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Yamazaki, M.
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THE ESSENTIAL SELF-ADJOINTNESS OF PSEUDODIFFERENTIAL OPERATORS ASSOCIATED WITH NON-ELLIPTIC WEYL SYMBOLS WITH LARGE POTENTIALS

MASAO YAMAZAKI

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1. Introduction

In this paper we consider the problem whether pseudodifferential operators associated with real Weyl symbols are essentially self-adjoint on $L^2(\boldsymbol{R}^n)$. There is a great deal of work concerning this problem, especially for Schrödinger operators. (Seen Kato [14], [16, V, 5], [17], [18], Simon [23], Nagase-Umeda [20], T. Ichinose [8], and Iwatsuka [13] and the papers cited therein for example.) Most of them treated elliptic symbols having positive potentials with little regularity. Here we limit our consideration to smooth symbols only, but we do not assume the ellipticity of symbols or the positivity of potentials. Our aim is to give a simple sufficient condition on the growth of potentials for the essential self-adjointness. Besides, we give a counter example which shows the sharpness of our condition.

As an application of the above result, we obtain the $L^2(\boldsymbol{R}^n)$ well-posedness of the Cauchy problem for evolution equations whose evolution operators are time-dependent pseudodifferential operators associated with pure imaginary valued symbols, which include dispersive partial differential equations. Here and in the following non-Kowalevskian non-parabolic partial differential equations of evolution are called dispersive.

By a similar argument, we can show that the above Cauchy problem is well-posed on a family of weighted Sobolev spaces introduced by Beals [1] as well. Petrovskii [22] investigated the well-posedness of these equations with coefficients depending only on the time variable. Volevich [28] and Gindikin [5] generalized the above result to the equations with small potentials. W. Ichinose [9], [10] and Takeuchi [24], [25] studied the $H^\infty(\boldsymbol{R}^n)$ well-posedness of these equations with evolution operators associated with not necessarily formally skew self-adjoint elliptic symbols, and obtained some necessary conditions and sufficient conditions on the growth of the self-adjoint part of the symbols along the classical orbits. They also investigated the $L^2(\boldsymbol{R}^n)$ well-posedness and the

$H^s(\boldsymbol{R}^n)$ well-posedness of these equations in [11] and [26]. So far the author knows no sufficient conditions for the $L^2(\boldsymbol{R}^n)$ well-posedness or $H^\infty(\boldsymbol{R}^n)$ well-posedness of evolution equations with evolution operators having unbounded self-adjoint part with non-elliptic symbols.

Concrete examples of our theory contain Schrödinger equations with bounded magnetic fields and first order electric fields as well as non-elliptic partial differential equations. The results of this paper will be applied in the forthcoming paper [29] to the uniform regularizing effect of linear dispersive partial differential equations.

We introduce some notions in order to state our main theorem. Let $\mathcal{M}$ be the set of C^∞ functions $\varphi(r)$ defined on $[1, \infty[$ satisfying $\varphi(r)>0$, $\varphi(1)=1$ and $|\varphi^{(k)}(r)|\leq C_k r^{-k}\varphi(r)$ with some constant C_k for every $k\in\boldsymbol{N}$, where $\boldsymbol{N}$ denotes the set of nonnegative integers. Further, let $\mathcal{M}_+$ denote the set of monotone-increasing functions in $\mathcal{M}$. In the following we write $\Phi(r)=\int_1^r \varphi(s)\,ds+1$ and $\Psi(r)=\int_1^r \psi(s)\,ds+1$ for $\varphi, \psi\in\mathcal{M}$.

Next we define our symbol classes as follows.

DEFINITION 1.1. For real numbers $\rho, \delta, \rho', \delta'$ satisfying

$$0\leq\delta\leq\rho\leq 1,\ \delta<1,\ 0\leq\delta'\leq\rho'\leq 1,\ \delta'<1 \tag{1.1}$$

and for $\varphi, \psi\in\mathcal{M}$, let $S_{\rho,\delta,\rho',\delta'}(\varphi, \psi)$ denote the set of C^∞-functions $A(x, \xi)$ defined on $\boldsymbol{R}^n_x\times\boldsymbol{R}^n_\xi$ such that

$$\{\partial_\xi^\alpha\partial_x^\beta A(x, \xi)\}\ \varphi(\langle\xi\rangle)^{-1}\langle\xi\rangle^{\rho|\alpha|-\delta|\beta|}\ \psi(\langle x\rangle)^{-1}\langle x\rangle^{\rho'|\beta|-\delta'|\alpha|}$$

is bounded for every $\alpha, \beta\in\boldsymbol{N}^n$, where $\langle x\rangle=\sqrt{|x|^2+1}$ for $x\in\boldsymbol{R}^n$. If $\varphi(r)=r^p$ and $\psi(r)=r^q$, we write $S_{\rho,\delta,\rho',\delta'}(r^p, r^q)=S^{p,q}_{\rho,\delta,\rho',\delta'}$. Functions $A(x, \xi)$ belonging to $S_{\rho,\delta,\rho',\delta'}(\varphi, \psi)$ are often called *symbols*.

REMARK 1.2. The class $S_{\rho,\delta,\rho',\delta'}(\varphi, \psi)$ above coincides with the class $S(m, g)$ in Hörmander [7] and Beals [1] with $m(x, \xi)=\varphi(\langle\xi\rangle)\psi(\langle x\rangle)$ and $g_{(x,\xi)}(y, \eta)=\langle\xi\rangle^{2\delta}\langle x\rangle^{-2\rho'}|y|^2+\langle\xi\rangle^{-2q}\langle x\rangle^{2\delta'}|\eta|^2$.

DEFINITION 1.3. Assume $A(x, \xi)\in S_{\rho,\delta,\rho',\delta'}(\varphi, \psi)$, and put ${}^tA(\xi, x)=A(x, \xi)$. Then, following Hörmander [7], we define the *pseudodifferential operators* $A(X, D)$ associated with the *symbol* $A(x, \xi)$, ${}^tA(D, X)$ associated with the *dual symbol* ${}^tA(\xi, x)=A(x, \xi)$ and $A^w(X, D)$ associated with the *Weyl symbol* $A(x, \xi)$ by the formulas

$$A(X, D)\,u=\int\exp(ix\cdot\xi)\,A(x, \xi)\,\hat{u}(\xi)\,\bar{d}\xi\,,$$

$${}^tA(D, X)\,u=\iint\exp(i(x-y)\cdot\xi)\,A(y, \xi)\,u(y)\,dy\,\bar{d}\xi\,,$$

and

$$A^w(X, D)\,u = \iint \exp\,(i(x-y)\cdot\xi)\,A\Big(\frac{x+y}{2}, \xi\Big)\,u(y)\,dy\,\bar{d}\xi$$

respectively where $\bar{d}\xi=(2\pi)^{-n}\,d\xi$. Here and hereafter we omit the domain of integration if it is the whole space $\boldsymbol{R}^n$.

Then our main theorem is the following.

Theorem 1.4. *Suppose that $\rho, \delta, \rho', \delta'$ satisfy (1.1) and that the functions $\varphi(r), \psi(r)\in\mathcal{M}_+$ satisfy*

$$\varphi(\psi(r))\leq r \quad \textit{for} \quad r\geq 1\,. \tag{1.2}$$

Let $B(x, \xi)$ be a real-valued symbol such that $\partial_{\xi_j}B(x, \xi)\in S^{0,1}_{\rho,\delta,\rho',\delta'}$, $\partial_{x_j}B(x, \xi)\in S^{1,0}_{\rho,\delta,\rho',\delta'}$ and $\partial_{x_j}\partial_{\xi_j}B(x, \xi)\in S^{0,0}_{\rho,\delta,\rho',\delta'}$ hold for every $j=1, \cdots, n$, and let $A(\xi)$ [resp. $C(x)$] be a real-valued symbol independent of the x-variables [resp. ξ-variables] such that $\partial_{\xi_j}A(\xi)$ belongs to the class $S_{0,0,0,0}(\varphi, 1)$ [resp. $\partial_{x_j}C(x)$ belongs to the class $S_{0,0,0,0}(1, \psi)$] for every $j=1, \cdots, n$. Then the pseudodifferential operator $A(D)+B^w(X, D)+C(X)$ defined on $C_0^\infty(\boldsymbol{R}^n)$ is essentially self-adjoint in $L^2(\boldsymbol{R}^n)$, and each of the operators $A(D)+B(X, D)+C(X)$ and $A(D)+{}^tB(X, D)+C(X)$, both defined on $C_0^\infty(\boldsymbol{R}^n)$, can be written as the sum of an essentially self-adjoint operater in $L^2(\boldsymbol{R}^n)$ and a bounded operator on $L^2(\boldsymbol{R}^n)$.

EXAMPLE 1.5. If $\varphi(r)=r^p$ and $\psi(r)=r^q$ for $p, q\geq 0$, the condition (1.2) is equivalent to $pq\leq 1$.

An example for $p=1$ is the symbol $\sum_{l=1}^n(-\xi_l-a_l(x))^2+V(x)$, where $a_l(x)$ and $V(x)$ are real-valued smooth functions satisfying $|\partial_x^\alpha a_l(x)|\leq C_\alpha\langle x\rangle^{1-|\alpha|}$ and $|\partial_x^\alpha V(x)|\leq C_\alpha\langle x\rangle^{2-|\alpha|}$ for every $\alpha\in\boldsymbol{N}^n$. This is the symbol of the Schrödinger operator with bounded magnetic fields and first order electric fields.

An example for $p=2$ is the symbol $\xi_1^3+\xi_2^3+V(x)$, where $V(x)$ is a real valued smooth function satisfying $|\partial_x^\alpha V(x)|\leq C_\alpha\langle x\rangle^{4/3-|\alpha|}$ for every $\alpha\in\boldsymbol{N}^n$. This example illustrates that the ellipticity is not necessary.

Without any positivity assumption, the condition (1.2) is necessary in general. In fact, we have the following proposition, which follows from a result in Dunford-Schwartz [2].

Proposition 1.6. *Let $b(x), C(x)\in C^2(\boldsymbol{R})$ be real-valued functions such that for some $x_0, M\in\boldsymbol{R}$, the function $f(x)=b(x)^2-C(x)$ satisfies $f(x)>-M$ if $x>x_0$,*

$$\int_{x_0}^\infty\Big\{\Big(\frac{f'(x)}{(f(x)+M)^{3/2}}\Big)'+\frac{1}{4}\,\frac{f'(x)^2}{(f(x)+M)^{5/2}}\Big\}\,dx<\infty\,, \tag{1.3}$$

and

$$\int_{x_0}^\infty\frac{1}{(f(x)+M)^{1/2}}\,dx<\infty\,. \tag{1.4}$$

Then, for $B(x, \xi)=2b(x)\xi$, *the operator* $D^2+B^w(X, D)+C(X)$ *defined on* $C_0^\infty(\boldsymbol{R})$ *is not essentially self-adjoint on* $L^2(\boldsymbol{R})$.

This proposition asserts that Theorem 1.4 does not hold if $\varphi(r)=r$ and $\psi(r)=r^q$ for $q>1$. Condition (1.4) is optimal in view of Ikebe-Kato [12].

We apply the previous result to well-posedness on weighted Sobolev spaces. For $\varphi(r), \psi(r)\in\mathcal{M}$, we define weighted Sobolev spaces $H[\varphi, \psi]$ as the set of tempered distributions $u(x)$ such that

$$\|u\|_{H[\varphi,\psi]}=\|\varphi(\langle D\rangle)\,\{\psi(\langle X\rangle)\,u(x)\}\|_{L^2(\boldsymbol{R}^n)}<\infty\,.$$

This corresponds to the class of $H(m, g)$ of Beals [1] with m and g as in Remark 1.2. (This fact will be verified in Section 2.)

For $\varphi, \psi\in\mathcal{M}_+$ satisfying (1.2) and $\sigma\in\boldsymbol{R}$, we put

$$\mathcal{H}^\sigma=\begin{cases} H[\Phi(r)^\sigma, 1]\cap H[1, \Psi(r)^\sigma] & \text{if } \sigma>0\,, \\ L^2(\boldsymbol{R}^n) & \text{if } \sigma=0\,, \\ H[\Phi(r)^\sigma, 1]+H[1, \Psi(r)^\sigma] & \text{if } \sigma<0\,, \end{cases}$$

and equip $\mathcal{H}^\sigma$ with a norm so that it becomes a Hilbert space. (Details will be given in Section 2.)

Next we introduce families of time-dependent symbols.

DEFINITION 1.7. For a positive number T, real numbers $\rho, \delta, \rho', \delta'$ satisfying (1.1) and $\varphi, \psi\in\mathcal{M}_+$ satisfying (1.2), let $S_{\rho,\delta,\rho',\delta'}(T; \varphi, \psi)$ denote the set of functions $A(t, x, \xi)$ defined on $[-T, T]\times\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n$ such that $A(t, \cdot, \cdot)\in S_{\rho,\delta,\rho',\delta'}(\varphi, \psi)$ holds for every $t\in[-T, T]$ and that the function

$$\{\partial_\xi^\alpha\,\partial_x^\beta\,A(t, x, \xi)\}\;\varphi(\langle\xi\rangle)^{-1}\langle\xi\rangle^{\rho|\alpha|-\delta|\beta|}\;\psi(\langle x\rangle)^{-1}\langle x\rangle^{\rho'|\beta|-\delta'|\alpha|}$$

is continuous with respect to $t\in[-T, T]$ uniformly in $(x, \xi)\in\boldsymbol{R}_x^n\times\boldsymbol{R}_\xi^n$ for every $\alpha, \beta\in\boldsymbol{N}^n$. If $\varphi(r)=r^p$ and $\psi(r)=r^q$, we write $S_{\rho,\delta,\rho',\delta'}(T; r^p, r^r)=S_{\rho,\delta,\rho',\delta'}^{p;q}(T)$.

Then we have the following result on the well-posedness of a class of evolution equations on the weighted Sobolev spaces.

Theorem 1.8. *Let* $B(t, x, \xi)$ *be a real-valued time-independent symbol such that* $\partial_{\xi_j}B(t, x, \xi)\in S_{\rho,\delta,\rho',\delta'}^{0;1}(T)$, $\partial_{x_j}B(t, x, \xi)\in S_{\rho,\delta,\rho',\delta'}^{1;0}(T)$ *and* $\partial_{x_j}\partial_{\xi_j}B(t, x, \xi)\in S_{\rho,\delta,\rho',\delta'}^{0;0}(T)$ *hold for every* $j=1, \cdots, n$, *and let* $A(t, \xi)$ *[resp.* $C(t, x)$*] be a real-valued time-dependent symbol independent of the x-variables [resp.* ξ*-variables] such that* $\partial_{\xi_j}A(t, \xi)\in S_{0,0,0,0}(T; \varphi, 1)$ *[resp.* $\partial_{x_j}C(t, x)\in S_{0,0,0,0}(T; 1, \psi)$*] for every* $j=1, \cdots, n$. *Further, let* $\mathcal{A}(t)$ *stand for either* $A(t, D)+B^w(t, X, D)+C(t, X)$, $A(t, D)+B(t, X, D)+C(t, X)$ *or* $A(t, D)+{}^tB(t, D, X)+C(t, X)$. *Then, for every* $u_0(x)\in\mathcal{H}^\sigma$ *and* $f(t, x)\in L^\infty([-T, T], \mathcal{H}^\sigma)$, *there uniquely exists a solution* $u(t, x)$ *of the evolution equation*

$$(1.5)\qquad \begin{cases} \dfrac{\partial u}{\partial t}(t, x) = i\mathcal{A}(t)\,u(t, x)+f(t, x) & \textit{for}\quad (t, x)\in[-T, T]\times\boldsymbol{R}^n \\ u(0, x) = u_0(x) & \textit{for}\quad x\in\boldsymbol{R}^n \end{cases}$$

belonging to the class $\bigcup_{\varphi_0,\psi_0\in\mathcal{M}} AC([-T, T], H[\varphi_0, \psi_0])$*, and it satisfies the following properties*:

$$u(t, x)\in C([-T, T], \mathcal{H}^\sigma)\,,$$
$$\|u(t, \cdot)\|_{\mathcal{H}^\sigma}$$
$$\leq C\{\exp(C\,|t|)\,\|u_0\|_{\mathcal{H}^\sigma}+\operatorname{sgn}(t)\int_0^t \exp(C\,|t-\tau|)\,\|f(\tau, \cdot)\|_{\mathcal{H}^\sigma}\,d\tau\}$$
$$\textit{for some}\quad C>0\,.$$

Here $AC([-T, T], X)$ *denotes the set of* X*-valued absolutely continuous functions defined on* $[-T, T]$ *for a separable Hilbert space* X.

This partly generalizes Theorem 1 of Ozawa [21] concerning the well-posedness of Schrödinger equations with potentials. (Ozawa treated non-smooth potentials as well.)

Suppose that $\psi(r)\in\mathcal{M}_+$ is bounded. Then, for every $\varphi(r)\in\mathcal{M}_+$, there exists a constant C such that (1.2) is satisfied with $\varphi(r)$ replaced by $\varphi(r)/C$. In this case we have well-posedness in more general function spaces. Let $\omega(r)$ be a function in $\mathcal{M}_+$ satisfying $\varphi(\omega(r))\leq r$ for $r\geq 1$, and put

$$\mathcal{K}^{\sigma,\tau} = \begin{cases} H[\Phi^{\sigma+\tau}, 1]\cap H[\Phi^\sigma, \Omega^\tau] & \text{if}\quad \tau>0\,, \\ H[\Phi^\sigma, 1] & \text{if}\quad \tau=0\,, \\ H[\Phi^{\sigma+\tau}, 1]+H[\Phi^\sigma, \Omega^\tau] & \text{if}\quad \tau<0 \end{cases}$$

for every σ, $\tau\in\boldsymbol{R}$, where $\Omega(r)=\int_1^r \omega(s)\,ds+1$. Then we have the following theorem.

Theorem 1.9. *Under the same assumptions as in Theorem* 1.8, *we have the same conclusions as in Theorem* 1.8 *with* $\mathcal{H}^\sigma$ *replaced by* $\mathcal{K}^{\sigma,\tau}$.

This paper is organized as follows. In Section 2 we give several properties of functions belonging to $\mathcal{M}$ and recall several properties of pseudodifferential operators and weighted Sobolev spaces given in Beals [1]. Sections 3 and 4 are devoted to the proof of Theorem 1.4, and Theorems 1.8 and 1.9 respectively. Finally we give a proof of Proposition 1.6 in the Appendix.

2. Some properties of measure functions and symbols

We start with the study of some properties of functions in $\mathcal{M}$.

Proposition 2.1.

1) *The set* $\mathcal{M}$ [*resp.* $\mathcal{M}_+$] *is closed under multiplication and raising to the power of real numbers.* [*resp. multiplication and raising to the power of nonnegative numbers.*]
2) *Every function* $\varphi(r)\in\mathcal{M}$ *is slowly varying, that is, there exists a constant* $C>0$ *such that* $\varphi(s)\leq C\varphi(r)$ *holds for every* $s\in[r/2, 2r]$.
3) *For* $\varphi(r)\in\mathcal{M}$, *we have* $\Phi(r)\in\mathcal{M}_+$, *and there exists a constnat* $C>0$ *such that* $r\varphi(r)\leq C\Phi(r)$.
4) *For* $\varphi(r)\in\mathcal{M}_+$, *we have* $\Phi(r)\leq r\varphi(r)\leq C\Phi(r)$.
5) *For* $\varphi(r), \psi(r)\in\mathcal{M}_+$, (1.2) *is equivalent ot* $\psi(\varphi(r))\leq r$ *for* $r\geq 1$.
6) *For* $\varphi(r), \psi(r)\in\mathcal{M}_+$, *the fact* $\varphi(\psi(r))\geq r/C$ *for* $r\geq 1$ *with some positive constant* C *is equivalent to* $\psi(\varphi(r))\geq r/C'$ *for* $r\geq 1$ *with some positive constant* C'.
7) *For* $\varphi(r), \psi(r)\in\mathcal{M}_+$ *satisfying the equivalent conditions of* 6), *there exists* $a>1$ *such that* $\varphi(2^s)\geq a^s/C$ *and* $\psi(2^s)\geq a^s/C'$ *hold for every* $s>0$.
8) *For* $\varphi(r), \psi(r)\in\mathcal{M}_+$ *satisfying* (1.2), *there exist* $\bar\varphi(r), \bar\psi(r)\in\mathcal{M}_+$ *such that* $\varphi(r)\leq C\bar\varphi(r)$, $\psi(r)\leq C\bar\psi(r)$ *and* $r/C\leq\bar\varphi(\bar\psi(r))\leq r$ *hold for* $r\geq 1$ *with some positive constant* C. *Furthermore, if* $\varphi(r), \psi(r)$ *satisfy the equivalent conditions of* 6), *then the estimates* $\bar\varphi(r)\leq C\varphi(r)$ *and* $\bar\psi(r)\leq C\psi(r)$ *hold with some positive constant* C.

Proof. For $\varphi, \psi\in\mathcal{M}$ and $a\in\boldsymbol{R}$, we have

$$\left|\frac{d^k\{\varphi(r)\,\psi(r)\}}{dr^k}\right|\leq\sum_{j=0}^{k}\binom{k}{j}|\varphi^{(j)}(r)\,\psi^{(k-j)}(r)|$$
$$\leq\sum_{j=0}^{k}\binom{k}{j}Cr^{-j}\,\varphi(r)\,Cr^{j-k}\,\psi(r)\leq Cr^{-k}\,\varphi(r)\,\psi(r)$$

and

$$\left|\frac{d^k\varphi(r)^a}{dr^k}\right|\leq\sum_{j=0}^{k}\sum_{k_1+\cdots+k_j=k}C_{k_1,\cdots,k_j}\,\varphi(r)^{a-j}\{\varphi^{(k_1)}(r)\cdots\varphi^{(k_j)}(r)\}$$
$$\leq\sum_{j=0}^{k}\sum_{k_1+\cdots+k_j=k}C_{k_1,\cdots,k_j}\,\varphi(r)^{a-j}\prod_{l=1}^{j}\{Cr^{-k_l}\,\varphi(r)\}\leq Cr^{-k}\,\varphi(r)^a\,,$$

which shows $\varphi(r)\,\psi(r), \varphi(r)^a\in\mathcal{M}$. It is also clear that if $\varphi, \psi\in\mathcal{M}_+$ and $a\geq 0$, then $\varphi(r)\,\psi(r)$ and $\varphi(r)^a$ are monotone-increasing, and this shows Assertion 1).

Next, integrating the inequality $|\varphi'(\rho)|/\varphi(\rho)\leq C/\rho$ on $[r, s]$, we obtain $|\int_r^s\varphi'(\rho)/\varphi(\rho)\,d\rho|\leq C|\int_r^s(1/\rho)\,d\rho|$, which yields

$$|\log\{\varphi(s)/\varphi(r)\}|\leq C|\log(s/r)|\leq C\log 2\quad\text{for}\quad s\in[r/2, 2r]\,.$$

This implies Assertion 2).

Further, for $\varphi(r)\in\mathcal{M}$, we have $\Phi'(r)=\varphi(r)>0$ and $\Phi(1)=1$. On the other hand, Assertion 2) yields

$$r\varphi(r)=2\varphi(r)\int_{r/2}^{r}dr\leq C\int_{r/2}^{r}\varphi(s)\,ds\leq C\Phi(r)\,.$$

This implies

$$|\Phi^{(k)}(r)|/\Phi(r)\leq C|\varphi^{(k-1)}(r)|/r\varphi(r)\leq Cr^{-k}\,.$$

These estimates prove Assertion 3).

For $\varphi(r)\in\mathcal{M}_{+}$, Assertion 4) follows from the inequality

$$\Phi(r)\leq\varphi(r)\int_{1}^{r}dr+1=(r-1)\,\varphi(r)+\varphi(1)\leq r\varphi(r)\,.$$

In order to prove Assertion 5), it suffices to deduce $\psi(\varphi(r))\leq r$ for $r\geq 1$ from (1.2). If $r\geq M=\sup\{\psi(s);\,s\geq 1\}$, the conclusion is trivial from $\psi(\varphi(r))\leq M$. On the other hand, if $1\leq r<M$, there exists $s\geq 1$ such that $r=\psi(s)$. It follows from (1.2) and the monotonicity of ψ that $\psi(\varphi(r))=\psi(\varphi(\psi(s)))\leq\psi(s)=r$.

Similarly, in order to prove Assertion 6), it suffices to deduce $\psi(\varphi(r))\geq r/C'$ for $r\geq 1$ from $\varphi(\psi(r))\geq r/C$ for $r\geq 1$. First, since $r\leq C\varphi(\psi(r))\leq C\varphi(M)$ holds for all $r\geq 1$, we see $M=\infty$. Hence, for every $r\geq 1$, there exists $s\geq 1$ such that $r=\psi(s)$. Now from the assumption, together with the monotonicity and Assertion 2) for ψ, we conclude $\psi(\varphi(r))=\psi(\varphi(\psi(s)))\geq\psi(s/C)\geq\psi(s)/C'=r/C'$.

We turn to Assertion 7). It follows from Assertion 2) that $\varphi(2^s),\psi(2^s)\leq A^s$ holds for every $s>0$ with some positive constant A. Putting $a=\exp((\log 2)^2/\log A)$, we obtain $\varphi(a^s)\leq 2^s$ and $\psi(a^s)\leq 2^s$ for every $s>0$. Hence we conclude $\varphi(2^s)\geq\varphi(\psi(a^s))\geq a^s/C$ and $\psi(2^s)\geq\psi(\varphi(a^s))\geq a^s/C'$.

Finally we shall prove Assertion 8). If $\lim_{r\to\infty}\psi(r)=\infty$, put $\varphi_0(r)=\sup\{s;\,\psi(s)\leq r\}$ for every $r\geq 1$. Then $\varphi_0(r)$ is monotone-increasing, and we have $\psi(\varphi_0(r))=r$ and $\varphi_0(\psi(r))\geq r$. Let $\chi(r)$ be a C^∞-function on $[1,\infty[$ such that $\chi(r)\geq 0$, $\operatorname{supp}\chi(r)\subset[1,2]$ and $\int_1^2\chi(r)\,dr=1$, and put $\bar{\varphi}(r)=\int_1^2\chi(s)\,\varphi_0(r/s)\,ds$ and $\bar{\psi}(r)=\psi(r)$. Then it is easy to see that $\bar{\varphi}(r)\in\mathcal{M}_{+}$ and

$$\varphi_0\left(\frac{r}{2}\right)=\int_1^2\chi(s)\,\varphi_0\left(\frac{r}{2}\right)ds\leq\bar{\varphi}(r)\leq\int_1^2\chi(s)\,\varphi_0(r)\,ds=\varphi_0(r)\,. \tag{2.1}$$

Applying ψ to (2.1), we obtain $r/2\leq\psi(\bar{\varphi}(r))\leq r$, which implies $s/C\leq\bar{\varphi}(\psi(s))\leq s$ with some constant C in view of Assertion 5).

If $\psi(r)\leq C$ holds for some C and $\lim_{r\to\infty}\varphi(r)=\infty$, we can proceed in the same way, by interchanging the role of φ and that of ψ. Finally, if $\varphi(r),\psi(r)\leq C$ holds for some C, then $\bar{\varphi}(r)=\bar{\psi}(r)=r$ satisfy all the conditions.

Assume that $\varphi(r)$ and $\psi(r)$ satisfy the conditions of 6). Then, for every $r\geq 1$, there exists $s\geq 1$ such that $r=\bar{\psi}(s)$. It follows from Assertion 2) for φ that

$$\bar{\varphi}(r)=\bar{\varphi}(\bar{\psi}(s))\leq s\leq C\varphi(\psi(s))\leq C\varphi(C'r)\leq C''\varphi(r)\,.$$

In the same way we can prove $\bar{\psi}(r)\leq C\psi(r)$ with some positive constant C. The proof is complete.

Next we recall several properties of weighted Sobolev spaces introduced by Beals [1], and add some remarks. We put $m(x,\xi)=\varphi(\langle\xi\rangle)\psi(\langle x\rangle)$ and $g_{(x,\xi)}(y,\eta)=\langle\xi\rangle^{2\delta}\langle x\rangle^{-2\rho'}|y|^2+\langle\xi\rangle^{-2\rho}\langle x\rangle^{2\delta'}|\eta|^2$, and observe some property of these functions. We obtain the inequality $\langle x\rangle-|y-x|\leq\langle y\rangle\leq\langle x\rangle+|y-x|$ by identifying x and y with $(1,x_1,\cdots,x_n)\in\boldsymbol{R}^{n+1}$ and $(1,y_1,\cdots,y_n)\in\boldsymbol{R}^{n+1}$ respectively, and making use of the triangle inequality on $\boldsymbol{R}^{n+1}$. It follows from this inequality that

$$\frac{\langle y\rangle}{\langle x\rangle}\leq 1+\frac{|y-x|}{\langle x\rangle}\,.$$

If $|y-x|\leq\langle x\rangle/2$, we also have

$$\frac{\langle x\rangle}{\langle y\rangle}\leq 1+2\frac{|y-x|}{\langle x\rangle}\,.$$

Hence, in general, we have

$$\frac{\langle x\rangle}{\langle y\rangle}\leq\{2(1+4\langle x\rangle^{-\rho'-\delta'}|y-x|^2\}^{1/(2-\rho'-\delta')}\,.$$

In fact, this inequality follows from the previous one if $|y-x|\leq\langle x\rangle/2$, and from the inequality $\langle x\rangle^{-\rho'-\delta'}|y-x|^2\geq\langle x\rangle^{2-\rho'-\delta'}/4$ if $|y-x|\geq\langle x\rangle/2$.

We can write

$$g^{\sharp}_{(x,\xi)}(y,\eta)=\langle\xi\rangle^{\rho+\delta}\langle x\rangle^{-\rho'-\delta'}|y|^2+\langle\xi\rangle^{-\rho-\delta}\langle x\rangle^{\rho'+\delta'}|\eta|^2\,,$$
$$g^{\sigma}_{(x,,\xi)}(x,\eta)=\langle\xi\rangle^{2\rho}\langle x\rangle^{-2\delta'}|y|^2+\langle\xi\rangle^{-2\delta}\langle x\rangle^{2\rho'}|\eta|^2$$

and $h(x,\xi)=\langle\xi\rangle^{\delta-\rho}\langle x\rangle^{\delta'-\rho'}$ in the notation of [1], and it is easy to see that these functions fulfill the assumptions (2.5), (2.6), (2.7) and (2.16) of [1]. (See Hörmander [7].)

Next, let φ, ψ, φ_0, ψ_0 be functions in $\mathcal{M}$, and let $A(x,\xi)$, $B(x,\xi)$ be symbols in $S_{\rho,\delta,\rho',\delta'}(\varphi,\psi)$ and $S_{\rho,\delta,\rho',\delta'}(\varphi_0,\psi_0)$ respectively. For every $N\in\boldsymbol{N}$ and $\theta\in[0,1]$, put

$$C_{\theta,1,N}(x,\xi)$$
$$=\sum_{|\alpha|=N}\frac{1}{\alpha!}\iint\exp(-iy\cdot\eta)(\partial_\xi^\alpha A)(x,\xi+\theta\eta)(D_y^\alpha B)(x+y,\xi)fy\bar{d}\eta\,,$$

$$C_{\theta,2,N}(x,\xi)$$
$$=\sum_{|\alpha|+|\beta|=N}\frac{(-1)^{|\beta|}}{2^N\alpha!\beta!}\iiiint\exp(-iy\cdot\eta+iz\cdot\zeta)$$
$$(\partial_\xi^\beta D_x^\alpha A)\left(x+\frac{z}{2},\xi+\theta\eta\right)(\partial_\xi^\alpha D_x^\beta B)\left(x+\frac{y}{2},\xi+\theta\zeta\right)dy\,\bar{d}\eta\,dz\,\bar{d}\zeta\,,$$

and

$$C_{\theta,3,N}(x,\xi)$$
$$=\sum_{|\alpha|=N}\frac{(-1)^N}{\alpha!}\iint \exp(-iy\cdot\eta)\,(D_x^\alpha A)(x,\xi+\theta\eta)\,(\partial_\xi^\alpha B)(x+y,\xi)\,dy\,\bar{d}\eta\,.$$

Then, from the above facts and Theorems 2.7 and 2.7′ of [1], we have

Proposition 2.2.

1) *For every* $j=1,2,3$, *every* $N\in\boldsymbol{N}$ *and every* $\theta\in[0,1]$, *the mapping* $(A(x,\xi), B(x,\xi))\mapsto C_{\theta,j,N}(x,\xi)$ *is continuous from* $S_{\rho,\delta,\rho',\delta'}(\varphi,\psi)\times S_{\rho,\delta,\rho',\delta'}(\varphi_0,\psi_0)$ *to* $S_{\rho,\delta,\rho',\delta'}(\langle\xi\rangle^{N(\delta-\tau)}\varphi\varphi_0,\langle x\rangle^{N(\rho'-\rho')}\psi\psi_0)$, *and the family* $\{C_{\theta,j,N}(x,\xi);\theta\in[0,1]\}$ *is bounded in the class* $S_{\rho,\delta,\rho',\delta'}(\langle\xi\rangle^{N(\delta-\rho)}\varphi\varphi_0,\langle x\rangle^{N(\delta'-\rho')}\psi\psi_0)$.

2) *For* $\theta=0$, *the following equalities hold*:

$$C_{0,1,k}(x,\xi)=\frac{1}{k!}(\partial_\xi D_y)^k A(x,\xi)\,B(y,\eta)|_{y=x,\eta=\xi}\,,$$

$$C_{0,2,k}(x,\xi)=\frac{1}{k!}\Big(\frac{\partial_\xi D_y-\partial_\eta D_x}{2}\Big)^k A(x,\xi)\,B(y,\eta)|_{y=x,\eta=\xi}\,,$$

and

$$C_{0,3,k}(x,\xi)=\frac{1}{k!}(-\partial_\eta D_x)^k A(x,\xi)\,B(y,\eta)|_{y=x\ \eta=\xi}\,.$$

3) *For every positive integer* N, *we have the following equalities as operators*:

$$A(X,D)\,B(X,D)$$
$$=\sum_{0\le k<N} C_{0,1,k}(X,D)+N\int_0^1(1-\theta)^{N-1}\,C_{\theta,1,N}(X,D)\,d\theta\,,$$
$$A^w(X,D)\,B^w(X,D)$$
$$=\sum_{0\le k<N}(C_{0,2,k})^w(X,D)+N\int_0^1(1-\theta)^{N-1}(C_{\theta,2,N})^w(X,D)\,d\theta\,,$$

and

$${}^tA(D,X)\,{}^tB(D,X)$$
$$=\sum_{0\le k<N}{}^tC_{0,3,k}(D,X)+N\int_0^1(1-\theta)^{N-1}\,{}^tC_{\theta,3,N}(D,X)\,d\theta\,,$$

For $m(x,\xi)=\varphi(\langle\xi\rangle)\,\psi(\langle x\rangle)$. Beals [1] introduced the space $H(m,g)$ as the set

$$\{u\in\mathcal{S}'(\boldsymbol{R}^n);\,A^w(X,D)\,u\in L^2(\boldsymbol{R}^n)\quad\text{for all}\quad a(x,\xi)\in S(m,g)\}$$

with natural locally convex topology by way of the symbol class $S(m,g)=S_{\rho,\delta,\rho',\delta'}(\varphi,\psi)$. However, for every $\varphi,\psi\in\mathcal{M}$, the functions $\varphi(\langle\xi\rangle)^{\pm1}$ and $\psi(\langle x\rangle)^{\pm1}$ belong to the class $S_{1,0,1,0}(\varphi^{\pm1},1)$ and $S_{1,0,1,0}(1,\psi^{\pm1})$ respectively. Hence, applying Theorem 3.1 of Beals [1] to the operator $\psi(\langle X\rangle)$ and to $\varphi(\langle D\rangle)$ successively, we see that $\varphi(\langle D\rangle)\,\psi(\langle X\rangle)$ is an isomorphism from $H(m,g)$ to $L^2(\boldsymbol{R}^n)$ for $m(x,\xi)$ as above. Therefore $H(m,g)$ is isomorphic to $H[\varphi,\psi]$ as locally convex spaces.

From this fact and Theorem 3.1 of [1] we see the following

Proposition 2.3. *For every $\varphi,\psi,\varphi_0,\psi_0\in\mathcal{M}$, the following properties hold*:

1) *For $A(x,\xi)\in S_{\rho,\delta,\rho',\delta'}(\varphi,\psi)$, the operator T denoting either $A(X,D)$, $A^w(X,D)$ or ${}^tA(D,X)$ is a bounded operator from $H[\varphi\varphi_0,\psi\psi_0]$ to $H[\varphi_0,\psi_0]$.*
2) *In addition to the assumption of* 1), *assume further that T is a topological isomorphism. Then there exist $B_1(x,\xi), B_2(x,\xi), B_3(x,\xi)\in S_{\rho,\delta,\rho',\delta'}(1/\varphi,1/\psi)$ such that $T^{-1}=B_1(X,D)=B_2^w(X,D)={}^tB_3(D,X)$.*
3) *$\mathcal{S}$ is dense in $H[\varphi,\psi]$.*
4) *$H[1/\varphi,1/\psi]$ can be regarded as the dual space of $H[\varphi,\psi]$ by way of the standard pairing.*
5) *$[H[\varphi,\psi],H[\varphi_0,\psi_0]]_\theta=H[\varphi^{1-\theta}\varphi_0^\theta,\psi^{1-\theta}\psi_0^\theta]$, where $[\cdot,\cdot]_\theta$ denotes the complex interpolation space.*

We proceed to the study of function spaces $H[\varphi_0\varphi,\psi_0]\cap H[\varphi_0,\psi_0\psi]$ and $H[\varphi_0/\varphi,\psi_0]+H[\varphi_0,\psi_0/\psi]$, where $\varphi_0,\psi_0\in\mathcal{M}$ and $\varphi,\psi\in\mathcal{M}_+$; in particular, the spaces $\mathcal{H}^\sigma$ and $\mathcal{K}^{\sigma,\tau}$. The following lemma equips these spaces with the Hilbert space structure, which we shall use throughout this paeer.

Lemma 2.4. *Let E and F be Hilbert spaces continuously imbedded in a common Hausdorff topological vector space. We introduce norms $\|\cdot\|_{E\cap F}$ on $E\cap F$ and $\|\cdot\|_{E+F}$ on $E+F$ defined by $\|u\|^2_{E\cap F}=\|u\|^2_E+\|u\|^2_F$ and $\|u\|^2_{E+F}=\inf\{\|u_1\|^2_E+\|u_2\|^2_F;\,u_1\in E,\,u_2\in F,\,u=u_1+u_2\}$ respectively. Then these norms make $E\cap F$ and $E+F$ Hilbert spaces, and induce them topologies which coincide with their natural locally convex topologies.*

Proof. We shall only show the equalities

$$\|u+v\|^2_{E\cap F}+\|u-v\|^2_{E\cap F}=2\{\|u\|^2_{E\cap F}+\|v\|^2_{E\cap F}\} \tag{2.2}$$

and

$$\|u+v\|^2_{E+F}+\|u-v\|^2_{E+F}=2\{\|u\|^2_{E+F}+\|v\|^2_{E+F}\}\,, \tag{2.3}$$

since other facts are well-known. (See Lions-Peetre [19].)

(2.2) can be shown by direct calculation from the similar equalities for $\|\cdot\|_E$ and $\|\cdot\|_F$.

Hence we shall show (2.3). For every $\varepsilon>0$, we can choose $u_1,v_1\in E$ and $u_2,v_2\in F$ such that $\|u_1\|^2_E+\|u_2\|^2_F<\|u\|^2_{E+F}+\varepsilon$ and $\|v_1\|^2_E+\|v_2\|^2_F<\|v\|^2_{E+F}+\varepsilon$. Then, since $u_1\pm v_1\in E$, $u_2\pm v_2\in E$ and $u\pm v=(u_1\pm v_1)+(u_2\pm v_2)$, we have

$$\begin{aligned}&\|u+v\|^2_{E+F}+\|u-v\|^2_{E+F}\\&\le\|u_1+v_1\|^2_E+\|u_1-v_1\|^2_E+\|u_2+v_2\|^2_F+\|u_2-v_2\|^2_F\\&=2\{\|u_1\|^2_E+\|v_1\|^2_E+\|u_2\|^2_F+\|v_2\|^2_F\}<2\{\|u\|^2_{E+F}+\|v\|^2_{E+F}\}+4\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$ in this inequality, we conclude

$$(2.4) \qquad \|u+v\|_{E+F}^2+\|u-v\|_{E+F}^2 \leq 2\{\|u\|_{E+F}^2+\|v\|_{E+F}^2\} .$$

Replacing u and v by $u+v$ and $u-v$ respectively in (2.4), we obtain the converse inequality. This inequality and (2.4) yield (2.3).

The following lemma provides a topological isomorphism between these spaces.

Lemma 2.5. *Let E, F and X be Hilbert spaces such that E and F are densely imbedded in X, and let S [resp. T] be a closed operator in X with domain E [resp. F] which is a topological isomorphism from E [resp. F] to X. Suppose that the spaces $E \cap F'$ and $E'+F'$ are topologized as in Lemma 2.4, and that $E \cap F$ is dense in E and also in F. Suppose further that the inequalities*

$$(2.5) \qquad \|(S+T)u\|_X^2 \geq C_1(\|Su\|_X^2+\|Tu\|_X^2)-C_2\|u\|_X^2$$

and

$$(2.6) \qquad \Re((S+T)u, u)_X \geq -C_3\|u\|_X^2$$

hold with some $C_1, C_2, C_3>0$ for all u belonging to a dense subset K of $E \cap F$. Then the operator U from $E \cap F$ to X defined by $Uu=Su+Tu$ satisfies the following:

1) *U is closed in X.*
2) *The space $E'+F'$ is topologically isomorphic to $(E \cap F)'$, and there exists a positive constant M_0 such that, for every $M \geq M_0$, the operator $U+M$ and its dual ${}^t(U+M)$ are topological isomorphisms from $E \cap F$ to X, and from X' to $E'+F'$ respectively, satisfying the inequalities*

$$C^{-1}\|(U+M)u\|_X \leq \|u\|_{E\cap F} \leq C\|(U+M)u\|_X ,$$

and

$$C^{-1}\|{}^t(U+M)u\|_{E'+F'} \leq \|u\|_{X'} \leq C\|{}^t(U+M)u\|_{E'+F'}$$

with some positive constant C.

Proof. First we shall show Assertion 1). Let $\{u_j\}$ be a sequence in $E \cap F$ such that $u_j \to u$ and $Uu_j \to v$ hold in X, and take a sequence $\{w_j\}$ in K such that $\|u_j-w_j\|_{E\cap F} \to 0$ holds as $j \to \infty$. Then, since U is a bounded operator from $E \cap F$ to X, we have $Uw_j=Uu_j+U(w_j-u_j) \to v$ in X as $j \to \infty$. Hence, substituting w_j-w_k into u in (2.5), we have

$$\begin{aligned}&\|U(w_j-w_k)\|_X^2+C_2\|w_j-w_k\|_X^2 \\ &\quad \geq C_1(\|S(w_j-w_k)\|_X^2+\|T(w_j-w_k)\|_X^2) .\end{aligned}$$

This implies that $\{w_j\}$ is a Cauchy sequence both in E and in F, and hence so is $\{u_j\}$. It follows that $u \in E \cap F$ and $Uu = v$, and this completes the proof of Assertion 1).

We turn to the proof of 2). Since $E' + F'$ can be canonically identified with $(E \cap F)'$, (See Yoshikawa [30, Proposition 1.6]) and since the first inequality yields the second one with $E' + F'$ replaced by $(E \cap F)'$, it suffices to prove the first inequality. From the above argument we also have

$$||Uu||_X^2 + C_2 ||u||_X^2 \geq C_1 (||Su||_X^2 + ||Tu||_X^2) \geq CC_1 ||u||_{E \cap F}^2 \tag{2.7}$$

and

$$\Re(Uu, u)_X \geq -C_3 ||u||_X^2 \tag{2.8}$$

for every $u \in E \cap F$ with some constant C. Now put $M_0 = C_3 + \sqrt{C_2 + C_3^2}$. Then, for $M \geq M_0$, (2.8) implies

$$\begin{aligned} ||(U+M)u||_X^2 &= ||Uu||_X^2 + 2M \Re(Uu, u)_X + M^2 ||u||_X^2 \\ &\geq ||Uu||_X^2 + (M^2 - 2MC_3) ||u||_X^2 \geq ||Uu||_X^2 + C_2 ||u||_X^2 . \end{aligned}$$

This inequality and (2.7) yield $||(U+M)u||_X \geq CC_1 ||u||_{E \cap F}$. On the other hand, it is easy to see that U is a bounded operator from $E \cap F$ to X, and satisfies the inequality

$$\begin{aligned} ||(U+M)u||_X &\leq ||Su||_X + ||Tu||_X + ||Mu||_X \\ &\leq C ||u||_E + C ||u||_F + M ||u||_X \leq (2C+M) ||u||_{E \cap F} \end{aligned}$$

for every $M \geq M_0$ with some constant C. This completes the proof.

Corollary 2.6. *Suppose $\varphi_0, \psi_0 \in \mathcal{M}$ and $\varphi, \psi \in \mathcal{M}_+$, and put $E = H[\varphi_0 \varphi, \psi_0]$, $F = H[\varphi_0, \psi_0 \psi]$, $X = H[\varphi_0, \psi_0]$, $Y = H[\varphi_0/\varphi, \psi_0]$ and $Z = H[\varphi_0, \psi_0/\psi]$. Then there exists a positive number M_0 such that, for every $M \geq M_0$, the operator $\varphi(\langle D \rangle) + \psi(\langle X \rangle) + M$ is an isomorphism from $E \cap F$ to X, and also from X to $Y+Z$.*

In order to prove this corollary, we introduce the following

Lemma 2.7. *Putting $Q = \varphi_0(\langle D \rangle) \psi_0(\langle X \rangle)$, we have*

$$\begin{aligned} &||Q(\varphi(\langle D \rangle) + \psi(\langle X \rangle)) Q^{-1} u||_{L^2(\boldsymbol{R}^n)}^2 \\ &\geq \frac{1}{2} (||Q\varphi(\langle D \rangle) Q^{-1} u||_{L^2(\boldsymbol{R}^n)}^2 + ||Q\psi(\langle X \rangle) Q^{-1} u||_{L^2(\boldsymbol{R}^n)}^2) - C ||u||_{L^2(\boldsymbol{R}^n)}^2 \end{aligned} \tag{2.9}$$

and

$$\Re(Q(\varphi(\langle D \rangle) + \psi(\langle X \rangle)) Q^{-1} u, u)_{L^2(\boldsymbol{R}^n)} \geq -C ||u||_{L^2(\boldsymbol{R}^n)}^2 \tag{2.10}$$

for every $u \in \mathcal{S}$ with some constant C.

Admitting this lemma for the moment, we complete the proof of Corollary 2.6. Prpposition 2.3 implies that $\varphi(\langle D\rangle)$ [resp. $\psi(\langle X\rangle)$] is a topological isomorphism from E [resp. from F] to X. It is also easy to see tahat these operators are closed in X.

Moreover, since Q is an isomorphism from X to $L^2(\boldsymbol{R}^n)$, Lemma 2.7 asserts that (2.5) and (2.6) hold for all $u\in\mathcal{S}$ with $S=\varphi(\langle D\rangle)$ and $T=\psi(\langle X\rangle)$. Hence we can apply Lemma 2.5 to conclude that there exists a positive constant M_0 such that, for every $M\geq M_0$, the operator $\varphi(\langle D\rangle)+\psi(\langle X\rangle)+M$ is an isomorphism from $E\cap F$ to X, and its dual is an isomorphism from X' to $E'+F'$.

On the other hand, in view of Proposition 2.3, 4), the spaces X', E' and F' can be identified with $H[1/\varphi_0, 1/\psi_0]$, $H[1/\varphi\varphi_0, 1/\psi_0]$ and $H[1/\varphi_0, 1/\psi\psi_0]$ respectively, and we have ${}^tS=\varphi(\langle D\rangle)$ and ${}^tT=\psi(\langle X\rangle)$ under this identification. Replacing φ_0 and ψ_0 by $1/\varphi_0$ and $1/\psi_0$, we conclude that $\varphi(\langle D\rangle)+\psi(\langle X\rangle)+M$ is an isomorphism from X to $Y+Z$ for every $M\geq M_0$. This completes the proof.

Proof of Lemma 2.7. Since $Q^{-1}=(1/\psi_0)(\langle X\rangle)(1/\varphi_0)(\langle D\rangle)$, Proposition 2.2 implies that there exist $B(x,\xi)\in S_{1,0,1,0}(\langle\xi\rangle^{-1}\varphi, \langle x\rangle^{-1}\psi)$, $R(x,\xi)\in S^{0;0}_{1,0,1,0}$ and $S(x,\xi)\in S^{0;0}_{1,0,1,0}$ such that

(2.11)
$$\begin{aligned}&Q^{-1}\psi(\langle X\rangle)Q^2\varphi(\langle D\rangle)Q^{-1}\\&\quad=Q^{-1}\sqrt{\psi}(\langle X\rangle)\sqrt{\varphi}(\langle D\rangle)Q^2\sqrt{\varphi}(\langle D\rangle)\sqrt{\psi}(\langle X\rangle)Q^{-1}+B(X,D),\end{aligned}$$

$$\langle D\rangle\varphi(\langle D\rangle)^{-1}B(X,D)=R(X,D)\langle X\rangle^{-1}\psi(\langle X\rangle) \tag{2.12}$$

and

$$\psi(X)=Q\psi(\langle X\rangle)Q^{-1}+S(X,D)\langle X\rangle^{-1}\psi(\langle X\rangle). \tag{2.13}$$

Then, from (2.11) and (2.12), we obtain

(2.14)
$$\begin{aligned}&-\|Q(\varphi(\langle D\rangle)+\psi(\langle X\rangle))Q^{-1}u\|^2_{L^2(\boldsymbol{R}^n)}\\&\quad+(\|Q\varphi(\langle D\rangle)Q^{-1}u\|^2_{L^2(\boldsymbol{R}^n)}+\|Q\psi(\langle X\rangle)Q^{-1}u\|^2_{L^2(\boldsymbol{R}^n)})\\&=-2\Re(Q\varphi(\langle D\rangle)Q^{-1}u, Q\psi(\langle X\rangle)Q^{-1}u)_{L^2(\boldsymbol{R}^n)}\\&\leq-2\|Q\sqrt{\varphi}(\langle D\rangle)\sqrt{\psi}(\langle X\rangle)Q^{-1}u\|^2_{L^2(\boldsymbol{R}^n)}-2\Re(B(X,D)u,u)_{L^2(\boldsymbol{R}^n)}\\&\leq2\|\langle D\rangle\varphi(\langle D\rangle)^{-1}B(X,D)u\|_{L^2(\boldsymbol{R}^n)}\cdot\|\langle D\rangle^{-1}\varphi(\langle D\rangle)u\|_{L^2(\boldsymbol{R}^n)}\\&\leq C_0\|\langle X\rangle^{-1}\psi(\langle X\rangle)u\|_{L^2(\boldsymbol{R}^n)}\cdot\|\langle D\rangle^{-1}\varphi(\langle D\rangle)u\|_{L^2(\boldsymbol{R}^n)}\end{aligned}$$

for some positive number C_0.

We wish to estimate the right-hand side of (2.14). First, Proposition 2.1, 2) implies the existence of a positive integer k and a positive unmber C_1 such that $\psi(t)\leq C_1t^k$ and $\varphi(t)\leq C_1t^k$. It follows that

(2.15)
$$\begin{aligned}||\langle X\rangle^{-1}\psi(\langle X\rangle)u||_{L^2(\mathbf{R}^n)} &\le C_1^{1/k}||(\psi(\langle X\rangle))^{(k-1)/k}u||_{L^2(\mathbf{R}^n)}\\ &\le C_1^{1/k}||\psi(\langle X\rangle)u||_{L^2(\mathbf{R}^n)}^{(k-1)/k}||u||_{L^2(\mathbf{R}^n)}^{1/k}\\ &\le \varepsilon||\psi(\langle X\rangle)u||_{L^2(\mathbf{R}^n)}+C_1\varepsilon^{1-k}||u||_{L^2(\mathbf{R}^n)}\end{aligned}$$

and

(2.16)
$$||\langle X\rangle^{-1/2}\sqrt{\psi}(\langle X\rangle)u||_{L^2(\mathbf{R}^n)}\le\varepsilon||\sqrt{\psi}(\langle X\rangle)u||_{L^2(\mathbf{R}^n)}+C_1\varepsilon^{1-k}||u||_{L^2(\mathbf{R}^n)}$$

for every $\varepsilon>0$. On the other hand, (2.13) yields the estimate

$$||\psi(\langle X\rangle)u||_{L^2(\mathbf{R}^n)}\le||Q\psi(\langle X\rangle)Q^{-1}u||_{L^2(\mathbf{R}^n)}+C_2||\langle X\rangle^{-1}\psi(\langle X\rangle)u||_{L^2(\mathbf{R}^n)}$$

with some constant C_2. This inequality and (2.15) imply

$$\begin{aligned}&||\langle X\rangle^{-1}\psi(\langle X\rangle)u||_{L^2(\mathbf{R}^n)}\\ &\le\frac{\varepsilon}{1-C_2\varepsilon}||Q\psi(\langle X\rangle)Q^{-1}u||_{L^2(\mathbf{R}^n)}+\frac{C_1\varepsilon^{1-k}}{1-C_2\varepsilon}||u||_{L^2(\mathbf{R}^n)}\,.\end{aligned}$$

Choosing ε so small that $C_0\varepsilon/(1-C_2\varepsilon)<1/2$, we obtain

$$C_0||\langle X\rangle^{-1}\psi(\langle X\rangle)u||_{L^2(\mathbf{R}^n)}\le\frac{1}{2}||Q\psi(\langle X\rangle)Q^{-1}u||_{L^2(\mathbf{R}^n)}+C||u||_{L^2(\mathbf{R}^n)}$$

with some positive constant C. In the same way we obtain

$$||\langle D\rangle^{-1}\varphi(\langle D\rangle)u||_{L^2(\mathbf{R}^n)}\le\frac{1}{2}||Q\varphi(\langle D\rangle)Q^{-1}u||_{L^2(\mathbf{R}^n)}+C||u||_{L^2(\mathbf{R}^n)}\,.$$

These estimates imply that the right-hand side of (2.14) is dominated by

$$\begin{aligned}&\left\{\frac{1}{2}||Q\psi(\langle X\rangle)Q^{-1}u||_{L^2(\mathbf{R}^n)}+C||u||_{L^2(\mathbf{R}^n)}\right\}\\ &\quad\times\left\{\frac{1}{2}||Q\varphi(\langle D\rangle)Q^{-1}u||_{L^2(\mathbf{R}^n)}+C||u||_{L^2(\mathbf{R}^n)}\right\}\\ &\quad\le\frac{1}{4}||Q\psi(\langle X\rangle)Q^{-1}u||_{L^2(\mathbf{R}^n)}^2+\frac{1}{4}||Q\varphi(\langle D\rangle)Q^{-1}u||_{L^2(\mathbf{R}^n)}^2+2C^2||u||_{L^2(\mathbf{R}^n)}^2\end{aligned}$$

Now (2.9) follows from this estimate and (2.14).

Next we shall prove (2.10). There exists a symbol $C(x,\xi)\in S^{-1,0}_{1,0,1,0}$ such that

$$Q^{-1}\psi(\langle X\rangle)Q=\psi(\langle X\rangle)+\langle X\rangle^{-1/2}\sqrt{\psi}(\langle X\rangle)C(X,D)\langle X\rangle^{-1/2}\sqrt{\psi}(\langle X\rangle)\,.$$

This equality and (2.16) with $\varepsilon=1/C$ yield

$$\begin{aligned}&\Re(Q^{-1}\psi(\langle X\rangle)Qu,u)\\ &\quad=\Re(\psi(\langle X\rangle)u,u)+\Re(\langle X\rangle^{-1/2}\sqrt{\psi}(\langle X\rangle)C(X,D)\langle X\rangle^{-1/2}\sqrt{\psi}(\langle X\rangle)u,u)\end{aligned}$$

$$\geq \|\sqrt{\psi}(\langle X\rangle)\, u\|_{L^2(\mathbf{R}^n)}^2 - C\, \|\langle X\rangle^{-1/2} \sqrt{\psi}(\langle X\rangle)\, u\|_{L^2(\mathbf{R}^n)}^2$$
$$\geq -C^k\, C_1\, \|u\|_{L^2(\mathbf{R}^n)}^2 .$$

In the same way we can prove

$$\Re(Q^{-1}\, \varphi(\langle D\rangle)\, Qu, u) \geq -C'\, \|u\|_{L^2(\mathbf{R}^n)}^2 .$$

Summing up these inequalities, we conclude (2.10).

3. Proof of Theorem 1.4

Since $\partial_{x_j} \partial_{\xi_j} B(x, \xi) \in S^{0;0}_{\rho,\delta,\rho',\delta'}$ holds for every $j=1, \cdots, n$, it follows that the operators $B^w(X, D) - B(X, D)$ and ${}^tB(D, X) - B(X, D)$ are bounded on $L^2(\mathbf{R}^n)$. Moreover, the operator $B^w(X, D)$ is symmetric on $C_0^\infty(\mathbf{R}^n)$ with respect to the L^2-norm. Hence it suffices to show that the operator $S = A(D) + B(X, D) + C(X)$ defined on $C_0^\infty(\mathbf{R}^n)$ is closable in $L^2(\mathbf{R}^n)$, and that the adjoint in $L^2(\mathbf{R}^n)$ of the operator $T = A(D) + {}^tB(D, X) + C(X)$ defined on $C_0^\infty(\mathbf{R}^n)$ coincides with $\bar{S}$, the closure of S in $L^2(\mathbf{R}^n)$.

First, assume that $\{u_j\}$ is a sequence of functions in $C_0^\infty(\mathbf{R}^n)$ such that $u_j \to 0$ and $Su_j \to v$ hold in $L^2(\mathbf{R}^n)$. Then $u_j \to 0$ holds in $\mathcal{D}'(\mathbf{R}^n)$, and hence $Su_j \to 0$ holds in $\mathcal{D}'(\mathbf{R}^n)$. This implies $v = 0$, and it follows that S is closable in $L^2(\mathbf{R}^n)$.

Next,, for $u, v \in C_0^\infty(\mathbf{R}^n)$, we have $(Su, v) = (u, Tv)$. Hence, for $v \in C_0^\infty(\mathbf{R}^n)$ and $u \in Dom(\bar{S})$, we see easily $(\bar{S}u, v) = (u, Tv)$ by approximating u by functions in $C_0^\infty(\mathbf{R}^n)$. This shows $T^* \supset \bar{S}$.

It remains only to show $T^* \subset \bar{S}$. We introduce a function $\zeta(x) \in C^\infty(\mathbf{R}^n)$ such that $\operatorname{supp} \zeta \subset \{x \in \mathbf{R}^n;\ |x| < 1\}$ and that $\zeta(x) \equiv 1$ for $x \in \mathbf{R}^n$ satisfying $|x| \leq 3/5$. Further,, let $\bar{\varphi}(r) \in \mathcal{M}_+$ be a function satisfying the properties of Proposition 2.1, 8), and put $Z_j(x, \xi) = \zeta(\bar{\varphi}(j)^{-1} x)\, \zeta(j^{-1} \xi)$ for $j = 1, 2, \cdots$. Then we have the following

Proposition 3.1. *The sequence of operators*

$$\{Z_j(X, D)\, (A(D) + B(X, D) + C(X)) -$$
$$(A(D) + B(X, D) + C(X))\, Z_j(X, D);\ \ j = 1, 2, \cdots\}$$

defined on $L^2(\mathbf{R}^n)$ *is uniformly bounded on* $L^2(\mathbf{R}^n)$, *and converges to* 0 *with respect to the strong topology of* $L(L^2(\mathbf{R}^n))$.

Admitting this proposition for the moment, we shall prove $T^* \subset \bar{S}$. For $u(x) \in \mathrm{Dom}(T^*)$, put $u_j(x) = Z_j(X, D)\, u(x) \in C_0^\infty(\mathbf{R}^n)$. Then we have

$$u_j - u = \{\zeta(\bar{\varphi}(j)^{-1} X) - 1\}\, u + \zeta(\bar{\varphi}(j)^{-1} X)\, \{\zeta(j^{-1} D) - 1\}\, u .$$

Since

$$\|\{\zeta(\bar{\varphi}(j)^{-1} X) - 1\}\, u\|_{L^2(\mathbf{R}^n)}^2 \leq C \int_{|x| \geq \bar{\varphi}(j)/2} |u(x)|^2\, dx \to 0 ,$$

and

$$\|\{\zeta(j^{-1}D)-1\}\,u\|^2_{L^2(\boldsymbol{R}^n)} \leq C\int_{|\xi|\geq j/2} |\hat{u}(\xi)|^2\,\bar{d}\xi \to 0$$

hold as $j\to\infty$, and since $\zeta(\bar{\varphi}(j)^{-1}X)$ is bounded on $L^2(\boldsymbol{R}^n)$ uniformly in j, we see that

$$u_j - u \to 0 \quad \text{in} \quad L^2(\boldsymbol{R}^n) \quad \text{as} \quad j\to 0\,. \tag{3.1}$$

If $u\in \mathrm{Dom}(T^*)$, we see

$$Z_j(X, D)\,T^*u \to T^*u \quad \text{in} \quad L^2(\boldsymbol{R}^n) \quad \text{as} \quad j\to 0 \tag{3.2}$$

in the same way.

On the other hand, Proposition 3.1 yields $Z_j(X, D)\,T^*u(x) - Su_j(x)\to 0$ in $L^2(\boldsymbol{R}^n)$ as $j\to\infty$, which together with (3.2) implies that Su_j converges to T^*u in $L^2(\boldsymbol{R}^n)$ as $j\to\infty$. This fact and (3.1) yield $T^*\subset\bar{S}$.

Proof of Proposition 3.1. We follow the method of Friedrichs [3]. First we have

$$\begin{aligned} Y_j\,u(x) &= \sum_{1\leq|\alpha|\leq N-1}(B^{(1)}_{j,\alpha,0}(X, D)\,u + C_{j,\alpha,0}(X, D)\,u) \\ &\quad - \sum_{1\leq|\alpha|\leq N-1}(A_{j,\alpha,0}(X, D)\,u + B^{(2)}_{j,\alpha,0}(X, D)\,u) \\ &\quad + N\int_0^1 (1-\theta)^{N-1}\sum_{|\alpha|=N}(B^{(1)}_{j,\alpha,\theta}(X, D)\,u + C_{j,\alpha,\theta}(X, D)\,u)\,d\theta \\ &\quad - N\int_0^1 (1-\theta)^{N-1}\sum_{|\alpha|=N}(A_{j,\alpha,\theta}(X, D)\,u + B^{(2)}_{j,\alpha,\theta}(X, D)\,u)\,d\theta \end{aligned} \tag{3.3}$$

for every positive integer N, where

$$A_{j,\alpha,\theta}(x, \xi) = \frac{1}{\alpha!}\iint \exp(-iy\cdot\eta)\,D^\alpha_x Z_j(x+y, \xi)\,\partial^\alpha_\xi A(\xi+\theta\eta)\,\bar{d}\eta\,dy\,,$$

$$B^{(1)}_{j,\alpha,\theta}(x, \xi) = \frac{1}{\alpha!}\iint \exp(-iy\cdot\eta)\,\partial^\alpha_\xi Z_j(x, \xi+\theta\eta)\,D^\alpha_x B(x+y, \xi)\,\bar{d}\eta\,dy\,,$$

$$B^{(2)}_{j,\alpha,\theta}(x, \xi) = \frac{1}{\alpha!}\iint \exp(-iy\cdot\eta)\,D^\alpha_x Z_j(x+y, \xi)\,\partial^\alpha_\xi B(x, \xi+\theta\eta)\,\bar{d}\eta\,dy$$

and

$$C_{j,\alpha,\theta}(x, \xi) = \frac{1}{\alpha!}\iint \exp(-iy\cdot\eta)\,\partial^\alpha_\xi Z_j(x, \xi+\theta\eta)\,D^\alpha_x C(x+y)\,\bar{d}\eta\,dy\,.$$

Then we have

$$\begin{aligned} &\partial^\alpha_\xi D^\gamma_x B^{(2)}_{j,\alpha,\theta}(x, \xi) \\ &= \sum_{\beta'\leq\beta}\sum_{\gamma'\leq\gamma}\binom{\beta}{\beta'}\binom{\gamma}{\gamma'}\iint \frac{\exp(-iy\cdot\eta)}{\langle y\rangle^{2L}}(1-\Delta_\eta)^L\{\frac{1}{\langle\eta\rangle^{2M}}(1-\Delta_y)^M \end{aligned}$$

$$(\partial_\xi^{\beta'} D_x^{\alpha+\gamma'} Z_j(x+y, \xi)\, \partial_\xi^{\alpha+\beta-\beta'} D_x^{\gamma-\gamma'} B(x, \xi+\theta\eta))\}\, \bar{d}\eta\, dy$$

for every $\alpha \neq 0$ and $L, M \in \boldsymbol{N}$. Since $\partial_\xi^\alpha B(x, \xi) \in S_{\rho,\delta,\rho',\delta'}^{\rho(1-|\alpha|), 1+\delta(|\alpha|-1)}$, this yields

$$\begin{aligned}
&|\partial_\xi^\beta D_x^\alpha B_{j,\alpha,\theta}^{(2)}(x, \xi)| \\
&\leq C \sum_{\beta' \leq \beta'} \sum_{\gamma' \leq \gamma} \iint \frac{\bar{\varphi}(j)^{-|\alpha|-|\gamma'|} j^{-|\beta'|}}{\langle y\rangle^{2L} \langle \eta\rangle^{2M}} \langle x\rangle^{1-\rho'|\gamma-\gamma'|+\delta'(|\alpha|+|\beta-\beta'|-1)} \\
&\quad \langle \xi+\theta\eta\rangle^{-\rho(|\alpha|+|\beta-\beta'|-1)+\delta|\gamma-\gamma'|} \chi_{E_j}(x, y, \xi, \eta)\, \bar{d}\eta\, dy
\end{aligned}$$

for every $\beta, \gamma \in \boldsymbol{N}^n$, where $\chi_{E_j}(x, y, \xi, \eta)$ is the characteristic function of the set

$$E_j = \{(x, y, \xi, \eta);\, \bar{\varphi}(j)/2 \leq |x+y| \leq \bar{\varphi}(j);\, |\xi| \leq j\}\,.$$

Now, given $\beta, \gamma \in \boldsymbol{N}^n$, choose $L, M \in \boldsymbol{N}$ satisfying

$$2L > n + \rho'|\gamma| + \delta'(|\alpha| + |\beta|) + 1$$

and

$$2M > n + \rho(|\alpha| + |\beta|) + \delta|\gamma| + 1\,.$$

Then, from the facts

$$\langle \xi+\theta\eta\rangle^{\pm 1} \leq \langle \xi\rangle^{\pm 1} \langle \theta\eta\rangle \leq \langle \xi\rangle^{\pm 1} \langle \eta\rangle \leq j^{1/2 \pm 1/2} \langle \eta\rangle$$

and

$$\langle x\rangle^{\pm 1} \leq \langle x+y\rangle^{\pm 1} \langle y\rangle \leq 2\bar{\varphi}(j)^{\pm 1} \langle y\rangle$$

for $(x, y, \xi, \eta) \in E_j$, we obtain the estimate

$$\begin{aligned}
&|\partial_\xi^\beta D_x^\alpha B_{j,\alpha,\theta}^{(2)}(x, \xi)| \\
&\leq C \sum_{\beta' \leq \beta} \sum_{\gamma' \leq \gamma} \int \frac{\langle \theta\eta\rangle^{|-\rho(|\alpha|-1+|\beta-\beta'|)+\delta|\gamma-\gamma'||}}{\langle \eta\rangle^{2M}}\, \bar{d}\eta \\
&\quad j^{-|\beta'|} \chi_{\{\xi\,;\,|\xi| \leq j\}}(\xi) \langle \xi\rangle^{-\rho(|\alpha|-1+|\beta-\beta'|)+\delta|\gamma-\gamma'|} \\
&\quad \int \langle y\rangle^{|1+\rho'|\gamma'|+\delta'(|\alpha|-|\beta'|-1)|-2L}\, dy \langle x\rangle^{-\rho'|\gamma|+\delta'|\beta|} \\
&\quad \bar{\varphi}(j)^{1-|\alpha|-|\gamma'|+\rho'|\gamma'|+\delta'(|\alpha|-1-|\beta'|)} \\
&\leq C \sum_{\beta' \leq \beta} \sum_{\gamma' \leq \gamma} C_{\beta,\beta',M}\, C_{\gamma,\gamma',L} \langle x\rangle^{-\rho'|\gamma|+\delta'|\beta|} \langle \xi\rangle^{-\rho|\beta|+\delta|\gamma|} \\
&\quad j^{-|\beta'|+\max\{0, -\rho'(|\alpha|-1-|\beta'|)-\delta|\gamma'|\}}\, \bar{\varphi}(j)^{(1-|\alpha|)(1-\delta')-(1-\rho')|\gamma'|-\delta'|\beta'|} \\
&\leq C_{\beta,\gamma} \langle x\rangle^{-\rho'|\gamma|+\delta'|\beta|} \langle \xi\rangle^{-\rho|\beta|+\delta|\gamma|}\, \bar{\varphi}(j)^{(1-|\alpha|)(1-\delta')}\,.
\end{aligned}$$

By virtue of the facts

$$\begin{aligned}
&\partial_\xi^\alpha A(\xi) \in S_{0,0,1,0}(\varphi, 1)\,, \\
&D_x^\alpha B(x, \xi) \in S_{\rho,\delta,\rho',\delta'}^{1+\delta'(|\alpha|-1), \rho'(1-|\alpha|)}
\end{aligned}$$

and

$$D_x^\alpha C(x) \in S_{1,0,0,0}(1, \psi),$$

together with the estimates

$$\varphi(\langle \xi+\theta\eta \rangle) \le \max\{\varphi(2\langle \xi \rangle), \varphi(2\langle \theta\eta \rangle)\} \le C\varphi(\langle \xi \rangle)\langle \eta \rangle^C$$

and

$$\psi(\langle x+y \rangle) \le C\psi(\langle x \rangle)\langle y \rangle^C$$

with some $C>0$, we obtain

$$\begin{aligned}
&|\partial_\xi^\beta D_x^\gamma A_{j,\alpha,\theta}(x,\xi)| \\
&\quad \le C_{\beta,\gamma}\,\chi_{\{\xi\,;\,|\xi|\le j\}}(\xi)\,\varphi(\langle \xi \rangle)\,\bar{\varphi}(j)^{-|\alpha|}\langle x \rangle^{-|\gamma|} \\
&\quad \le C_{\beta,\gamma}\,\bar{\varphi}(j)^{1-|\alpha|}, \\
&|\partial_\xi^\beta D_x^\gamma B_{j,\alpha,\theta}^{(1)}(x,\xi)| \\
&\quad \le \sum_{\beta'\le\beta}\sum_{\gamma'\le\gamma} C'_{\alpha',\beta'}\langle \xi \rangle^{-\rho|\beta|+\delta|\gamma|}\, j^{-|\alpha|-|\beta'|+1+\rho|\beta'|+\delta(|\alpha|-1-|\gamma'|)} \\
&\qquad \chi_{\{x\,;\,|x|\leqq\bar{\varphi}(j)\}}(x)\,\langle x \rangle^{\delta'|\beta-\beta'|+\rho'(1-|\alpha|-|\gamma-\gamma'|)}\,\bar{\varphi}(j)^{-|\gamma'|} \\
&\quad \le C_{\beta,\gamma}\langle \xi \rangle^{-\rho|\beta|+\delta|\gamma|}\langle x \rangle^{-\rho'|\gamma|+\delta'|\beta|}\, j^{(1-|\alpha|)(1-\delta)} \\
&\qquad \bar{\varphi}(j)^{-|\gamma'|+\max\{0,-\delta|\beta'|+\rho'(1-|\alpha|+|\gamma'|)\}} \\
&\quad \le C_{\beta,\gamma}\langle \xi \rangle^{-\rho|\beta|+\delta|\gamma|}\langle x \rangle^{-\rho'|\gamma|+\delta'|\beta|}\, j^{(1-|\alpha|)(1-\delta)}
\end{aligned}$$

and

$$\begin{aligned}
&|\partial_\xi^\beta D_x^\gamma C_{j,\alpha,\theta}(x,\xi)| \\
&\quad \le C_{\beta,\gamma}\, j^{-|\alpha|}\langle \xi \rangle^{-|\beta|}\chi_{\{x\,;\,|x|\leqq\bar{\varphi}(j)\}}(x)\,\psi(\langle x \rangle) \\
&\quad \le C_{\beta,\gamma}\langle \xi \rangle^{-|\beta|} j^{-|\alpha|}\psi(\bar{\varphi}(j)+1) \\
&\quad \le C_{\beta,\gamma}\, j^{1-|\alpha|}
\end{aligned}$$

in the same way. These facts imply

$$(3.4)\qquad \begin{cases} A_{j,\alpha,\theta} \text{ and } C_{k,\alpha,\theta} & \text{are bounded in } S^{0,0}_{0,0,0,0} \\ B^{(1)}_{j,\alpha,\theta} \text{ and } B^{(2)}_{j,\alpha,\theta} & \text{are bounded in } S^{0,0}_{\rho,\delta,\rho',\delta'} \end{cases}$$

uniformly in $\theta\in[0,1]$ and $j=1,2,\cdots$ for every $\alpha\in \boldsymbol{N}^n$ satisfying $|\alpha|=1$, and

$$(3.5)\qquad \begin{cases} A_{j,\alpha,\theta} \text{ and } C_{j,\alpha,\theta}\to 0 & \text{in } S^{0,0}_{0,0,0,0} \quad \text{as } j\to\infty \\ B^{(1)}_{j,\alpha,\theta} \text{ and } B^{(2)}_{j,\alpha,\theta}\to 0 & \text{in } S^{0,0}_{\rho,\delta,\rho',\delta'} \quad \text{as } j\to\infty \end{cases}$$

uniformly in $\theta\in[0,1]$ for every $\alpha\in\boldsymbol{N}^n$ satisfying $|\alpha|\ge 2$.

Fix $\alpha\in\boldsymbol{N}^n$ satisfying $|\alpha|=1$, and choose a function $\phi(x)\in C_0^\infty(\boldsymbol{R}^n)$ such that $\phi(x)\equiv 1$ holds for $x\in\boldsymbol{R}^n$ satisfying $1/3<|x|<4/3$ and that $\operatorname{supp}\phi\subset\{x\in\boldsymbol{R}^n;\ 1/4<|x|<3/2\}$.

Then, from the fact

$$\operatorname{supp} C_{j,\alpha,0}(x,\xi),\ \operatorname{supp} B^{(1)}_{j,\alpha,0}(x,\xi)\subset\{(x,\xi);\ j/2\le|\xi|\le j\},$$

we see

$$B^{(1)}_{j,\alpha,0}(X, D)\, u = B^{(1)}_{j,\alpha,0}(X, D)\, \phi(j^{-1} D)\, u\,,$$
$$C_{j,\alpha,0}(X, D)\, u = C_{j,\alpha,0}(X, D)\, \phi(j^{-1} D)\, u$$

for every $u \in L^2(\boldsymbol{R}^n)$. It follows from (3.4) and the fact $\phi(j^{-1} D)\, u \to 0$ in $L^2(\boldsymbol{R}^n)$ as $j \to \infty$ that

$$(3.6) \qquad B^{(1)}_{j,\alpha,0}(X, D)\, u,\ C_{j,\alpha,0}(X, D)\, u \to 0 \quad \text{in} \quad L^2(\boldsymbol{R}^n) \quad \text{as} \quad j \to \infty\,.$$

In the same way we have

$$(3.7) \qquad A_{j,\alpha,0}(X, D)\, \phi(\bar{\varphi}(j)^{-1} X)\, u,\ B^{(2)}_{j,\alpha,0}(X, D)\, \phi(\bar{\varphi}(j)^{-1} X)\, u \to 0 \quad \text{in} \quad L^2(\boldsymbol{R}^n) \quad \text{as} \quad j \to \infty\,.$$

Finally, let $E_{j,\alpha}(x, \xi)$ and $F_{j,\alpha}(x, \xi)$ be symbols such that

$$(3.8) \qquad E_{j,\alpha}(X, D) = A_{j,\alpha,0}(X, D)\ \{1 - \phi(\bar{\varphi}(j)^{-1} X)\}$$

and

$$(3.9) \qquad F_{j,\alpha}(X, D) = B^{(2)}_{j,\alpha,0}(X, D)\ \{1 - \phi(\bar{\varphi}(j)^{-1} X)\}\,.$$

Since $\phi(\bar{\varphi}(j)^{-1} x) \equiv 1$ if $(x, \xi) \in \operatorname{supp} A_{j,\alpha,0}(x, \xi) \cup \operatorname{supp} B^{(2)}_{j,\alpha,0}(x, \xi)$, we have

$$(3.10) \qquad E_{j,\alpha}(x, \xi) = -N \bar{\varphi}(j)^{-N} \int_0^1 (1-\theta)^{N-1} \sum_{|\gamma|=N} \frac{1}{\gamma\,!} \iint \exp(-iy \cdot \eta)\ (\partial_\xi^\gamma A_{j,\alpha,0})\,(x, \xi + \theta\eta)\,(D_x^\gamma \phi)\,(\bar{\varphi}(j)^{-1}(x+y))\, dy\, \bar{d}\eta\, d\theta$$

and

$$(3.11) \qquad F_{j,\alpha}(x, \xi) = -N \bar{\varphi}(j)^{-N} \int_0^1 (1-\theta)^{N-1} \sum_{|\gamma|=N} \frac{1}{\gamma\,!} \iint \exp(-iy \cdot \eta)\ (\partial_\xi^\gamma B^{(2)}_{j,\alpha,0})\,(x, \xi + \theta\eta)\,(D_x^\gamma \phi)\,(\bar{\varphi}(j)^{-1}(x+y))\, dy\, \bar{d}\eta\, d\theta\,.$$

Then, from the fact that the families $\{A_{j,\alpha,0}(x, \xi);\, j=1, 2, \cdots\}$ and $\{B^{(2)}_{j,\alpha,0}(x, \xi);\, j=1, 2, \cdots\}$ are bounded in $S^{0,0}_{0,0,0,0}$ and $S^{0,0}_{\rho,\delta,\rho',\delta'}$ with $\delta, \delta' < 1$ respectively, we conclude that

$$(3.12) \qquad E_{j,\alpha}(X, D)\, u,\ F_{j,\alpha}(X, D)\, u \to 0 \quad \text{in} \quad L^2(\boldsymbol{R}^n) \quad \text{as} \quad j \to \infty\,.$$

Now the conclusion follows from (3.3)—(3.12).

4. Proof of Theorems 1.8 and 1.9

First we state two propositions.

Proposition 4.1. *Let X denote a function space of the form $H[\varphi_0, \psi_0]$, where $\varphi_0(r), \psi_0(r) \in \mathcal{M}$. Suppose that $\varphi(r), \psi(r) \in \mathcal{M}_+$ satisfy* (1.2), *and put $S = \Phi(\langle D \rangle) + \Psi(\langle X \rangle)$. Next, let M be a positive constant such that $S+M$ gives a topological* *isomorphism from* $H[\Phi\varphi_0, \psi_0]\cap H[\varphi_0, \Psi\psi_0]$ *to* X, *and also* X *to* $H[\varphi^0/\Phi, \psi_0]+H[\varphi_0, \psi_0/\Psi]$, *and put*

$$\mathcal{B}(t)=(S+M)\,\mathcal{A}(t)\,(S+M)^{-1}-\mathcal{A}(t)$$

and

$$\mathcal{C}(t)=(S+M)^{-1}\,\mathcal{A}(t)\,(S+M)-\mathcal{A}(t)\,.$$

Then $\{\mathcal{B}(t); t\in[-T, T]\}$ *and* $\{\mathcal{C}(t); t\in[-T, T]\}$ *are uniformly bounded families of bounded operators on* X, *and they are continuous with respect to the uniform topology as operator-valued functions of* t. *Further, if* $r/C\leq\varphi(\psi(r))$ *holds, then* $\{\mathcal{A}(t)\,(S+M)^{-1}; t\in[-T, T]\}$ *is also a uniformly bounded family of bounded operators on* X, *and it is continuous with respect to the uniform topology as an operator-valued function of* t.

Proof. Putting

$$\begin{aligned}
&P(t, x, \xi)\\
&=\sum_{h=1}^{n}\int_0^1\iint(\partial_{\xi_h}\Phi(\langle\xi+\theta\eta\rangle)\,D_{x_h}\{B(t, x+y, \xi)+C(t, x+y)\}\\
&\quad -D_{x_h}\Psi(\langle x+y\rangle)\,\partial_{\xi_h}\{A(t, \xi+\theta\eta)+B(t, x, \xi+\theta\eta)\})\,dy\,\bar{d}\eta\,d\theta\\
&=P_1(t, x, \xi)+P_2(t, x, \xi)+P_3(t, x, \xi)+P_4(t, x, \xi)\,,
\end{aligned}$$

we have $\mathcal{B}(t)\,(S+M)=-(S+M)\,\mathcal{C}(t)=[S+M, \mathcal{A}(t)]=P(t, X, D)$. Hence, in the same way as in the proof of Proposition 3.1, we can prove that the family $\{P_j(t, x, \xi); t\in[-T, T]\}$ is bounded and continuous in the symbol class S_j for every $j=1, 2, 3, 4$, where $S_2=S_{\rho,\delta,\rho',\delta'}(T; \Phi(r), 1)$, $S_2=S_3=S_{0,0,0,0}(T; \varphi(r), \psi(r))$ and $S_4=S_{\rho,\delta,\rho',\delta'}(T; 1, \Psi(r))$. Hence, putting $E_1=H[\Phi\varphi_0, \psi_0]$, $E_2=E_3=H[\varphi\varphi_0, \psi\psi_0]$, $E_1=H[\varphi_0, \Psi\psi_0]$, $F_1=H[\varphi_0/\Phi, \psi_0]$, $F_2=F_3=H[\varphi_0/\varphi, \psi_0/\psi]$, and $F_4=H[\varphi_0, \psi_0/\Psi]$, we easily see that $\{P_j(t, X, D); t\in[-T, T]\}$ is a uniformly bounded family of operators bounded from E_j to X and also from X to F_k for every $j=1, 2, 3, 4$. Further, the set $\{\mathcal{A}(t); t\in[-T, T]\}$ is a uniformly bounded family of operators bounded from $E_0=E_1\cap E_4\cap H[r\varphi_0, r\psi_0]$ to X. Moreover, as operator-valued functions of t, they are continuous with respect to the uniform topology. Since $(S+M)^{-1}$ is an isomorphism from X to $E_1\cap E_4$ and also from F_1+F_4 to X, it suffices to show $E_1\cap E_4\subset E_2$ and $F_2\subset F_1+F_4$, and also $E_1\cap E_4\subset H[r\varphi_0, r\psi_0]$ under the extra assumption $r/C\leq\varphi(\psi(r))$. In view of Proposition 2.3, we may assume $\varphi_0(r)\equiv\psi_0(r)\equiv 1$; namely, $X=L_2(\boldsymbol{R}^n)$.

Let $Z_{2^j}(X, D)$ be the same as in Section 3. For $u\in E_1\cap E_4$, put

$$\begin{aligned}
u_j&=Z_{2^j}(X, D)\,u\,,\\
u_j^{(1)}&=\varphi(\langle D\rangle)\,\psi(\langle X\rangle)\,(\zeta(\bar{\varphi}(2^{j+1})^{-1}X)-\zeta(\bar{\varphi}(2^j)^{-1}X))\,\zeta(2^{-j}D)\,u
\end{aligned}$$

and

$$u_j^{(2)} = \varphi(\langle D\rangle)\,\psi(\langle X\rangle)\,\zeta(\bar{\varphi}(2^{j+1})^{-1}X)\,(\zeta(2^{-j-1}D)-\zeta(2^{-j}D))\,u\,.$$

Then we have $\varphi(\langle D\rangle)\,\psi(\langle X\rangle)\,(u_{j+1}-u_j)=u_j^{(1)}+u_j^{(2)}$. Next, put $F_j=\{x\in \boldsymbol{R}^n;\ \bar{\varphi}(2^{j-1})\le |x| \le \bar{\varphi}(2^{j+1})\}$ and $\phi_j(\xi)=\chi_{F^j}(\xi)$, and take $k\in \boldsymbol{N}$ such that $\partial_\xi^\alpha\,\varphi(\xi)$ and $\partial_x^\alpha\,\psi(x)$ are bounded for every $\alpha\in\boldsymbol{N}^n$ satisfying $|\alpha|=k$. Then we can write

$$u_j^{(1)} = \sum_{0\le|\alpha|\le k-1} P_j^\alpha(D)\,Q_j^\alpha(X)\,\phi_j(X)\,u+R_j(X,D)\,u\,, \tag{4.1}$$

where

$$P_j^\alpha(\xi) = \varphi(\langle\xi\rangle)\,\partial_\xi^\alpha\,\zeta(2^{-j}\xi)\,,$$
$$Q_j^\alpha(x) = i^{|\alpha|}\,\partial_x^\alpha\{\psi(\langle x\rangle)\,\{\zeta(\bar{\varphi}(2^{j+1})^{-1}x)-\zeta(\bar{\varphi}(2^j)^{-1}x)\}\}$$

and

$$\{\bar{\varphi}(2^{j-1})^{k+1}R_j(x,\xi);\, j=1,2,\cdots\}\quad \text{is bounded in}\quad S^{0;0}_{1,0,1,0}\,.$$

Since supp $Q_j^\alpha(x)\subset F_j$, we can apply the Plancherel equality to obtain

$$\begin{aligned}
&||P_j^\alpha(D)\,Q_j^\alpha(X)\,\phi_j(X)\,u||_{L^2(\boldsymbol{R}^n)}\\
&\le\Big(\sup_{\zeta\in\boldsymbol{R}^n}|P_j^\alpha(\xi)|\Big)\Big(\sup_{x\in F_j}\Big|\frac{Q_j^\alpha(x)}{\Psi(\langle x\rangle)}\Big|\Big)\,||\phi_j(x)\,\Psi(\langle x\rangle)\,u(x)||_{L^2(\boldsymbol{R}^n)}\\
&\le C_\alpha\,\varphi(2^{j+1})\,C_\alpha\,\bar{\varphi}(2^{1-j})^{-|\alpha|-1}\,||\phi_j(x)\,\Psi(\langle x\rangle)\,u(x)||_{L^2(\boldsymbol{R}^n)}\\
&\le C'_\alpha\,||\phi_j(x)\,\Psi(\langle x\rangle)\,u(x)||_{L^2(\boldsymbol{R}^n)}\,.
\end{aligned}\tag{4.2}$$

On the other hand, the operators $\bar{\varphi}(2^{j-1})^{k+1}R_j(X,D)$ are bounded on $L^2(\boldsymbol{R}^n)$ uniformly in j. Substituting this fact and the estimate (4.2) into (4.1), we conclude

$$\begin{aligned}
&\sum_{j=h}^{l-1}||u_j^{(1)}||_{L^2(\boldsymbol{R}^n)}\\
&\le C\sum_{j=h}^{l-1}(\bar{\varphi}(2^{j-1})^{-k-1}\,||u||_{L^2(\boldsymbol{R}^n)}+||\phi_j(x)\,\Psi(\langle x\rangle)\,u(x)||_{L^2(\boldsymbol{R}^n)})\,.
\end{aligned}$$

Since $\Psi(\langle x\rangle)\,u(x)\in L^2(\boldsymbol{R}^n)$ and since

$$\sum_{j=1}^\infty \bar{\varphi}(2^{j-1})^{-k-1}\le \sum_{j=1}^\infty\Big(\frac{a^{j-1}}{C}\Big)^{-k-1}<\infty\,,$$

the right-hand side of this formula converges to 0 as $h, l\to\infty$. Hence $\sum_{j=1}^\infty u_j^{(1)}$ converges in $L^2(\boldsymbol{R}^n)$. In the same way we can show that $\sum_{j=1}^\infty u_j^{(2)}$ converges in $L^2(\boldsymbol{R}^n)$, by using the fact $\psi(\bar{\varphi}(2^j))\le 2^j$. We thus conclude that $\varphi(\langle D\rangle)\,\psi(\langle X\rangle)\,u_j$ converges in $L^2(\boldsymbol{R}^n)$, which implies $\varphi(\langle D\rangle)\,\psi(\langle X\rangle)\,u\in L^2(\boldsymbol{R}^n)$. This proves $E_1\cap E_4\subset E_2$, which yields $F_2\subset F_1+F_4$ by duality argument.

Next we assume $r/C\le\varphi(\psi(r))$. Then we have $\langle D\rangle\,\langle X\rangle\,(u_{j+1}-u_j)=u_j^{(3)}+u_j^{(4)}$, where

$$u_j^{(3)} = \langle D\rangle\,\langle X\rangle\,(\zeta(\bar{\varphi}(2^{j+1})^{-1}X)-\zeta(\bar{\varphi}(2^j)^{-1}X))\,\zeta(2^{-j}D)\,u$$

and

$$u_j^{(4)} = \langle D\rangle \langle X\rangle \zeta(\bar{\varphi}(2^{j+1})^{-1} X)\,(\zeta(2^{-j-1} D)-\zeta(2^{-j} D))\, u\,.$$

Putting

$$A_j(\xi) = \langle\xi\rangle\, \zeta(2^{-j}\,\xi)$$

and

$$B_j(x) = \langle x\rangle\ \{\zeta(\bar{\varphi}(2^{j+1})^{-1}\,x)-\zeta(\bar{\varphi}(2^j)^{-1}\,x))\}\ ,$$

we can write

$$u_j^{(3)} = A_j(D)\, B_j(X)\, \phi_j(X)\, u+C_j(X, D)\, u\ ,$$

where $\{\bar{\varphi}(2^{j-1})\, C_j(x, \xi); j=1, 2, \cdots\}$ is bounded in $S^{0;0}_{1;0,1,0}$. Since supp $Q_j^{\alpha}(\xi)\subset F_j$ and since

$$2^{j-1}\,\bar{\varphi}(2^{j-1})\leq C\,\psi(\bar{\varphi}(2^{j-1}))\,\bar{\varphi}(2^{j-1})\leq C'\,\Psi(\bar{\varphi}(2^{j-1}))\,,$$

we can apply the Plancherel equality to obtain

$$\begin{aligned}&\|A_j(D)\, B_j(X)\, \phi_j(X)\, u\|_{L^2(\boldsymbol{R}^n)}\\ &\leq C2^{j+1}\cdot C\,\frac{\bar{\varphi}(2^{j+1})}{\Psi(\bar{\varphi}(2^{j-1}))}\,\|\varphi_j(x)\,\Psi(\langle x\rangle)\,u(x)\|_{L^2(\boldsymbol{R}^n)}\\ &\leq C'\,\|\phi_j(x)\,\Psi(\langle x\rangle)\,u(x)\|_{L^2(\boldsymbol{R}^n)}\,.\end{aligned}$$

Using this fact, we can conclude as before that $\sum_{j=i}^{\infty} u_j^{(3)}$ converges in $L^2(\boldsymbol{R}^n)$. In the same way we can show that $\sum_{j=1}^{\infty} u_j^{(4)}$ converges in $L^2(\boldsymbol{R}^n)$, by using the fact $2^j\,\bar{\varphi}(2^j)\leq C\Phi(2^j)$ with some positive constant C independent of $j\in\boldsymbol{N}$. These facts imply the convergence of $\langle D\rangle\langle X\rangle u_j$ in $L^2(\boldsymbol{R}^n)$. Now $E_1\cap E_4\subset H[r, r]$ follows from this similarly as before.

Proposition 4.2. *Let X denote a function space of the form* $H[\varphi_0, \psi_0]$, *where* $\varphi_0(r), \psi_0(r)\in\mathcal{M}$. *Suppose that* $\varphi(r), \psi(r)\in\mathcal{M}_+$ *such that* $\varphi(r)$ [*resp.* $\psi(r)$] *is bounded, and put* $R=\Psi(X)$. [*resp.* $R=\Phi(D)$.] *Tnen* $\{R\mathcal{A}(t)\,R^{-1}-\mathcal{A}(t); t\in[-T, T]\}$ *and* $\{R^{-1}\,\mathcal{A}(t)\,R-\mathcal{A}(t); t\in[-T, T]\}$ *are uniformly bounded families of bounded operators on X, and* $\{\mathcal{A}(t)\,R^{-1}; t\in[-T, T]\}$ *is a uniformly bounded family of bounded operators from* $X\cap H[r\varphi_0, \psi_0/\psi]$ *to X, and also from X to* $X+H[\varphi_0/r, \psi_0\,\psi]$. [*resp. from* $X\cap H[\varphi_0/\varphi, r\psi_0]$ *to X, and also from X to* $X+H[\varphi_0\,\varphi, \psi_0/r]$.] *Further, these families are continuous with respect to the uniform topology as operator-valued funciions of t.*

Proof. We consider only the case where $\varphi(r)$ is bounded and $R=\Psi(X)$. We proceed in the same way as in the proof of Proposition 4.1. Putting

$$\begin{aligned}P'(t, x, \xi) &= P_3(t, x, \xi)+P_4(t, x, \xi)\\ &= -\sum_{h=1}^{n}\int_0^1\iint D_{x^h}\,\Psi(\langle x+y\rangle)\,\partial_{\xi^h}\{A(t, \xi+\theta\eta)+B(t, x, \xi+\theta\eta)\})\,dy\,\bar{d}\eta\,d\theta\ ,\end{aligned}$$

we have $[R, \mathcal{A}(t)]=P'(t, X, D)$. In this case we have $E_3=H[\varphi_0, \psi\psi_0]\supset E_1=H[\varphi_0, \Psi\psi_0]$ and $F_3=H[\varphi_0, \psi_0/\psi]\subset F_4=H[\varphi_0, \psi_0/\Psi]$. Further, R^{-1} is an isomorphism from X to E_4 and also from F_4 to X. Hence it is easy to see that $\{[R, \mathcal{A}(t)]R^{-1}; t\in[-T, T]\}$ and $\{R^{-1}[R, \mathcal{A}(t)]; t\in[-T, T]\}$ are uniformly bounded families of bounded operators on X. Moreover, as operator-valued functions of t, they are continuous with respect to the uniform topology.

On the other hand, the family $\{\mathcal{A}(t); t\in[-T, T]\}$ is a uniformly bounded family of bounded operators from $H[r\varphi_0, r\psi_0]\cap H[\varphi_0, \psi_0\Psi]$ to X, and also from X to $H[\varphi_0/r, \psi_0/r]+H[\varphi_0, \psi_0/\Psi]$. Further, as an operator-valued function of t, it is continuous with respect to the uniform topology. The assertion for $\mathcal{A}(t)R^{-1}$ is an immediate consequence of this fact and the isomorphism property of R^{-1}.

Let X denote either $\mathcal{H}^{\sigma}$ or $\mathcal{K}^{\sigma,\tau}$. Take $\bar{\varphi}(r), \bar{\psi}(r)\in\mathcal{M}_+$ as in Proposition 2.1, 8), and put $\bar{\Phi}(r)=1+\int_1^r\bar{\varphi}(s)\,ds$, $\bar{\Psi}(r)=1+\int_1^r\bar{\psi}(s)\,ds$, and $Q=\bar{\Phi}(D)+\bar{\Psi}(X)$. Then, by Corollary 2.6, we see that there exists a positive constant M such that $Q+M$ is an isomorphism from $H[\Phi(r)^{\sigma}\bar{\Phi}(r), 1]\cap H[\Phi(r)^{\sigma}, \bar{\Psi}(r)]$ to $H[\Phi(r)^{\sigma}, 1]$ and also from $H[\bar{\Phi}(r), \Psi(r)^{\sigma}]\cap H[1, \Psi(r)^{\sigma}\bar{\Psi}(r)]$ to $H[1, \Psi(r)^{\sigma}]$ if $X=\mathcal{H}^{\sigma}$, and that $Q+M$ is an isomorphism from $H[\Phi(r)^{\sigma+\tau}\bar{\Phi}(r), 1]\cap H[\Phi(r)^{\sigma+\tau}, \bar{\Psi}(r)]$ to $H[\Phi(r)^{\sigma+\tau}, 1]$ and also from $H[\Phi(r)^{\sigma}\bar{\Phi}(r), \Omega(r)^{\tau}]\cap H[\Phi(r)^{\sigma}, \Omega(r)^{\tau}\bar{\Psi}(r)]$ to $H[\Phi(r)^{\sigma}, \Omega(r)]^{\tau}]$ if $X=\mathcal{K}^{\sigma,\tau}$. Then, by using Lemma 2.4 repeatedly, we can equip the space $Y=(Q+M)^{-1}X$ with the Hilbert space structure with which $Q+M$ becomes an isomorphism from Y to X. It follows immediately from the previous propositions that the family $\{\mathcal{A}(t); t\in[-T, T]\}$ is a uniformly bounded family of bounded operators from Y to X, and that $\{(Q+M)\mathcal{A}(t)(Q+M)^{-1}-\mathcal{A}(t); t\in[-T, T]\}$ is a uniformly bounded family of bounded operators on X. Further, as operator-valued functions of t, they are continuous with respect to the uniform topology. Hence, in view of Proposition 6.1 of Kato [15], Theorems 1.8 and 1.9 will follow from the following

Proposition 4.3. *The family $\{\pm i\mathcal{A}(t); t\in[-T, T]\}$ is stable in X; that is, there exist constants $L, N>0$ such that, for every $k\in N$, $t_1, t_2, \cdots, t_k\in[-T, T]$ and $\lambda>N$, the following estimate holds*:

$$\|\prod_{j=1}^{k}(\pm i\mathcal{A}(t_j)-\lambda)^{-1}\|_{L(X)}\leq L(\lambda-N)^{-k}.$$

For the proof of this proposition we use the following two lemmas.

Lemma 4.4.

1) *Put $S=\Phi(\langle D\rangle)+\Psi(\langle X\rangle)$. Then, for every $\sigma, \tau\in N$, there exists a positive number M such that $(S+M)^{\sigma}$ is an isomorphism from $\mathcal{H}^{\sigma+\tau}$ to $\mathcal{H}^{\tau}$.*

2) *Asssume that* $\psi(r)$ *is bounded, and put* $Q=\Phi(\langle D\rangle)+\Omega(\langle X\rangle)$. *Then, for every* $\sigma\in\boldsymbol{R}$ *and every* $\rho, \tau\in\boldsymbol{N}$, *there exists a positive number* M *such that* $(Q+M)^{\rho}$ *is an isomorphism from* $\mathcal{K}^{\sigma,\tau+\rho}$ *to* $\mathcal{K}^{\sigma,\tau}$.

Proof. In order to prove Assertion 1), put $\mathcal{H}^{(\tau)}=\bigcap_{j=0}^{\tau} H[\Phi^j, \Psi^{\tau-j}]$ for every $\tau\in\boldsymbol{N}$. Then we can take $M>0$ so large that the mapping $S+M$ gives an isomorphism from $H[\Phi^{j+1}, \Psi^k]\cap H[\Phi^j, \Psi^{k+1}]$ to $H[\Phi^j, \bar{\Psi}^k]$ for every $j, k\in\boldsymbol{N}$ such that $j+k\leq\sigma+\tau$. Then $(S+M)^{\sigma}$ is an isomorphism from $\mathcal{H}^{(\sigma+\tau)}$ to $\mathcal{H}^{(\tau)}$.

On the other hand, Proposition 2.3, 5) yields

$$H[\Phi^j, \Psi^{\sigma-j}]=[H[1, \Psi^{\sigma}], H[\Phi^{\sigma}, 1]]_{j/\sigma}\supset H[\Phi^{\sigma}, 1]\cap H[1, \Psi^{\sigma}]$$

for every $j\in\boldsymbol{N}$ such that $j\leq\sigma$. This implies $\mathcal{H}^{(\sigma)}=\mathcal{H}^{\sigma}$ for every $\sigma\in\boldsymbol{N}$, which completes the proof of Assertion 1).

We can prove Assertion 2) in the same way, by considering the space $\bigcap_{j=0}^{\tau} H[\Phi^{\sigma+j}, \Omega^{\tau-j}]$ in place of $\mathcal{H}^{(\tau)}$.

Lemma 4.5.

1) *For every positive integer* h *and every positive number* σ *less than* h, *we have* $\mathcal{H}^{\sigma}=[L^2(\boldsymbol{R}^n), \mathcal{H}^h]_{\sigma/h}=(L^2(\boldsymbol{R}^n), \mathcal{H}^h)_{\sigma/h,2}$, *where* $(\cdot, \cdot)_{t,2}$ *denotes the real interpolation space.*
2) *Assume that* $\psi(r)$ *is bounded. Then, for every* $\sigma\in\boldsymbol{R}$, *every positive integer* h *and every positive number* τ *less than* h, *we have* $\mathcal{K}^{\sigma,\tau}=[\mathcal{K}^{\sigma,0}, \mathcal{K}^{\sigma,h}]_{\tau/h}=(\mathcal{K}^{\sigma,0}, \mathcal{K}^{\sigma,h})_{\tau/h,2}$,

Proof. For every $h\in\boldsymbol{N}$, there exists a positive constant C such that the family

$$\{\Psi(\langle X\rangle)^h\,(1+\frac{1}{\lambda}\,\Phi(\langle D\rangle)^h)^{-1}\,\Psi(\langle X\rangle)^{-h};\,\lambda>C\}$$

is uniformly bounded on $L^2(\boldsymbol{R}^n)$ with respect to the operator norm. Hence we can apply the main theorem of Grisvard [6] to conclude that

$$\begin{aligned}(4.3)\qquad &(L^2(\boldsymbol{R}^n), H[\Phi^h, 1]\cap H[1, \Psi^h])_{\sigma/h,2}\\ &\quad=(L^2(\boldsymbol{R}^n), H[\Phi^h, 1])_{\sigma/h,2}\cap(L^2(\boldsymbol{R}^n), H[1, \Psi^h])_{\sigma/h,2}\,.\end{aligned}$$

On the other hand, Triebel [27, §1.8.10, Remark 3] implies

$$\begin{aligned}(4.4)\qquad &(L^2(\boldsymbol{R}^n), H[\Phi^h, 1]\cap H[1, \Psi^h])_{\sigma/h,2}\\ &\quad=[L^2(\boldsymbol{R}^n), H[\Phi^h, 1]\cap H[1, \Psi^h]]_{\sigma/h}\,,\end{aligned}$$

and

$$\begin{aligned}(4.5)\qquad &(L^2(\boldsymbol{R}^n), H[\Phi^h, 1])_{\sigma/h,2}\cap(L^2(\boldsymbol{R}^n), H[1, \Psi^h])_{\sigma/h,2}\\ &\quad=[L^2(\boldsymbol{R}^n), H[\Phi^h, 1]]_{\sigma/h}\cap[L^2(\boldsymbol{R}^n), H[1, \Psi^h]]_{\sigma/h}\,.\end{aligned}$$

Finally, Proposition 2.3, 5) yields

$$\begin{aligned}(4.6)\qquad &[L^2(\boldsymbol{R}^n), H[\Phi^h, 1]]_{\sigma/h} \cap [L^2(\boldsymbol{R}^n), H[1, \Psi^h]]_{\sigma/h} \\ &= H[\Phi^\sigma, 1] \cap H[1, \Psi^\sigma] = \mathcal{H}^\sigma .\end{aligned}$$

Now Assertion 1) follows from (4.3)-(4.6). Assertion 2) can be proved exactly in the same way, by replacing $L^2(\boldsymbol{R}^n)$ and $\mathcal{H}^h$ by $\mathcal{K}^{\sigma,0}$ and $\mathcal{K}^{\sigma,h}$ respectively.

Proof of Proposition 4.3. We begin with the case $X=\mathcal{H}^\sigma$. First, assume $\sigma\in\boldsymbol{N}$. Then Lemma 4.4, 1) assures the existence of positive constants M and $C_1, \cdots, C_\sigma$ such that

$$\frac{1}{C_j}\|T\|_{L(\mathcal{H}^j)} \le \|(S+M)^j T(S+M)^{-j}\|_{L(L^2(\boldsymbol{R}^n))} \le C_j \|T\|_{L(\mathcal{H}^j)}$$

for every $j=1, 2, \cdots, \sigma$, where $S=\Phi(\langle D\rangle)+\Psi(\langle X\rangle)$. Making use of these constants and observing Proposition 4.1, we obtain

$$\begin{aligned}(4.7)\quad &\frac{1}{C_\sigma}\| \prod_{j=1}^{k} (\pm i\mathcal{A}(t_j)-\lambda)^{-1}\|_{L(\mathcal{H}^\sigma)} \\ &\le \|(S+M)^\sigma \{ \prod_{j=1}^{k} (\pm i\mathcal{A}(t_j)-\lambda)^{-1}\} (S+M)^{-\sigma}\|_{L(L^2(\boldsymbol{R}^n))} \\ &= \| \prod_{j=1}^{k} \{(S+M)^\sigma (\pm i\mathcal{A}(t_j)-\lambda) (S+M)^{-\sigma}\}^{-1}\|_{L(L^2(\boldsymbol{R}^n))} \\ &= \| \prod_{j=1}^{k} \{(S+M)^{\sigma-1}(\pm i\mathcal{A}(t_j)\pm i\mathcal{B}(t_j)-\lambda) (S+M)^{1-\sigma}\}^{-1}\|_{L(L^2(\boldsymbol{R}^n))} \\ &= \| \prod_{j=2}^{k} (\pm i\mathcal{A}(t_j)\pm \sum_{j=0}^{\sigma-1} (S+M)^j i\mathcal{B}(t_j) (S+M)^{-j}-\lambda)^{-1}\|_{L(L^2(\boldsymbol{R}^n))} \\ &= \| \prod_{j=1}^{k} \left(\pm i\frac{\mathcal{A}(t_j)+(\mathcal{A}(t_j))^*}{2} \pm \mathcal{R}(t_j)-\lambda\right)^{-1}\|_{L(L^2(\boldsymbol{R}^n))}\end{aligned}$$

for every $k\in\boldsymbol{N}$ and $t_1, t_2, \cdots, t_k\in[-T, T]$, where

$$\mathcal{R}(t_j) = i\sum_{j=0}^{\sigma-1} (S+M)^j \mathcal{B}(t_j) (S+M)^{-j} + i\frac{\mathcal{A}(t_j)-(\mathcal{A}(t_j))^*}{2} .$$

It follows from Proposition 4.1 and Theorem 1.4 that

$$\begin{aligned}&\|\mathcal{R}(t_j)\|_{L(L^2(\boldsymbol{R}^n))} \\ &\le \sum_{j=0}^{\sigma-1} C_j \sup_{-T\le t\le T} \|\mathcal{B}(t)\|_{L(\mathcal{H}^j)} + \sup_{-T\le t\le T} \|\frac{\mathcal{A}(t)-(\mathcal{A}(t))^*}{2}\|_{L(L^2(\boldsymbol{R}^n))} \\ &= N .\end{aligned}$$

From this fact and (4.6) we conclude

$$\| \prod_{j=1}^{k} (\pm i\mathcal{A}(t_j)-\lambda)^{-1}\|_{L(\mathcal{H}^\sigma)} \le C_\sigma(\lambda-N)^{-k}$$

for $\lambda>N$. This estimate shows the conclusion in the case $\sigma\in\boldsymbol{N}$. We can treat the case $\sigma>0$ and $\sigma<0$ by interpolation and duality argument, making use of Lemma 4.5, 1) and Proposition 2.3, 4) respectively.

We turn to the case $X=\mathcal{K}^{\sigma,\tau}$. We first consider the case $\tau=0$.

First, suppose $\sigma\in\boldsymbol{N}$. Then, mrking use of R in place of $S+M$ and applying Proposition 4.2 in place of Proposition 4.1, we can prove the well-posedness by an argument similar to that for $\mathcal{H}^{\tau}$ with $\sigma\in\boldsymbol{N}$. The case $\sigma>0$ and $\sigma<0$ can be handled by way of interpolation and duality argument respectively.

Next we consider the case $\tau\in\boldsymbol{N}$. Starting from the well-posedness on $\mathcal{K}^{\sigma,0}$ in place of that on $L^2(\boldsymbol{R}^n)$ and making use of $Q+M$ in place of $S+M$, we can treat this case similarly as in the case $X=\mathcal{H}^{\tau}$ with $\sigma\in\boldsymbol{N}$, by virtue of Lemma 4.4, 2) in place of Lemma 4.4, 1). We can handle the case $\tau>0$ by interpolation argument, using Lemma 4.5, 2) in place of Lemma 4.5, 1). Finally, we obtain the conclusion for $\tau<0$ by way of the duality $\mathcal{K}^{\sigma,\tau}=(\mathcal{K}^{-\sigma,-\tau})'$. This completes the proof of Proposition 4.3.

Appendix. Proof of Proposition 1.6

Defining a unitary transformation U on $L^2(\boldsymbol{R})$ mapping $C_0^2(\boldsymbol{R})$ onto itself by the formula $(Uu)(x)=\exp\left(-i\int_0^x b(t)\,dt\right)u(x)$ and introducing an operator $\mathcal{A}$ with domain $C_0^2(\boldsymbol{R})$ by

$$\mathcal{A}u(x)=(U^{-1}(D^2+B^w(X,D)+C(X))\,Uu)(x)\,,$$

we can write

$$\begin{aligned}\mathcal{A}u(x)&=-u''(x)+2ib(x)\,u'(x)+ib'(x)\,u(x)+b(x)^2\,u(x)+C(x)\,u(x)\\&\quad+2\iint\exp\left(i(x-y)\,\xi-i\int_x^y b(t)\,dt\right)b\left(\frac{x+y}{2}\right)\xi v(y)\,dy\,\bar{d}\xi\\&=-u''(x)+2ib(x)\,u'(x)+ib'(x)\,u(x)+b(x)^2\,u(x)+C(x)\,u(x)\\&\quad+2D_y\{\exp\left(-i\int_x^y b(t)\,dt\right)b\left(\frac{x+y}{2}\right)v(y)\}\,|_{y=x}\\&=-u''(x)-f(x)\,u(x)\,,\end{aligned}$$

and the essential self-adjointness of the operator $D^2+B^w(X,D)+C(X)$ on $C_0^\infty(\boldsymbol{R})$ implies that of $\mathcal{A}$. But, in view of (1.3) and (1.4), we can apply Dunford-Schwartz [2, XIII. 6.22] to conclude that the point $+\infty$ is a boundary point of limit circle type of $\mathcal{A}$. It follows that $\mathcal{A}$ has positive deficiency indices, and hence $\mathcal{A}$ cannot be essentially self-adjoint.

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Department of Mathematics,
Hitotsubashi University,
Kunitachi, Tokyo 186 Japan