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L^p–L^q ESTIMATES FOR THE WAVE EQUATION WITH A TIME-DEPENDENT POTENTIAL

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1. Introduction

We shall study the following Cauchy problem:

$$\text{(CP)} \qquad \begin{cases} (\Box + m)u(t,x) = 0, \\ u(0,x) = 0, \\ \partial_t u(0,x) = f(x). \end{cases}$$

Here the potential $m = m(t,x)$ is non-negative. Our interest on this problem is in

(i) estimates for the solution,
and
(ii) the existence and the uniqueness of the solution,

on as weak conditions of the potential m as possible.

The equations which we have in mind include the free wave equation $(m = 0)$ and the Klein-Gordon equation (m = positive constant). For these special equations, L^2-estimates, together with the existence and the uniqueness of the solution, are well-known. Furthermore, L^p–L^q estimates for the solutions

$$\|u(t)\|_q = \|u(t,x)\|_{L^q_x} \le C_{pq}(t)\|f\|_p, \tag{1.1}$$

which play an important role in the semi-linear problems, have been proved by many authors; for the free wave equation, by Strichartz [6] and Peral [4] for the Klein-Gordon equation, by Marshall-Strauss-Wainger [3]. Here $(1/p, 1/q)$ is in the triangle $T_1T_2T_3$ (see Figure 1), and

$$C_{pq}(t) = Ct^{1-n(1/p-1/q)} \tag{1.2}$$

for the free wave equation (similar constant $C_{pq}(t)$ for the Klein-Gordon equation).

On the other hand, in the case when m is not necessarily a constant, we do not have so many results. As for the existence, Strichartz [7] proved that the Cauchy

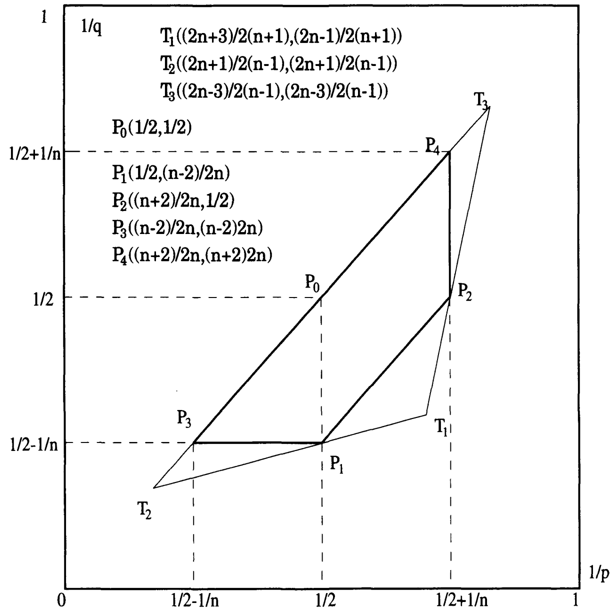

Figure 1: the region of $(1/p, 1/q)$ for L^p–L^q estimate

problem (CP) has the unique solutions $u \in L^{2(n-1)/(n-3)}(\mathbb{R}^n)$ for initial data $f \in L^{2(n-1)/(n+1)}(\mathbb{R}^n)$ and potentials $m = m(t,x) \in L^{(n-1)/2}(\mathbb{R}^n)$. As for L^p–L^q estimates, Beals-Strauss [1] proved that the solution $u(t)$ satisfies the estimate (1.1) at T_1, that is, the estimate with $(1/p, 1/q) = (1/2 + 1/(n+1), 1/2 - 1/(n+1)) = T_1$. They proved it with the same $C_{pq}(t)$ as that of the free wave equation when m is independent of the time variable t and its derivatives up to a certain order are bounded and decay rapidly enough at infinity. We remark here that the estimate (1.1) at T_1 is not always true without the boundedness of m. In fact, if the estimate could be true, the norm $\|u(t)\|_q$ should decay as $t \to \infty$. But if we take $m = |x|^2$, $-\Delta + m$ has positive eigenvalues, and the Cauchy problem (CP) allows a time periodic solution for some initial data. That is a contradiction.

Recently Zhong [8] proved the estimate (1.1) for more general time-independent smooth potentials, including $m = |x|^2$, where the constant $C_{pq}(t)$ is the same as that of the free wave equation, but the region for $(1/p, 1/q)$ is in the trapezoid $P_1P_2P_3P_4$.

The objective of this paper is to extend Zhong's result to the case

(i) when m depends not only on the space variable x, but also on the time variable t, and

(ii) when $m(t,x)$ is not a smooth potential of x.

In general, $C_{pq}(t)$ is not always the same as that of Zhong's (Theorem 2.1). But if m fulfills some more assumptions, it turn out to be the same (Theorem 2.2).

This paper is organized as follows: In Section 2, we shall state our main results. In Sections 3 and 4, we shall prove Theorem 2.1, and in Section 5 Theorem 2.2.

2. Results

Throughout this paper, we assume the following:

Assumption 2.1. *The measurable function* $m(t,x)$ *on* $\mathbb{R}_+^{n+1} = \mathbb{R}_+ \times \mathbb{R}^n$ $(n \geq 3)$ *fulfills the following:*

(i) The function $m(t,x)$ is non-negative.

(ii) There exist the derivative $\partial_t m(t,x)$ and non-negative functions $\mu_+(t), \mu_-(t) \in L^1_{loc}(\mathbb{R}_+)$ such that $-\mu_-(t)m(t,x) \leq \partial_t m(t,x) \leq \mu_+(t)m(t,x)$.

Then we have

Theorem 2.1. *Let* $(1/p, 1/q)$ *be in the trapezoid* $P_1P_2P_3P_4$. *Suppose* $m(t) = m(t,\cdot) \in C^0(\mathbb{R}_+; L^r_{loc}(\mathbb{R}^n))$. *Then, for any* $f \in L^p = L^p(\mathbb{R}^n)$, *there exists a unique weak solution* $u(t) = u(t,\cdot) \in L^\infty_{loc}(\mathbb{R}_+; L^q(\mathbb{R}^n))$ *to* (CP) *which satisfies*

$$\|u(t)\|_q \leq C^1_{pq}(t) t^{1-n(1/p-1/q)} \|f\|_p. \tag{2.1}$$

Here

$$C^1_{pq}(t) = \begin{cases} C \exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right) & \textit{for} \quad (1/p,1/q) \in \triangle P_0P_1P_3, \\ C \exp\left(\frac{1}{2}\int_0^t \mu_-(s)ds\right) & \textit{for} \quad (1/p,1/q) \in \triangle P_0P_2P_4, \\ C \max\left\{\exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right), \exp\left(\frac{1}{2}\int_0^t \mu_-(s)ds\right)\right\} & \\ \qquad \textit{for} \quad (1/p,1/q) \in \triangle P_0P_1P_2 & \end{cases} \tag{2.2}$$

and C *is a constant which depends only on the dimension* n.

We remark that Theorem 2.1 with time-independent C^2 potentials $m = m(x)$ and smooth data $f \in C_0^\infty$, which guarantees the solution to be sufficiently smooth, has been given by Zhong [8]. In the case, $C^1_{pq}(t)$ is just a constant.

On the other hand, in the case when $m(t,x)$ is decreasing with respect to t, $m(t,x) = \exp(-t)$ for example, we have that $\mu_+ = 0$ and $C^1_{pq}(t)$ is a constant function for $(1/p, 1/q) \in \triangle P_0P_1P_3$ while it is not necessarily true for $(1/p, 1/q) \notin \triangle P_0P_1P_3$. We run up against a similar situation in the case when m is increasing with respect to t. But the following theorem says that it is true for $(1/p, 1/q)$ in the whole trapezoid $P_1P_2P_3P_4$ if $|\nabla_x m(t)| = |\nabla_x m(t, \cdot)|$ satisfies some integrability condition.

Theorem 2.2. *Let $(1/p, 1/q)$ be in the trapezoid $P_1P_2P_3P_4$. Suppose $m(t) = m(t, \cdot) \in C^0(\mathbb{R}_+; L^r_{loc}(\mathbb{R}^n) \cap L^1(\mathbb{R}_+; \dot{H}^{n/2})$.*

$$\sup_{t\geq 1} \frac{1}{t^2} \int_0^t \|\nabla_x m(s)\|_{n/2} ds < \infty. \tag{2.3}$$

Then, for any $f \in L^p$, there exists a unique weak solution $u(t) = u(t, \cdot) \in L^\infty_{loc}(\mathbb{R}_+; L^q(\mathbb{R}^n))$ to (CP) *which satisfies*

$$\|u(t)\|_q \leq C^2_{pq}(t) t^{1-n(1/p-1/q)} \|f\|_p. \tag{2.4}$$

Here

$$\begin{aligned} C^2_{pq}(t) = C \exp\left(\int_0^t \|\nabla_x m(s)\|_{n/2} ds\right) \\ \times \min\left\{\exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right), \exp\left(\frac{1}{2}\int_0^t \mu_-(s)ds\right)\right\} \end{aligned} \tag{2.5}$$

and C is a constant which depends only on the dimension n.

We shall consider several examples of $m(t, x)$ which our theorems can be applied to.

EXAMPLE 2.1. Let $m = \exp(\exp(-t))$. Then $C^1_{pq}(t)$ in Theorem 2.1 is a bounded function since both $\mu_+ = 0$ and $\mu_- = \exp(-t)$ are integrable.

EXAMPLE 2.2. Let $m = \exp(t)$ [$m = \exp(-t)$ resp.]. Then $C^2_{pq}(t)$ in Theorem 2.2 is a constant function since $\nabla_x m = 0$ and $\mu_- = 0$ [$\mu_+ = 0$ resp.].

EXAMPLE 2.3. Let $m = \exp(-t)\phi(x)$ where $\phi \in L^r_{loc}(\mathbb{R}^n)$ and $\nabla\phi \in L^{n/2}(\mathbb{R}^n)$. Then $C^2_{pq}(t)$ is a bounded function since $\nabla_x m = \exp(-t)\nabla_x\phi$ and $\mu_+ = 0, \mu_- = 1$. We remark that $m = \exp(t)\phi(x)$ does not have the same property.

EXAMPLE 2.4. Let $m = \psi((t^2+1)^a x)$ where $a > 1/2$, $\psi \in C^1(\mathbb{R}^n)$, $\psi(x) \geq 0$, $\nabla\psi(x) \cdot x \leq 0$ and $\nabla\psi \in L^{n/2}(\mathbb{R}^n)$. Then $C^2_{pq}(t)$ is a bounded function since $\mu_+ = 0$.

3. Proof of Theorem 2.1 (smooth case)

The proof of Theorem 2.1 is divided into two parts. The first part (Section 3) is devoted to prove the estimate (2.1) for smooth potentials and initial data, and the second part (Section 4) the existence and the uniqueness for non smooth ones. Throughout this section, we assume that $m = m(t,x)$ is of $C^0(\mathbb{R}_+; C^\infty(\mathbb{R}^n))$ and $f = f(x)$ is of $C_0^\infty(\mathbb{R}^n)$.

The strategy to prove the estimate for smooth data is as follows: For smooth potential and initial data as assumed above, the problem (CP) has unique smooth solutions $u \in C^2(\mathbb{R}_+^{n+1}) \cap C^1(\mathbb{R}_+; L^2(\mathbb{R}^n)) \cap C^0(\mathbb{R}_+; H^1(\mathbb{R}^n))$ so as to justify the argument below. First we prove the energy estimate (Lemma 3.1), which yields the desired estimate (2.1) at P_0 with $t = 1$. Next, following the same argument as in Zhong [8], we derive the estimate at P_1 with $t = 1$ from the energy estimate as well. The estimate at P_3 is derived from that at P_1. Duality argument and the interpolation theorem imply the estimate at the rest with $t = 1$. Scaling argument yields the estimate with general $t > 0$.

Before proving the theorem, we define some symbols: We write derivatives as $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \partial_{x_j} = \frac{\partial}{\partial x_j}$, and $u_t = \partial_t u$, $u_j = u_{x_j} = \partial_{x_j} u$ for a function u on $\mathbb{R}_+^{n+1}$ and $j = 1, \cdots, n$. We denote the characteristic function of the set E by χ_E, and use the notation $\chi_t(s) = \chi_{[0,t]}(s)$. For $T, R \geq 0$, we define

$$B_R := \{x \in \mathbb{R}^n : |x| \leq R\}, \tag{3.1}$$

$$\Omega_{T,R} := \{(t,x) \in \mathbb{R}_+^{n+1} : 0 \leq t \leq T\ , |x| \leq T + R - t\}, \tag{3.2}$$

$$\Omega'_{T,R} := \{(t,x) \in \mathbb{R}_+^{n+1} : 0 \leq t \leq T, |x| \leq R + t\}. \tag{3.3}$$

For $u \in C_0^\infty(\mathbb{R}_+^{n+1})$, $B \subset \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, we define the energy density $\varepsilon(t,x)$ and the energy $E(t,B)$ on $\{t\} \times B$ by

$$\varepsilon(t,x) := \frac{1}{2}\left\{|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2 + m(t,x)|u(t,x)|^2\right\}, \tag{3.4}$$

$$E(t,B) := \int_B \varepsilon(t,x)dx. \tag{3.5}$$

A constant which depends only on n is denoted by letters C or C', which we are not necessarily the same at different occurrence.

Lemma 3.1. *Let $t, R \geq 0$. Suppose that u satisfies* (CP). *Then*

$$E(t, B_R) \leq \exp(\|\chi_t \mu_+\|_1) E(0, B_{R+t}), \tag{3.6}$$

$$E(t, B_{R+t}) \geq \exp(-\|\chi_t \mu_-\|_1) E(0, B_R). \tag{3.7}$$

REMARK. By mollifying u in a usual way, we extend this estimate for the non-smooth case, when $u \in C^1(\mathbb{R}_+; L^2(\mathbb{R}^n)) \cap C^0(\mathbb{R}_+; H^1(\mathbb{R}^n))$. We shall use this fact in Section 4.

Proof. We may assume that u is real-valued. Since u satisfies (CP), we have

$$\begin{aligned}\partial_s \varepsilon(s,x) &= u_s\,(u_{ss} - \Delta u + mu) + \sum_{j=1}^{n} (u_s u_j)_j + \frac{1}{2} m_s u^2 \\ &= \sum_{j=1}^{n} (u_s u_j)_j + \frac{1}{2} m_s u^2.\end{aligned}$$

By $m_s \leq \mu_+ m$, we obtain

$$\begin{aligned}&\partial_s \left(\exp\left(-\int_0^s \mu_+(\sigma) d\sigma \right) \varepsilon(s,x) \right) - \sum_{j=1}^{n} \partial_j \left(\exp\left(-\int_0^s \mu_+(\sigma) d\sigma \right) u_s u_j \right) \\ =& \exp\left(-\int_0^s \mu_+(\sigma) d\sigma \right) \cdot \left(\frac{1}{2} m_s u^2 - \mu_+ \varepsilon \right) \\ =& \frac{1}{2} \exp\left(-\int_0^s \mu_+(\sigma) d\sigma \right) \cdot \left(-(\mu_+ m - m_s) u^2 - \mu_+ (u_s^2 + |\nabla_x u|^2) \right) \\ \leq& 0.\end{aligned}$$

Integrating this over $\Omega_{t,R}$, we have (3.6) since we obtain, from the Stokes formula,

$$\begin{aligned}0 &\geq \exp\Big(-\int_0^t \mu_+(\sigma) d\sigma \Big) E(t, B_R) - E(0, B_{R+t}) \\ &\quad + \frac{1}{\sqrt{2}} \int_{\text{side}} \exp\left(-\int_0^s \mu_+(\sigma) d\sigma \right) \left(\varepsilon - u_s n(\nu) \cdot (\nabla_x u) \right) d\nu \\ &\geq \exp\Big(-\int_0^t \mu_+(\sigma) d\sigma \Big) E(t, B_R) - E(0, B_{R+t}),\end{aligned}$$

where "side" $= \partial\Omega_{t,R} - (\{t\} \times B_R \cup \{0\} \times B_{R+t})$ and $(1/\sqrt{2}, n(\nu)/\sqrt{2})$ is the exterior unit normal at $\nu \in$ side. Here non-negativity of the integrand

$$\varepsilon - u_s n(\nu) \cdot (\nabla_x u) = \frac{1}{2} \sum_{j=1}^{n} \left(u_j - \frac{x_j}{|x|} u_s \right)^2 + \frac{1}{2} m u^2 \geq 0$$

at $\nu \in$ side has been used. The estimate (3.7) is carried by the same argument if we replace μ_+ by $-\mu_-$ and $\Omega_{t,R}$ by $\Omega'_{t,R}$.

From this lemma, we have the estimate (2.1) at P_0 with $t = 1$. In fact, since $u(0) = 0$,

$$\begin{aligned}\|u(1)\|_2 &= \int_0^1 \partial_t \|u(t)\|_2 dt \\ &\leq \int_0^1 \|\partial_t u(t)\|_2 dt \\ &\leq \int_0^1 \sqrt{2E(t, \mathbb{R}^n)} dt \\ &\leq \sqrt{2E(0, \mathbb{R}^n)} \exp\left(\frac{1}{2}\int_0^1 \mu_+(s)ds\right) \\ &\leq \exp\left(\frac{1}{2}\int_0^1 \mu_+(s)ds\right) \|f\|_2.\end{aligned}$$

Here we have used

$$\partial_t \|u(t)\|_2 \leq \|u_t(t)\|_2, \tag{3.8}$$

which is derived from

$$\begin{aligned}2\|u(t)\|_2 \partial_t \|u(t)\|_2 &= \partial_t \|u(t)\|_2^2 \\ &= \partial_t \int u(t) \cdot u(t) dx \\ &= 2 \int u(t) \partial_t u(t) dx \\ &\leq 2\|u(t)\|_2 \|u_t(t)\|_2.\end{aligned}$$

Next we shall prove the estimate (2.1) at P_1. For $p = 2, q = (\frac{1}{2} - \frac{1}{n})^{-1}$, we have, by Lemma 3.1 and Sobolev's lemma [5, Theorem 2 p.124],

$$\begin{aligned}\|u(1)\|_q &\leq C\|\nabla_x u(1)\|_p \\ &\leq C\{2E(1, \mathbb{R}^n)\}^{1/2} \\ &\leq C \exp\left(\frac{1}{2}\int_0^1 \mu_+(s)ds\right) \{2E(0, \mathbb{R}^n)\}^{1/2} \\ &\leq C \exp\left(\frac{1}{2}\int_0^1 \mu_+(s)ds\right) \|f\|_p.\end{aligned} \tag{3.9}$$

In order to derive the estimate at P_3, we shall prepare a partition of unity and a covering lemma. Let $\{B^j\}_{j \in \mathbb{N}}$ be a covering of $\mathbb{R}^n$ of finite multiplicity by closed unit balls: $B^j = \{x : |x - x_j| \leq 1\}$, $\cup_{j\in\mathbb{N}} B^j = \mathbb{R}^n$, $\exists A < \infty$ such that

$$\sum_{j=1}^{\infty} \chi_{B^j}(x) \leq A. \tag{3.10}$$

Let $\{\phi^j\}_{j\in\mathbb{N}}$ be a smooth function which is subordinate to $\{B^j\}_{j\in\mathbb{N}}$ which satisfies $0 \leq \phi_j \leq 1$ and $\sum_{j\in\mathbb{N}} (\phi^j)^q = 1$.

We decompose f by $\{\phi^j\}_{j\in\mathbb{N}}$ as $f^j := \phi^j f$. We denote by u^j the solution of (CP) with $f = f^j$. Then we have,

$$f = \sum_{j=1}^{\infty} f^j, \qquad u = \sum_{j=1}^{\infty} u^j, \tag{3.11}$$

$$\operatorname{supp} f^j \subset \{y : |y - x_j| \leq 1\}, \qquad \operatorname{supp} u^j \subset \{(t, y) : |y - x_j| \leq t + 1\}. \tag{3.12}$$

Here we have noticed that Lemma 3.1 shows the finite speed propagation. For $r > 0$ we set $B^j(r) = \{y \in \mathbb{R}^n : |y - x_j| \leq r\}$ and $\Lambda_j(r) = \{i : B^i \cap B^j(r) \neq \emptyset\}$. Then $\Lambda_j(r)$ satisfies

Lemma 3.2. *We have*

$$\sup_{j\in\mathbb{N}} \#\Lambda_j(r) \leq A(r + 2)^n, \tag{3.13}$$

where A is the same constant as in (3.10) *and $\#S$ denotes the cardinal number of the set S.*

Proof. If $i \in \Lambda_j(r)$, then $|x_j - x_i| \leq r + 1$ and therefore $B^i \subset B^j(r + 2)$. If $x \in B^j(r + 2)$, then the number of B^i which contains x is, at most, A. From these facts, we obtain

$$\begin{aligned}
\#\Lambda_j(r) &= \sum_{i\in\Lambda_j(r)} |B^i|/|B_1| \\
&\leq A \left|\cup_{i\in\Lambda_j(r)} B^i\right| / |B_1| \\
&\leq A|B^j(r + 2)|/|B_1| \\
&\leq A(r + 2)^n,
\end{aligned}$$

where B_1 is a unit ball.

Now we shall derive the estimate (2.1) at P_3 with $t = 1$ from that at P_1. For

$(1/p, 1/q) = (1/2 - 1/n, 1/2 - 1/n) = P_3$, we have by the estimate at P_1

$$
\begin{aligned}
\|\phi^j u^i(1)\|_q &\le \|u^i(1)\|_q \\
&\le C \exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right)\|f^i\|_2 \\
&= C \exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right)\|\chi_{B_i} f^i\|_2 \\
&\le C' \exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right)\|f^i\|_p.
\end{aligned}
\tag{3.14}
$$

On the other hand, by the finite speed propagation, we have

$$
\phi^j u(1) = \sum_{i\in\Lambda_j(2)} \phi^j u^i(1). \tag{3.15}
$$

From Lemma 3.2, (3.14) and the Hölder inequality we obtain

$$
\begin{aligned}
\|\phi^j u(1)\|_q &\le \sum_{i\in\Lambda_j(2)} \|\phi^j u^i(1)\|_q \\
&\le C \exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right)\sum_{i\in\Lambda_j(2)} \|f^i\|_p \\
&\le C' \exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right)\left(\sum_{i\in\Lambda_j(2)} \|f^i\|_p^p\right)^{1/p}
\end{aligned}
$$

Then we have by Lemma 3.2 again

$$
\begin{aligned}
\|u(1)\|_q^q &= \sum_{j\in\mathbb{N}} \|\phi^j u(1)\|_q^q \\
&\le C \exp\left(\frac{p}{2}\int_0^t \mu_+(s)ds\right)\sum_{j\in\mathbb{N}}\sum_{i\in\Lambda_j(2)} \|f^i\|_p^p \\
&= C \exp\left(\frac{p}{2}\int_0^t \mu_+(s)ds\right)\sum_{i\in\mathbb{N}}\sum_{j\in\Lambda_i(2)} \|f^i\|_p^p \\
&\le C' \exp\left(\frac{p}{2}\int_0^t \mu_+(s)ds\right)\sum_{i\in\mathbb{N}} \|f^i\|_p^p \\
&= C' \exp\left(\frac{1}{2}\int_0^t \mu_+(s)ds\right)^p \|f\|_p^p.
\end{aligned}
$$

Thus we have obtained the estimate at P_3.

Next we shall prove the estimate at P_2 and P_4 with $t=1$. Let $(1/p, 1/q)$ be at P_2 or P_4. For this purpose, we shall calculate the formal adjoint of the solution operator $S(1,0)$. Here the operator $S(t,s)$ is defined in a usual way as

$$S(t,s)\colon f \longmapsto u(t) \tag{3.16}$$

for the problem

$$(\Box + m(t))u(t) = 0, \quad (u, \partial_t u)|_{t=s} = (0, f). \tag{3.17}$$

Lemma 3.3. *Let ϕ be an element of $C_0^\infty(\mathbb{R}^n)$ and $w(\tau) = w^t(\tau, x)$ a solution of the following Cauchy problem,*

$$(\partial_\tau^2 - \Delta + m(t-\tau))w(\tau) = 0 \quad in \quad \mathbb{R}_+^{n+1}, \tag{3.18}$$

$$\big(w(0), \partial_\tau w(0)\big) = (0, \phi). \tag{3.19}$$

Then

$$(S(t,s)f, \phi) = (f, w(t-s)). \tag{3.20}$$

Proof. We have

$$\begin{aligned}
&\partial_\tau\big\{(\partial_\tau S(\tau,s)f, w(t-\tau)) + (S(\tau,s)f, \partial_\tau w(t-\tau))\big\}\\
=&(\partial_\tau^2 S(\tau,s)f, w(t-\tau)) - (S(\tau,s)f, \partial_\tau^2 w(t-\tau))\\
=&((\Delta - m(\tau))S(\tau,s)f, w(t-\tau)) - (S(\tau,s)f, \partial_\tau^2 w(t-\tau))\\
=&-(S(\tau,s)f, (\Box + m(\tau))w(t-\tau))\\
=&0.
\end{aligned}$$

Here we have used the fact that we obtain $(\Box + m(\tau))w(t-\tau) = 0$ from the equation (3.18) by changing the variable t to $t-\tau$. Integrating this equation from s to t inτ, we have (3.20).

Since the function w is the solution of (3.18) and (3.19), the estimates (2.1) at P_1 and P_3 with $t=1$ imply

$$\|S(1,0)^*\phi\|_{p'} = \|w(1)\|_{p'} \le C\exp\left(\frac{1}{2}\int_0^1 \mu_-(\tau)d\tau\right)\|\phi\|_{q'}, \tag{3.21}$$

where $1/p + 1/p' = 1/q + 1/q' = 1$. Here we have used Lemma 3.3 and the facts

$$-\mu_+(1-\tau)m(1-\tau,x) \le \partial_\tau\{m(1-\tau,x)\} \le \mu_-(1-\tau)m(1-\tau,x), \tag{3.22}$$

$$\int_0^1 \mu_\pm(\tau)d\tau = \int_0^1 \mu_\pm(1-\tau)d\tau. \tag{3.23}$$

By the duality argument, we have the estimate

$$\|S(1,0)u\|_q \le C\exp\left(\frac{1}{2}\int_0^1 \mu_-(\tau)d\tau\right)\|f\|_p \tag{3.24}$$

at P_2 and P_4 with $t=1$. Thus we have obtained the estimate (2.1) at all the vertex of the trapezoid $P_1P_2P_3P_4$ with $t=1$. By interpolation [2, Theorem1.1.1 p.2] we have the estimate at the rest of the points.

In the last place, we shall calculate the dependence of the estimate on time variable t by the scaling argument. We set $\tilde{u}^\sigma(\tau,x)=u(\sigma\tau,\sigma x)$, $\tilde{m}^\sigma(\tau,x)=\sigma^2 m(\sigma\tau,\sigma x)$, $\tilde{f}^\sigma(x)=\sigma f(\sigma x)$. Then these functions satisfy

$$(\partial_\tau^2+\tilde{m}^\sigma(\tau,x))\tilde{u}^\sigma=0 \qquad \text{in} \qquad \mathbb{R}^{n+1}_+, \tag{3.25}$$

$$\tilde{u}^\sigma(0,\cdot)=0, \partial_\tau\tilde{u}^\sigma(0,\cdot)=\tilde{f}^\sigma, \tag{3.26}$$

and

$$-\tilde{\mu}^\sigma_-(\tau)\tilde{m}^\sigma(\tau,x)\le\partial_\tau\tilde{m}^\sigma(\tau,x)\le\tilde{\mu}^\sigma_+(\tau)\tilde{m}^\sigma(\tau,x), \tag{3.27}$$

where $\tilde{\mu}^\sigma_\pm(\tau)=\sigma\mu_\pm(\sigma\tau)$. Then we obtain from the estimate (2.1) with $\tau=1$

$$\|\tilde{u}^\sigma(1)\|_q\le\tilde{C}^\sigma_{pq}(1)\|\tilde{f}^\sigma\|_p, \tag{3.28}$$

where

$$\tilde{C}^\sigma_{pq}(1)=\int_0^1\tilde{\mu}^\sigma_\pm(\tau)d\tau=\int_0^\sigma\mu_\pm(\tau)d\tau=C^1_{pq}(\sigma). \tag{3.29}$$

On the other hand, we have

$$\|\tilde{u}^\sigma(1)\|_q=\sigma^{-n/q}\|u(\sigma)\|_q, \tag{3.30}$$

$$\|\tilde{f}^\sigma\|_p=\sigma^{1-n/p}\|f\|_p. \tag{3.31}$$

Letting $\sigma=t$ in (3.28), (3.30) and (3.31), we obtain (2.1). Thus we have proved the required estimate in Theorem 2.1 in a smooth case.

4. Proof of Theorem 2.1 (non-smooth case)

In this section, we shall consider the uniqueness and the existence for non-smooth potentials and initial data.

First we shall consider the uniqueness of the weak solution. It is enough to show that a solution u of (CP) with $f = 0$ satisfies $\chi_{B_R} u(t) = 0$ for $0 \le t \le T$ with any $R > 0$ and $T > 0$.

Let t be in $(0, T)$ for any fixed T. We have the integral equation

$$u(t) = -\int_0^t E(t-s) * (mu)(s)ds, \tag{4.1}$$

where $E(\tau)* = \sin\tau|D_x|/|D_x|$ is the convolution with the fundamental solution for the free wave equation. The supporting property of $E(\tau)$ yields

$$\chi_{B_R} u(t) = -\chi_{B_R} \int_0^t E(t-s) * \chi_{B_{R+t-s}}(mu)(s)ds. \tag{4.2}$$

Taking the L^q-norm, we have for $0 \le t \le T$

$$\begin{aligned} \|\chi_{B_R} u(t)\|_q &\le C_T \int_0^t \|\chi_{B_{R+t-s}} mu(s)\|_p ds \\ &\le C_T \int_0^t \|\chi_{B_{T+R}} m(s)\|_r \|\chi_{B_{R+t-s}} u(s)\|_q ds \\ &\le C_T \sup_{0\le s\le T} \|\chi_{B_{T+R}} m(s)\|_r \int_0^t \|\chi_{B_{R+t-s}} u(s)\|_q ds. \end{aligned}$$

Here we have used the Hölder inequality and the L^p–L^q estimate

$$\|E(s) * \|_{p,q} = \sup_{\|g\|_p=1} \|E(s) * g\|_q \le C s^{1-n(1/p-1/q)} \tag{4.3}$$

obtained by Strichartz [6]. By Gronwall's inequality, we have $\|\chi_{B_{R+t-s}} u(t)\|_q = 0$ for $0 \le s \le t$ and therefore $\|u(t)\|_q = 0$ for $0 \le t \le T$.

Next we shall prove the existence of the weak solution and the estimate (2.1) for u. We may assume $f \in C_0^\infty(\mathbb{R}^n)$ because the general case can be obtained from it by the approximation argument. We may also assume $r \ne \infty$, because the case where $m \in C^0(\mathbb{R}_+; L^\infty_{loc})$ guarantees existence of a solution $u \in C^0(\mathbb{R}_+; H^1) \cap C^1(\mathbb{R}_+; L^2)$, which is smooth enough to satisfy the energy estimate Lemma 3.1.

We take a smooth non-negative function ϕ of x which satisfies $\|\phi\|_1 = 1$ and is supported in $\{|x| < 1\}$. We define $\phi^\eta(x) := \eta^{-n}\phi(\eta^{-1}x)$ and approximate m by $m^\eta(t,x) = m * \phi^\eta(t,x) \in C^0(\mathbb{R}_+; C^\infty(\mathbb{R}^n))$.

Now we shall consider an approximated Cauchy problem

$$(\Box + m^\eta)u^\eta = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \tag{4.4}$$

$$(u^\eta, \partial_t u^\eta)|_{t=0} = (0, f), \tag{4.5}$$

which is also expressed in the integral form

$$u^\eta(t) = E(t) * f - \int_0^t E(t-s) * (m^\eta u^\eta)(s) ds. \tag{4.6}$$

We shall show the existence of $\lim_{\eta \searrow 0} u^\eta(t)$ in $L^q(\mathbb{R}^n)$, which is to be the weak solution to (CP). Since $u^\eta - u^\theta$ is the solution of the approximated problem

$$(\Box + m^\eta)(u^\eta - u^\theta) = (m^\theta - m^\eta)u^\theta \tag{4.7}$$

$$(u^\eta - u^\theta, \partial_t(u^\eta - u^\theta))(0) = (0,0), \tag{4.8}$$

we have

$$u^\eta(t) - u^\theta(t) = \int_0^t S^\eta(t,s)(m^\theta - m^\eta)u^\theta(s) ds. \tag{4.9}$$

Here $S^\eta(t,s)$ denotes the solution operator $f \mapsto u$ to (3.17) with m replaced by m^η.

Assume $\theta, \eta < 1$ so that $\operatorname{supp} \phi^\theta \subset \{|x| \le 1\}$ and $\operatorname{supp} \phi^\eta \subset \{|x| \le 1\}$. By the finite speed propagation and the compactness of $\operatorname{supp} f$, we have for some $R > 0$

$$\begin{aligned} u^\eta(t) - u^\theta(t) &= \chi_{B_{t+R}}(u^\eta(t) - u^\theta(t)) \\ &= \chi_{B_{t+R}} \int_0^t S^\eta(t,s)(m^\theta(s) - m^\eta(s))u^\theta(s) ds \\ &= \chi_{B_{t+R}} \int_0^t S^\eta(t,s)\chi_{B_{2t+R}}(m^\theta(s) - m^\eta(s))u(s) ds \\ &= \chi_{B_{t+R}} \int_0^t S^\eta(t,s)\chi_{B_{2t+R}}(\phi^\eta - \phi^\theta) * (\chi_{B_{1+2t+R}} m(s))u^\theta(s) ds. \end{aligned}$$

Hence, by the Hölder inequality,

$$\begin{aligned} &\|u^\eta(t) - u^\theta(t)\|_q \\ &\le \int_0^t \|S^\eta(t,s)\|_{p,q} \|(\phi^\eta - \phi^\theta) * (\chi_{B_{1+2t+R}} m(s))u^\theta(s)\|_p ds \\ &\le \int_0^t \|S^\eta(t,s)\|_{p,q} \|(\phi^\eta - \phi^\theta) * (\chi_{B_{1+2t+R}} m(s))\|_r \|u^\theta(s)\|_q ds. \end{aligned}$$

On the other hand, applying the estimate (2.1) to the problem (4.4) and (4.5) with the potential m^η, we have the L^p–L^q estimate

$$\|u^\eta(t)\|_q \le C^1_{pq}(t) t^{1-n(1/p-1/q)} \|f\|_p, \tag{4.10}$$

or more generally,

$$\|S^\eta(t,s)\|_{p,q} \le C^1_{pq}(t,s)(t-s)^{1-n(1/p-1/q)}, \tag{4.11}$$

for $(1/p, 1/q)$ in the trapezoid $P_1P_2P_3P_4$. Here $C^1_{pq}(t,s)$ is defined by (2.2) with $\int_0^t$ replaced by $\int_s^t$. Then for fixed T, (4.10) and (4.11) lead to

$$\|u^\eta(t) - u^\theta(t)\|_q \le C_T \int_0^t \|(\phi^\eta - \phi^\theta) * (\chi_{B_{1+2t+R}} m)(s)\|_r ds \|f\|_p \tag{4.12}$$

for $0 \le t \le T$. We remark here that C_T can be taken independently of η and $theta$ by the definition of m^η and C^1_{pq}. Futhermore, we have

$$\|(\phi^\eta - \phi^\theta) * (\chi_{B_{1+2t+R}} m(s))\|_r \to 0 \tag{4.13}$$

as $\eta, \theta \searrow 0$ since $\phi^\eta * g \to g$ in L^r as $\eta \searrow 0$ for $g \in L^r(\mathbb{R}^n)$ $(r \ne \infty)$. Hence $\{u^\eta(t)\}_\eta$ is a Cauchy sequence in L^q and has a limit

$$u^0(t) \in L^\infty_{loc}(\mathbb{R}_+; L^q(\mathbb{R}^n)) \tag{4.14}$$

as $\eta \searrow 0$, which satisfies, by (4.10),

$$\begin{aligned} \|u^0(t)\|_q &= \lim_{\eta \searrow 0} \|u^\eta(t)\|_q \\ &\le C^1_{pq}(t) t^{1-n(1/p-1/q)} \|f\|_p. \end{aligned} \tag{4.15}$$

Thus we have finished the proof of Theorem 2.1.

5. Proof of Theorem 2.2

In this section, we shall prove Theorem 2.2. We have only to show the estimate for $f \in C_0^\infty(\mathbb{R}^n)$. In fact,

$$\|u(1)\|_q \le C \exp\left(\int_0^1 \|\nabla_x m(s)\|_{n/2} ds + \frac{1}{2}\int_0^1 \mu_-(s) ds\right) \|f\|_p, \tag{5.1}$$

from (5.1) and (2.1), we obtain

$$\begin{aligned} \|u(1)\|_q \le C \exp&\left(\int_0^1 \|\nabla_x m(s)\|_{n/2} ds\right) \\ &\min\left\{\exp\left(\frac{1}{2}\int_0^1 \mu_+(s) ds\right), \exp\left(\frac{1}{2}\int_0^1 \mu_-(s) ds\right)\right\} \|f\|_p. \end{aligned} \tag{5.2}$$

This estimate is extended to all $(1/p, 1/q)$ in the whole trapezoid $P_1P_2P_3P_4$ and we have the estimate (2.4) by scaling argument.

Let $(1/p, 1/q) = (1/2, 1/2 - 1/n)$ and u be the solution to (CP). Then $\partial_j u$ is equal to the solution v of the problem

$$(\Box + m)v = -(\partial_j m)u,$$
$$(v, \partial_t v)|_{t=0} = (0, \partial_j f).$$

Hence we have an integral equation

$$\partial_j u(t) = S(t,0)(\partial_j f) - \int_0^t S(t,s)(\partial_j m(s))u(s)ds, \tag{5.3}$$

where the solution operator $S(t,s)$ is defined as in (3.16). Operating the Riesz potential of order 1 on both sides of this equality, we get

$$\begin{aligned} R_j u(t) =& I^1 \partial_j u(t) \\ =& I^1 S(t,0)\partial_j f - \int_0^t I^1 S(t,s)(\partial_j m(s))u(s)ds \\ =& I^1 S(t,0) I^{-1} R_j f - \int_0^t I^1 S(t,s) I^{-1} I^1\big((\partial_j m(s))u(s)\big)ds. \end{aligned} \tag{5.4}$$

Here we call the operator $I^\alpha = \mathcal{F}^{-1}|\xi|^{-\alpha}\mathcal{F}$ the Riesz potential of order α, and $R_j = \mathcal{F}^{-1}\xi_j/|\xi|\mathcal{F}$ the Riesz transform.

In order to estimate $\|u(1)\|_q$, we shall consider the L^p–L^q operator norm $\|I^1 S(t,s) I^{-1}\|_{p,q}$. Applying Lemma 3.1 to the Cauchy problem (3.18) and (3.19), we obtain from Lemma 3.3

$$\|\nabla_x S(t,s)^* \phi\|_2 \le C \exp\left(\frac{1}{2}\int_s^t \mu_-(\tau)d\tau\right) \|\phi\|_2. \tag{5.5}$$

Hence we have for $t > s$

$$\begin{aligned} \|I^{-1} S(t,s)^* I^1 u(s)\|_2 \le& C\|\nabla_x S(t,s)^* I^1 u(s)\|_2 \\ \le& C \exp\left(\frac{1}{2}\int_s^t \mu_-(\tau)d\tau\right) \|I^1 u(s)\|_2 \\ \le& C' \exp\left(\frac{1}{2}\int_s^t \mu_-(\tau)d\tau\right) \|u(s)\|_{(1/2+1/n)^{-1}}. \end{aligned}$$

Here we have used the equivalence of the operator norms of I^{-1} and ∇_x and the L^2–$L^{(1/2+1/n)^{-1}}$ bondedness of the Riesz potential. This implies, by duality,

$$\|I^1 S(t,s) I^{-1}\|_{p,q} \le \exp\left(\frac{1}{2}\int_s^t \mu_-(\tau)d\tau\right). \tag{5.6}$$

On the other hand, we have

$$
\begin{aligned}
\|I^1((\nabla_x m)(s)u(s))\|_p &\le C\|(\nabla_x m)(s)u(s))\|_{(1/2+1/n)^{-1}} \\
&\le C'\|\nabla_x m(s)\|_{n/2}\|u(s)\|_q,
\end{aligned} \tag{5.7}
$$

by the mapping property of the Riesz potential again and Hölder's inequality. Applying (5.6) and (5.7) to (5.4), we have

$$
\begin{aligned}
\|u(1)\|_q &\le C\exp\left(\frac{1}{2}\int_0^1 \mu_-(\tau)d\tau\right)\|f\|_p \\
&+ C\int_0^1 \exp\left(\frac{1}{2}\int_s^1 \mu_-(\tau)d\tau\right)\|\nabla_x m(s)\|_{n/2}\|u(s)\|_q ds,
\end{aligned} \tag{5.8}
$$

where we have used the equivalence of the norms $\|g\|$ and $\sum_{j=1}^n \|R_j g\|$. Denoting $\exp\left(-\frac{1}{2}\int_0^1 \mu_-(\tau)d\tau\right)\|u(1)\|_q$ by $J(1)$, we rewrite the above inequality as

$$
J(1) \le \|f\|_p + \int_0^1 \|\nabla_x m(s)\|_{n/2}J(s)ds. \tag{5.9}
$$

Then we have, by Gronwall's inequality,

$$
J(1) \le C\exp\left(\int_0^1 \|\nabla_x m(s)\|_{n/2}ds\right)\|f\|_p, \tag{5.10}
$$

that is (5.1). Thus we have proved Theorem 2.2.

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