energy. The semi-norm

$$||\mu-\nu|| = \sqrt{\int K\mu d\mu - 2\int K\mu d\nu + \int K\nu d\nu}$$

defines the strong topology on $\mathfrak{E}^+$. When any strong Cauchy net in $\mathfrak{E}^+$ converges to an element of $\mathfrak{E}^+$ with respect to the strong topology, $\mathfrak{E}^+$ is, by definition, *strongly complete.* When K satisfies the balayage principle, it is of positive type by Theorem A and we can prove

Theorem 3.3. *Suppose that K satisfies the balayage principle and that $\mathfrak{E}^+$ is strongly complete. Then K satisfies the strong lower envelope principle.*

Proof. Let μ and ν be positive measures with compact support and $\mu \in \mathfrak{E}^+$. We shall show that $f = K\mu \wedge K\nu$ coincides nearly everywhere in X with a potential $K\lambda$ of a positive measure. Take a compact set F_0 which contains $S_\mu \cup S_\nu$, and denote by D the directed set of compact sets F containing F_0. To each F of D a positive measure λ_F supported by F corresponds, by Theorem 3.2, in such a way:

$$K\lambda_F = f \quad \text{nearly everywhere on } F.$$

Put $T = \{\lambda_F;\ F \in D\}$. Then T is a strong Cauchy net, since for any $F \subset F'$, $K\lambda_F \leq K\lambda_{F'}$ nearly everywhere in X and

$$\int K\lambda_F d\lambda_F \leq \int K\lambda_{F'} d\lambda_F = \int K\lambda_F d\lambda_{F'} \leq \int K\lambda_{F'} d\lambda_{F'} \leq \int K\mu d\mu < \infty,$$

$$0 \leq \int K\lambda_F d\lambda_F - 2\int K\lambda_F d\lambda_{F'} + \int K\lambda_{F'} d\lambda_{F'} \leq \int K\lambda_{F'} d\lambda_{F'} - \int K\lambda_F d\lambda_F.$$

Thus, by the strong completeness of $\mathfrak{E}^+$, T converges strongly to a positive measure $\lambda \in \mathfrak{E}^+$. As is easily seen

$$\int K\lambda d\tau = \lim \int K\lambda_F d\tau \qquad \text{for any } \tau \in \mathfrak{E}^+.$$

and hence $K\lambda = \lim K\lambda_F$ nearly everywhere in X. Consequently $K\lambda = f$ nearly everywhere in X.

As to the strong completeness of $\mathfrak{E}^+$ we refer to Fuglede [8] and Ohtsuka [12]. They gave sufficient conditions for the strong completeness. The following example is contained in [12]. Exclude the closed unit ball with center at the origin from the 3-dimensional Euclidean space and take the rest for X, and consider the Newtonian kernel K in X. Then K satisfies the domination principle and $\mathfrak{E}^+$ is not strongly complete. But as easily seen, it satisfies the strong lower envelope principle.

Now we state relations among the lower envelope principle and the equilibrium principle.

First we prove

Theorem 3.4. *Suppose that K is of positive type and that if $K\mu$ is the potential of a positive measure with finite energy, then the lower envelope $K\mu \wedge 1$ coincides nearly everywhere on a given compact set with the potential $K\nu$ of a positive measure supported by the compact set. Then K satisfies the equilibrium principle and Frostman's maximum principle.*

Proof. We shall show that K satisfies Frostman's maximum principle. Let μ be a positive measure supported by a compact set F and $K\mu(x) \leq 1$ on F. If there exists a point $x_1 \notin F$ such that $K\mu(x_1) > 1$, then the same inequality holds in a neighborhood U of x_1 with compact closure such that $F \cap \bar{U} = \emptyset$. By our assumption the lower envelope $f(x) = K\mu(x) \wedge 1$ coincides nearly everywhere on $F' = F \cup \bar{U}$ with a potential $K\nu(x)$ of a positive measure supported by F'. This leads to a contradiction as in the proof of Theorem 3.1.

The converse of Theorem 3.4 is not true; there exists a kernel K which satisfies the equilibrium principle but the lower envelope $K\mu \wedge 1$ is not a potential. Let X be a compact space consisting of four points, x_1, x_2, x_3 and x_4, and let a kernel K be given by a matrix

$$\begin{pmatrix} \frac{3}{2}, & \frac{1}{2}, & 1, & 1 \\ \frac{1}{2}, & \frac{3}{2}, & 1, & 1 \\ 1, & 1, & \frac{3}{2}, & \frac{1}{2} \\ 1, & 1, & \frac{1}{2}, & \frac{3}{2} \end{pmatrix}.$$

As easily seen, this kernel satisfies the equilibrium principle, but the lower envelope $K\mu \wedge 1$, $\mu = \varepsilon_{x_4}$, is not a potential, since the determinant of the matrix vanishes. This kernel does not satisfy the lower envelope principle. We shall come back to this example in § 4.

Theorem 3.5. *Let K be a kernel satisfying the equilibrium principle. If it satisfies the compact lower envelope principle or the domination principle, then it satisfies the lower envelope principle.*

Proof. By Theorem 3.2, it is sufficient to prove that if K satisfies the equilibrium and compact lower envelope principles, then it satisfies the lower envelope principle. Let F be a compact set and $\sigma=\mu_1-\mu_2$ be a signed measure such that μ_i $(i=1,2)$ are positive measure with finite energy and supported by F. Then

$$K\sigma \wedge 1 = K\mu_1 \wedge (1+K\mu_2)-K\mu_2$$

and 1 coincides with an equilibrium potential $K\lambda$ of F nearly everywhere on F. Hence, by the compact lower envelope principle, $K\mu_1 \wedge (1+K\mu_2)$ coincides nearly everywhere on F with a potential $K\nu$ of a positive measure supported by F. Consequently K satisfies the lower envelope principle. Note that if σ is a positive measure, then $K\tau=K\sigma\wedge 1$ is a potential of a positive measure.

Naturally there exists a kernel which satisfies the equilibrium and lower envelope principles but not the compact lower envelope principle. For example, let $X=\{x_1, x_2, x_3\}$ and K be given by

$$\begin{pmatrix} 2, & 2, & 2 \\ 2, & 3, & 1 \\ 2, & 1, & 4 \end{pmatrix}.$$

As easily seen, K satisfies the equilibrium principle but not the domination principle. Hence by Theorem 3.1, it does not satisfy the compact lower envelope principle. Since the determinant of the matrix does not vanish, K satisfies the lower envelope principle.

Closing this section we state the relations among three envelope principles in the following diagram:

$$\begin{tikzcd}
\text{strong lower env. prin.} \arrow[rr, shift left, "(1)"] \arrow[dr, shift right, "|" marking, "(3)"'] & & \text{compact lower env. prin.} \arrow[ll, shift left, "|" marking, "(2)"] \arrow[dl, shift left, "|" marking, "(5)"] \\
& \text{lower env. prin.} \arrow[ul, shift right, "|" marking, "(4)"'] \arrow[ur, shift left, "|" marking, "(6)"] &
\end{tikzcd}$$

The relations (1), (2) and (6) have been already shown; (3) is shown by a kernel on a compact space, which satisfies the domination principle but not the equilibrium principle; from this and (1) follows (5); (4) is shown by a kernel given to show (2).

§ 2. Regular kernels

In the potential theory "regularity" is the important notion, from which it follows that if a potential is continuous on a compact set F, its balayaged potential to F is continuous.

DEFINITION. We shall say that a compact set F is *regular* with respect to a kernel K or K-regular when, for any positive continuous function $h(x)$ on F and for any potential $K\mu(x)$ of a positive measure μ, if $K\mu(x) \geq h(x)$ nearly everywhere on F, then the same inequality holds everywhere on F.

DEFINITION. We shall say that a kernel K is regular[2)] when, for any compact set F and for any open neighborhood G of F, there exists a K-regular compact set F' such that $F \subset F' \subset G$.

From the classical potential theory it is seen that the kernels of order α on the Euclidean spaces and the Green kernels on open Riemann spaces are regular. Evidently a kernel K is regular if $K(x, x)$ is finite at every point x. The following is essentially contained in Frostman [7].

Theorem 3.6. *Let X be the m-dimensional Euclidean space. In order that a kernel K is regular, it is sufficient that it satisfies the condition:*

For any point $x_0 \in X$, there exist an open neighborhood U of x_0 and a positive constant $A = A(x_0)$ such that

$$\int_U \int_U K(x, y)\, dx\, dy \quad \textit{is finite,}$$

and for any positive measure μ whose support S_μ is contained in U and for any ball $B(x_0, r)$ in U, with center at x_0 and radius r,

$$\frac{1}{v_r} \int_{B(x_0, r)} K\mu(x)\, dx \leq A \cdot K\mu(x_0),$$

where v_r denotes the volume of $B(x_0, r)$.

Proof. Let us assume our condition. We first show that a compact set F is K-regular, if it satisfies the condition of Poincaré, that is, for every point x of F, there exists in F a cone with vertex at x. Let $h(x)$ be a positive continuous function on F and let $K\mu(x)$ be a potential of a positive measure. Suppose that

$$(5) \qquad K\mu(x) \geq h(x) \quad \text{nearly everywhere on } F.$$

What we have to show is the validity of $K\mu(x) \geq h(x)$ everywhere on F. Let x_0 be an arbitrarily fixed point of F, and let c be a cone in F with vertex at x_0. We denote by c_r the intersection $c \cap B(x_0, r)$, which is contained in $U \cap F$ for every sufficiently small r, and by u_r the volume of c_r. Then $u_r / v_r \equiv a < 1$. Since we may suppose that $K\mu(x_0)$ is finite, we can choose, for any positive number ε, a positive number r_1 such that

2) In the sense of Choquet regular kernels are those which satisfy the continuity principle.

$$(6) \qquad K\mu'(x_0) < \frac{a\varepsilon}{3A}.$$

where μ' is the restriction of μ to $B(x_0, r_1)$. Because of the continuity of $h(x)$ and $K\mu''(x)$, $\mu''=\mu-\mu'$, there exists a positive number $r_2<r_1$ such that

$$(7) \qquad h(x_0) < h(x)+\frac{\varepsilon}{3} \quad \text{in} \quad B(x_0, r_2)\cap F,$$

$$(8) \qquad K\mu''(x) < K\mu''(x_0)+\frac{\varepsilon}{3} \quad \text{in} \quad B(x_0, r_2).$$

By (7) and (5), we have

$$h(x_0) < \frac{1}{u_{r_2}}\int_{c_{r_2}} h(x)\,dx+\frac{\varepsilon}{3} = \frac{1}{u_{r_2}}\int_{c_{r_2}} K\mu(x)+\frac{\varepsilon}{3}$$
$$= \frac{1}{u_{r_2}}\int_{c_{r_2}} K\mu'(x)\,dx+\frac{1}{u_{r_2}}\int_{c_{r_2}} K\mu''(x)\,dx+\frac{\varepsilon}{3}.$$

Here the second term $\frac{1}{u_{r_2}}\int_{c_{r_2}} K\mu''(x)\,dx$ is not greater than $K\mu''(x_0)+\frac{\varepsilon}{3}$ by (8), and the first term

$$\frac{1}{u_{r_2}}\int_{c_{r_2}} K\mu'(x)\,dx = \frac{1}{av_{r_2}}\int_{c_{r_2}} K\mu'(x)\,dx$$
$$\leq \frac{1}{av_{r_2}}\int_{B(x_0,r_2)} K\mu'(x)\,dx \leq \frac{A}{a}K\mu'(x_0) < \frac{\varepsilon}{3},$$

by the conditions in our theorem and (6). Thus we obtain

$$h(x_0) \leq K\mu''(x_0)+\varepsilon \leq K\mu(x_0)+\varepsilon.$$

Consequently $h(x_0)\leq K\mu(x_0)$, and hence F is K-regular.

Now we show that for any compact set F and for any open neighborhood G of F, there exists a K-regular compact set F' such that $F\subset F'\subset G$. For this purpose we take a compact set F' which is enclosed by surfaces, pararell to the coordinate-axes. This F' is K-regular, since it satisfies the condition of Poincaré. This completes the proof of the theorem.

Corollary. *The α-kernels, $0<\alpha<m$, on the m-dimensional Euclidean space are regular.*

Now we state some consequences of "regularity".

Theorem 3.7. *Let K be balayable. If a potential $K\mu$ of a positive*

$K\mu(x)$ of positive measures with finite energy, and let $\mathfrak{D}(F)$ be a totality of continuous functions $f=f_1-f_2$ with $f_i \in \mathfrak{D}^+(F)$ $(i=1,2)$.

Lemma 3.5. *Let K be regular and balayable, and let F be K-regular. Then $\mathfrak{D}(F)$ is closed with respect to the operations $\vee$ and $\wedge$.*

Proof. It is sufficient to show that $\mathfrak{D}(F)$ is closed with respect to the operation $\wedge$, since $f\vee g=-\{(-f)\wedge(-g)\}$. Let f and g be in $\mathfrak{D}(F)$. Then $f=f_1-f_2$, $g=g_1-g_2$ with f_1 and g_1 in $\mathfrak{D}^+(F)$ $(i=1,2)$, and

$$f\wedge g=(f_1+g_2)\wedge(f_2+g_1)-(f_2+g_2).$$

Hence, $\mathfrak{D}^+(F)$ being a half vector space, it is sufficient to show that if h_1 and h_2 are in $\mathfrak{D}^+(F)$, then $h_1\wedge h_2$ is also in $\mathfrak{D}^+(F)$. Write

$$h_i=K\mu_i \qquad \text{on} \quad F\,(\mu_i\geqq 0).$$

Then μ_i have finite energy and, by Theorem 3.9, $h_1\wedge h_2$ coincides on F with a potential $K\lambda$ of a positive measure supported by F, and hence $h_1\wedge h_2$ belongs to $\mathfrak{D}^+(F)$.

Lemma 3.6. *Let K be regular and balayable, and let F be a compact set. If K is non-degenerate, then for any distinct points x_1 and x_2 of F and for any pair of real numbers a_1 and a_2, there exists a function f in $\mathfrak{D}(F)$ such that $f(x_i)=a_i$ $(i=1,2)$.*

Proof. We first show that there exist positive measures μ_1 and μ_2 with finite energy such that $K\mu_j(x)$ $(j=1,2)$ are continuous in X and the determinant

$$\begin{vmatrix} K\mu_1(x_1) & K\mu_1(x_2) \\ K\mu_2(x_1) & K\mu_2(x_2) \end{vmatrix} \neq 0.$$

Since K is non-degenerate, there exist distinct points y_j $(j=1,2)$ such that

$$\frac{K(x_1, y_1)}{K(x_2, y_1)} \neq \frac{K(x_1, y_2)}{K(x_1, y_1)}.$$

When $K(y_j, y_j)$ $(j=1$ or $2)$ is finite, we put $\mu_j=\varepsilon_{y_j}$. Then μ_j is a positive measure with finite energy and $K\mu_j(x)$ is continuous and $K\mu_j(x)=K(x, y_j)$ in X. When $K(y_j, y_j)$ $(j=1$ or $2)$ is infinite, we take a neighborhood G_j of y_j, such that $x_i \notin G_j$ $(i=1, 2)$. Then the compact set $F-G_j$ contains x_i $(i=1,2)$ but not y_j. By Theorem 3.8 there exists a positive measure μ_j with finite energy such that $K\mu_j$ is continuous in X and $K\mu_j(x)=K(x, y_j)$ everywhere on $F-G_j$. Consequently $K\mu_j(x_i)=K(x_i, y_j)$ $(i=1,2)$. Thus we have constructed positive measures μ_j

measure μ is continuous on a K-regular compact set F', then a balayaged potential $K\mu'$ of $K\mu$ onto F' is continuous on F'.

Theorem 3.7′. *Let K satisfy the equilibrium principle. If a compact set F' is K-regular, then the equilibrium potential of F' is continuous on F'.*

These two theorems follow immediately from the definition of "regularity".

Theorem 3.8. *Let K be regular and balayable, and let x_0 be a point which is not contained in a compact set F. Then there exists a positive measure μ with finite energy such that $K\mu(x)$ is continuous in X and $K\mu(x)=K(x, x_0)$ everywhere on F.*

Proof. Let G be an open neighborhood of F which does not contain x_0. The kernel K being regular, there exists a K-regular compact set F' such that $F\subset F'\subset G$. Let μ be a balayaged measure of ε_{x_0} onto F'. Then by Theorem 3.7, $K\mu(x)$ is continuous on $F'\supset S_\mu$ and $K\mu(x)=K(x, x_0)$ everywhere on F'. Since K satisfies the continuity principle by Theorem F, $K\mu(x)$ is continuous in X.

Theorem 3.9. *Let K be regular and balayable, and let F' be a K-regular compact set. If μ and ν are positive measures, one of which has finite energy, and if $K\mu(x)$ and $K\nu(x)$ are continuous on F', then the lower envelope $K\mu(x)\wedge K\nu(x)$ coincides everywhere on F' with a potential $K\lambda(x)$ of a positive measure λ supported by F'.*

Proof. Put $f(x)=K\mu(x)\wedge K\nu(x)$. Then $f(x)$ is continuous on F'. Since K is balayable and hence it satisfies the domination principle, it satisfies the compact lower envelope principle by Theorem 3.2. Hence there exists a positive measure λ supported by F' such that

$$K\lambda(x) = f(x) \quad \text{nearly everywhere on } F'.$$

Again by the domination principle,

$$K\lambda(x) \leq f(x) \quad \text{everywhere in } X.$$

Then by the K-regularity of F', it is shown that

$$K\lambda(x) = f(x) \quad \text{everywhere on } F'.$$

§3. Sufficient conditions for the U- and BU-principles

In this section we give sufficient conditions in order that balayable kernels satisfy the U- and BU-principles.

Let F be a compact set, and let $\mathfrak{D}^+(F)$ be a totality of non-negative finite continuous functions $f(x)$ on F which coincide on F with potentials

$(j=1,2)$ with finite energy such that $K\mu_j(x)$ are continuous in X and the determinant

$$\begin{vmatrix} K\mu_1(x_1) & K\mu_1(x_2) \\ K\mu_2(x_1) & K\mu_2(x_2) \end{vmatrix} \neq 0 .$$

Now, for any real numbers ξ_1 and ξ_2, $\xi_1 K\mu_1(x)+\xi_2 K\mu_2(x)$ belongs to the family $\mathfrak{D}(F)$ and, since the determinant above does not vanish, there exists a function $f(x)=\xi_1 K\mu_1(x)+\xi_2 K\mu_2(x)$ of $\mathfrak{D}(F)$ such that $h(x_i)=a_i$ $(i=1,2)$.

Now we can prove an approximation theorem.

Theorem 3.10. *Let K be regular, balayable and non-degenerate, and let F be a K-regular compact set. Then $\mathfrak{D}(F)$ is dense in $\mathfrak{C}(F)$ with respect to the uniform convergence topology.*

Proof. By Lemmas 3.5 and 3.6, $\mathfrak{D}(F)$ has the properties (a) and (b) of Theorem 2.1. Hence our theorem follows from Theorem 2.1.

By this theorem we obtain a sufficient condition for the U- and BU-principles.

Theorem 3.11. *Let K be regular and balayable. In order that it satisfies the U-principle, it is sufficient that it is non-degenerate.*

Proof. Let μ_1 and μ_2 be positive measures supported by a compact set F, and $K\mu_1=K\mu_2$ nearly everywhere in X. We shall show the equality $\mu_1=\mu_2$. Take a K-regular compact set F' containing F. Then by Theorem 3.10, $\mathfrak{D}(F')$ is dense in $\mathfrak{C}(F')$ with respect to the uniform convergence topology on F'. Hence, for any finite continuous function f and for any positive number ε, there exists a function $h=K\nu_1-K\nu_2$ of $\mathfrak{D}(F')$ such that

$$|f(x)-h(x)|<\varepsilon \qquad \text{on } F',$$

and hence

$$\left|\int (f(x)-h(x))\,d\mu_i(x)\right| < \varepsilon \int d\mu_i < \varepsilon' \qquad (i=1,2),$$

where ε' tends to 0 with ε. Since ν_i $(i=1,2)$ have finite energy, $\int K\mu_1 d\nu_i=\int K\mu_2 d\nu_i$. Consequently

$$\begin{aligned}\int h(x)\,d\mu_1(x) &= \int K\nu_1 d\mu_1-\int K\nu_2 d\mu_1 \\ &= \int K\mu_1 d\nu_1-\int K\mu_1 d\nu_2 \\ &= \int K\mu_2 d\nu_1-\int K\mu_2 d\nu_2\end{aligned}$$

$$= \int K\nu_1 d\mu_2 - \int K\nu_2 d\mu_2$$
$$= \int h(x) d\mu_2(x) .$$

Therefore $\left| \int f(x) d\mu_1(x) - \int f(x) d\mu_2(x) \right| < 2\varepsilon'$ and hence $\int f(x) d\mu_1(x) = \int f(x) d\mu_2(x)$ for any finite-valued continuous function f on F'. Thus $\mu_1 = \mu_2$. This completes the proof of the theorem.

Corollary 1. *If a regular kernel K satisfies the* BU-*principle, then it satisfies the* U-*principle and the energy principle.*

Proof. If a regular kernel K satisfies the BU-principle, then, by Theorem 1.1, it is non-degenerate and by Theorem 3.11, it satisfies the U-principle, and hence the energy principle.

Corollary 2. *Let K be regular and balayable. In order that it satisfies the* BU-*principle, it is sufficient that it is non-degenerate.*

Proof. This follows immediately from Theorem 3.11, since the U-principle implies the BU-principle.

The following was proved by Ninomiya [11].

Theorem 3.12. *Let K be regular and balayable. In order that it satisfies the* U- *and* BU-*principles, it is sufficient that is satisfies Ninomiya's condition* [S] (in Chap. I, § 3).

Proof. This follows from the remark in § 3 of Chap. I; if K satisfies Condition [S], it is non-degenerate.

In case that K satisfies both the balayage and equilibrium principles, a weaker condition than non-degeneracy is sufficient for the U- and BU-principles.

Theorem 3.13. *Let K be regular and satisfy both the balayage and equilibrium principles. In order that it satisfies the* U- *and* BU-*principles, it is sufficient that K-potentials separate points.*

Proof. It is sufficient to prove the sufficiency for the U-principle. Let μ_1 and μ_2 be positive measures with compact support such that $K\mu_1 = K\mu_2$ nearly everywhere in X. By the regularity of K we may suppose that they are supported by a K-regular compact set F. Evidently the half vector space $\mathfrak{D}^+(F)$ prossesses the property (1′) in Corollary 2 of Theorem 2.1. By Lemma 3.4, it possesses (2′) of the corollary. We verify the property (3′). By our assumption there exists a potential $K\nu$ of a positive measure with finite energy which separates distinct points

x_1 and x_2, and by Lemma 3.3, there exists a smoothed measure ν' of ν such that $K\nu'(x_1) \neq K\nu'(x_2)$. Consequently by the corollary, $\mathfrak{D}(F)$ in dense in $\mathfrak{C}(F)$ with respect to the uniform convergence topology. Then we obtain the equality $\mu_1 = \mu_2$ by the same way as in the proof of Theorem 3.11.

REMARK 1. If a regular kernel satisfies only the balayage principle, the condition "K-potentials separate points" is not sufficient for the U- nor BU-principles. In fact, if $X = \{x_1, x_2\}$ and K is given by

$$\begin{pmatrix} 4, & 2 \\ 2, & 1 \end{pmatrix},$$

then K is regular and balayable, and K-potentials separate points, but K does not satisfy the U- nor BU-principles.

REMARK 2. In case that K is not regular, the author does not know whether the statements in Theorems 3.11 and 3.12 are true. We can prove only that *if K satisfies both the balayage and equilibrium principles and K-potentials separate points, then it satisfies the following restricted unicity principle* (*and hence the analogous restricted balayage-unicity principle*): if μ_1 and μ_2 are positive measures *with finite energy* such that $K\mu_1 = K\mu_2$ nearly everywhere in X, then $\mu_1 = \mu_2$. This is verified as follows. Denote by $\mathfrak{D}_0^+(F)$ the totality of non-negative finite continuous functions f which coincide *nearly everywhere* on a compact set F with potentials $K\mu$ of positive measures *with finite energy*, and put

$$\mathfrak{D}_0(F) = \{f = f_1 - f_2 \,;\, f_i \in \mathfrak{D}_0^+(F) \quad i = 1, 2\}.$$

Evidently $\mathfrak{D}^+(F) \subset \mathfrak{D}_0^+(F)$ and $\mathfrak{D}(F) \subset \mathfrak{D}_0(F)$. By our assumptions $\mathfrak{D}_0(F)$ has the properties (1′), (2′) and (3′) of Corollary 2 of Theorem 2.1. Hence $\mathfrak{D}_0(F)$ is dense in $\mathfrak{C}(F)$ with respect to the uniform convergence topology. Consequently for any finite continuous function f on F by which μ_1 and μ_2 are supported and for any positive number ε, there exists a function h in $\mathfrak{D}_0(F)$ such that

$$|f - h| < \varepsilon \qquad \text{on } F \text{ and}$$
$$h = K\nu_1 - K\nu_2 \quad \text{nearly everywhere on } F.$$

Since μ_i and ν_i $(i = 1, 2)$ have finite energy, we obtain

$$\int h\,d\mu_1 = \int h\,d\mu_2 \quad \text{and} \quad \int f\,d\mu_1 = \int f\,d\mu_2\,.$$

Hence $\mu_1 = \mu_2$.

Now we state sufficient conditions in order that a composition kernel satisfies U- and BU-principles.

Theorem 3.14. *Let K be a regular composition kernel satisfying the domination principle. Each of the following conditions is sufficient for the* U- *and* BU-*principles*:

(1) *K is non-degenerate,*

(2) *K-potentials separate points,*

(3) *$k(x)$ is not periodic,*

(4) *$k(0)>k(x)$ at every point $x\neq 0$,*

(5) *K satisfies Ninomiya's condition* [S] (in Chap. I, § 2).

Proof. This follows immediately from Theorems 1.5, 1.6 and 3.11 and the remark in § 2 of Chapter I.

Choquet-Deny [5] announced the condition (3) without proof. The condition (4) was obtained by Ninomiya [11] assuming both the balayage and equilibrium principles. The sufficiency of (4) was proved by the author assuming only the balayage principle [10].

Here we prove the statement in Remark 2 in Chapter I: if a balayable composition kernel is non-degenerate, it satisfies Condition [S]. We may suppose that $k(0)$ is finite, since otherwise it satisfies Condition [S]. Then K is regular. Hence by Theorem 3.14 it satisfies the BU-principle, and by Theorem 1.3 it satisfies Condition [S].

Summing up the results obtained in Chapters I and III, we state

Theorem. *Let K be a symmetric regular balayable kernel. In order that K satisfies the* U- *or* BU-*principle, it is necessary and sufficient that K is non-degenerate. It is also necessary and sufficient that K satisfies Ninomiya's condition* [S].

Theorem. *Let K be a symmetric regular kernel satisfying both the balayage and equilibrium principles. In order that K satisfies the* U- *or* BU-*principle, it is necessary and sufficient that K-potentials separate points.*

Theorem. *Let K be a symmetric regular balayable composition kernel. Each of the five conditions in Theorem* 3.14 *is necessary and sufficient for the* U- *or* BU-*principle.*

§ 4. Sufficient condition for the U- and EU-principles

In this section we assume that K satisfies the equilibrium principle and we give a sufficient condition in order that K satisfies the U- and EU-principles.

First we remark that *any condition considered in the preceding section is not sufficient.* In fact, the kernel given soon after Theorem 3.4 does by Theorem C′, and

$$\left|\int f d\mu_1 - \int f d\mu_2\right| < 2\varepsilon \int d\mu_1 .$$

Therefore $\int f d\mu_1 = \int f d\mu_2$, and hence $\mu_1 = \mu_2$.

Corollary. *Let K satisfy the equilibrium principle and the lower envelope principle, and let K-potentials separate points. Then K satisfies the energy principle and the restricted unicity principle* (in Remark 2 to Theorem 3. 14).

REMARK. In § 3, we have shown that the following implication is valid for regular kernels: the BU-principle $\Longrightarrow$ the U-principle. The analogous implication, the EU-principle $\Longrightarrow$ the U-principle, seems to be false, but the author has not yet any example which shows the EU-principle $\not\Longrightarrow$ the U-principle.

Mathematical Institute, Nagoya University

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not satisfy the U- nor EU-principle, since the determinant of the matrix vanishes, but it is non-degenerate and it satisfies Ninomiya's Conditions [S] and potentials separate points.

This remark is an answer to the following Ninomiya's problem, raised in his thesis: Is it true that Condition [S] is sufficient in order that a symmetric kernel of positive type satisfies the U-principle? By the above remark the answer is negative. Thus we need to give an additional condition to [S], which should be weaker than the balayage principle, since, by Theorem 3. 13, a regular kernel satisfies the U-principle if it satisfies Condition [S] and the balayage and equilibrium principles; it is also seen that it satisfies the EU-principle.

Denote by $\mathfrak{G}(F)$ the family consisting of all non-negative finite-valued continuous functions on a compact set F which coincide *nearly everywhere* on F with potentials $K\sigma$ of signed measures $\sigma=\mu-\nu$ such that μ and ν are positive measures with finite energy.

Theorem 3. 15. *If K satisfies the equilibrium principle and the lower envelope principle and if K-potentials separate points, then, for any compact set F, $\mathfrak{G}(F)$ is dense in $\mathfrak{C}^+(F)$ with respect to the uniform convergence topology.*

Proof. It is sufficient to verify that $\mathfrak{G}(F)$ has the properties (i), (ii) and (iii) of Theorem 2. 2. The properties (i) and (ii) are immediate. We shall show that $\mathfrak{G}(F)$ has the property (iii). Since K-potentials separate points, there exists, for any given distinct points x_1 and x_2 of F, a potential $K\mu(x)$ of a positive measure with finite energy such that $K\mu(x_1)\neq K\mu(x_2)$. Then by Lemma 3. 3, there exists a smoothed measure μ' of μ such that $K\mu'$ is finite continuous in X and $K\mu'(x_1)\neq K\mu'(x_2)$. Thus $\mathfrak{G}(F)$ has the property (iii). Consequently, by Theorem 2. 2, $\mathfrak{G}(F)$ is dense in $\mathfrak{C}^+(F)$.

Theorem 3. 16. *Let K satisfy the equilibrium principle and the lower envelope principle. In order that K satisfies the* EU-*principle it is sufficient that K-potentials separate points.*

Proof. Let μ_1 and μ_2 be equilibrium measures of a compact set F, and let f be an arbitrary finite continuous function on F. Then by Theorem 3. 15, for any positive number ε, there exists a function $h\in\mathfrak{G}(F)$ such that

$$h = K\nu_1-K\nu_2 \quad \text{and} \quad |f-h|<\varepsilon \quad \text{on } F.$$

Since μ_i and ν_i $(i=1,2)$ have finite energy, we obtain

$$\int h d\mu_1 = \int h d\mu_2$$