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S⁴ DOES NOT HAVE ONE FIXED POINT ACTIONS

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1. Introduction

In this paper we mean smooth actions on manifolds of compact Lie groups simply by actions.

Several authors found one fixed point actions on spheres [9] (or [10]), [12], [13] and [14]. Those spheres have dimensions greater than 5. It is easy to see that the spheres S^n of dimension $n \leq 2$ do not have one fixed point actions of compact Lie groups. Further it is conjectured among topologists dealing with 3-dimensional manifolds that S^3 has no one fixed point actions of compact Lie groups. The purpose of this paper is to show:

Theorem A. *The 4-dimensional homotopy spheres have no one fixed point actions of compact Lie groups.*

Special cases of this theorem were proved by M. Furuta and W.-Y. Hsiang-E. Straume. Let Σ be an oriented 4-dimensional homotopy sphere.

Theorem (M. Furuta [4]). *Any finite group G can not act on Σ in such a way that (1) Σ^G consists of exactly one point and (2) each element of G preserves the orientation of Σ .*

Corollary to Theorem 1 of W.-Y. Hsiang-E. Straume [6]. *Any compact connected Lie group can not act on Σ with exactly one fixed point.*

Our proof of Theorem A goes on by showing the following lemmas. For a compact manifold X and for an integer $k \geq 0$, we denote by X_k the totality of k -dimensional connected components of X . For a set Y , we denote by $|Y|$ the cardinality of Y . Let Ξ be an oriented 4-dimensional homology sphere.

Lemma B. *If a compact Lie group G of dimension ≥ 1 acts effectively on Ξ , then Ξ^G is empty or diffeomorphic to S^n with $n \leq 2$. Especially one has $|\Xi_0^G| = 0$ or 2.*

Lemma C. *If a finite group G acts on Ξ , then one has $|\Xi_0^G| \leq 2$.*

For a G -action on Ξ we define $K=K(G, \Xi)$ to be the subgroup of G of elements preserving the orientation of Ξ . If a finite group G acts on Σ^4 with $|\Sigma^G|=1$, then by Furuta's theorem we have $G \neq K$, moreover we will see $|\Sigma_0^K| \geq 3$ in Section 5. This contradicts Lemma C.

We wish to express our gratitude to M. Furuta for informing us of his result.

2. Preliminary

Let G be a compact Lie group, H a subgroup of G and X a compact G -manifold of dimension n . If X_k^H is non-empty, then take an H -equivariant normal bundle of X_k^H in X . The fibers of it are $(n-k)$ -dimensional real H -representations. We call them the *normal representations of X_k^H in X* . We remark that if the G -action on X is effective, then the normal representations are faithful.

We frequently use the following well known result.

Theorem (P.A. Smith [1, Theorem 5.1]). *If G is a p -group (p prime) and if it acts on a mod p homology sphere X , then X^G is empty or a mod p homology sphere.*

The following lemma is well known and easily proved.

Lemma 2.1. *If a compact Lie group G acts on S^n with $n \leq 2$, then S^G is empty or diffeomorphic to S^m with $m \leq 2$.*

3. Proof of Lemma B

Let G be a compact Lie group of dimension ≥ 1 and Ξ an oriented 4-dimensional homology sphere with G -action. Suppose that the G -action is effective. Let G_0 be the identity component of G .

Proposition 3.1. *If G_0 has an abelian normal subgroup $A \neq \{1\}$, then Ξ^G is empty or diffeomorphic to S^m with $m \leq 2$.*

Proof. Each element of G_0 preserves the orientation on Ξ . Since the G -action is effective, we have $\dim \Xi^B \leq 2$ for any subgroup $B \neq 1$ of G_0 . Let C be a cyclic subgroup of A of prime order. By Smith's theorem Ξ^C is a sphere of dimension ≤ 2 . By Lemma 2.1 $\Xi^G = (((\Xi^C)^A)^H)^G$, $H = G_0$, is empty or also a sphere.

Proposition 3.2. *It holds that*

- (1) *if $G_0 = SO(3)$, then Ξ^G is empty or diffeomorphic to S^m with $m \leq 1$,*
- (2) *if $G_0 = SU(2)$, then $|\Xi^G| = 0$ or 2, and*

(3) if $G_0 = SO(4)$, then $|\Xi^G| = 0$ or 2.

Proof. The proof is done under the assumption $\Xi^G \neq \phi$ and the notation $H = G_0$.

(1) Since $SO(3)$ has no irreducible k -dimensional representations for $k=2$ and 4, we have $\dim \Xi^H = 1$. Take a dihedral subgroup D of G_0 of order 4. Then we have $\dim \Xi^D = 1$. By Smith's theorem Ξ^D is a circle. Thus Ξ^H coincides with Ξ^D . By Lemma 2.1 we have that $\Xi^G = (\Xi^H)^G$ is a sphere of dimension ≤ 1 .

(2) Since $SU(2)$ has no faithful representations of dimension ≤ 3 , Ξ^H is a finite set. Furthermore the normal H -representations of Ξ^H in Ξ are unique up to isomorphisms. For any cyclic subgroup C of H of prime order, Ξ^C is a sphere and includes Ξ^H . Observing the normal representations of Ξ^H , we see that Ξ^C consists of exactly two points. For any non-trivial subgroup B of H , we have $1 \leq |\Xi^B| \leq 2$. Let T be a maximal toral subgroup of H . We have $|\Xi^T| = 2$ by Smith's theorem. If $\Xi^T - \Xi^H$ is non-empty, then denote the point by x . There is a subgroup L of H such that (i) L has a normal subgroup Q of order 8 and L/Q has order 3 and (ii) $L \cap T \neq \{1\}$. By Oliver's theorem [11, Proposition 2] we have that $|\Xi^L| = 2$, hence $\Xi^L = \Xi^T$. Since the smallest subgroup of H which includes T and L is H , we have $H_x = H$. This contradicts the assumption $\{x\} = \Xi^T - \Xi^H$. Thus $\Xi^T = \Xi^H$ and Ξ^G also consists of exactly two points.

(3) The conclusion follows from (2) and the fact that $SO(4)$ has a normal subgroup isomorphic to $SU(2)$.

Proof of Lemma B. Suppose that $|\Xi^G| \neq 0$ nor 2. Then G is a subgroup of $O(4)$ and G_0 is a subgroup of $SO(4)$. By Proposition 3.1 G_0 does not have an abelian normal subgroup except $\{1\}$. Hence G_0 is isomorphic to either one of $SO(3)$, $SU(2)$ and $SO(4)$. This contradicts Proposition 3.2.

4. Proof of Lemma C

Let G be a finite group, Ξ an oriented 4-dimensional homology sphere with G -action and $K = K(G, \Xi)$ the subgroup of G defined in Section 1. Our proof of Lemma C is done under the assumption that the G -action on Ξ is effective and $\Xi^G \neq \phi$.

First we note that G is a subgroup of $O(4)$, K a subgroup of $SO(4)$ and $\dim \Xi^H \leq 2$ for any non-trivial subgroup H of K .

Proposition 4.1. *Let H be a subgroup of K . Then it holds that*

(1) if $\Xi_2^H \neq \phi$, then H is cyclic, and

- (2) if $\Xi_1^H \neq \phi$, then $\Xi_2^H = \phi$ and H is dihedral or isomorphic to one of A_4 , S_4 and A_5 .

Here S_4 stands for the symmetric group on four letters, and A_n , $n=4$ and 5, stand for alternating groups on n letters.

Proof. (1) It follows from the fact that a finite subgroup of $SO(2)$ is cyclic.

(2) A finite subgroup of $SO(3)$ is cyclic, dihedral or isomorphic to one of A_4 , S_4 and A_5 (see [5]). Suppose that H is cyclic. Then the normal H -representations have even dimensions. This contradicts $\Xi_1^H \neq \phi$.

Proposition 4.2. *Let H be a non-trivial solvable subgroup of K . Then Ξ^H is (empty or) diffeomorphic to S^m with $m \leq 2$.*

Proof. Take a normal series of subgroups $H(i)$ of H : $\{1\} = H(0) \trianglelefteq H(1) \trianglelefteq \dots \trianglelefteq H(n) = H$ with $H(i)/H(i-1)$ of prime order. By Smith's theorem and Proposition 4.1, $\Xi^{H(1)}$ is a sphere of dimension ≤ 2 . Since $\Xi^{H(i)} = (\Xi^{H(i-1)})^{H(i)}$, by induction on i $\Xi^{H(i)}$ are spheres of dimension ≤ 2 .

Proposition 4.3. *Let H be a subgroup of K and suppose H is isomorphic to A_5 . Then it holds that*

- (1) if $\Xi_0^H \neq \phi$, then $|\Xi^H| = 1$ or 2, and
 (2) if $\Xi^H \neq \phi$ and $\Xi_0^H = \phi$, then Ξ^H is diffeomorphic to S^1 .

Proof. (1) Let V be a normal representation of Ξ_0^H in Ξ . Since V is faithful, $V^H = 0$ and $\dim V = 4$, V is an irreducible H -representation. Let C be a cyclic subgroup of H of order 5. Then we have $V^C = 0$, hence $\Xi_0^C \supset \Xi_0^H (\neq \phi)$. By Proposition 4.2, Ξ^C consists of exactly two points. The relation $\Xi^C \supset \Xi^H$ implies that $|\Xi^H| = 1$ or 2.

(2) In the case Ξ^H is a disjoint union of circles. Let D be a dihedral subgroup of H of order 4. Then Ξ^D is a circle by Smith's theorem. Immediately we have $\Xi^H = \Xi^D \cong S^1$.

Proposition 4.4. *Provided $|\Xi^K| \geq 3$, then every Sylow subgroup of K is either cyclic or dihedral.*

Proof. Let P be a Sylow subgroup of K . Since P is solvable, Ξ^P is a sphere of dimension 1 or 2 by Proposition 4.2. The conclusion follows from Proposition 4.1.

Now we prove Lemma C. We suppose that $|\Xi_0^G| \geq 3$, and we will meet with a contradiction.

We note that $K \neq \{1\}$ and $|\Xi^K| \geq 3$. If K is solvable, then Ξ^K is a sphere, hence $\Xi^G = (\Xi^K)^G$ is also a sphere. We have $|\Xi_0^G| = 0$ or 2. This contradicts

the above assumption. Thus K is non-solvable. By Suzuki's theorem [15, p. 671, Theorem B] and Proposition 4.4, there exist subgroups H, L and Z of K such that (1) $[K:H] \leq 2$, (2) $H = Z \times L$, (3) Z is solvable and (4) L is isomorphic to $PSL(2, q)$. Here q is a prime greater than 4. Since L is non-solvable and $\Xi^L \neq \phi$, L has an irreducible representation of dimension 3 or 4. By Tables 3 and 4 of [8], $PSL(2, q) \cong L$ is nothing but $PSL(2, 5)$. In other words, L is isomorphic to A_5 . By Proposition 4.3 it holds that $\Xi^L \cong S^1$ or $|\Xi^L| \leq 2$. From the assumption that $|\Xi_0^G| \geq 3$, we have $\Xi^L \cong S^1$. Since $\Xi^H = (\Xi^L)^H$, it is isomorphic to S^0 or S^1 , so is Ξ^K . Then Ξ^G is also diffeomorphic to S^m , $m \leq 1$. This is a contradiction.

5. Proof of Theorem A

By Lemma B, it is sufficient to prove the case in which G is a finite group acting effectively on Σ , an oriented 4-dimensional homotopy sphere. The following arguments go on in this case.

Proposition 5.1. *Provided $|\Sigma^G| = 1$, then $|\Sigma^K|$ is finite and an odd number, where K is the subgroup of G defined in Section 1.*

Proof. Suppose $|\Sigma^G| = 1$. By Proposition 4.2, K is non-solvable. It follows from Proposition 4.1 that $\Sigma^K = \Sigma_0^K \amalg \Sigma_1^K$. It holds that

$$\begin{aligned} 1 = \chi(\Sigma^G) &= \chi((\Sigma^K)^G) = \chi((\Sigma_0^K)^G) + \chi((\Sigma_1^K)^G) \\ &\equiv \chi((\Sigma_0^K)^G) \pmod{2} \\ &\equiv \chi(\Sigma_0^K) \pmod{2}. \end{aligned}$$

Thus $|\Sigma_0^K|$ is an odd number. Especially Σ_0^K is non-empty. If Σ_1^K is non-empty, then K is isomorphic to A_5 by Proposition 4.1. In this case, Proposition 4.3 gives that either Σ_0^K or Σ_1^K is empty. This is a contradiction. Hence we have $\Sigma^K = \Sigma_0^K$.

Now we prove Theorem A. Provided $|\Sigma^G| = 1$, then by Furuta's theorem and Proposition 5.1 we have $|\Sigma_0^K| \geq 3$. This, however, contradicts Lemma C. Thus we get the conclusion of Theorem A.

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