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ON THE STABLE JAMES NUMBERS OF THOM COMPLEXES

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1. Introduction

Let X be a connected finite CW-complex with a base point, and ξ be a (real) vector bundle over X. We have the natural inclusion map

$$i: S^n = (pt)^n \to X^\xi$$

of Thom complexes, where $n=\dim \xi$ and the trivial n-dimensional bundle is also denoted by n for brevity. Consider the homomorphism

$$i^*: \{X^\xi, S^n\} \to \{S^n, S^n\} = Z$$

of stable cohomotopy groups. Then the stable James number of X^ξ, which we shall denote by $d(X, \xi)$, is defined to be the non-negative generator of image i^* (see [7]). Thus $d(X, \xi)$ is the least positive integer r such that a map $S^n \to S^n$ of degree r can be stably extended to X^ξ, if it exists, or zero otherwise. For a map $f: X^\xi \to S^n$, we shall call the degree of $f\circ i$ the degree of f simply.

Suppose, for example, that X is the projective space FP^{k-1} ($F=C$ or H), and ξ is n-fold Whitney sum of the canonical line bundle η over FP^{k-1}, then $X^\xi = FP^{k+n-1}/FP^{n-1}$ and $d(FP^{k-1}, n\eta)$ is the same as $F\{n, k\}$ in [9]. In that paper, Ōshima determined $F\{n, k\}$ for several small k's (see also [3], [7] and [8] for $F\{1, k\}$).

Now let X and ξ be as before. Let $J(X)$ denote the group of stable fibre homotopy equivalence classes of real vector bundles over X, and $J(\xi)$ the class of ξ in $J(X)$. Since a stable fibre homotopy equivalence of bundles induces a stable homotopy equivalence of their Thom complexes, we may regard $d(X, -)$ as a function from $J(X)$ to Z. We shall abuse notations, and not distinguish $d(X, J(\xi))$ from $d(X, \xi)$. Our main result is as follows:

Theorem 1.1. *Let p be a prime number, and suppose that ξ is an orientable vector bundle over X. Then,*

(1) *$d(X, \xi)$ is not zero*

(2) *p is a divisor of $d(X, \xi)$ if and only if p is a divisor of the order of $J(\xi)$.*

As an immediate corollary, we have

Corollary 1.2. *$d(X, \xi)=0$ if and only if ξ is non-orientable.*

A few other corollaries, and propositions concerning the properties of $d(X, \xi)$ will be found in §3.

2. Proof of the main theorem

Let ξ be an n-dimensional vector bundle over X as in §1. We may give a Riemannian metric on ξ and regard ξ as an orthogonal bundle as usual. Let $S(\xi)$ (and $D(\xi)$) denote the associated sphere (and disk, respectively) bundle.

Lemma 2.1. *Let k be an integer, and let $n \geqq \dim X+3$. Then there exists a map f: $S(\xi) \to S^{n-1}$ such that f has degree $\pm k$ on each fibre, if and only if there exists a map from X^{ξ} to S^n of degree k.*

This follows immediately from the commutative diagram of cohomotopy groups (see [4])

$$\begin{array}{ccc} \pi^{n-1}(S(\xi)) & \xrightarrow[\cong]{\delta} & \pi^n(D(\xi)/S(\xi)) \\ {\scriptstyle i^*}\downarrow & & {\scriptstyle i^*}\downarrow \\ \pi^{n-1}(S(\xi)_{pt}) & \xrightarrow[\cong]{\delta} & \pi^n(D(\xi)_{pt}/S(\xi)_{pt}) \end{array}$$

and the fact that X is connected.

We remark that, if such a map f as in (2.1) (with $k \neq 0$) exists, we can orient ξ so that f has degree k on each fibre. Hence, if $d(X, \xi) \neq 0$, it follows that ξ is orientable since we may suppose dim ξ is sufficiently large. The proof of (1.2) is completed by this, and (1) of (1.1).

Now, by (2.1), "if" part of (2) in (1.1) is an immediate consequene of the following Adams' mod k-Dold theorem ([1], (3.2)).

Theorem 2.2 (Adams). *Suppose that $k>0$ and there is a map f: $S(\xi) \to S^{n-1}$ of degree $\pm k$ on each fibre. Then there exists an integer e such that $k^e\xi$ is fibre homotopy equivalent to a trivial bundle.*

For the proof of "only if" part of (2) in (1.1), we need the lemmas which Adams used in the proof of (2.2). Let $G(n, m)$ be the space of maps from S^{n-1} to S^{n-1} of degree m. For $g \in G(n, l)$ and a positive integer k, we define the following maps:

$$c(g)\colon\ G(n, m) \to G(n, lm)\,, \quad c(g)(f) = g \circ f \qquad \text{(composition)}$$
$$j(k)\colon\ G(n, m) \to G(kn, m^k)\,, \quad j(k)(f) = f * f * \cdots * f \quad (k\text{-fold join})$$

As is well-known, if $n \geqq r+3$, there is an isomorphism θ: $\pi_r(G(n, m)) \cong \pi_r{}^S$.

Then Adams' lemmas we shall use are summarized as follows:

Lemma 2.3. *If $n \geqq r+3$, the following diagrams commute.*

(1)
$$\begin{array}{ccc} \pi_r(G(n, m)) & \xrightarrow{c(g)_*} & \pi_r(G(n, lm)) \\ \theta \downarrow & & \theta \downarrow \\ \pi_r^S & \xrightarrow{\times l} & \pi_r^S \end{array}$$

(2)
$$\begin{array}{ccc} \pi_r(G(n, m)) & \xrightarrow{j(k)_*} & \pi_r(G(kn, m^k)) \\ \theta \downarrow & & \theta \downarrow \\ \pi_r^S & \xrightarrow{\times km^{k-1}} & \pi_r^S \end{array}$$

Hereafter we shall express an element of $G(n, l)$ by l. We prove the following.

Theorem 2.4. *Let ξ be an oriented bundle of dimension n over a connected finite CW-complex X, and let $n \geqq \dim X+3$. Suppose that t: $S(k\xi) \rightarrow X \times S^{kn-1}$ is a fibre homotopy equivalence of degree 1 on each fibre. Then there exist an integer $e \geqq 0$, and a fibrewise map f: $S(\xi) \rightarrow X \times S^{n-1}$ of degree k^e on each fibre such that the following diagram is fibre homotopy commutative*:

$$\begin{tikzcd} S(k\xi) \arrow[r, "kf"] \arrow[dr, "t"'] & X \times S^{kn-1} \\ & X \times S^{kn-1} . \arrow[u, "1 \times (k^e)^k"'] \end{tikzcd}$$

Proof. We proceed by an induction on the number of cells of X. If X consists of only one cell, the result is clearly true (with $e=0$). Suppose that $X = Y \cup e^r$ with $r \geqq 1$, and the result is true for Y. Let t: $S(k\xi) \rightarrow X \times S^{kn-1}$ be a fibre homotopy equivalence of degree 1. By the inductive hypothesis, we can find a fibrewise map g: $S(\xi | Y) \rightarrow Y \times S^{n-1}$ of degree m, which is a power of k, such that the following diagram is fibre homotopy commutative:

$$\begin{tikzcd} S(k\xi | Y) \arrow[r, "kg"] \arrow[dr, "t | Y"'] & Y \times S^{kn-1} \\ & Y \times S^{kn-1} . \arrow[u, "1 \times m^k"'] \end{tikzcd}$$

We wish to extend $(1 \times l) \circ g$ to a fibrewise map from $S(\xi)$ to $X \times S^{n-1}$ for some l, a power of k. Let c: $(D^r, S^{r-1}) \rightarrow (X, Y)$ be the characteristic map of the cell e^r. Since the induced bundle $c^*(\xi)$ over D^r is trivial, g defines a fibrewise map from $S^{r-1} \times S^{n-1}$ to $Y \times S^{n-1}$ of degree m, and hence, a map $\langle g \rangle$: $S^{r-1} \rightarrow G(n, m)$. We can find such an extension as stated above, if and only if $\langle (1 \times l) \circ g \rangle = c(l) \circ \langle g \rangle$: $S^{r-1} \rightarrow G(n, lm)$ is null-homotopic. The case $r=1$ is trivial, since $G(n, m)$ is path-connected. Now let $r>1$. From the above fibre homotopy commutative diagram, we have the following homotopy commutative diagram:

$$\begin{tikzcd} S^{r-1} \arrow[r, "\langle kg\rangle"] \arrow[dr, "\langle t\mid Y\rangle"'] & G(kn, m^k) \\ & G(kn, 1) . \arrow[u, "c(m^k)"'] \end{tikzcd}$$

Since $(1\times m^k)\circ(t\mid Y)$ can be extended over X (in fact, to the map $(1\times m^k)\circ t$), $\langle(1\times m^k)\circ(t\mid Y)\rangle = c(m^k)\circ\langle t\mid Y\rangle$ is trivial, and hence $\langle kg\rangle = j(k)\circ\langle g\rangle$ is also trivial. On the other hand, according to (2.3), we have

$$\begin{aligned}\theta(j(k)\circ\langle g\rangle) &= km^{k-1}\theta(\langle g\rangle) \\ &= \theta(c(km^{k-1})\circ\langle g\rangle) .\end{aligned}$$

Therefore $c(km^{k-1})\circ\langle g\rangle$ is trivial, and it follows that we can find a fibrewise extension of $(1\times km^{k-1})\circ g$ over X.

So, let h: $S(\xi)\rightarrow X\times S^{n-1}$ be a fibrewise extension of $(1\times km^{k-1})\circ g$ over X. Note that h has degree km^k on each fibre. We now have the following diagram which is fibre homotopy commutative over Y:

$$\begin{tikzcd} S(k\xi) \arrow[r, "kh"] \arrow[dr, "t"'] & X\times S^{kn-1} \\ & X\times S^{kn-1} . \arrow[u, "1\times(\deg h)^k"'] \end{tikzcd}$$

To extending the fibre homotopy over the cell e^r, we have an obstruction, which is a map ϕ: $\partial(D^r\times I)=S^r\rightarrow G(kn, (km^k)^k)$. If we take $f=(1\times l)\circ h$ instead of h, the obstruction becomes $c(l^k)\circ\phi$. On the other hand, we can alter f (over e^r) by using any element of $\pi_r(G(n, lkm^k))$. Therefore the obstruction $c(l^k)\circ\phi$ can be altered by any element of $j(k)_*(\pi_r(G(n, lkm^k)))$. In other words, the obstruction to making the homotopy $kf\simeq(1\times(\deg f)^k)\circ t$, up to an alteration of f, is

$$c(l^k)\circ\phi \mod j(k)_*(\pi_r(G(n, lkm^k))) .$$

Under the isomorphism θ: $\pi_r(G(kn, (lkm^k)^k))\cong\pi_r{}^S$, this corresponds, by (2.3), to

$$l^k\theta(\phi) \mod k(lkm^k)^{k-1}\pi_r{}^S .$$

If l is a multiple of $k^k m^{k(k-1)}$, this class is zero. It follows that we can find a map f: $S(\xi)\rightarrow X\times S^{n-1}$ with the desired property, by taking $l=k^k m^{k(k-1)}$. Note that the degree of f, being $lkm^k=k^{k+1}m^{k^2}$, is a power of k. This completes the proof of (2.4).

It is now easy to prove (1.1). By (2.4) and the fact that $J(X)$ is a finite group, it follows that $d(X, \xi)\neq 0$ for an orientable bundle ξ. The rest of (1.1) is an immediate consequence of (2.4) and (2.2).

3. Applications and remarks

Let M be a connected, orientable closed manifold of dimension n. By the S-duality, there exists a stable map f: $S^n \to M^0$ of degree k, if and only if there exists a stable map g: $M^{\nu} \to S^m$ of degree k (in our sense). Here ν is a normal bundle of M, and $m=\dim \nu$. Then (1.1) implies that, for a prime p, there exists a stable map f: $S^n \to M^0$ of degree prime to p, if and only if the order of $J(\nu)$ is prime to p. Thus we have the following theorem, which is a "mod p" analogue of Milnor-Spanier's theorem [6].

Theorem 3.1. *Let M be as above, and let p be a prime. Then there exists a stable map f: $S^n \to M$ such that*

$$f_*: H_n(S^n; Z_p) \cong H_n(M; Z_p)$$

if and only if the order of $J(\nu)$ is prime to p.

Our next application is concerned with the James numbers of Stiefel manifolds ([5]). Let $O_{n,k}$ denote the Stiefel manifold over $F(=C$ or $H)$, with the projection p: $O_{n,k} \to O_{n,1}=S^{dn-1}$ ($d=2$ or 4). The James number $O_F\{n, k\}$ is defined to be the non-negative integer r such that r generates the image of p_*: $\pi_{dn-1}(O_{n,k}) \to \pi_{dn-1}(S^{dn-1})=Z$ ([5]). $O_{n,k}$ has the stunted quasi-projective space $Q_{n,k}$ as a subcomplex, which is the S-dual of $(FP^{k-1})^{-n\eta}$ (η is the canonical line bundle). Since the inclusion $Q_{n,k}\subset O_{n,k}$ is $2d(n-k)+3(d-1)$ equivalence, and $Q_{n,k}$ is $d(n-k+1)-2$ connected, it follows, by the S-duality and the suspension theorem, that we have $O_F\{n, k\}=d(FP^{k-1}, -n\eta)$ for $n\geqq 2k-1$ ([9], (4.6)). Hence, as a corollary of (1.1), we have the following, which is a (stable) portion of Sigrist's result ([10], Théoréme I).

Proposition 3.2. *If $n\geqq 2k-1$ and p is a prime, $O_F\{n, k\}$ is prime to p if and only if the order of $J(-n\eta)$ is prime to p, that is, if and only if $\nu_p(n)\geqq\nu_p(b_k)$. Here b_k denotes the order of $J(\eta)$ in $J(FP^{k-1})$, and $\nu_p(n)$ the exponent of p in the prime factorization of n.*

We note that we did not use the intrinsic join, while Sigrist's proof depends on James' theorems proved by using the intrinsic join.

Finally, we give some properties of $d(X, \xi)$ for a general ξ over X.

Proposition 3.3. *$d(X, \xi_1)\cdot d(X, \xi_2)$ is a multiple of $d(X, \xi_1+\xi_2)$.*

Proof. If f_i: $X^{\xi_i} \to S^{n_i}$ has degree r_i ($i=1, 2$), the composition of maps

$$X^{\xi_1+\xi_2} \xrightarrow{\Delta} (X\times X)^{\xi_1\times\xi_2} = X^{\xi_1}\wedge X^{\xi_2} \xrightarrow{f_1\wedge f_2} S^{n_1}\wedge S^{n_2} = S^{n_1+n_2}$$

clearly has degree $r_1 r_2$. Thus the result follows.

In case that X is FP^{k-1}, and ξ_1, ξ_2 are Whitney sums of the canonical line bundle, the above proposition corresponds by the S-duality to the James' result that $O_F\{m, k\}\cdot O_F\{n, k\}$ is a multiple of $O_F\{m+n, k\}$, which was obtained by using the intrinsic join ([5]).

Now, in case that $X=S^m$, as is well-known, $(S^m)^\xi$ has the same homotopy type as $S^n\underset{J(\xi)}{\cup} e^{n+m}$. Hence $d(S^m, \xi)$ is equal to the order of $J(\xi)$. More generally, we have the following proposition:

Proposition 3.4. *If X is a double suspension, then $d(X, \xi)$ is equal to the order of $J(\xi)$ in $J(X)$.*

Proof. Suppose $X=\Sigma Y$, with Y also a suspension. Then there is an isomorphism θ: $[\Sigma Y, BG_n]\cong[\Sigma^n Y, S^n]$ for a sufficiently large n. Here BG_n denotes the classifying space for $(n-1)$-spherical fibrations. By the result of Wall ([11], (3.7), see also [2], (2.2)), we have the following cofibration

$$\Sigma^n Y \xrightarrow{\theta(J(\xi))} S^n \longrightarrow (\Sigma Y)^\xi$$

from which we obtain

$$\begin{aligned} d(X, \xi) &= \text{order of } \theta(J(\xi)) \\ &= \text{order of } J(\xi)\,. \end{aligned}$$

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Added in proof. After this paper was accepted, an article of I. Dibag was published in which Theorem 2.4 was proved in a little more general setting.
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