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S4 DOES NOT HAVE ONE FIXED POINT ACTIONS

MASAHARU MORIMOTO

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1. Introduction

In this paper we mean smooth actions on manifolds of compact Lie groups simply by actions.

Several authors found one fixed point actions on spheres [9] (or [10]), [12], [13] and [14]. Those spheres have dimensions greater than 5. It is easy to see that the spheres $S^n$ of dimension $n \leq 2$ do not have one fixed point actions of compact Lie groups. Further it is conjectured among topologists dealing with 3-dimensional manifolds that $S^3$ has no one fixed point actions of compact Lie groups. The purpose of this paper is to show:

**Theorem A.** The 4-dimensional homotopy spheres have no one fixed point actions of compact Lie groups.

Special cases of this theorem were proved by M. Furuta and W.-Y. Hsiang-E. Straume. Let $\Sigma$ be an oriented 4-dimensional homotopy sphere.

**Theorem** (M. Furuta [4]). Any finite group $G$ can not act on $\Sigma$ in such a way that (1) $\Sigma^G$ consists of exactly one point and (2) each element of $G$ preserves the orientation of $\Sigma$.

**Corollary to Theorem 1 of W.-Y. Hsiang-E. Straume** [6]. Any compact connected Lie group can not act on $\Sigma$ with exactly one fixed point.

Our proof of Theorem A goes on by showing the following lemmas. For a compact manifold $X$ and for an integer $k \geq 0$, we denote by $X_k$ the totality of $k$-dimensional connected components of $X$. For a set $Y$, we denote by $|Y|$ the cardinality of $Y$. Let $\Xi$ be an oriented 4-dimensional homology sphere.

**Lemma B.** If a compact Lie group $G$ of dimension $\geq 1$ acts effectively on $\Xi$, then $\Xi^G$ is empty or diffeomorphic to $S^k$ with $k \leq 2$. Especially one has $|\Xi^G| = 0$ or 2.

**Lemma C.** If a finite group $G$ acts on $\Xi$, then one has $|\Xi^G| \leq 2$.

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For a $G$-action on $\Xi$ we define $K=K(G, \Xi)$ to be the subgroup of $G$ of elements preserving the orientation of $\Xi$. If a finite group $G$ acts on $\Sigma^4$ with $|\Sigma^G|=1$, then by Furuta's theorem we have $G\not\cong K$, moreover we will see $|\Sigma^K|\geq 3$ in Section 5. This contradicts Lemma C.

We wish to express our gratitude to M. Furuta for informing us of his result.

2. Preliminary

Let $G$ be a compact Lie group, $H$ a subgroup of $G$ and $X$ a compact $G$-manifold of dimension $n$. If $X^H$ is non-empty, then take an $H$-equivariant normal bundle of $X^H$ in $X$. The fibers of it are $(n-k)$-dimensional real $H$-representations. We call them the normal representations of $X^H$ in $X$. We remark that if the $G$-action on $X$ is effective, then the normal representations are faithful.

We frequently use the following well known result.

**Theorem** (P.A. Smith [1, Theorem 5.1]). *If $G$ is a $p$-group ($p$ prime) and it acts on a mod $p$ homology sphere $X$, then $X^G$ is empty or a mod $p$ homology sphere.*

The following lemma is well known and easily proved.

**Lemma 2.1.** *If a compact Lie group $G$ acts on $S^n$ with $n\leq 2$, then $S^G$ is empty or diffeomorphic to $S^m$ with $m\leq 2$.*

3. Proof of Lemma B

Let $G$ be a compact Lie group of dimension $\geq 1$ and $\Xi$ an oriented 4-dimensional homology sphere with $G$-action. Suppose that the $G$-action is effective. Let $G_0$ be the identity component of $G$.

**Proposition 3.1.** *If $G_0$ has an abelian normal subgroup $A\neq \{1\}$, then $\Xi^G$ is empty or diffeomorphic to $S^m$ with $m\leq 2$.*

**Proof.** Each element of $G_0$ preserves the orientation on $\Xi$. Since the $G$-action is effective, we have $\dim \Xi^B\leq 2$ for any subgroup $B\neq 1$ of $G_0$. Let $C$ be a cyclic subgroup of $A$ of prime order. By Smith's theorem $\Xi^C$ is a sphere of dimension $\leq 2$. By Lemma 2.1 $\Xi^G=((\Xi^C)^A)^G$, $H=G_0$, is empty or also a sphere.

**Proposition 3.2.** *It holds that*

1. *if $G_0=SO(3)$, then $\Xi^G$ is empty or diffeomorphic to $S^m$ with $m\leq 1$,*
2. *if $G_0=SU(2)$, then $|\Xi^G|=0$ or 2, and*
(3) if \( G_0 = SO(4) \), then \(|\Xi^c| = 0 \text{ or } 2\) .

Proof. The proof is done under the assumption \( \Xi^c \neq \phi \) and the notation \( H = G_0 \).

(1) Since \( SO(3) \) has no irreducible \( k \)-dimensional representations for \( k = 2 \) and 4, we have \( \dim \Xi^H = 1 \). Take a dihedral subgroup \( D \) of \( G_0 \) of order 4. Then we have \( \dim \Xi^D = 1 \). By Smith's theorem \( \Xi^D \) is a circle. Thus \( \Xi^H \) coincides with \( \Xi^D \). By Lemma 2.1 we have that \( \Xi^G = (\Xi^H)^G \) is a sphere of dimension \( \leq 1 \).

(2) Since \( SU(2) \) has no faithful representations of dimension \( \leq 3 \), \( \Xi^H \) is a finite set. Furthermore the normal \( H \)-representations of \( \Xi^H \) in \( \Xi \) are unique up to isomorphisms. For any cyclic subgroup \( C \) of \( H \) of prime order, \( \Xi^C \) is a sphere and includes \( \Xi^H \). Observing the normal representations of \( \Xi^H \), we see that \( \Xi^C \) consists of exactly two points. For any non-trivial subgroup \( B \) of \( H \), we have \( 1 \leq |\Xi^B| \leq 2 \). Let \( T \) be a maximal toral subgroup of \( H \). We have \( |\Xi^T| = 2 \) by Smith's theorem. If \( \Xi^T = \Xi^H \) is non-empty, then denote the point by \( x \). There is a subgroup \( L \) of \( H \) such that (i) \( L \) has a normal subgroup \( Q \) of order 8 and \( L/Q \) has order 3 and (ii) \( L \cap T \neq \{1\} \). By Oliver's theorem [11, Proposition 2] we have that \( |\Xi^L| = 2 \), hence \( \Xi^L = \Xi^T \). Since the smallest subgroup of \( H \) which includes \( T \) and \( L \) is \( H \), we have \( H_x = H \). This contradicts the assumption \( \{x\} = \Xi^T - \Xi^H \). Thus \( \Xi^T = \Xi^H \) and \( \Xi^G \) also consists of exactly two points.

(3) The conclusion follows from (2) and the fact that \( SO(4) \) has a normal subgroup isomorphic to \( SU(2) \).

Proof of Lemma B. Suppose that \( |\Xi^G| \neq 0 \) nor 2. Then \( G \) is a subgroup of \( O(4) \) and \( G_0 \) is a subgroup of \( SO(4) \). By Proposition 3.1 \( G_0 \) does not have an abelian normal subgroup except \( \{1\} \). Hence \( G_0 \) is isomorphic to either one of \( SO(3) \), \( SU(2) \) and \( SO(4) \). This contradicts Proposition 3.2.

4. Proof of Lemma C

Let \( G \) be a finite group, \( \Xi \) an oriented 4-dimensional homology sphere with \( G \)-action and \( K = K(G, \Xi) \) the subgroup of \( G \) defined in Section 1. Our proof of Lemma C is done under the assumption that the \( G \)-action on \( \Xi \) is effective and \( \Xi^G \neq \phi \).

First we note that \( G \) is a subgroup of \( O(4) \), \( K \) a subgroup of \( SO(4) \) and \( \dim \Xi^H \leq 2 \) for any non-trivial subgroup \( H \) of \( K \).

Proposition 4.1. Let \( H \) be a subgroup of \( K \). Then it holds that

(1) if \( \Xi^H \neq \phi \), then \( H \) is cyclic, and
(2) If $\Xi_1^H \neq \phi$, then $\Xi_2^H = \phi$ and $H$ is dihedral or isomorphic to one of $A_4$, $S_4$ and $A_5$.

Here $S_4$ stands for the symmetric group on four letters, and $A_n$, $n=4$ and $5$, stand for alternating groups on $n$ letters.

Proof. (1) It follows from the fact that a finite subgroup of $SO(2)$ is cyclic.
(2) A finite subgroup of $SO(3)$ is cyclic, dihedral or isomorphic to one of $A_4$, $S_4$ and $A_5$ (see [5]). Suppose that $H$ is cyclic. Then the normal $H$-representations have even dimensions. This contradicts $\Xi_1^H \neq \phi$.

**Proposition 4.2.** Let $H$ be a non-trivial solvable subgroup of $K$. Then $\Xi^H$ is (empty or) diffeomorphic to $S^m$ with $m \leq 2$.

Proof. Take a normal series of subgroups $H(i)$ of $H$: $\{e\}=H(0) \triangleleft H(1) \triangleleft \cdots \triangleleft H(n)=H$ with $H(i)/H(i-1)$ of prime order. By Smith's theorem and Proposition 4.1, $\Xi^{H(i)}$ is a sphere of dimension $\leq 2$. Since $\Xi^{H(i)}=(\Xi^{H(i-1)})^{H(i)}$, by induction on $i$ $\Xi^{H(i)}$ are spheres of dimension $\leq 2$.

**Proposition 4.3.** Let $H$ be a subgroup of $K$ and suppose $H$ is isomorphic to $A_5$. Then it holds that

(1) if $\Xi_0^H \neq \phi$, then $|\Xi^H|=1$ or $2$, and

(2) if $\Xi^H \neq \phi$ and $\Xi_0^H = \phi$, then $\Xi^H$ is diffeomorphic to $S^1$.

Proof. (1) Let $V$ be a normal representation of $\Xi_0^H$ in $\Xi$. Since $V$ is faithful, $V^H=\{e\}$ and $\dim V=4$, $V$ is an irreducible $H$-representation. Let $C$ be a cyclic subgroup of $H$ of order 5. Then we have $V^C=0$, hence $\Xi_0^C \supseteq \Xi_0^H (\neq \phi)$. By Proposition 4.2, $\Xi^C$ consists of exactly two points. The relation $\Xi^C \supseteq \Xi^H$ implies that $|\Xi^H|=1$ or $2$.

(2) In the case $\Xi^H$ is a disjoint union of circles. Let $D$ be a dihedral subgroup of $H$ of order 4. Then $\Xi^D$ is a circle by Smith's theorem. Immediately we have $\Xi^H=\Xi^D \cong S^1$.

**Proposition 4.4.** Provided $|\Xi^K| \geq 3$, then every Sylow subgroup of $K$ is either cyclic or dihedral.

Proof. Let $P$ be a Sylow subgroup of $K$. Since $P$ is solvable, $\Xi^P$ is a sphere of dimension 1 or 2 by Proposition 4.2. The conclusion follows from Proposition 4.1.

Now we prove Lemma C. We suppose that $|\Xi_0^K| \geq 3$, and we will meet with a contradiction.

We note that $K \neq \{e\}$ and $|\Xi^K| \geq 3$. If $K$ is solvable, then $\Xi^K$ is a sphere, hence $\Xi^\circ=(\Xi^K)^\circ$ is also a sphere. We have $|\Xi_0^K|=0$ or $2$. This contradicts
the above assumption. Thus \( K \) is non-solvable. By Suzuki's theorem [15, p. 671, Theorem B] and Proposition 4.4, there exist subgroups \( H, L \) and \( Z \) of \( K \) such that (1) \([K:H] \leq 2\), (2) \( H = Z \times L \), (3) \( Z \) is solvable and (4) \( L \) is isomorphic to \( PSL(2, q) \). Here \( q \) is a prime greater than 4. Since \( L \) is non-solvable and \( \Xi^L = \phi \), \( L \) has an irreducible representation of dimension 3 or 4. By Tables 3 and 4 of [8], \( PSL(2, q) \approx L \) is nothing but \( PSL(2, 5) \). In other words, \( L \) is isomorphic to \( A_5 \). By Proposition 4.3 it holds that \( \Xi^L = S^1 \) or \( |\Xi^L| \leq 2 \). From the assumption that \( |\Xi^L| \geq 3 \), we have \( \Xi^L \approx S^1 \). Since \( \Xi^L = (\Xi^L)^p \), it is isomorphic to \( S^0 \) or \( S^1 \), so is \( \Xi^E \). Then \( \Xi^E \) is also diffeomorphic to \( S^m \), \( m \leq 1 \). This is a contradiction.

5. Proof of Theorem A

By Lemma B, it is sufficient to prove the case in which \( G \) is a finite group acting effectively on \( \Sigma \), an oriented 4-dimensional homotopy sphere. The following arguments go on in this case.

**Proposition 5.1.** Provided \( |\Sigma^G| = 1 \), then \( |\Sigma^K| \) is finite and an odd number, where \( K \) is the subgroup of \( G \) defined in Section 1.

Proof. Suppose \( |\Sigma^G| = 1 \). By Proposition 4.2, \( K \) is non-solvable. It follows from Proposition 4.1 that \( \Sigma^K = \Sigma_0^K \coprod \Sigma_1^K \). It holds that

\[
1 = \chi(\Sigma^G) = \chi((\Sigma^K)^G) = \chi((\Sigma_0^K)^G) + \chi((\Sigma_1^K)^G)
\]

\[
\equiv \chi((\Sigma_0^K)^G) \quad \text{(mod. 2)}
\]

\[
\equiv \chi(\Sigma^K_0) \quad \text{(mod. 2)}.
\]

Thus \( |\Sigma^K_0| \) is an odd number. Especially \( \Sigma^K_0 \) is non-empty. If \( \Sigma^K_1 \) is non-empty, then \( K \) is isomorphic to \( A_5 \) by Proposition 4.1. In this case, Proposition 4.3 gives that either \( \Sigma^K_0 \) or \( \Sigma^K_1 \) is empty. This is a contradiction. Hence we have \( \Sigma^K = \Sigma^K_0 \).

Now we prove Theorem A. Provided \( |\Sigma^G| = 1 \), then by Furuta's theorem and Proposition 5.1 we have \( |\Sigma^K_0| \geq 3 \). This, however, contradicts Lemma C. Thus we get the conclusion of Theorem A.

References


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