Title	Lorentz invariant cutting off of Lorentz invariant distributions by a space-like hyperplane
Author(s)	Kasuga, Takashi
Citation	Osaka Mathematical Journal. 1963, 15(1), p. 99-107
Version Type	VoR
URL	https://doi.org/10.18910/6533
rights	
Note	

Kasuga T.
Osaka Math. J.
15 (1963), 99-107.

LORENTZ INVARIANT CUTTING OFF OF LORENTZ INVARIANT DISTRIBUTIONS BY A SPACE-LIKE HYPERPLANE

Dedicated to Professor K. Shoda on his sixtieth birthday

By

TAKASHI KASUGA

Introduction

We shall use the following notations.

$\mathfrak{M}=\{x \mid x=(x^0, x^1, x^2, x^3)\}$: Minkowski's space-time, the real variables x^0 and (x^1, x^2, x^3) being the time coordinate and the space coordinates respectively.

$O = (0, 0, 0, 0)$: the origin of $\mathfrak{M}$.

$(x)^2 = (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2$.

$\Gamma^+ = \{x \mid x \in \mathfrak{M},\ (x)^2 \geqq 0,\ x^0 \geqq 0\}$.

$\Gamma^- = \{x \mid x \in \mathfrak{M},\ (x)^2 \geqq 0,\ x^0 \leqq 0\}$.

$\mathfrak{L} = \{\Lambda\}$: the homogeneous proper Lorentz group i.e. the group of homogeneous linear transformations of $\mathfrak{M}$ onto $\mathfrak{M}$ which leave $(x)^2$ invariant, have determinants equal to 1 and transform each of Γ^+ and Γ^- onto itself. In the following we shall call $\mathfrak{L}$ simply *Lorentz group*. The terms Lorentz invariance etc. are used in this sense.

In the relativistic quantum theory of fields, one frequently meets the following situation. Let a Lorentz invariant[1] field $S(x)$ in $\mathfrak{M}$ of quantities obeying some representation of the Lorentz group be given. If the field $S(x)$ vanishes in the space-like region $\mathfrak{M}-\Gamma^+-\Gamma^-$, then one forms a new field $\theta(x)S(x)$ by multiplying $S(x)$ by a function $\theta(x)$ defined by

$$\theta(x) = \begin{cases} 1 & \text{for } x^0 \geqq 0 \\ 0 & \text{for } x^0 < 0 \end{cases}$$

---

1) A field $S(x)$ in $\mathfrak{M}$ is called Lorentz invariant if $P_\Lambda S(\Lambda^{-1}x)=S(x)$ for all $\Lambda \in \mathfrak{L}$ where P_Λ is the linear transformation induced by Λ in the linear space of quantities constituting the field. The mapping $\Lambda \rightarrow P_\Lambda$ is a representation of $\mathfrak{L}$.

and claims $\theta(x)S(x)$ to be a Lorentz invariant field[2].

But the components of the fields which appear in the relativistic quantum theory of fields are usually distributions in the sense of L. Schwartz which are highly singular in the neighbourhood of the origin O of $\mathfrak{M}$. Hence the above procedure can not always be justified. So we put the following question. Given a Lorentz invariant distribution field[3] $S(x)$ in $\mathfrak{M}$ which vanishes in the space-like region $\mathfrak{M}-\Gamma^+-\Gamma^-$, is there always a Lorentz invariant distribution field $S^*(x)$ in $\mathfrak{M}$ which coincides with $S(x)$ in $\mathfrak{M}-\Gamma^-$ and vanishes in $\mathfrak{M}-\Gamma^+$? The answer is "yes" as it is shown in Theorem 2.1[4] though in general such $S^*(x)$ is not uniquely determined.

For simplicity, we shall state the proof of this result only for a scalar distribution field. But the proof of the result goes quite similarly for a distribution field of quantities obeying any representation of the Lorentz group. Also similar result can be established for the extended Lorentz group which is generated by the Lorentz group $\mathfrak{L}$ and the space inversion.

In a fixed Lorentz frame, a scalar distribution field can be regarded as a distribution in the sense of L. Schwartz and we shall denote it by S instead of $S(x)$ in the following.

§ 1. In this section, we prove Theorem 1.1.

We begin with some notations.

$\mathfrak{D}$: The linear space of infinitely differentiable complex-valued functions with compact carriers in $\mathfrak{M}$. Its topology is that given in L. Schwartz [1].

$\mathfrak{D}'$: the linear space of distributions in $\mathfrak{M}$ i.e. the dual space of $\mathfrak{D}$.

$\mathfrak{E}$: the linear space of infinitely differentiable complex-valued functions in $\mathfrak{M}$. Its topology is that given in L. Schwartz [1].

$\mathfrak{E}'$: the linear space of distributions in $\mathfrak{M}$ with compact carriers. It is well-known that $\mathfrak{E}'$ can be regarded as the dual space of $\mathfrak{E}$.

$\mathfrak{E}_0$: the linear subspace of $\mathfrak{E}$ which is composed of the functions $\varphi(x)$ belonging to $\mathfrak{E}$ and vanishing in some neighbourhoods of the origin O of $\mathfrak{M}$. Of course this neighbourhood depends on $\varphi(x)$. $\mathfrak{E}_0$ is endowed with the topology induced on it by $\mathfrak{E}$.

---

2) In the relativistic quantum theory of fields, generally, Lorentz invariant fields in the direct sum $\mathfrak{M}\oplus\mathfrak{M}\cdots\oplus\mathfrak{M}$ come into question and the manners of cutting off are more complicated. We treat here the simplest case.

3) We call a field $S(x)$ in $\mathfrak{M}$ a distribution field if its components in a coordinate system are distributions in the sense of L. Schwartz.

4) Also Cf. Theorem 1.1.

Lemma 1.1. *Given a distribution $S_1 \in \mathfrak{E}'$ with its carrier contained in $\Gamma^+ \cup \Gamma^-$, we can form a distribution $S_1^* \in \mathfrak{E}'$ with its carrier contained in Γ^+ which coincides with S_1 in $\mathfrak{M}-\Gamma^-$.*

Proof. The order m of the distribution S_1 is finite since S_1 belongs to $\mathfrak{E}'$. We take an open sphere $U(O)$ in $\mathfrak{M}$ with its center O and with a sufficiently large radius so that it contains the carrier of S_1.

Let $\rho(u)$ be an infinitely differentiable real function defined for $-1 \leqq u \leqq 1$ such that

$$\rho(u) = 1 \quad \text{for} \quad 1 \geqq u \geqq (\sqrt{2})^{-1} - \varepsilon$$
$$\rho(u) = 0 \quad \text{for} \quad -(\sqrt{2})^{-1} + \varepsilon \geqq u \geqq -1$$

where ε a positive number $<(\sqrt{2})^{-1}$. We put

$$\sigma(x) = \begin{cases} (x^0/r) & \text{if} \quad r \neq 0 \\ 0 & \text{if} \quad r = 0 \end{cases}$$

where $r=((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)^{1/2}$. Then $\sigma(x)$ is infinitely differentiable in $\mathfrak{M}-(O)$ and the absolute values of all its partial derivatives of order $\leqq m$ can be bounded from above by Mr^{-q} on $U(O)$ except at the origin O of $\mathfrak{M}$ where M is a positive constant and q is a natural number.

Now we prove that the linear functional $\tilde{S}_1$ on $\mathfrak{E}_0$ defined by

$$\tilde{S}_1: \quad \varphi \to S_1(\sigma\varphi) \quad (\varphi \in \mathfrak{E}_0)$$

is continuous.

Let $\varphi_i \in \mathfrak{E}_0$ $(i=1, 2, \cdots)$ converge in $\mathfrak{E}_0$ to 0 as $i \to \infty$. Then $\varphi_i(x)$ and their partial derivatives of order $\leqq m+q$ converge to 0 uniformly on $U(O)$ as $i \to \infty$. Also each of $\varphi_i(x)$ vanishes in some neighbourhood of the origin O of $\mathfrak{M}$. Hence by the formula of Taylor, we can easily prove that the absolute values of all partial derivatives of order $\leqq m$ of $\varphi_i(x)$ can be bounded from above on $U(O)$ by $M_i' r^q$ where the constants $M_i' \geqq 0$ depend only on i and $M_i' \to 0$ as $i \to \infty$. Therefore by the formula of Leibniz and by the properties of the function $\sigma(x)$ previsouly stated, the absolute values of all partial derivatives of order $\leqq m$ of $\sigma(x)\varphi_i(x)$ can be bounded from above by C_i on $U(O)$ where the constants $C_i \geqq 0$ depend only on i and $C_i \to 0$ as $i \to \infty$. Hence $S_1(\sigma\varphi_i) \to 0$ as $i \to \infty$ because the order of S_1 is m and the open sphere $U(O)$ contains the carrier of S_1. Thus the linear functional $\tilde{S}_1$ on $\mathfrak{E}_0$ is continuous since $\mathfrak{E}_0$ is a metrizable locally convex linear space.

Now by Hahn-Banach's extension theorem we can extend the continuous linear functional $\tilde{S}_1$ on $\mathfrak{E}_0$ to a continuous linear functional on $\mathfrak{E}$ i. e. a distribution $S_1^* \in \mathfrak{E}'$. $\tilde{S}_1$ can be regarded as a distribution in $\mathfrak{M}-(O)$

and S_1^* coincides with $\tilde{S}_1$ in $\mathfrak{M}-(O)$. Also $\tilde{S}_1$ vanishes in $\mathfrak{M}-\Gamma^+$ and coincides with S_1 in $\mathfrak{M}-\Gamma^-$ by the properties of S_1 and the function $\sigma(x)$. Therefore S_1^* has the desired properties. q. e. d.

Theorem 1. 1.[5] *Given a distribution S which vanishes in the space-like region $\mathfrak{M}-\Gamma^+-\Gamma^-$, we can form a distribution S^* which coincides with S in $\mathfrak{M}-\Gamma^-$ and vanishes in $\mathfrak{M}-\Gamma^+$.*

Proof. The carrier of the distribution S is contained in $\Gamma^+\cup\Gamma^-$. We take an open sphere $U(O)$ whose center is the origin O of $\mathfrak{M}$. The sets $\mathfrak{M}-\Gamma^+$, $\mathfrak{M}-\Gamma^-$, $U(O)$ are open and $(\mathfrak{M}-\Gamma^+)\cup(\mathfrak{M}-\Gamma^-)\cup U(O)=\mathfrak{M}$. Also we have $(\mathfrak{M}-\Gamma^-)\cap(\Gamma^+\cup\Gamma^-)=\Gamma^+-(O)$ and $(\mathfrak{M}-\Gamma^+)\cap(\Gamma^+\cup\Gamma^-)=\Gamma^- -(O)$. Therefore by Theorem 29, chapter 3 of L. Schwartz [1], we can form three distributions S^+, S^- and S_1 such that their carriers are contained in $\Gamma^+-(O)$, $\Gamma^- -(O)$ and $U(O)\cap(\Gamma^+\cup\Gamma^-)$ respectively and such that $S=S^++S^-+S_1$.

The distribution S_1 satisfies the premises of Lemma 1. 1. Hence we can form a distribution S_1^* with its carrier contained in Γ^+ which coincides with S_1 in $\mathfrak{M}-\Gamma^-$. Then we can easily see that the distribution $S^*=S^++S_1^*$ has the desired properties. q. e. d.

§ **2.** Let $\varphi\in\mathfrak{D}$ and $\Lambda\in\mathfrak{L}$.

Then we define $\varphi\Lambda\in\mathfrak{D}$ by

$$(\varphi\Lambda)(x) = \varphi(\Lambda x) \quad \text{for all} \quad x\in\mathfrak{M}.$$

Also let $S\in\mathfrak{D}'$ and $\Lambda\in\mathfrak{L}$. Then we define $S\Lambda\in\mathfrak{D}'$ by

$$(S\Lambda)(\varphi) = S(\varphi\Lambda^{-1}) \quad \text{for all} \quad \varphi\in\mathfrak{D}.$$

We shall call $S\Lambda^{-1}$[6] the transformed of S by Λ and shall call $S\in\mathfrak{D}'$ *Lorentz invariant* if $S\Lambda^{-1}=S$ for all $\Lambda\in\mathfrak{L}$.

Lemma 2. 1. *Let $G=\{g\}$ be a connected semi-simple Lie group and let $P=\{P_g\}$ be a continuous representation of G by linear transformations of a vector space $V=\{v\}$ of finite dimension over the field of complex numbers. If f is an infinitely differentiable*[7] *V-valued function on G*

5) This theorem follows from Theorem 34, chapter 3 of L. Schwartz [1]. But the proof of Theorem 34 is not given in L. Schwartz [1]. Hence we prove Theorem 1. 1 independently of it for completeness.

6) $S\Lambda^{-1}$ is a rigorous definition of the distribution $S(\Lambda^{-1}x)$.

7) A connected Lie group G has a unique real analytic structure which makes it an analytic group. The differentiability or the analyticity of functions on G is considered with respect to this structure.

satisfying the following condition

$$f(g_2g_1) = P_{g_2}f(g_1)+f(g_2) \tag{2.1}$$

for all $g_1, g_2 \in G$, *then there is an element* $v \in V$ *such that*

$$f(g) = P_g v - v \quad \textit{for all} \quad g \in G.$$

Proof.[8] Let T_g be the operation of left translation assigned to $g \in G$ on a V-valued differential form ω' on G and

$$T_g: \quad \omega' \to \omega' T_g.$$

If we regard f as a V-valued differential form of order 0 on G, the condition (2.1) can be written as

$$f T_g = P_g f + f(g) \quad \text{for all} \quad g \in G. \tag{2.2}$$

If we differentiate both sides of (2.2) fixing $g \in G$ and put $df = \omega$, then we have

$$\omega T_g = P_g \omega \quad \text{for all} \quad g \in G$$

since the operation d commutes with the operations T_g and P_g for a fixed $g \in G$. Also we have $d\omega = d(df) = 0$. Hence ω is a closed equivariant V-valued differential form of order 1 on G.

Now we denote the Lie algebra of the group G by $L = \{y\}$ and denote the localization of the differential form ω by $[\omega]$ i. e. $[\omega]$ is a V-valued linear form on L defined by

$$[\omega](y) = \omega_e(y_e) \quad \text{for all} \quad y \in L$$

where ω_e and y_e are the values of ω and y at the identity element e of G respectively. We denote by the same notation P the representation of the Lie algebra L induced in V by the representation P of the Lie group G and also denote by $P(y)$ the linear transformation in V assigned to $y \in L$ in the representation P of the Lie algebra L. Then by Theorem 10.1[9] of C. Chevalley and S. Eilenberg [2], we have

$$\frac{1}{2}[\omega]([y_1, y_2]) + \frac{1}{2}\{P(y_1)[\omega](y_2) - P(y_2)[\omega](y_1)\} = 0 \tag{2.3}$$

for all $y_1, y_2 \in L$.

---

8) In the proof of Lemma 2.1, we shall follow the terminology and the notations in C. Chevalley and S. Eilenberg [2].

9) In C. Chevalley and S. Eilenberg [2], Theorem 10.1 is proved for a representation of G in a real vector space V but the proof goes quite similarly for a representation of G in a complex vector space V.

We denote by L_c the Lie algebra obtained from L by extending the ground field to the field of complex numbers. L_c is semi-simple since the Lie group G is semi-simple. Also we extend the representation P of L to that of L_c and extend the V-valued linear form $[\omega]$ on L to that on L_c. For simplicity, we shall denote the P and the $[\omega]$ thus extended by the same notations P and $[\omega]$. The ground field being thus extended, we can easily prove that (2.3) holds also for all $y_1, y_2 \in L_c$. Hence $[\omega]$ is a one-dimensional P-cocyle in L_c. The cohomology group $H^1(L_c, P)$ of L_c over P reduces to $\{0\}$ by Theorem 25.1 of C. Chevalley and S. Eilenberg [2] since L_c is semi-simple. Therefore $[\omega]$ is a one-dimensional P-coboundary in L_c. Hence there is an element $v \in V$ such that

$$[\omega](y) = P(y)v \quad \text{for all} \quad y \in L.$$

Now we put

$$\tilde{f}(g) = P_g v - v$$

and regard $\tilde{f}$ as a V-valued differential form of order 0 and we put

$$\tilde{\omega} = d\tilde{f}.$$

Then we can easily prove that

$$P_g \tilde{\omega} = \tilde{\omega} T_g \qquad \text{for all} \quad g \in G$$
$$[\tilde{\omega}]y = P(y)v \quad \text{for all} \quad y \in L.$$

Hence $\tilde{\omega}$ is a one-dimensional equivariant V-valued differential form of order 1 and

$$[\omega] = [\tilde{\omega}].$$

Since the correspondence between an equivariant V-valued differential form ω' and its localization $[\omega']$ is one to one, we have

$$df = \omega = \tilde{\omega} = d\tilde{f}. \tag{2.4}$$

Putting $g_2 = e$ in (2.1), we have

$$f(e) = 0$$

Also from the definition of $\tilde{f}$, we have

$$\tilde{f}(e) = 0 = f(e)$$

Hence from (2.4), we have $f = \tilde{f}$, since the Lie group G is connected. Therefore we have

$$f(g) = P_g v - v \quad \text{for all} \quad g \in G. \quad \text{q. e. d.}$$

Theorem 2.1. *Given a Lorentz invariant distribution* $S \in \mathfrak{D}'$ *which vanishes in the space-like region* $\mathfrak{M}-\Gamma^{+}-\Gamma^{-}$, *we can form a Lorentz invariant distribution* $S^{*} \in \mathfrak{D}'$ *which coincides with* S *in* $\mathfrak{M}-\Gamma^{-}$ *and vanishes in* $\mathfrak{M}-\Gamma^{+}$.

Proof. We shall use the following abbreviation. If a^{l} is a symmetric contravariant[10] tensor of order l whose $(i_1 \cdots i_l)$-component in (x^0, x^1, x^2, x^3)-coordinate system is a complex number $\alpha^{i_1 \cdots i_l}$, we write

$$\sum_{(i_1 \cdots i_l)} \alpha^{i_1 \cdots i_l} \frac{\partial^{l}}{\partial x^{i_1} \cdots \partial x^{i_l}} = a^{l} D_{l}.$$

A Lorentz transformation Λ induces a linear transformation $P_{\Lambda}(l)$ in the complex linear space of symmetric contravariant tensors of order l. The mapping

$$\Lambda \rightarrow P_{\Lambda}(l)$$

is a continuous representation of the Lorentz group $\mathfrak{L}$. If R is a distribution in $\mathfrak{M}$, we can easily prove that

$$(a^{l} D_{l} R)\Lambda^{-1} = (P_{\Lambda}(l)\, a^{l}) D_{l}(R\Lambda^{-1})$$

for all $\Lambda \in \mathfrak{L}$.

By Theorem 1.1, we can form a distribution S^{**} with its carrier in Γ^{+} which coincides with S in $\mathfrak{M}-\Gamma^{-}$. $S^{**}\Lambda^{-1}$ which arises from S^{**} by a Lorentz transformation $\Lambda \in \mathfrak{L}$ coincides with S^{**} in $\mathfrak{M}-(O)$, since S is Lorentz invariant and a Lorentz transformation transforms each of the regions $\mathfrak{M}-\Gamma^{+}$ and $\mathfrak{M}-\Gamma^{-}$ onto itself. Also if m is the order of S^{**} in a relatively compact neighbourhood $V(O)$ of O, then m is finite and we can easily prove that $S^{**}\Lambda^{-1}$ is of order m in the neighbourhood $\Lambda V(O)$ of O. Hence by Theorem 35, chapter 3 of L. Schwartz [1], we have

$$S^{**}\Lambda^{-1} = S^{**} + \sum_{l=0}^{m} a^{l}(\Lambda) D_{l} \delta \qquad (2.5)$$

[11]

where $a^{l}(\Lambda)$ is a symmetric contravariant tensor of order l depending on $\Lambda \in \mathfrak{L}$.

We prove that each of $a^{l}(\Lambda)$ is uniquely determined by Λ and an infinitely differentiable function of Λ on the Lorentz group $\mathfrak{L}$[12]. We put

$$\psi(x) = \rho(x) \times x^{i_1} \times \cdots \times x^{i_l}$$

10) We shall regard (x^0, x^1, x^2, x^4) as components of a contravariant vector.

11) Here δ is the four-dimensional Dirac's δ.

12) Cf. foot note 7).

where $\rho(x)$ is an infinitely differentiable function of x with compact carrier which is equal to 1 in a neighbourhood of the origin O of $\mathfrak{M}$. Then we have from (2. 5)

$$S^{**}(\psi\Lambda) = S^{**}\Lambda^{-1}(\psi) = S^{**}(\psi) + (-1)^l l! \alpha^{i_1 \cdots i_l}(\Lambda)$$

where $\alpha^{i_1 \cdots i_l}(\Lambda)$ is the $(i_1 \cdots i_l)$-component of $a^l(\Lambda)$. $S^{**}(\psi\Lambda)$ is an infinitely differentiable function of Λ on $\mathfrak{L}$ from Theorem 2, chapter 4 of L. Schwartz [1]. From this, we have the desired result.

Now we have from (2. 5)

$$\begin{aligned} &S^{**}(\Lambda_2\Lambda_1)^{-1} = (S^{**}\Lambda_1^{-1})\Lambda_2^{-1} = S^{**}\Lambda_2^{-1} + \\ (2.6) \qquad &\sum_{l=0}^{m} (a^l(\Lambda_1) D_l \delta) \Lambda_2^{-1} = S^{**} + \sum_{l=0}^{m} (a^l(\Lambda_2) + \\ &P_{\Lambda_2}(l) a^l(\Lambda_1)) D_l \delta \end{aligned}$$

for all Λ_1, $\Lambda_2 \in \mathfrak{L}$, since δ is a Lorentz invariant distribution. On the other hand, we have from (2. 5)

$$(2.7) \qquad S^{**}(\Lambda_2\Lambda_1)^{-1} = S^{**} + \sum_{l=0}^{m} a^l(\Lambda_2\Lambda_1) D_l \delta$$

From (2. 6) and (2. 7), we have

$$a^l(\Lambda_2\Lambda_1) = P_{\Lambda_2}(l) a^l(\Lambda_1) + a^l(\Lambda_2)$$

for all $l \geqq 0$ and for all Λ_1, $\Lambda_2 \in \mathfrak{L}$ since $a^l(\Lambda)$ is uniquely determined by Λ for each $l \geqq 0$ in (2. 5). Therefore by Lemma 2. 1, for each $l \geqq 0$, there is a symmetric contravariant tensor c^l of order l such that

$$(2.8) \qquad a^l(\Lambda) = P_\Lambda(l) c^l - c^l \quad \text{for all} \quad \Lambda \in \mathfrak{L}$$

since the Lorentz group $\mathfrak{L}$ is a connected semi-simple Lie group and $a^l(\Lambda)$ are infinitely differentiable on $\mathfrak{L}$.

Now we put

$$S^* = S^{**} - \sum_{l=0}^{m} c^l D_l \delta$$

Then we have by (2. 5) and (2. 8)

$$\begin{aligned} S^*\Lambda^{-1} &= S^{**}\Lambda^{-1} - \sum_{l=0}^{m} ((c^l D_l)\delta)\Lambda^{-1} \\ &= S^{**} + \sum_{l=0}^{m} (a^l(\Lambda) - P_\Lambda(l) c^l) D_l \delta \\ &= S^{**} - \sum_{l=0}^{m} c^l D_l \delta = S^* \end{aligned}$$

for all $\Lambda \in \mathfrak{L}$. Also S^* coincides with S^{**} in $\mathfrak{M}-(O)$. Therefore S^* has the desired properties. q. e. d.

REMARK. S^* in Theorem 2.1 is not uniquely determined by S. If S^* is such one, any other distribution having the desired properties is given by

$$S^* + \sum_{p=0}^{n} d_p \Box^p \delta$$

where $\Box$ is the d'Alembertian $-\partial^2/(\partial x^0)^2 + \partial^2/(\partial x^1)^2 + \partial^2/(\partial x^2)^2 + \partial^2/(\partial x^3)^2$ and d_p are constants and n is a non-negative integer.

OSAKA UNIVERSITY

(Received March 5, 1963)

References

[1] L. Schwartz: Théorie des distributions I, Paris, Hermann, 1950.
[2] C. Chevalley and S. Eilenberg: *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85-124.