<table>
<thead>
<tr>
<th>Title</th>
<th>Positive linear functionals on ideals of continuous functions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Wada, Junzo</td>
</tr>
<tr>
<td>Citation</td>
<td>Osaka Mathematical Journal. 11(2) P.173–P.185</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1959</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/6516">https://doi.org/10.18910/6516</a></td>
</tr>
<tr>
<td>DOI</td>
<td>10.18910/6516</td>
</tr>
<tr>
<td>rights</td>
<td></td>
</tr>
</tbody>
</table>
Positive Linear Functionals on Ideals of Continuous Functions

By Junzo WADA

Let \( N \) be the set of all continuous functions on a compact Hausdorff space or the set of all continuous functions whose carriers are compact on a locally compact Hausdorff space. Then any positive linear functional \( T \) on \( N \) has an integral representation (Kakutani [12] and Halmos [8]), so any \( T \) has the condition \( (MA') \), i.e. \( T(f_n) \) converges to \( T(f) \) for any \( f \in N \) and for any sequence \( \{f_n\} \subseteq N \) with \( f_n \uparrow f \). Let \( X \) be a locally compact space and let \( Y \) be the one-point compactification of \( X \) ([1], p. 93). Then we can regard the set of all continuous functions whose carriers are compact on \( X \) as an ideal (\( = l \)-ideal. § 1) of \( C(Y) \), the set of all real-valued continuous functions of \( Y \). V.S. Varadarajan [16] raised the following question: Let \( X \) be a compact Hausdorff space and let \( N \) be an ideal of \( C(X) \). When can we say that all non-negative linear functionals on \( N \) satisfy the condition \( (MA') \)? An ideal \( N \) is said to satisfy the property \( (A) \) if \( T \) satisfies the condition \( (MA') \) for any non-negative linear functional \( T \) on \( N \) (§ 1). In this paper we consider more generalized problems. After some preliminaries in § 1 we consider in § 2 the above problem in the case where \( X \) is a completely regular space. We characterize ideals which satisfy the property \( (A) \) under some conditions (Theorem 4). In § 3 we prove that any \( m \)-closed (ring-) ideal satisfies \( (A) \) (Theorem 5), and in § 4 we show that an \( \alpha \)-ideal satisfies the stronger property \( (B) \) (§ 1) if it satisfies \( (A) \) in the case where \( X \) is a normal \( Q \)-space (Theorem 6).

§ 1. Preliminaries.

Throughout this paper, spaces are always completely regular Hausdorff spaces.

For a space \( X \), a subset \( N \) in \( C(X) \) will be called an \( l \)-ideal \( \Phi \) (or, briefly, an ideal) if the following conditions are satisfied:

(i) if \( f, g \in N \), then \( f + g \in N \),
(ii) if \( f \in N \) and \( t \) is any real number, then \( tf \in N \),
(iii) if \( f \in N \), \( |g| \leq f \), then \( g \in N \).

1) See, Bourbaki [3]. Varadarajan [16] used the term "\( \sigma \)-smooth" in place of "the condition \( (MA') \)."
2) See, Birkhoff [2].
3) For any function \( f \), \( |f| = |f(x)| \).
Let $X$ be a space and let $f$ be in $C(X)$. Then we put
\[ Z(f) = \{ x \mid x \in X, \quad f(x) = 0 \}, \]
\[ P(f) = \{ x \mid x \in X, \quad f(x) > 0 \}, \]
\[ \mathcal{P}(X) = \{ P(f) \mid f \in C(X) \}. \]

Let $N$ be an ideal. Then we put
\[ Z(N) = \bigcap_{f \in N} Z(f), \]
\[ \mathcal{P}(N) = \{ P(f) \mid f \in N \}. \]

Let $N$ be an ideal and let $T$ be a non-negative linear functional. Then $T$ is said to satisfy the condition $(MA')$ (resp. $(MA)$) if $T(f_n)$ (resp. $T(f_{\alpha})$) converges to $T(f)$ for any $f \in N$ and for any sequence $\{ f_n \} \subseteq N$ (resp. for any directed set $\{ f_\alpha \} \subseteq N$ with $f_\alpha \uparrow f$ (resp. $f_\alpha \uparrow f)$).

An ideal $N$ is said to satisfy the property $(A)$ (resp. $(B)$) if $T$ satisfies the condition $(MA')$ (resp. $(MA)$) for any non-negative linear functional $T$ on $N$.

Let $N$ be an ideal. Then we put
\[ K = \{ f \mid f \in C(X), \quad \varphi_p(f) \leq \text{some } h \in N \}, \]
\[ K^* = K \cap C^*(X), \]
where $C^*(X)$ is the set of all bounded continuous functions. We denote by $\varphi_A$ the characteristic function of a set $A$. We easily see that $K$ and $K^*$ are ideals and both are contained in $N$. If $X$ is compact, then we have that $K = K^* = \{ f \mid f \in C(X) \}$, the carrier of $f$ is contained in some compact subset of $Y = X - Z(N)$.

Let $N$ be an ideal. Then $N$ is called an $\alpha$-ideal if $f \in N$ for any $f \in C(X)$ with $|f| \wedge n \in N$ ($n=1, 2, 3, \cdots$). If $X$ is compact (or pseudo compact), then any ideal is an $\alpha$-ideal, and if $X$ is locally compact, the set of all continuous functions whose carriers are compact on $X$ is an $\alpha$-ideal. If $N$ is an $\alpha$-ideal and if $f \in K$, then we have that $K = K^* = \{ g \mid g \in C(X), \quad P(|g|) \subseteq P(f) \}$.

Let $X$ be any space. Then E. Hewitt [10] introduced a Baire measure on $\mathcal{P}(X)$. Let $N$ be an $\alpha$-ideal and let $T$ be a non-negative linear functional. Similarly, we can introduce a countably additive measure on $\mathcal{P}(K)$ as follows.

Let $G$ be any set in $\mathcal{P}(K)$. We define the measure $\gamma(G)$ as $\sup T(f)$,
where \( f \) runs through the set of all functions in \( K \) such that \( 0 \leq f \leq \varphi_G \).

By the similar method as Hewitt [10], we have

1. a) \( G \subseteq H \) implies that \( \gamma(G) \leq \gamma(H) \),
   b) \( 0 \leq \gamma(G) < +\infty \)
   c) \( \gamma(0) = 0 \),
   \( G \) and \( H \) being arbitrary sets in \( \Psi(K) \).
2. \( \gamma(G \cup H) \leq \gamma(G) + \gamma(H) \) for any \( G, H \) in \( \Psi(K) \).
3. If \( G, H \in \Psi(K) \) and \( G \cap H = 0 \), then \( \gamma(G \cup H) = \gamma(G) + \gamma(H) \).
4. Let \( G_n, G \) be in \( \Psi(K) \) and let \( G \subseteq \bigcap_{n=1}^{\infty} G_n \). Then \( \gamma(G) \leq \sum_{n=1}^{\infty} \gamma(G_n) \).
   For any subset \( A \subseteq X \), we put
   \[
   \gamma^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \gamma(G_n), A \subseteq \bigcap_{n=1}^{\infty} G_n, G_n \in \Psi(K) \right\}
   \]
   if this set is non-empty, and \( \gamma^*(A) = +\infty \) otherwise.

   Then we have
5. a) \( 0 \leq \gamma^*(A) \) for any \( A \subseteq X \),
   b) \( \gamma^*(A) \leq \gamma^*(B) \) if \( A \subseteq B \),
   c) \( \gamma^*\left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \gamma^*(A_n) \) for all \( \{A_1, A_2, \ldots, A_n, \ldots\} \),
   d) \( \gamma^*(G) = \gamma(G) \) for any \( G \in \Psi(K) \).
6. Every set in \( \Psi(K) \) is measurable with respect to the outer measure \( \gamma^* \).
7. The outer measure \( \gamma^* \) is countably additive on the family \( \overline{\Psi(K)} \),
   where \( \overline{\Psi(K)} \) is the smallest family which contains \( \Psi(K) \) and closed under the formation of complements and of countable unions.
8. For any non-negative function \( f \in K \), there exists some \( a > 0 \) such that \( \gamma[x \in X, 0 < f(x) \leq a] = \gamma(P(f)) \).
   If \( X \) is a locally compact space and if \( N \) is the set of all continuous functions on \( X \) whose carriers are compact, then we easily see that \( \gamma(G) = \mu(G) \) for any \( G \in \Psi(N) = \Psi(K) \), whose \( \mu \) is the measure introduced by Halmos ([8], p. 247, Theorem 8).

   By the similar method as Hewitt, we have that for any \( \alpha \)-ideal \( N \) and for any \( f \in K \), \( T(f) = \int f(x) \, d\gamma(x) \). If \( T \) satisfies the condition \( (MA') \) and if \( f \) is a non-negative function in \( N \), then \( g_n = f - f \wedge n^{-1} \uparrow f \) and \( g_n \in K \), so \( T(f) = \int f(x) \, d\gamma(x) \). Therefore we have

   Let \( N \) be an \( \alpha \)-ideal. Then a non-negative linear functional \( T \) satisfies the condition \( (MA') \) if and only if there exists a countably additive measure \( \gamma \) on \( \overline{\Psi(K)} \) for which

\[
T(f) = \int f(x) \, d\gamma(x) \quad (f \in N).
\]
Let $X$ be a space and let $\mathcal{O}(X)$ be the set of all open subsets in $X$. By a Borel measure, we shall mean a real-valued function $\gamma$ defined on $\mathcal{O}(X)$ which is countably additive, where $\mathcal{O}(X)$ is the smallest family which contains $\mathcal{O}(X)$ and closed under the formation of complements and of countable unions.

Let $N$ be a set of continuous functions such that (i) $N$ is a linear lattice, (ii) if $f \in N$, then $1 \land f \in N$ and (iii) for any closed subset $F$ and for any point $p$ with $p \notin F$, there is an $f \in N$ such that $f(F) = 0$, $f(p) = 1$ and $0 \leq f(x) \leq 1$. Then Ishii [11] proved the following: Let $T$ be a positive linear functional on $N$ having the condition (MA). Then there is a reducible $^5$ Borel measure $\gamma$ on $X$ such that $T(f) = \int f(x) d\gamma(x) \ (f \in N)$.

Similarly, we have

Let $N$ be an ideal and let $T$ be a positive linear functional on $N$ having the condition (MA). Then there is a reducible Borel measure $\gamma$ on $Y = X - Z(N)$ such that

$$T(f) = \int_Y f(x) d\gamma(x) \quad (f \in N).$$

§ 2. Property (A).

We first prove the following lemmas.

**Lemma 1.** Let $N$ be an $\alpha$-ideal and let $T$ be a non-negative functional on $N$. Then the restriction $T_0$ on $K$ of $T$ satisfies the condition (MA').

Proof. By § 1, there is a measure $\gamma$ such that for any $f \in K$ $T_0(f) = \int f(x) d\gamma(x)$, so the lemma is clear.

This lemma can also be proved directly.

**Lemma 2.** If $N$ is an $\alpha$-ideal, then it is a ring, i.e. if $f, g \in N$, then $fg \in N$.

Proof. Let $f \in N$ $f \geq 0$ and let $m$ be a natural number. Then $mf - (f^2 \land m) \geq mf - (f \land m) = (m - f) \land m \geq 0$, or $mf \geq f^2 \land m$. Since $f \in N$ and $N$ is an $\alpha$-ideal, $f^2 \in N$. If $f, g \in N$, then $fg \in N$ since $(|f| + |g|)^2 \geq 4|fg|.$

We can prove the following theorem.

**Theorem 1.** Let $N$ be an $\alpha$-ideal. Then the following conditions are equivalent:

5) A measure $\gamma$ on $X$ is said to be reducible if there is a closed subset $F$ in $X$ such that $F$ is measurable and $\gamma(X - F) = 0$. (Cf. [13]).
(1) $N$ satisfies the property (A).

(2) If $T$ is a non-negative functional on $N$ such that $T(K^*)=0$, then $T$ is identically zero.

(3) If $T$ is a non-negative functional on $N$ such that $T(K)=0$, then $T$ is identically zero.

Proof. (1)$\rightarrow$(2). Suppose that there is a positive functional $T$ on $N$ such that $T(K^*)=0$ and $T(f)=1$ for some $f\in N$, $f\geq 0$. Put $f_n=(f-n)\lor(f\land n^{-1})$. Then $f_n\downarrow 0$. We easily see that $\varphi_{p,f-f_n}<nf\in N$ and $0\leq f-f_n\leq n$, so $f-f_n\in K^*$ and $T(f)-T(f_n)=0$, or $T(f_n)=T(f)=1$ for any $n$. This shows that (1) does not hold.

(2)$\rightarrow$(3). Clear.

(3)$\rightarrow$(1). Let $T$ be a non-negative linear functional on $N$. For any $f\in N$, $f\geq 0$, we put $T'(f)=\inf\lim_{n,\to\infty}T(f_n)$, where $f_n\geq 0$ ($n=1,2,3,\ldots$) and $f_n\uparrow f$, and the infimum is taken for all sequences $\{f_n\}$ such that $f_n\uparrow f$, $f_n\geq 0$ and $f_n\in N$. Then we have that for any $f$, $g\in N$, $g\geq 0$, $T'(f+g)=T'(f)+T'(g)$ and for any real number $t\geq 0$, $T'(tf)=tT'(f)$. For any arbitrary function $f\in N$, we define $T'(f)=T'(f^+)-T'(f^-)$, where $f^+$ and $f^-$ denotes $f\lor 0$ and $(-f)\lor 0$ respectively. Then $T'$ is a linear functional on $N$ and $T\geq T'$. Put $T''=T-T'$, then $T''$ is non-negative linear functional on $N$. But, by Lemma 1, $T''(K)=0$ and by (2) $T''\equiv 0$, so $T'=T$. This shows that $N$ has the property $(A)^0$.

If $N$ is an ideal which is not an $\alpha$-ideal, we can easily see that Theorem 1 does not always hold.

DEFINITION. An ideal $N(=N_{f_0})$ will be called a principal ideal if there exist a non-negative function $f_0\in N$ such that $N=\{g|g\in C(X), |g|\leq \alpha f_0$ for some $\alpha\geq 0\}$. An ideal $N$ will be called a $0$-principal (resp. $\infty$-principal) if there exists a non-negative $s$-function (resp. an unbounded function) $f_0$ such that $N=\{g|g\in C(X), |g(x)|\leq \alpha f_0(x)$ on $U_m$ for some $\alpha>0$ and some natural number $m\}$ (resp. $N=\{g|g\in C(X), |g(x)|\geq \alpha f_0(x)$ on $V_m$ for some $\alpha>0$ and $m\}$), where $U_m=\{x|x\in X, 0\leq f(x)<m^{-1}\}$ and $V_m=\{x|x\in X, f(x)>m\}$. A positive function $f$ is said to be an $s$-function if it admits any small value, i.e. $U_m$ is not empty for any $m$. If $X$ is compact, then any 0-principal ideal is principal, but it is not true in general.

Theorem 2. (1) A principal ideal $N(=N_{f_0})$ fulfills the condition $(A)$ if and only if $Z(f_0)$ is open, $Y=X-Z(f_0)$ is pseudo-compact\(^6\) and $N=\{f|f\in C(X), f(Z(f_0))=0\}$ (it is lattice-isomorphic to $C(Y)$).

\(^6\) A topological space $X$ is said to be pseudo-compact if any continuous function on $X$ is bounded.
(2) Any 0-principal (or ∞-principal) ideal \( N (=N_{f_0}) \) does not fulfill the condition (A).

Proof. (1) Suppose that \( N \) fulfills (A). Then we put \( U_n = \{ x \mid x \in X, 0 < f_0(x) < n^{-1} \} \). If for any \( n \) \( U_n \) is not empty, we can select a point \( x_n \) in \( U_n \). We put \( M = \{ g \mid g \in N, \lim_{n \to \infty} g(x_n)/f_0(x_n) \text{ exists} \} \). For any \( g \in M \), we define \( T(g) = \lim_{n \to \infty} g(x_n)/f_0(x_n) \). Then \( T \) is a positive linear functional on \( M \). For any \( g \in N_{f_0} \) there exists an \( m > 0 \) such that \( |g| \leq mf_0 \). Since \( mf_0 \in M \), \( T \) is extended to a positive linear functional on \( N_{f_0} \) (Cf. [4] p. 20). We denote it again with \( T \). If \( f_n = f_0 \wedge 1/n \), we have that \( f_n \downarrow 0 \) and \( T(f_n) = 1 \) for any \( n \). Since \( T \) satisfies (MA'), it is a contradiction. This fact shows that \( U_m \) is empty for some \( m \), or \( f(x) \geq m^{-1} \) for any \( x \) with \( f(x) = 0 \). Therefore \( Z(f_0) \) is open, so \( Y = X \setminus Z(f_0) \) is open and closed. Let \( f' \) be the restriction of \( f \) on \( Y \). Then \( N_{f'} (\subset C(Y)) \) satisfies the property (A). For any non-negative linear functional \( T^* \) on \( C^*(Y) \) and for any \( h \in N_{f'} \), we define \( T(h) = T^*(h/f') \). Then \( T^*(g) = T_1(f'g) \) for any \( g \in C^*(Y) \). We easily see that \( C^*(Y) \) satisfies (A). By Glucksberg [5], \( Y \) is pseudo-compact and \( N_{f'} = C^*(Y) = C(Y) \). The converse is clear by [5].

(2) We define \( U_n, M \) and \( T \) as (1). Then \( T \) is a positive linear functional on \( M \). For any \( g \in N \), there are a positive integer \( m \) and \( \alpha > 0 \) such that \( |g(x)| \leq \alpha f_0(x) \) on \( U_m \). We put \( h = \alpha f_0 \wedge |g| \). Then \( h \in M \) and \( |g| \leq h \), so \( T \) is extended to a positive functional on \( N \). If \( f_n = f_0 \wedge n^{-1} \), then \( T(f_n) = 1 \) for any \( n \) and \( f_n \downarrow 0 \). This is a contradiction.

REMARK. If \( X \) is an infinite (completely regular) space, then there is an \( \alpha \)-ideal in \( C(X) \) which does not satisfy (A). For, if \( X \) is infinite, then there is an \( \alpha \)-function \( f \in C(X) \), so the 0-principal ideal \( N_f \) does not satisfy (A) (Theorem 2. (2)). We easily see that \( N_f \) is an \( \alpha \)-ideal.

DEFINITION. A directed set \( \{ f_\alpha \}_{\alpha \in A} \) of positive functions (\( \subset N \)) is called a base of an ideal \( N \) if for any \( f \in N \) there is an \( f_\alpha \) such that \( |f| \leq mf_\alpha \) for some \( m \).

Let \( f \) be a positive \( s \)-function in \( C(X) \) and let \( g \) be any function in \( C(X) \). Then we define

\[
\lim_{f \to 0} g/f = \lim_{n \to \infty} \sup_{x \in U_n} g(x)/f(x),
\]

\[
\lim_{f \to 0} g/f = \lim_{n \to \infty} \inf_{x \in U_n} g(x)/f(x),
\]

where \( U_n = \{ x \mid x \in X, 0 < f(x) < n^{-1} \} \).

If \( \lim_{f \to 0} g/f = \lim_{f \to 0} g/f \), we write simply \( \lim_{f \to 0} g/f \) (admits \( +\infty \)).
Theorem 3. Let \( N \) be an \( \alpha \)-ideal and let \( \{f_\alpha\}_{\alpha \in A} \) be a base in \( N \). If for any \( s \)-function \( f_\alpha \) there is an \( f_\beta \) such that \( \lim_{f_\alpha \to 0} f_\beta/f_\alpha = \infty \), then \( N \) satisfies the property (A).

Proof. Suppose that \( N \) does not satisfy (\( A \)). By Theorem 1 there exists a positive functional \( T \) such that \( T(K) = 0 \) and \( T(f) = 1 \) for some positive function \( f \in N \). Since \( \{f_\alpha\}_{\alpha \in A} \) is a base in \( N \), there is an \( f_\alpha \) and a positive constant \( c \) such that \( 0 \leq f_\alpha \leq cf_\alpha \). Now let \( f_\alpha \) be an \( s \)-function. Then by the hypothesis, there is an \( f_\beta \) such that \( \lim_{f_\alpha \to 0} f_\beta/f_\alpha = \infty \).

Therefore, for any positive number \( M \) there is an \( m \) such that \( f_\beta(x) \geq Mf_\alpha(x) \) if \( x \in U_m \). We set \( W_m = \{x \mid x \in X, 0 \leq f_\alpha(x) < m^{-1}\} \) and \( F = X - W_m \). Then if \( x \in W_m \), \( f_\beta(x) \leq Mf_\alpha(x) \). Let \( h \) be a function in \( K^* \) such that \( h(F) = 1 \). Then we easily see that \( Mf_\alpha h + f_\beta \geq Mf_\alpha \), or \( cMf_\alpha h + cf_\beta \geq cf_\alpha \geq Mf_\alpha \). Since \( f_\alpha h \in K \), \( T(f_\alpha h) = 0 \), so \( cT(f_\beta) \geq M \). But \( M \) is an arbitrary positive number. This is a contradiction.

Next, let \( f_\alpha \) be not an \( s \)-function. Then if \( f_\alpha(x) \neq 0 \), \( f_\alpha(x) \geq \delta \) for some positive number \( \delta \). The set \( P = \{x \mid x \in X, f_\alpha(x) > 0\} \) is open and closed and \( N \supset \{f \mid f \in C(X), f(Z(f_\alpha)) = 0\} \). Since \( N \) is an \( \alpha \)- ideal, \( N \supset N_0 = \{f \mid f \in C(X), f(Z(f_\alpha)) = 0\} \) and \( K \supset N_0 \). Since \( T(K) = 0 \), \( T(N_0) = 0 \). But \( f \in N_0 \) and \( T(f) = 1 \). This is a contradiction.

Finally, we characterize ideals which satisfy the property (\( A \)) under some conditions. We see that these conditions are necessary as the later example shows.

Theorem 4. Let \( N \) be an \( \alpha \)-ideal and let it have a base \( \{f_\alpha\} \) such that for any \( s \)-function \( f_\alpha \) and for any \( f_\beta \) with \( \beta \geq \alpha \) (\( \alpha \) depends on \( f \), \( \lim_{f_\alpha \to 0} f_\beta/f_\alpha \) exists (admits \( + \infty \)). Then \( N \) satisfies the property (\( A \)) if and only if \( N \) is not \( 0 \)-principal.

Proof. If \( N \) satisfies (\( A \)), then by Theorem 2. (2), \( N \) is not \( 0 \)-principal. Conversely, suppose that \( N \) is not \( 0 \)-principal. Then for any \( f_\alpha \) which is an \( s \)-function, there exists an \( f_\beta \) such that \( \lim_{f_\alpha \to 0} f_\beta/f_\alpha = \infty \). For, otherwise, there would exist an \( s \)-function \( f_\alpha \) such that for any \( f_\gamma \in \{f_\alpha\} \), \( \lim_{f_\alpha \to 0} f_\gamma/f_\alpha \leq \text{some } M_\gamma < + \infty \), i.e. if \( x \in U_m \), then \( f_\gamma(x) \leq M_\gamma f_\alpha(x) \) for some \( m \) and \( M_\gamma > 0 \), so \( N \) would be a \( 0 \)-principal ideal \( N_{f_\alpha} \). This is a contradiction. We can here assume that for any \( \alpha \) the above \( \beta \geq \alpha \). Therefore, by the hypothesis, for any \( s \)-function \( f_\alpha \) there is an \( f_\beta \) such that \( \lim_{f_\alpha \to 0} f_\beta/f_\alpha = \infty \). By Theorem 4 \( N \) satisfies (\( A \)).

Let \( X \) be a locally compact space and let \( N \) be the set of all continuous functions on \( X \) whose carriers are compact. Then \( N^c = \{f \mid f \in N, \)
$f \geq 0\} \text{ forms a base which satisfies the hypothesis of Theorem 4. The ordering of the directed system for the base can be defined as follows: } \alpha \succ \beta \text{ if } f_{\alpha} \geq \varphi_{P(f_{\beta})} \text{ for any } f_{\alpha}, f_{\beta} \text{ in } N^+.$

**Example.** The hypothesis in Theorem 4 is necessary. The following example shows it. Let $X$ be the closed interval $[0, 1]$ and let $N$ be an ideal having a base $\{f_n\}$. For any $n$ we define: $f_n(x) = x$ if $x = 2^{-2m}$ or $x = 0 \ (m = 0, 1, 2, \ldots)$, $f_n(x) = x^{1/n}$ if $x = 2^{-(2m^2 + 1)} \ (m = 0, 1, 2, \ldots)$ and it is linear on the intervals $[2^{-(m^2 + 1)}, 2^{-m}] \ (m = 0, 1, 2, \ldots)$. We see that $N$ is an $\alpha$-ideal (since $X$ is compact) and is not 0-principal. But $N$ does not satisfy (A). For, Put $M = \{f|f \in N, \lim_{n \to \infty} 2^{2n}f(2^{-2n}) \text{ exists}\}$. Define $T(f) = \lim_{n \to \infty} 2^{2n}f(2^{-2n})$ for any $f \in M$. $T$ is extended to a positive linear functional on $N$ (Cf. [4]. p. 20) Set $g_m = f_1 \wedge m^{-1}$. Then we have that $g_m \downarrow 0$ and $T(g_m) = 1$ for any $m$, so $N$ does not satisfy (A).

§ 3. Ring-ideals.

A subset $N$ in $C(X)$ is called a ring-ideal\(^7\) if it satisfies the following conditions:

(i) if $f, g \in N$, then $f + g \in N$,

(ii) if $f \in N$ and if $h \in C(X)$, then $hf \in N$.

A ring-ideal $N$ is said to be $m$-closed if $N$ is closed in the $m$-topology $C(X)$. Any neighborhood of $f \in C(X)$ in the $m$-topology is the set $\{g|g \in C(X), |g - f| < \pi\}$ for some everywhere positive function $\pi \in C(X)$ according to Hewitt [9]. Shirota [15], and Gillman, Henrikson, and Jerison [7] proved that any $m$-closed ring-ideal is an intersection of some maximal ring-ideals. We shall show that any $m$-closed ring-ideal is an $\alpha$-ideal and it satisfies (A) (Cf. Theorem 5).

The following lemma is proved by [16] in the case where $X$ is compact.

**Lemma 3.** Let $N$ be an $\alpha$-ideal and let it have the property such that if $f \in N$ then $|f|^{1/\pi} \in N$. Then $N$ satisfies the property (A).

**Proof.** Suppose that a positive functional $T$ on $N$ satisfies the property such that $T(K) = 0$ and $T(f) = 1$ for some positive $f \in N$ (Cf. Theorem 1). We put $g_n = (nf - f^{1/\pi}) \vee 0$. Then $nf \geq \varphi_{P(g_n)}$ and $g_n \in K$. $0 = T(g_n) \geq T(nf - f^{1/\pi})$, or $T(f^{1/\pi}) \geq nT(f) = n$ for any $n$. This contradiction proves the lemma.

We can easily prove the following lemmas.

---

\(^7\) We use the word "ring-ideal" to avoid the confusion.
**Lemma 4.** If $N$ is a maximal ideal (=l-ideal), then it satisfies the property (A).

Proof. By Lemma 3, it is sufficient prove that (i) for any positive $f$ in $N$, $f^{1/2} \in N$ and (ii) $N$ is an $\alpha$-ideal.

(i) Suppose that $f \in N$ and $f^{1/2} \notin N$. Since $N$ is maximal, the set \{h \in C(X), \lambda f^{1/2} + g \geq |h| \text{ for some positive } g \in N \text{ and for some } \lambda > 0 \} is identical to $C(X)$. Therefore $\lambda f^{1/2} + g \geq f^{1/2}$ for some positive $g \in N$ and for some $\lambda > 0$, or $g \geq f^{1/2} - \lambda f^{1/2} = f^{1/2}(1 - \lambda f^{1/2})$. For any $x$ in $X$ with $f(x) \leq (2\lambda)^{-4}$, $g(x) \geq 1/2 f^{1/2}(x)$, or $2g(x) \geq f^{1/2}(x)$. For any $x$ in $X$ with $f(x) \geq (2\lambda)^{-4}$, $(2\lambda)^3 f(x) - f^{1/2}(x) = f^{1/2}(x)((2\lambda)^3 f^{1/2}(x) - 1) \geq 0$, or $(2\lambda)^3 f(x) \geq f^{1/2}(x)$. Therefore $2g \vee (2\lambda)^3 f \geq f^{1/2}$, and so $f^{1/2} \in N$. By Lemma 2, we have $f^{1/2} \in N$. This contradication proves (i).

(ii) Let $f$ be a positive function in $C(X)$ such that for any $n$ $f \wedge n \in N$ and $f \notin N$. Since $N$ is a maximal ideal, the set \{h \mid \lambda f + g \leq |h| \text{ for some positive } g \in N \text{ and for some } \lambda > 0 \} is identical to $C(X)$. Therefore $\lambda f + g \geq f$ for some positive $g \in N$ and $\lambda > 0$, or $g \geq f - \lambda f = f(\lambda - 1)$. For $x \in X$ with $f(x) \geq 1 + \lambda$, we have $g(x) \geq f(x)$. For $x \in X$ with $f(x) \leq 1 + \lambda$, we can select a natural number $n$ such that $n \geq 1 + \lambda$. If we put $f_n = f \wedge n$, then $(1 + \lambda)^{1/2} f_n^{1/2}(x) \geq f(x)$. Therefore $g \vee (1 + \lambda)^{1/2} f_n^{1/2} \geq f$. Since $f_n^{1/2} \in N$ by (i), $f \in N$.

**Lemma 5.** A maximal ring-ideal is a maximal ideal.

Proof. Let $M$ be a maximal ring-ideal. Then we must first prove that it is an ideal. We put $M_0 = \{f \mid f \in C(X), |f| \leq \alpha g \text{ for some positive } g \in M \text{ and some } \alpha > 0 \}$. Then $M_0$ is a proper ring-ideal (for, $M_0 \nsubseteq 1$ since $M \nsubseteq 1$), and $M \subseteq M_0$, so $M_0 = M$, i.e. $M$ is an ideal. To prove the lemma, it is sufficient to show that if $N$ is a maximal ideal, then it is a proper ring-ideal. We put $N_0 = \{f \mid f \in C(X), |f| \leq h g \text{ for some positive } h \in C(X) \text{ and some } g \in N \}$. Then $N_0$ is an ideal and $N \subseteq N_0$. Therefore it is sufficient to prove that $N_0$ is proper. Suppose that $N_0 \subseteq C(X)$. Then there exist $h \in C(X)$ and $g \in N$ such that $hg \geq 1$, so $g$ is everywhere positive. If we put $N' = \{fg^{-1} \mid f \in N \}$, then $N'$ is a maximal ideal and $N' \supseteq 1$. By the proof of Lemma 4, $N'$ is an $\alpha$-ideal, so $N' = C(X)$ and $N = C(X)$. This is a contradiction.

Now we can prove the following theorem.

**Theorem 5.** Any m-closed ring-ideal is an $\alpha$-ideal and it satisfies the property (A).

Proof. Let $N$ be an $m$-closed ring-ideal. Then $N$ is an intersection of some maximal ideals $M_\alpha$ ([15] or [7]). Any $M_\alpha$ is a maximal ideal
(Lemma 5) and by the proof of Lemma 4, any $M_\alpha$ is an $\alpha$-ideal and has the property such that for any positive $f \in M_\alpha$, $f^{1/2} \in M_\alpha$. Therefore $N$ is an $\alpha$-ideal and has the property such that for any positive $f \in N$, $f^{1/2} \in N$. By Lemma 3, $N$ satisfies (A).

**Remark.** If $X$ is a $P$-space (Cf. Gillman and Henriksen [6]), then any ring-ideal in $C(X)$ satisfies (A) since any ring-ideal is $m$-closed ([6], p. 345).

**Example.** An $m$-closed ideal (not a ring-ideal) does not always satisfy the property (A). Such an example is the following: Let $X$ be the semi-line $[0, \infty)$ and let $N = \{ f | f \in C(X), |f(x)| \leq \alpha x$ for some $\alpha > 0$ and for $x \geq 1 \}$. Then we easily see that $N$ is an $m$-closed ideal but it does not satisfy (A) since $N$ is $\infty$-principal (Cf. Theorem 2. (2)).

§ 4. Property (B)

Let $X$ be a locally compact space and let $N$ be the set of all continuous functions whose carriers are compact on $X$. McShane [14] proved that $N$ has the property (B). We can regard $N$ as an ideal in $C(X)$, where $X_0$ is the one-point compactification of $X$. We here consider ideals in $C(X)$, where $X$ is a $Q$-space. $Q$-spaces are considered in [9]. Any separable metric space or any locally compact Hausdorff space which is sum of countable compact subsets is always a $Q$-space [9]. We here show that an $\alpha$-ideal satisfies the property (B) if it satisfies (A) in the case $X$ is a normal $Q$-space.

We first prove the following

**Lemma 6.** Let $X$ be a normal $Q$-space and let $F$ be a closed subset in $X$. Let $Y$ be the decomposition space\(^8\) consisting of $F$ and all elements in $X - F$. Then $Y$ is also a $Q$-space.

**Proof.** Let $\varphi$ be the mapping such that $\varphi^{-1}(y_0) = F$ and $\varphi^{-1}(y)$ is a set consisting of only one point for any $y \in Y$, $y \neq y_0$. Then $\varphi$ is continuous. Now suppose that $Y$ is not a $Q$-space. Then there exists a family

---

8) Let $X$ be a topological space and let $\{ F_\alpha \}$ be a division by closed sets of $X$, i.e. $X = \bigcup F_\alpha$, any $F_\alpha$ is closed and elements of $\{ F_\alpha \}$ are mutually disjoint. We can consider new space $Y$ whose points are $\{ F_\alpha \}$. This space is called the decomposition space of $X$ if for any open set $U \supseteq F_\alpha$, there exists an open set $V \supseteq F_\alpha$ such that $F_\beta \cap V = \emptyset$ implies that $F_\beta \subseteq U$. For any point $y_0 = F_\alpha$ in the decomposition space $Y$, any neighborhood of $y_0$ is the set $\{ y | y = F_\beta$, $F_\beta \subseteq U \}$ for some open set $U$ in $X$ (Cf. [1]). Then there exists a continuous mapping from $X$ onto $Y$. 

such that (i) \( \mathcal{F} \) is \( Z \)-maximal, (ii) any countable family of \( \mathcal{F} \) has a non-void intersection and (iii) \( \bigcap B = 0 \) (Cf. [9]). Let \( \mathcal{D} = \{ A \mid A \in \mathcal{B}, A \supseteq \varphi^{-1}B \} \). Then we shall first prove that \( \mathcal{D} \) is \( Z \)-maximal. If \( \mathcal{D} \) is not \( Z \)-maximal, then there exists an \( A_0 \notin \mathcal{D} \) such that \( \varphi^{-1}B \cap A_0 = 0 \) for any \( B \in \mathcal{F} \). Since \( A_0 \notin \mathcal{D} \), \( A_0 \nsubseteq \varphi^{-1}B \) for any \( B \in \mathcal{F} \), so \( \varphi A_0 \supseteq B \) for any \( B \in \mathcal{F} \). For, if \( \varphi A_0 \supseteq B \) for some \( B \in \mathcal{F} \), by (iii), there exists a \( B_1 \in \mathcal{F} \) such that \( \varphi A_0 \supseteq B_1 \) and \( B_1 \nsubseteq y_0 \). Therefore \( A_0 \nsubseteq \varphi^{-1}B_1 \). This is a contradiction, so \( \varphi A_0 \supseteq B \) for any \( B \in \mathcal{F} \). Let \( A_0 = Z(f) \) (\( f \in C^*(C) \)) and let \( V(y_0) \) be a neighborhood of \( y_0 \) in \( Y \). Then \( f \varphi^{-1} \) is continuous on \( Y - V(y_0) \). Let \( g \) be an extended continous function of \( f \varphi^{-1}(Y - V(y_0)) \) on \( Y \) (\( Y \) is a normal space). Then we have \( Z(g) \subseteq \varphi A_0 \lor V(y_0) \). We take a \( B_1 \in \mathcal{F} B_1 \nsubseteq y_0 \) and \( V(y_0) \) such that \( V(y_0) \cap B_1 = 0 \). To prove that \( Z(g) \notin \mathcal{F} \), we suppose that \( Z(g) \in \mathcal{F} \). Then \( \varphi A_0 \supseteq \varphi A_0 \cap B_1 = (\varphi A_0 \lor V(y_0)) \cap B_2 \supseteq Z(g) \cap B_2 \) and \( Z(g) \cap B_2 \in \mathcal{F} \), this is a contradiction. Since \( \mathcal{F} \) is \( Z \)-maximal, there exists a \( B \in \mathcal{F} \) such that \( B \cap V(y_0) = 0 \). We can assume that \( B \nsubseteq y_0 \) and \( B \cap V(y_0) = 0 \). Then \( \varphi^{-1}B \cap \varphi^{-1}(Z(g)) \supseteq \varphi^{-1}B \cap \varphi^{-1}(\varphi A_0 \lor V(y_0)) \cap \varphi^{-1}(B \cap (\varphi A_0 \lor V(y_0))) = \varphi^{-1}(B \cap \varphi A_0) = \varphi^{-1}B \cap A_0 \). But \( \varphi^{-1}B \cap A_0 = 0 \). This contradiction shows that \( \mathcal{D} \) is \( Z \)-maximal. We easily see that any countable family of \( \mathcal{D} \) has non-empty intersection and \( \bigcap A = 0 \). Therefore \( X \) is not a \( Q \)-space. This shows that \( Y \) is a \( Q \)-space.

By this lemma, we have

**Theorem 6.** Let \( X \) be a normal \( Q \)-space and let \( N \) be an \( \alpha \)-ideal. Then \( N \) satisfies the property (B) if and only if \( N \) satisfies the property (A).

Proof. It is sufficient to prove that if \( N \) satisfies (A), then it satisfies (B). Suppose that \( N \) satisfied (A). Let \( T \) be a positive linear functional on \( N \) and let \( f \) be a positive function \( N \). Then we define \( T'(f) = \inf \lim \alpha T(f_\alpha) \), where any \( f_\alpha \) of a directed set \( \{ f_\alpha \} \) is non-negative and \( f_\alpha \uparrow f \), and the infimum is taken for all directed sets \( \{ f_\alpha \} \) such that \( f_\alpha \uparrow f \), \( f_\alpha \geq 0 \) and \( f_\alpha \in N \). Then we have that for any \( f, g \geq 0 \) in \( N \), \( T'(f + g) = T'(f) + T'(g) \) and \( T'(tf) = t T'(f) \) for any \( t \geq 0 \). For any \( f \in N \), we put \( T'(f) = T'(f^+) - T'(f^-) \). Then \( T' \) is linear functional on \( N \). If we put \( T'' = T - T' \), then \( T'' \geq 0 \). To prove that \( T = T' \), it is sufficient to show that \( T''(K) = 0 \) (Theorem 1). Therefore we have only to show that for any positive function \( f \in K \), \( T|_K \) satisfies the condition (MA), where \( K_f = \{ g \mid g \in K, P(|g|) \subseteq P(f) \} \). Since \( N \) is an \( \alpha \)-ideal, \( K_f = \{ g \mid g \in C(X), P(|g|) \subseteq P(f) \} \). If we put \( F = P(f)^{10} \), we can regard \( K_f \) as the set of

---

9) For any topological space \( X \), we denote by \( \mathcal{Z}(X) \) the family \( \{ Z(f) \mid f \in C(X) \} \).

10) For any subset \( A \), \( \overline{A} \) denotes the closure of \( A \).
all functions in \( C(F) \) vanishing of \( F-P(f) \). Therefore to prove the
Theorem, we have only to show that if \( M \) is the set of all functions in
\( C(F) \) (\( F \) is a \( Q \)-space) vanishing on a fixed closed subset \( A \) in \( F \), then
\( M \) satisfies \((B)\).

(i) If \( A \) is the empty set, then \( M=C(F) \). If \( T \) is a positive linear
functional on \( M \), then there are a Baire measure \( \gamma \) on \( F \) and a compact
set \( C \subset F \) with \( T(f)=\int_C f(x)d\gamma \) (\( f \in M \)) ([10], Theorem 18). Therefore
\( M \) fulfills \((B)^{11}\).

(ii) Let \( A \) be a set consisting of one point and let \( A=(p) \). Let \( T \) be
a positive linear functional. Then \( T \) is continuous, i.e. \( ||T||=\sup_{f \in M} T(f) < +\infty \). We can assume that \( ||T||=1 \). We put for any \( f \in C(F) \), \( T^*(f)=T(f-f(p))+f(p) \). We easily see that \( T^* \) is a positive linear functional
on \( C(F) \). By (i) \( T^* \) satisfies \((MA)\) and so does \( T \).

(iii) If \( A \) is an arbitrary closed subset in \( F \), let \( Y \) be the decom-
position space consisting of \( A \) and \( \{x\}_{x \in F-A} \). Then by Lemma 6 \( Y \) is a
\( Q \)-space. Let \( \phi \) be the mapping such that \( \phi^{-1}(y_0)=A \) and \( \phi^{-1}(y) \) is a set
consisting of only one point for any \( y \in Y, y \neq y_0 \). For any \( f \in M \) we put
\( f'(y)=f(\phi^{-1}y) \), then \( f' \) is continuous on \( Y \). \( M^* = \{f'|f \in M\} \) is the set of
all continuous functions vanishing at \( y_0 \). Let \( T \) be a positive linear functional on \( M \) and let \( T_1(f')=T(f) \) for any \( f' \in M^* \). Then \( T_1 \) is positive
on \( M^* \). By (ii) \( T_1 \) satisfies \((MA)\) and so does \( T \).

Let \( N \) be an ideal and let \( T \) be a non-negative linear functional on
\( N \). \( T \) is said to has the property \((D)\) if \( T(f)=T(g) \) for any \( f,g \in N \)
where \( f=g \) on some open set \( U \supset Z(N) \).

In the case where \( X \) is compact, we have

**Theorem 7.** Let \( X \) be a compact space and let \( N \) be an ideal. Then
\( N \) satisfies the property \((B)\) (or \( (A) \)) if and only if any non-negative linear
functional on \( N \) which has the property \((D)\) is identically zero.

Proof. By Theoreme 1 and 6, we have only to show that \( T \) has
the property \((D)\) if and only if \( T(K)=0 \). Since \( X \) is compact, it is clear.

**Remark.** Any non-negative linear functional \( T \) on \( N \) which has the
property \((D)\) is of the following form. For any \( s \)-function \( f \in N \) we put
\( M=\{g|g \in N, Z(g) \supset Z(f) \} \) and \( \lim_{g \not\to f} g/f \) exists}. Then we have that for
any \( g \in M \)

\[ T(g) = c \lim_{f \not\to a} g/f, \text{ where } c \geq 0 \]

For, if we put \( \lim_{f \not\to a} g/f=0 \), then for any \( \varepsilon > 0 \) there is an \( m \) such that

---

11) This fact is pointed out by [11].
\[(a - \varepsilon)f(x) \leq g(x) \leq (a + \varepsilon)f(x) \text{ if } f(x) \leq m^{-1}. \quad (a - \varepsilon)g \leq g \leq (a + \varepsilon)g \vee g,\]

so \((a - \varepsilon)T(f) = T((a - \varepsilon)f \wedge g) \leq T(g) \leq T((a + \varepsilon)f \vee g) = (a + \varepsilon)T(f)\). Since \(\varepsilon\) is an arbitrary positive number, \(T(g) = aT(f) = c \lim_{f \to g} g/f\), where \(c = T(f)\).

(Received September 17, 1959)

Bibliography
