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FINITE DIMENSIONAL REPRESENTATIONS OF QUANTUM GROUPS

Toshiyuki TANISAKI

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0. Introduction

In [2] and [4] Drinfeld and Jimbo independently noticed that there exists an algebraic object behind the theory of the quantum Yang-Baxter equation. This is the *quantum algebra*, which is a quantization of the enveloping algebra $U(\mathfrak{g})$ of a finite dimensional semisimple Lie algebra \mathfrak{g} . Their formulations, however, are slightly different; Drinfeld's $U_{\hbar}(\mathfrak{g})$ is an algebra over the formal power series ring $k[[\hbar]]$, and Jimbo's $U_q(\mathfrak{g})$ is an algebra over the rational function field $k(q^{1/2})$, where k is a field of characteristic zero. The indeterminates \hbar and q are related by $q^{1/2} = e^{\hbar/4}$.

One of the purposes of this paper is to give a counter part for $U_{\hbar}(\mathfrak{g})$ of the results of Lusztig [6] and Rosso [7] concerning finite diemnsional $U_{\mathfrak{q}}(\mathfrak{g})$ -modules, by fully using the advantage that $U_{\hbar}(\mathfrak{g})$ is a topologically free $k[[\hbar]]$ -module satisfying $U_{\hbar}(\mathfrak{g})|_{\hbar=0} \simeq U(\mathfrak{g})$ (the indeterminate \hbar can be directly specialized to 0).

Let P (resp. P^{++}) be the set of integral (resp. dominant integral) weights. As in the case for $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$ we can construct a "finite dimensional highest weight module" $L(\lambda)$ for $\lambda \in P^{++}$. It is a $U_{\hbar}(\mathfrak{g})$ -module which is free of finite rank over $R = k[[\hbar]]$ such that $L(\lambda)|_{\hbar=0}$ is isomorphic to the finite dimensional irreducible $U(\mathfrak{g})$ -module with highest weight λ . Let \mathcal{A} be the category of $U_{\hbar}(\mathfrak{g})$ -modules which are free of finite rank over R. A $U_{\hbar}(\mathfrak{g})$ -module in \mathcal{A} is said to be \mathcal{A} -irreducible if it has no nontrivial quotients which belong to \mathcal{A} .

Theorem A. (i) Any $U_{\hbar}(\mathfrak{g})$ -module in \mathcal{A} is a direct sum of \mathcal{A} -irreducible $U_{\hbar}(\mathfrak{g})$ -modules.

(ii) A $U_{\hbar}(g)$ -module in \mathcal{A} is \mathcal{A} -irreducible if and only if it is isomorphic to $L(\lambda)$ for some $\lambda \in P^{++}$.

Let G be a connected split semisimple algebraic group defined over k with $\text{Lie}(G)=\mathfrak{g}$. Assume that G is simply connected for simplicity. Let $A_{\hbar}[G]$ be the "dual" Hopf algebra of $U_{\hbar}(\mathfrak{g})$ (see Section 2 below). Since it can be regarded as a quantization of the coordinate algebra k[G] of G, we call it the quantum group (see Drinfeld [3], Woronowicz [10]). We have the following

analogue of the Peter-Weyl theorem:

Theorem B. The matirx coefficients of $L(\lambda)$ for $\lambda \in P^{++}$ form a free *R*-basis of $A_{\hbar}[G]$. Hence we have an isomorphism

$$A_{\hbar}[G] \simeq \bigoplus_{\lambda \in P^{++}} (L(\lambda) \otimes L(\lambda)^*)$$

of $(U_{\hbar}(g), U_{\hbar}(g))$ -bimodules.

Let B be a Borel subgroup of G and let k[B] be its coordinate algebra. We have an induction functor Ind from the category of k[B]-comodules to that of k[G]-comodules. We can define a quantization $A_{\hbar}[B]$ of k[B] as a quotient Hopf algebra of $A_{\hbar}[G]$ and the induction functor Ind_{\hbar} from the category of $A_{\hbar}[B]$ -comodules to that of $A_{\hbar}[G]$ -comodules is similarly defined. For $\mu \in P$ let R_{μ} (resp. k_{μ}) be the one dimensional $A_{\hbar}[B]$ (resp. k[B])-comodule corresponding to μ . The following theorem implies that the analouge of the Borel-Weil-Bott theorem holds for quantum groups.

Theorem C. For $\mu \in P$ the $A_{\hbar}[G]$ -comodule $R^i \operatorname{Ind}_{\hbar}(R_{\mu})$ is a free R-module and the k[G]-comodule $k \otimes_R R^i \operatorname{Ind}_{\hbar}(R_{\mu})$ is isomorphic to $R^i \operatorname{Ind}(k_{\mu})$, where R^i denotes the right derived functors.

1. Irreducible Highest Weight Modules

1.1. Let g be a finite dimensional split semisimple Lie algebra over a field k of characteristic zero. Let $A = (a_{ij})_{1 \le i, j \le l}$ be the Cartan matrix of g and choose positive integres d_1, \dots, d_l satisfying $d_i a_{ij} = d_j a_{ji}$. The quantum algebra $U_{\hbar}(g)$ is the associative algebra over the formal power series ring $R = k[[\hbar]]$ with 1, which is \hbar -adically generated by 3l elements t_i, e_i, f_i $(i=1, \dots, l)$ satisfying the following fundamental relations:

(1.1.1)
$$t_i t_j = t_j t_i$$
 $(i, j = 1, \dots, l),$

(1.1.2)
$$t_i e_j - e_j t_i = d_i a_{ij} e_j$$
 $(i, j = 1, \dots, l),$

(1.1.3)
$$t_i f_j - f_j t_i = -d_i a_{ij} f_j \quad (i, j = 1, \dots, l),$$

(1.1.4)
$$e_i f_j - f_j e_i = \delta_{ij} \frac{\sinh(\hbar t_i/2)}{\sinh(\hbar d_i/2)}$$
 $(i, j = 1, \dots, l),$

(1.1.5)
$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} e_i^{1-a_{ij}-m} e_j e_i^m = 0 \qquad (i \neq j),$$

(1.1.6)
$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} f_i^{1-a_{ij}-m} f_j f_i^m = 0 \quad (i \neq j),$$

where $q_i = \exp(\hbar d_i/2) \in R^*$ $(i=1, \dots, l)$, and

$$\begin{bmatrix} n \\ m \end{bmatrix}_{q} = \frac{\prod_{r=1}^{n} (q^{r} - q^{-r})}{\prod_{r=1}^{m} (q^{r} - q^{-r}) \prod_{r=1}^{n-m} (q^{r} - q^{-r})} \qquad (n \ge m \ge 0)$$

(Drinfeld [2], Jimbo [4]).

Let N^+ (resp. N^- , resp. T) be the subalgebra of $U_{\hbar}(\mathfrak{g})$ generated by e_1, \dots, e_l (resp. f_1, \dots, f_l , resp. t_1, \dots, t_l) and let $U_{\hbar}^f(\mathfrak{g})$ be the subalgebra generated by N^+ , N^- , \overline{T} , where barring denotes the \hbar -adic closure.

Proposition 1.1.1 ([9], see also [2]). (i) N^+ (resp. N^-) is a free R-module, and the relation (1.1.5) (resp. (1.1.6)) is a fundamental relation among the generators e_1, \dots, e_l (resp, f_1, \dots, f_l) of N^+ (resp. N^-).

(ii) T is naturally isomorphic to the polynomial ring $R[t_1, \dots, t_l]$, and the inclusion $T \hookrightarrow \overline{T}$ is the \hbar -adic completion.

(iii) We have the following isomorphism of R-modules :

(1.1.7)
$$N^{-} \otimes T \otimes N^{+} \simeq U_{\hbar}^{f}(\mathfrak{g}) \qquad (u \otimes v \otimes w \leftrightarrow uvw).$$

(iv) The inclusion $U^{f}_{\hbar}(\mathfrak{g}) \hookrightarrow U_{\hbar}(\mathfrak{g})$ is the \hbar -adic completion.

By [8] we see that the k-algebra $k \otimes_R U_{\hbar}(\mathfrak{g})$ is naturally isomorphic to the enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} , where the ring homomorphism $R \to k$ is given by $\hbar \mapsto 0$. The natural Hopf algebra structure on $U(\mathfrak{g})$ lifts to the topological Hopf algebra structure on $U_{\hbar}(\mathfrak{g})$ given by the following:

(1.1.8) $\Delta(t_i) = t_i \otimes 1 + 1 \otimes t_i \qquad (i = 1, \dots, l),$

(1.1.9)
$$\Delta(e_i) = e_i \otimes \exp(-\hbar t_i/4) + \exp(\hbar t_i/4) \otimes e_i \qquad (i = 1, \dots, l)$$

(1.1.10)
$$\Delta(f_i) = f_i \otimes \exp(-\hbar t_i/4) + \exp(\hbar t_i/4) \otimes f_i \qquad (i = 1, \dots, l),$$

(1.1.11)
$$\mathcal{E}(t_i) = \mathcal{E}(e_i) = \mathcal{E}(f_i) = 0 \quad (i = 1, \dots, l),$$

(1.1.12) $S(t_i) = -t_i \quad (i = 1, \dots, l),$

(1.1.13)
$$S(e_i) = -q_i^{-1}e_i \quad (i = 1, \dots, l),$$

(1.1.14) $S(f_i) = -q_i f_i$ $(i = 1, \dots, l),$

where Δ , \mathcal{E} , S are the coproduct, the counit and the antipode, respectively (see [2], [4]).

Lemma 1.1.2. $U_{\hbar}(g)$ is a noetherian ring; i.e., the ascending chain conditions for left and right ideals are satisfied.

Proof. It is known that the enveloping algebra $U(\mathfrak{g})$ is a noetherian ring and by Proposition 1.1.1 we have

$$U_{\hbar}(\mathfrak{g}) = \lim_{\mathfrak{h}} U_{\hbar}(\mathfrak{g})/\hbar^{n} U_{\hbar}(\mathfrak{g}), \quad \hbar^{n} U_{\hbar}(\mathfrak{g})/\hbar^{n+1} U_{\hbar}(\mathfrak{g}) \simeq U_{\hbar}(\mathfrak{g})/\hbar U_{\hbar}(\mathfrak{g}) \simeq U(\mathfrak{g}).$$

Hence the assertion is proved similarly to the well known fact that the formal power series ring over a noetherian ring is noetherian. \Box

Let t (resp. t_0) be the *R*-submodule (resp. *k*-subspace) of *T* generated by t_1, \dots, t_l . By Proposition 1.1.1 $\{t_1, \dots, t_l\}$ is a basis of the *k*-vector space t_0 and

we have $t=R\otimes_k t_0$. Set $t^*=\operatorname{Hom}_R(t, R)$ and $t_0^*=\operatorname{Hom}_k(t_0, k)$. Via the ring homomorphisms $k \hookrightarrow R \to k$ we have:

$$\mathbf{t}^* \simeq R \otimes_k \mathbf{t}^*_0 \quad \mathbf{t}^*_0 \simeq k \otimes_R \mathbf{t}^*$$

We will identify \mathbf{t}_0^* with a subspace of \mathbf{t}^* and the natural homomorphism $\mathbf{t}^* \to \mathbf{t}_0^*$ is denoted by $\lambda \mapsto \lambda^0$. We will also identify $\mathbf{t}_0 (= k \bigotimes_R \mathbf{t})$ with a split Cartan subalgebra of \mathfrak{g} . We define $\alpha_i \in \mathbf{t}_0^*$ $(i=1, \dots, l)$ by $\alpha_i(t_j) = d_i a_{ij}$. Then $\{\alpha_1, \dots, \alpha_l\}$ is a set of simple roots of the root system Δ of $(\mathfrak{g}, \mathfrak{t}_0)$. We denote the set of positive roots by Δ^+ . Set

$$(1.1.15) Q = \bigoplus_{i=1}^{l} \mathbf{Z}\alpha_i,$$

$$(1.1.16) Q^+ = \bigoplus_{i=1}^l \boldsymbol{Z}_{\geq 0} \alpha_i,$$

$$(1.1.17) P = \{\lambda \in \mathfrak{t}_0^* \mid \lambda(2t_i/\alpha_i(t_i)) \in \mathbb{Z} \quad (i = 1, \dots, l)\},\$$

$$(1.1.18) P^{++} = \{ \lambda \in \mathfrak{t}_0^* | \lambda(2t_i/\alpha_i(t_i)) \in \mathbb{Z}_{\geq 0} \quad (i = 1, \dots, l) \} .$$

We denote by W the Weyl group of (g, t_0) . It is a finite subgroup of $GL(t_0)$ generated by the reflections s_i $(i=1, \dots, l)$ given by

$$s_i(t) = t - \frac{2\alpha_i(t)}{\alpha_i(t_i)} t_i \qquad (t \in \mathfrak{t}_0) \; .$$

The **Z**-lattices P, Q in t_0^* are preserved under the contragredient action of W on t_0^* .

1.2. Let \mathcal{A} be the category of $U_{\hbar}(\mathfrak{g})$ -modules which are free of finite rank as *R*-modules. This is not an abelian category but an exact category. Let M be a $U_{\hbar}(\mathfrak{g})$ -module in \mathcal{A} . A $U_{\hbar}(\mathfrak{g})$ -submodule M_1 of M is called a strict submodule if M/M_1 belongs to \mathcal{A} . A non-zero $U_{\hbar}(\mathfrak{g})$ -module in \mathcal{A} is said to be \mathcal{A} -irreducible if it does not contain non-zero proper strict submodules.

Lemma 1.2.1. If M_1 , M_2 are $U_{\hbar}(g)$ -modules in \mathcal{A} , we have

$$\operatorname{Ext}^1_{U_{t}(\mathfrak{g})}(M_1, M_2) = 0$$

Proof. By the exact sequence:

$$0 \to M_2 \xrightarrow{\hbar} M_2 \to M_2/\hbar M_2 \to 0$$

of $U_{\hbar}(\mathfrak{g})$ -modules we have:

$$\operatorname{Ext}^{1}_{U_{\hbar}(\mathfrak{g})}(M_{1}, M_{2}) \xrightarrow{\hbar} \operatorname{Ext}^{1}_{U_{\hbar}(\mathfrak{g})}(M_{1}, M_{2}) \to \operatorname{Ext}^{1}_{U_{\hbar}(\mathfrak{g})}(M_{1}, M_{2}/\hbar M_{2}) \quad (\operatorname{exact}).$$

Since $U_{\hbar}(\mathfrak{g})$ is a noetherian ring, we see that $\operatorname{Ext}^{1}_{U_{\hbar}(\mathfrak{g})}(M_{1}, M_{2})$ is a finitely generated *R*-module. Thus it is sufficient to show $\operatorname{Ext}^{1}_{U_{\hbar}(\mathfrak{g})}(M_{1}, M_{2}/\hbar M_{2})=0$. By the exact sequence:

$$0 \to U_{\hbar}(\mathfrak{g}) \xrightarrow{\hbar} U_{\hbar}(\mathfrak{g}) \to U(\mathfrak{g}) \to 0,$$

we have

$$\operatorname{Tor}_{q}^{U_{\hbar}(\mathfrak{g})}(U(\mathfrak{g}), M_{1}) = \begin{cases} M_{1}/\hbar M_{1} & q = 0\\ 0 & q \neq 0 \end{cases}.$$

Therefore, by the spectral sequence:

$$E_2^{pq} = \operatorname{Ext}_{U(\mathfrak{g})}^p(\operatorname{Tor}_q^{U_{\hbar}(\mathfrak{g})}(U(\mathfrak{g}), M_1), M_2/\hbar M_2) \Rightarrow \operatorname{Ext}_{U_{\hbar}(\mathfrak{g})}^{p+q}(M_1, M_2/\hbar M_2)$$

we have

$$\operatorname{Ext}^{1}_{U_{\hbar}(\mathfrak{g})}(M_{\mathfrak{l}}, M_{\mathfrak{l}}/\hbar M_{\mathfrak{l}}) = \operatorname{Ext}^{1}_{U(\mathfrak{g})}(M_{\mathfrak{l}}/\hbar M_{\mathfrak{l}}, M_{\mathfrak{l}}/\hbar M_{\mathfrak{l}}) \,.$$

The right-hand side is zero since any finite dimensional $U(\mathfrak{g})$ -module is completely reducible. We are done. \Box

Corollary 1.2.2. Any $U_{\hbar}(\mathfrak{g})$ -module in \mathcal{A} is a direct sum of \mathcal{A} -irreducible $U_{\hbar}(\mathfrak{g})$ -modules.

1.3. For $\lambda \in t^*$ let $\xi_{\lambda} \colon \overline{T} \to R$ be the unique algebra homomorphism satisfying $\xi_{\lambda}(t) = \lambda(t)$ for $t \in t$. For a \overline{T} -module M and $\lambda \in t^*$ we set

$$M_{\lambda} = \{ m \in M \mid t \cdot m = \xi_{\lambda}(t) m \ (t \in T) \} .$$

We define an ordering on t* by

$$\lambda \ge \mu$$
 if and only if $\lambda - \mu \in Q^+$.

For $\lambda \in \mathfrak{t}^*$ we define a $U^f_{\hbar}(\mathfrak{g})$ -module $M(\lambda)$, called the Verma module with highest weight λ , by

(1.3.1)
$$M(\lambda) = U_{\hbar}^{f}(\mathfrak{g})/(\sum_{i=1}^{l} U_{\hbar}^{f}(\mathfrak{g})e_{i} + U_{\hbar}^{f}(\mathfrak{g}) \ker \xi_{\lambda}) = U_{\hbar}^{f}(\mathfrak{g})m_{\lambda},$$

where m_{λ} is the canonical generator corresponding to the class of 1. By Proposition 1.1.1 we have $M(\lambda) = \bigoplus_{\mu \leq \lambda} M(\lambda)_{\mu}$ and each $M(\lambda)_{\mu}$ is a free *R*-module of finite rank. Moreover we have the character formula:

(1.3.2)
$$\sum_{\mu \leq \lambda} (\operatorname{rank}_R M(\lambda)_{\mu}) e^{\mu} = \frac{e^{\lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}$$

Lemma 1.3.1. If K is a $U^f_{\hbar}(\mathfrak{g})$ -submodule of $M(\lambda)$, we have:

$$K = \bigoplus_{\mu \leq \lambda} (K \cap M(\lambda)_{\mu})$$
 .

Proof. Assume that we have $m = \sum_{i=1}^{n} m_i \in K$ with $m_i \in M(\lambda)_{\mu_i}, \mu_i \neq \mu_j$ $(i \neq j)$. We will show that each m_i is an element of K by induction on n. The case n=1 being trivial, we assume that $n \ge 2$ and the assertion holds for n-1. Since μ_1^0, \dots, μ_n^0 are mutually different elements of t_0^* , there exists some $t \in k[t_1, \dots, t_l] \subset \overline{T}$ satisfying $\xi_{\mu_1^0}(t) = \dots = \xi_{\mu_{n-1}^0}(t) = 0$ and $\xi_{\mu_n^0}(t) = 1$. Then we have $\xi_{\mu_i}(t) = \hbar a_i$ $(i=1, \dots, n-1)$ and $\xi_{\mu_n}(t) = 1 + \hbar a_n$ for some $a_1, \dots, a_n \in R$. Hence we have

$$(1+\hbar a_n)m-t\cdot m = \sum_{i=1}^{n-1} (1+\hbar (a_n-a_i))m_i \in K$$

Since $1+\hbar(a_n-a_i)$ is an invertible element of R, we have $m_i \in K$ $(i=1, \dots, n-1)$ and hence $m_n \in K$. \square

Let $K(\lambda)$ be the sum of all $U^{f}_{\hbar}(\mathfrak{g})$ -submodules of $M(\lambda)$ contained in $\bigoplus_{\mu < \lambda} M(\lambda)_{\mu}$, and set

(1.3.3)
$$L(\lambda) = M(\lambda)/K(\lambda) .$$

Lemma 1.3.2. (i) We have $L(\lambda) = \bigoplus_{\mu \leq \lambda} L(\lambda)_{\mu}$ and each $L(\lambda)_{\mu}$ is a free *R*-module of finite rank.

(ii) If K is a proper $U_{\hbar}^{f}(\mathfrak{g})$ -submodule of $L(\lambda)$ such that $L(\lambda)/K$ is a torsion free R-module, we have K=0.

Proof. (i) Set $K' = \{m \in M(\lambda) \mid \hbar m \in K(\lambda)\}$. Then K' is a $U_{\hbar}^{f}(\mathfrak{g})$ -submodule satisfying $K(\lambda) \subset K' \subset \bigoplus_{\mu < \lambda} M(\lambda)_{\mu}$. Hence we have $K' = K(\lambda)$ and $L(\lambda)$ is a torsion free *R*-module. Therefore the assertion follows from Lemma 1.3.1.

(ii) Let K_1 be a proper $U_{\hbar}^{f}(\mathfrak{g})$ -submodule of $M(\lambda)$ such that $M(\lambda)/K_1$ is a torsion free *R*-module. By Lemma 1.3.1 we have $K_1 = \bigoplus_{\mu \leq \lambda} (M(\lambda)_{\mu} \cap K_1)$ and hence $M(\lambda)_{\mu}/M(\lambda)_{\mu} \cap K_1$ is a torsion free *R*-module for each $\mu \leq \lambda$. Since $M(\lambda)_{\lambda}$ is a free *R*-module of rank 1, we have $M(\lambda)_{\lambda} \cap K_1 = M(\lambda)_{\lambda}$ or 0. If $M(\lambda)_{\lambda} \cap K_1 = M(\lambda)_{\lambda}$, then K_1 contains the generator m_{λ} , and hence we have $K_1 = M(\lambda)$, which contradicts with the assumption. Therefore we have $M(\lambda)_{\lambda} \cap K_1 = 0$ and hence $K_1 \subset K(\lambda)$. The assertion is proved. \Box

We define $\rho \in t_0^*$ by $\rho(2t_i | \alpha_i(t_i)) = 1$ $(i=1, \dots, l)$. For $w \in W$ set

 $l(w) = \min \{ p | w = s_{i_1}, \dots, s_{i_p} \text{ for some } i_1, \dots, i_p \in [1, l] \}.$

Lemma 1.3.3 ([6]). (i) $L(\lambda)$ is finitely generated as an R-module if and only if $\lambda \in P^{++}$.

(ii) For $\lambda \in P^{++}$, $k \otimes_R L(\lambda)$ is an irreducible $U(\mathfrak{g})$ -module and we have

(1.3.4)
$$\sum_{\mu \leq \lambda} (\operatorname{rank}_{R} L(\lambda)_{\mu}) e^{\mu} = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)-\mu}}{\prod_{\alpha \in \Delta^{+}} (1-e^{-\alpha})}$$

(iii) If $\lambda \in P^{++}$, any $U^{f}_{\hbar}(\mathfrak{g})$ -submodule of $L(\lambda)$ is of the form $\hbar^{n}L(\lambda)$ for some non-negative integer n.

Proof. By the arguments of [6] we see that $L(\lambda)$ is integrable (i.e., the elements e_i , f_i $(i=1, \dots, l)$ act on $L(\lambda)$ locally nilpotently) if and only if $\lambda \in P^{++}$. If $L(\lambda)$ is finitely generated as an *R*-module, then it is integrable and hence we have $\lambda \in P^{++}$. If $\lambda \in P^{++}$, then $L(\lambda)$ is integrable, and hence $k \otimes_R L(\lambda)$ is an integrable highest weight module of $U(\mathfrak{g})$ with highest weight λ . Thus $k \otimes_R L(\lambda)$ is the (finite dimensional) irreducible $U(\mathfrak{g})$ -module with highest weight λ . Thus $k \otimes_R L(\lambda)$ is finitely generated as an *R*-module, and Weyl's character formula implies (1.3.4). The statements (i) and (ii) are proved. Let us show (iii). Let $\lambda \in P^{++}$ and let K be a non zero $U_{\hbar}^{f}(\mathfrak{g})$ -submodule of $L(\lambda)$. Take a non-negative integer *n* such that $K \subset \hbar^n L(\lambda)$ and $K \subset \hbar^{n+1}L(\lambda)$. Then we have $K = \hbar^n K_1$ for some $U_{\hbar}^{f}(\mathfrak{g})$ -submodule $L(\lambda)/\hbar L(\lambda) = k \otimes_R L(\lambda)$, we have $L(\lambda) = K_1 + \hbar L(\lambda)$. Since $L(\lambda)$ is a finitely generated *R*-module, we have $K_1 = L(\lambda)$, and hence $K = \hbar^n L(\lambda)$.

For $\lambda \in P^{++}$ the action of $U_{\hbar}^{f}(\mathfrak{g})$ on $L(\lambda)$ uniquely lifts to that of $U_{\hbar}(\mathfrak{g})$ on $L(\lambda)$. In the following we regard $L(\lambda)$ for $\lambda \in P^{++}$ as a $U_{\hbar}(\mathfrak{g})$ -module.

Corollary 1.3.4. $L(\lambda)$ is an A-irreducible $U_{\hbar}(\mathfrak{g})$ -module for $\lambda \in P^{++}$.

2. Quantum Groups

2.1. Define a $(U(\mathfrak{g}), U(\mathfrak{g}))$ -bimodule structure on $U(\mathfrak{g})^* = \operatorname{Hom}_k(U(\mathfrak{g}), k)$ by

$$((u_1 \cdot f \cdot u_2))(u) = f(u_2 u u_1)$$
 $(f \in U(g)^*, u, u_1 u_2 \in U(g)),$

and set

$$U(\mathfrak{g})^{\circ} = \{f \in U(\mathfrak{g})^* | \dim_k (U(\mathfrak{g})fU(\mathfrak{g})) < \infty \}.$$

It is an elementary fact concerning Hopf algebras that $U(\mathfrak{g})^{\circ}$ is also endowed with a Hopf algebra structure whose product, coproduct, unit, counit are induced by the coproduct, the product, the counit, the unit of $U(\mathfrak{g})$, respectively.

Set $U_{\hbar}(\mathfrak{g})^* = \operatorname{Hom}_{R}(U_{\hbar}(\mathfrak{g}), R)$. By Proposition 1.1.1. we see that $U_{\hbar}(\mathfrak{g})$ is the \hbar -adic completion of a free *R*-submodule $M = N^- \otimes T \otimes N^+$. Hence we have $U_{\hbar}(\mathfrak{g})^* \simeq \operatorname{Hom}_{R}(M, R) = (a \text{ product of rank 1 free } R \text{-modules})$. Therefore any *R*-submodule of $U_{\hbar}(\mathfrak{g})^*$ is torsion free and separated. Define a $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -bimodule structure on $U_{\hbar}(\mathfrak{g})^*$ by

$$(u_1 \cdot f \cdot u_2)(u) = f(u_2 u u_1) \qquad (f \in U_{\hbar}(\mathfrak{g})^*, u, u_1, u_2 \in U_{\hbar}(\mathfrak{g})),$$

and set

 $U_{\hbar}(\mathfrak{g})^{\circ} = \{ f \in U_{\hbar}(\mathfrak{g})^{*} | U_{\hbar}(\mathfrak{g}) f U_{\hbar}(\mathfrak{g}) \text{ is a finitely generated } R \text{-module} \}.$

For a $U_{\hbar}(\mathfrak{g})$ -module V in \mathcal{A} we have a natural right $U_{\hbar}(\mathfrak{g})$ -module structure on $V^* = \operatorname{Hom}_{R}(V, R)$ by

$$\langle v^* \cdot u, v \rangle = \langle v^*, u \cdot v \rangle$$
 $(v^* \in V^*, v \in V, u \in U_{\hbar}(\mathfrak{g})),$

where \langle , \rangle is the natural pairing. Define $\Phi_V \colon V \otimes V^* \to U_{\hbar}(\mathfrak{g})^*$ by $(\Phi_V(v \otimes v^*))(u) = \langle v^*, u \cdot v \rangle$. Then it is easily seen that Φ_V is a homomorphism of $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -bimodules and that $U_{\hbar}(\mathfrak{g})^\circ$ is the sum of Image (Φ_V) for $U_{\hbar}(\mathfrak{g})$ -modules V in \mathcal{A} (i.e., the *R*-module $U_{\hbar}(\mathfrak{g})^\circ$ is generated by the matrix coefficients of $U_{\hbar}(\mathfrak{g})$ -modules in \mathcal{A}). Moreover the topological Hopf algebra structure on $U_{\hbar}(\mathfrak{g})^\circ$.

2.2. The purpose of this subsection is to prove the following:

Proposition 2.2.1. For $\lambda \in P^{++}$ the homomorphism $\Phi_{L(\lambda)}$ is injective and we have

$$U_{\hbar}(\mathfrak{g})^{\circ} = \bigoplus_{\lambda \in \mathbb{P}^{++}} \operatorname{Image} \Phi_{L(\lambda)}$$
.

Since $U_{\hbar}(\mathfrak{g})$ is topologically free and since $U_{\hbar}(\mathfrak{g})/\hbar U_{\hbar}(\mathfrak{g}) \simeq U(\mathfrak{g})$, we have $U_{\hbar}(\mathfrak{g})^*/\hbar U_{\hbar}(\mathfrak{g})^* \simeq U(\mathfrak{g})^*$ We denote the natural homomorphism $U_{\hbar}(\mathfrak{g})^* \rightarrow U(\mathfrak{g})^*$ by $f \rightarrow f$. We first show the following:

Lemma 2.2.2. For $\lambda \in P^{++}$ the homomorphism $\Phi_{L(\lambda)}$ is injective and we have

$$\sum_{\lambda \in P^{++}} \operatorname{Image} \Phi_{L(\lambda)} = \bigoplus_{\lambda \in P^{++}} \operatorname{Image} \Phi_{L(\lambda)}$$
.

Proof. For $\lambda \in P^{++}$ let $\{f_{ij}^{\lambda} | 1 \leq i, j \leq \text{rank } L(\lambda)\}$ be the set of matrix coefficients of $L(\lambda)$ with respect to some *R*-basis of $L(\lambda)$. It is sufficient to show that

$$\{f_{ij}^{\lambda} | \lambda \in P^{++}, 1 \leq i, j \leq \operatorname{rank} L(\lambda)\}$$

is linearly independent over R. The set

$$\{f_{ij}^{\lambda} | \lambda \in P^{++}, 1 \leq i, j \leq \operatorname{rank} L(\lambda)\}$$

is linearly independent over k, since it consists of the matrix coefficients of irreducible $U(\mathfrak{g})$ -modules. Therefore the assertion follows from the fact that $U_{\hbar}(\mathfrak{g})^*$ is torsion free. \square

Set $V(\lambda) = L(\lambda) \otimes L(\lambda)^*$ for $\lambda \in P^{++}$.

Lemma 2.2.3. Let $\lambda_1, \dots, \lambda_n$ be mutually different elements in P^{++} and let

$$p_j \colon \bigoplus_{i=1}^n V(\lambda_i) \to V(\lambda_j)$$

be the projection. If V is a $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -submodule of $\bigoplus_{i=1}^{n} V(\lambda_i)$ such that $p_j(V) \neq 0$ for each j, there exist non-negative integers m_1, \dots, m_n satisfying $V = \bigoplus_{i=1}^{n} \hbar^{m_i} V(\lambda_i)$.

Proof. Set $U = \bigoplus_{i=1}^{n} V(\lambda_i)$. Since $p_j(V)$ is a non zero $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -submodule of $V(\lambda_j)$ and since $V(\lambda_j)/\hbar V(\lambda_j)$ is an irreducible $(U(\mathfrak{g}), U(\mathfrak{g}))$ -bimodule, the argument in the proof of Lemma 1.3.3 (iii) implies that there exists some non-negative integer m_j such that $p_j(V) = \hbar^{m_j} V(\lambda_j)$. Let $F: U \to U$ be the $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -homomorphism defined by

$$F(\sum_{i=1}^n v_i) = \sum_{i=1}^n \hbar^{m_i} v_i \qquad (v_i \in V(\lambda_i)).$$

Then there exist a $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -submodule V_1 of $\bigoplus_{i=1}^{n} V(\lambda_i)$ such that $F(V_1) = V$ and $p_j(V_1) = V(\lambda_j)$. Since $V(\lambda_i)/\hbar V(\lambda_i)$ $(i = 1, \dots, n)$ are mutually non-isomorphic $(U(\mathfrak{g}), U(\mathfrak{g}))$ -bimodules, we see that $(V_1 + \hbar U)/\hbar U = U/\hbar U$, and hence $U = V_1 + \hbar U$. Since U is a finitely generated R-module we have $V_1 = U$ and hence $V = F(V_1) = \bigoplus_{i=1}^{n} \hbar^{m_i} V(\lambda_i)$. \Box

Set $D = \bigoplus_{\lambda \in P^{++}}$ Image $\Phi_{L(\lambda)}$.

Lemma 2.2.4. (i) $U_{\hbar}(\mathfrak{g})^{\circ} = D + \hbar^{n} U_{\hbar}(\mathfrak{g})^{\circ}$ for any n. (ii) $\hbar^{n} D = \hbar^{n} U_{\hbar}(\mathfrak{g}) \cap D$ for any n.

Proof. (i) Let $f \in U_{\hbar}(\mathfrak{g})^{\circ}$. Then we have $f \in U(\mathfrak{g})^{\circ}$ and hence there exists some $f_1 \in D$ such that $f = f_1$. Therefore we have $f = f_1 + \hbar f_2$ for some $f_2 \in U_{\hbar}(\mathfrak{g})^*$. Since f and f_1 are elements of $U_{\hbar}(\mathfrak{g})^{\circ}$ and since $U_{\hbar}(\mathfrak{g})^*$ is torsion free, we have $f_2 \in U_{\hbar}(\mathfrak{g})^{\circ}$. Thus we have

$$U_{\hbar}(\mathfrak{g})^{\circ} = D + \hbar U_{\hbar}(\mathfrak{g})^{\circ} = D + \hbar (D + \hbar U_{\hbar}(\mathfrak{g}))^{\circ} = \cdots = D + \hbar^{n} U_{\hbar}(\mathfrak{g})^{\circ}$$

(ii) Let f be an element of $U(\mathfrak{g})^{\circ}$ such that $\hbar^{n} f \in D$. Set $V = U_{\hbar}(\mathfrak{g}) f U_{\hbar}(\mathfrak{g})$. Let $\{v_{1}, \dots, v_{p}\}$ be an *R*-basis of *V* and let $\{v_{1}^{*}, \dots, v_{p}^{*}\}$ be the dual basis of V^{*} . Regarding *V* as a left $U_{\hbar}(\mathfrak{g})$ -module we have

$$v_i = \sum_{j=1}^p v_j(1) \Phi_V(v_i \otimes v_j^*) \in \text{Image } \Phi_V$$
.

Especially we have $f \in \text{Image } \Phi_V$. Since $\hbar^n V$ is a $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -submodule of D, we see from Lemma 2.2.3 that

$$V \simeq \hbar^n V = \bigoplus_{i=1}^r \hbar^m (L(\lambda_i) \otimes L(\lambda_i)^*) \simeq \bigoplus_{i=1}^r L(\lambda_i) \otimes L(\lambda_i)^*$$

for some $\lambda_1, \dots, \lambda_r \in P^{++}$ and $m_1, \dots, m_r \in \mathbb{Z}_{\geq 0}$. Hence we have $f \in \text{Image } \Phi_V = \sum_{i=1}^r \text{Image } \Phi_{L(\lambda_i)} \subset D$. \Box

Proof of Proposition 2.2.1. By Lemma 2.2.2 it is sufficient to show $D = U_{\hbar}(g)^{\circ}$. By Lemma 2.2.4 the natural *R*-homomorphism

$$\hat{D}(=\varprojlim D/\hbar^*D) \to (U_{\hbar}(\mathfrak{g})^{\circ})^{\wedge}(=\varprojlim U_{\hbar}(\mathfrak{g})^{\circ}/\hbar^*U_{\hbar}(\mathfrak{g}))^{\circ})$$

is an isomorphism. Therefore we can regard $U_{\hbar}(\mathfrak{g})^{\circ}$ as an *R*-submodule of \hat{D} containing *D*. Since the $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -bimodule structures on *D* and $U_{\hbar}(\mathfrak{g})^{\circ}$ uniquely lift to the same $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -bimodule structure on \hat{D} , it suffices to prove that if *V* is a $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -submodule of $(\bigoplus_{\lambda \in P^{++}} V(\lambda))^{\wedge}$ which is finitely generated over *R*, then *V* is contained in $\bigoplus_{\lambda \in P^{++}} V(\lambda)$. Let $q_{\mu}: (\bigoplus_{\lambda \in P^{++}} V(\lambda))^{\wedge} \rightarrow V(\mu)$ be the unique extention of the projection $\bigoplus_{\lambda \in P^{++}} V(\lambda) \rightarrow V(\mu)$. We have only to show that $q_{\mu}(V)=0$ except for finitely many $\mu \in P^{++}$. Assume that there exists an infinite sequence μ_1, μ_2, \cdots , of mutually different elements in P^{++} such that $q_{\mu_i}(V) \neq 0$. Let $r_n: (\bigoplus_{\lambda \in P^{++}} V(\lambda))^{\wedge} \rightarrow \bigoplus_{i=1}^{n} V(\mu_i)$ be the unique extension of the projection. By Lemma 2.2.3 we have $r_n(V) = \bigoplus_{i=1}^{n} \hbar^{m_i} V(\mu_i)$ for some non-negative integers m_1, \cdots, m_n . Therefore we have rank $V \geq \operatorname{rank} r_n(V) \geq n$ for any *n*. This contradicts with the assumption. We are done. \Box

Corollary 2.2.5. Any *A*-irreducible $U_{\hbar}(\mathfrak{g})$ -module is isomorphic to $L(\lambda)$ for some $\lambda \in P^{++}$.

Proof. Let V be an \mathcal{A} -irreducible $U_{\hbar}(\mathfrak{g})$ -module. Take a non-zero element v^* of V^* and define $F: V \to U_{\hbar}(\mathfrak{g})^\circ$ by $(F(v))(u) = \langle v^*, u \cdot v \rangle$. Then F is a non-zero homomorphism of left $U_{\hbar}(\mathfrak{g})$ -modules. By Proposition 2.3.1 the left $U_{\hbar}(\mathfrak{g})$ -module $U_{\hbar}(\mathfrak{g})^\circ$ is a direct sum of $L(\lambda)$ for $\lambda \in P^{++}$. Considering the projections we see that there exists a non zero $U_{\hbar}(\mathfrak{g})$ -homomorphism $V \to L(\lambda)$ for some $\lambda \in P^{++}$. It is seen by Lemma 1.3.3 (iii) that $L(\lambda)$ is a quotient of V. Since V is \mathcal{A} -irreducible, we have $V = L(\lambda)$. \Box

2.3. Let G be a connected split semisimple algebraic group defined over k such that the Lie algebra consisting of k-rational points of Lie (G) coincides with \mathfrak{g} . Then the coordinate algebra k[G] is naturally endowed with a Hopf algebra structure and we have a natural injective Hopf algebra homomorphism from $k[G] \rightarrow U(\mathfrak{g})^{\circ}$ via the pairing

$$\langle f, u \rangle = (d_u f)(1) \qquad (f \in k[G], u \in U(\mathfrak{g})),$$

where d_u is the left invariant differential operator on G corresponding to u. The image of this homomorphism is described as follows. Let L_G be the set of elements of t_0^* consisting of weights of finite dimensional $U(\mathfrak{g})$ -modules coming from G-modules. L_G is a \mathbb{Z} -lattice satisfying $Q \subset L_G \subset P$. Then the image of $k[G] \rightarrow U_{\hbar}(\mathfrak{g})^\circ$ is spanned by the matrix coefficients of finite dimensional irreducible $U(\mathfrak{g})$ -modules with highest weight in $L_G \cap P^{++}$.

Set

(2.3.1)
$$A_{\hbar}[G] := \bigoplus_{\lambda \in L_G \cap P^{++}} \operatorname{Image} \Phi_{L(\lambda)}.$$

It is easily checked that $A_{\hbar}[G]$ is a Hopf subalgebra of $U_{\hbar}(\mathfrak{g})^{\circ}$. We call this

Hopf algebra the quantum group associated to G (see [3], [10]).

Let \mathcal{A}° be the category of right $U_{\hbar}(\mathfrak{g})^{\circ}$ -comodules which are free of finite rank over R and let \mathcal{A}_{G} be the category of right $A_{\hbar}[G]$ -comodules which belong to \mathcal{A}° as right $U_{\hbar}(\mathfrak{g})^{\circ}$ -comodules. Then the natural functor $\mathcal{A}^{\circ} \to \mathcal{A}$ gives an equivalence of categories $\mathcal{A}^{\circ} \simeq \mathcal{A}$. Moreover the category \mathcal{A}_{G} is equivalent to the full subcategory of \mathcal{A} consisting of $U_{\hbar}(\mathfrak{g})$ -modules in \mathcal{A} whose \mathcal{A} -irreducible factors are of the form $L(\lambda)$ for some $\lambda \in L_{G} \cap P^{++}$.

Lemma 2.3.1. Let V be a $U_{\hbar}(\mathfrak{g})$ -module in \mathcal{A} such that the Z-submodule of \mathfrak{t}^* generated by the weights of V coincides with L_G . Then $A_{\hbar}[G]$ is generated by Image Φ_V as an R-algebra.

Proof. Let H be the subalgebra of $A_{\hbar}[G]$ generated by Image Φ_{ν} . We see by definition that H is a $(U_{\hbar}(\mathfrak{g}), U_{\hbar}(\mathfrak{g}))$ -submodule of $A_{\hbar}[G]$. Hence by Lemma 2.2.3 we have

$$(2.3.2) H = \bigoplus_{\lambda \in \Gamma} \hbar^{n_{\lambda}} \operatorname{Image} \Phi_{L(\lambda)}$$

for a subset Γ of P^{++} and non-negative integers n_{λ} . On the other hand we see by the assumption on V that the representation $G \rightarrow GL(V/\hbar V)$ is injective and hence the k-algebra $k[G] (\simeq A_{\hbar}[G]/\hbar A_{\hbar}[G])$ is generated by the matrix coefficients of the G-module $V/\hbar V$. Therefore we have $A_{\hbar}[G]/\hbar A_{\hbar}[G] = (H + \hbar A_{\hbar}[G])/\hbar A_{\hbar}[G]$ and hence

The assertion follows from (2.3.2) and (2.3.3).

3. Borel-Weil-Bott Theorem

3.1. Let $U_{\hbar}^{f}(\mathfrak{b})$ be the subalgebra of $U_{\hbar}^{f}(\mathfrak{g})$ generated by \overline{T} , N^{+} and let $U_{\hbar}(\mathfrak{b})$ be its \hbar -adic closure in $U_{\hbar}(\mathfrak{g})$. By Proposition 1.1.1 we have $U_{\hbar}^{f}(\mathfrak{b}) \simeq \overline{T} \otimes N^{+}$ and the inclusion $U_{\hbar}^{f}(\mathfrak{b}) \simeq U_{\hbar}(\mathfrak{b})$ is the \hbar -adic completion. Moreover we have $k \otimes_{\mathbb{R}} U_{\hbar}(\mathfrak{b}) \simeq k \otimes_{\mathbb{R}} U_{\hbar}^{f}(\mathfrak{b}) \simeq U(\mathfrak{b})$, where \mathfrak{b} is a Borel subalgebra of \mathfrak{g} . Define $U_{\hbar}(\mathfrak{b})^{\circ} (\subset U_{\hbar}(\mathfrak{b})^{*})$ similarly to $U_{\hbar}(\mathfrak{g})^{\circ}$. Since $U_{\hbar}(\mathfrak{b})$ is a topological Hopf subalgebra of $U_{\hbar}(\mathfrak{g})$, we also have a natural Hopf algebra structure on $U_{\hbar}(\mathfrak{b})^{\circ}$.

Let G be a connected semisimple split algebraic group defined over k with $\operatorname{Lie}(G) = \mathfrak{g}$ and let B be the Borel subgroup of G corresponding to b. We denote by $F: U_{\hbar}(\mathfrak{g})^{\circ} \to U_{\hbar}(\mathfrak{b})^{\circ}$ the natural Hopf algebra homomorphism. Then $A_{\hbar}[B] = F(A_{\hbar}[G])$ is endowed with a natural Hopf algebra structure and it can be regarded as a quantization of the coordinate algebra k[B] of B by the following:

Lemma 3.1.1. $A_{\hbar}[B]$ is free *R*-module satisfying $k \otimes_{\mathbb{R}} A_{\hbar}[B] \simeq k[B]$.

Proof. It is easily verified using the results in Section 1 that we have

$$A_{\hbar}[B] = (\bigoplus_{\lambda \in L_{\mathcal{G}}} R\xi_{\lambda}) \otimes (\bigoplus_{\beta \in Q^+} (N_{\beta}^+)^*)$$

under the identification $U_{\hbar}^{f}(\mathfrak{b}) = \overline{T} \otimes N^{+}$. Hence the assertion follows from the corresponding fact for k[B]. \Box

3.2. Let $\operatorname{tirv}_{h[B]}$ (resp. $\operatorname{triv}_{A_{\hbar}[B]}$) be the right k[B] (resp. $A_{\hbar}[B]$)-comodule given by the unit and let $\operatorname{triv}_{U(\mathfrak{b})}$ (resp. $\operatorname{triv}_{U_{\hbar}(\mathfrak{b})}$) be the left $U(\mathfrak{b})$ (resp. $U_{\hbar}(\mathfrak{b})$)-module given by the counit. For a right k[B] (resp. $A_{\hbar}[B]$)-comodule V we set

(3.2.1)
$$\operatorname{Ind}(V) = \operatorname{Hom}(\operatorname{triv}_{k[B]}, k[G] \otimes_{k} V)$$

(3.2.2) (resp. Ind_ħ(V) = Hom(triv_{Aħ[B]}, Aħ[G] $\otimes_R V$)).

Here $k[G] \otimes_k V$ (resp. $A_{\hbar}[G] \otimes_R V$) is endowed with a right k[B] (resp. $A_{\hbar}[B]$)comodule structure via the right k[B] (resp. $A_{\hbar}[B]$)-comodule structure on k[G](resp. $A_{\hbar}[G]$) and Hom is taken in the category of right k[B] (resp. $A_{\hbar}[G]$)comodules. Then the left k[G] (resp. $A_{\hbar}[G]$)-comodules structure on k[G]) induces a left k[G] (resp. $A_{\hbar}[G]$)-comodule structure on Ind (V) (resp. Ind_{\hbar}(V)). Hence Ind (resp. Ind_{\hbar}) is a left exact functor from the category of right k[B](resp. $A_{\hbar}[B]$)-comodules to that of left k[G] (resp. $A_{\hbar}[G]$)-comodules. We denote by R^i Ind (resp. R^i Ind_{\hbar}) its right derived functors.

By the Peter-Weyl theorem for k[G] and by (2.3.1) we have

$$Ind (V) = \bigoplus_{\lambda \in L_{\mathcal{G}} \cap P^{++}} Hom (triv_{k[B]}, L^{0}(\lambda) \otimes_{k} V) \otimes_{k} L^{0}(\lambda)^{*},$$

$$= \bigoplus_{\lambda \in L_{\mathcal{G}} \cap P^{++}} Hom_{U(\mathfrak{b})} (triv_{U(\mathfrak{b})}, L^{0}(\lambda) \otimes_{k} V) \otimes_{k} L^{0}(\lambda)^{*},$$

$$Ind_{\hbar}(V) = \bigoplus_{\lambda \in L_{\mathcal{G}} \cap P^{++}} Hom (triv_{A_{\hbar}[B]}, L(\lambda) \otimes_{R} V) \otimes_{R} L(\lambda)^{*},$$

$$= \bigoplus_{\lambda \in L^{\mathcal{G}} \cap P^{++}} Hom_{U_{\hbar}(\mathfrak{b})} (triv_{U_{\hbar}(\mathfrak{b})}, L(\lambda) \otimes_{R} V) \otimes_{R} L(\lambda)^{*},$$

where $L^{0}(\lambda) = k \bigotimes_{R} L(\lambda)$. Hence we have

$$(3.2.3) \quad R^{i} \operatorname{Ind}(V) = \bigoplus_{\lambda \in L_{\mathfrak{g}} \cap P^{++}} \operatorname{Ext}^{i}_{U(\mathfrak{h})}(\operatorname{triv}_{U(\mathfrak{h})}, L^{0}(\lambda) \otimes_{k} V) \otimes_{k} L^{0}(\lambda)^{*},$$

(3.2.4)
$$R^{i} \operatorname{Ind}_{\hbar}(V) = \bigoplus_{\lambda \in L_{\mathcal{G}} \cap P^{++}} \operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}_{U_{\hbar}(\mathfrak{b})}, L(\lambda) \otimes_{R} V) \otimes_{R} L(\lambda)^{*}$$

For $\mu \in L_G$ we denote by $\hat{\xi}_{\mu} : U_{\hbar}(b) \to R$ the *R*-algebra homomorphism given by $\hat{\xi}_{\mu}(t) = \xi_{\mu}(t)$ for $t \in \overline{T}$ and $\hat{\xi}_{\mu}(e_i) = 0$ for $i = 1, \dots, l$. It is seen that the one dimensional left $U_{\hbar}(b)$ -module induced by ξ_{μ} comes from a one dimensional right $A_{\hbar}[B]$ -comodule R_{μ} . Set $k_{\mu} = k \bigotimes_R R_{\mu}$.

Proposition 3.2.1. For $\mu \in L_G$ the left $A_{\hbar}[G]$ -comodule $R^i \operatorname{Ind}_{\hbar}(R_{\mu})$ is free of finite rank as an R-module and we have

$$k \otimes_{R} R^{i} \operatorname{Ind}_{\hbar}(R_{\mu}) \simeq R^{i} \operatorname{Ind}(k_{\mu})$$

as a left k[G]-comodule.

Proof. Set $V(\lambda, \mu) = L(\lambda) \bigotimes_{R} R_{\mu}$ and $V^{0}(\lambda, \mu) = L^{0}(\lambda) \bigotimes_{k} k_{\mu}$. By (3.2.3) and (3.2.4) it is sufficient to show that $\operatorname{Exit}_{U_{\sharp}(\mathfrak{b})}^{i}(\operatorname{triv}_{U_{\sharp}(\mathfrak{b})}, V(\lambda, \mu))$ is a free Rmodule of rank dim_k (Extⁱ_{U(b)}(triv_{U(b)}, $V^{0}(\lambda, \mu)$). By the argument in the proof of Lemma 1.2.1 we have

(3.2.5)
$$\operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}_{U_{\hbar}(\mathfrak{b})}, V^{0}(\lambda, \mu)) = \operatorname{Ext}^{i}_{U(\mathfrak{b})}(\operatorname{triv}_{U(\mathfrak{b})}, V^{0}(\lambda, \mu))$$

Since $U_{\hbar}(\mathfrak{b})$ is noetherian, the *R*-module $\operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}_{U_{\hbar}(\mathfrak{b})}, V(\lambda, \mu))$ is finitely generated, and hence the exact sequence

$$\operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}, V(\lambda, \mu)) \xrightarrow{\hbar} \operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}, V(\lambda, \mu)) \to \operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}, V^{0}(\lambda, \mu))$$

implies that $\operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}_{U_{\hbar}(\mathfrak{b})}, V(\lambda, \mu)) = 0$ if $\operatorname{Ext}^{i}_{U(\mathfrak{b})}(\operatorname{triv}_{U(\mathfrak{b})}, V^{0}(\lambda, \mu)) = 0$. Assume that $\operatorname{Ext}^{i}_{U(b)}(\operatorname{triv}_{U(b)}, V^{0}(\lambda, \mu)) \neq 0$. By the Borel-Weil-Bott theorem for k[G] there exists at most one such i for each λ , μ . Hence the natural Rhomomorphism

$$\operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}, V(\lambda, \mu)) \xrightarrow{\hbar} \operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}, V(\lambda, \mu))$$

is injective and we have

$$\operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}, V(\lambda, \mu))/\hbar \operatorname{Ext}^{i}_{U_{\hbar}(\mathfrak{b})}(\operatorname{triv}, V(\lambda, \mu)) = \operatorname{Ext}^{i}_{U(\mathfrak{b})}(\operatorname{triv}, V^{0}(\lambda, \mu)).$$

This proves the assertion. \Box

Appendix

In [3] Drinfeld has given an explicit description of the quantum group $A_{\pi}[SL_n]$ by generators and relations. Since [3] contains no proof, we will give a proof here.

Set $q = e^{\hbar/2}$. The quantum algebra $U_{\hbar}(\mathfrak{gl}_n)$ is an *R*-algebra \hbar -adically generated by the elements $h_1, \dots, h_n, e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}$ satisfying the following fundamental relations:

(A.1)
$$h_i h_j = h_j h_i,$$

(A.2)
$$h_i e_i - e_i h_i = e_i,$$

(A.3)
$$h_i e_{i-1} - e_{i-1} h_i = -e_{i-1},$$

 $h_i e_j - e_j h_i = 0$ $(j \neq i, j-1)$, (A.4)

(A.5)
$$h_i f_i - f_i h_i = -f_i,$$

 $h_i f_i - f_i h_i = -f_i ,$ $h_i f_{i-1} - f_{i-1} h_i = f_{i-1} ,$ (A.6)

(A.7)
$$h_i f_j - f_j h_i = 0$$
 $(j \neq i, i-1),$

(A.8)
$$e_i f_j - f_j e_i = \delta_{ij} \frac{\sinh(\hbar (h_i - h_{i+1})/2)}{\sinh(\hbar/2)}$$

(A.9)
$$e_i^2 e_j - (q+q^{-1})e_i e_j e_i + e_j e_i^2 = 0$$
 $(|i-j|=1),$

(A.10)
$$e_i e_j = e_j e_i \qquad (|i-j| \ge 2)$$

(A.11)
$$f_i^2 f_j - (q+q^{-1}) f_i f_j f_i + f_j f_i^2 = 0$$
 $(|i-j| = 1),$

(A.12)
$$f_i f_j = f_j f_i$$
 $(|i-j| \ge 2)$.

Then $U_{\hbar}(\mathfrak{gl}_n)$ is naturally identified with the \hbar -adic closure of the *R*-subalgebra of $U_{\hbar}(\mathfrak{gl}_n)$ generated by $t_i = h_i - h_{i+1}$, e_i , f_i $(i=1, \dots, n-1)$, and the topological Hopf algebra structure of $U_{\hbar}(\mathfrak{gl}_n)$ is extended to that of $U_{\hbar}(\mathfrak{gl}_n)$ by $\Delta(h_i) =$ $h_i \otimes 1 + 1 \otimes h_i$, $\mathcal{E}(h_i) = 0$, $S(h_i) = -h_i$ (see [9]).

,

Define an R-algebra homomorphism $\rho: U_{\hbar}(\mathfrak{gl}_n) \to M_n(R)$ by $\rho(h_i) = E_{i,i} - E_{i+1,i+1}, \rho(e_i) = E_{i,i+1}, \rho(f_i) = E_{i+1,i}$, where $E_{i,j} \in M_n(R)$ is the matrix whose (r, s)entry is $\delta_{ir} \delta_{js}$. Let $\hat{\rho}_{ij} \in U_{\hbar}(\mathfrak{gl}_n)^*$ and $\rho_{ij} \in U_{\hbar}(\mathfrak{gl}_n)^*$ be the matrix coefficients
of ρ . They are elements of the Hopf algebras $U_{\hbar}(\mathfrak{gl}_n)^\circ$ and $U_{\hbar}(\mathfrak{gl}_n)^\circ$ $(U_{\hbar}(\mathfrak{gl}_n)^\circ)$ is defined similarly). We see by a direct calculation that

$$(A.13) \qquad \qquad \hat{\rho}_{ij}\hat{\rho}_{is} = q\hat{\rho}_{is}\hat{\rho}_{ij} \qquad (j < s),$$

(A.14)
$$\hat{\rho}_{ij}\hat{\rho}_{rj} = q\hat{\rho}_{rj}\hat{\rho}_{ij} \qquad (i < r),$$

(A.15)
$$\hat{\rho}_{ij}\hat{\rho}_{rs} = \hat{\rho}_{rs}\hat{\rho}_{ij}$$
 $(i < r, j > s),$

(A.16)
$$\hat{\rho}_{ij}\hat{\rho}_{rs}-\hat{\rho}_{rs}\hat{\rho}_{ij}=(q-q^{-1})\hat{\rho}_{is}\hat{\rho}_{rj}$$
 $(i < r, j < s)$.

Since ρ_{ij} is the image of $\hat{\rho}_{ij}$ under the natural algebra homomorphism $U_{\hbar}(\mathfrak{gl}_n)^{\circ} \rightarrow U_{\hbar}(\mathfrak{gl}_n)^{\circ}$, we have

(A.17)
$$\rho_{ij}\rho_{is} = q\rho_{is}\rho_{ij} \qquad (j < s),$$

(A.18)
$$\rho_{ij}\rho_{rj} = q\rho_{rj}\rho_{ij} \qquad (i < r),$$

(A.19)
$$\rho_{ij} \rho_{rs} = \rho_{rs} \rho_{ij}$$
 $(i < r, j > s)$,

(A.20)
$$\rho_{ij}\rho_{rs}-\rho_{rs}\rho_{ij}=(q-q^{-1})\rho_{is}\rho_{rj}$$
 $(i < r, j < s)$.

It is also checked directly that

(A.21)
$$\sum_{\sigma \in \mathfrak{S}_n} (-q)^{l(\sigma)} \rho_{1\sigma(1)} \rho_{2\sigma(n)} \cdots \rho_{n\sigma(n)} = 1 ,$$

where \mathfrak{S}_n is the symmetric group and $l(\sigma)$ for $\sigma \in \mathfrak{S}_n$ is the number of the elements of the set $\{(i, j) | i < j, \sigma(i) > \sigma(j)\}$.

Our purpose is to give a proof of the following:

Proposition A.1 (Drinfeld [3]). The R-algebra $A_{\hbar}[SL_n]$ is generated by the elements ρ_{ij} $(i, j=1, \dots, n)$ satisfying the fundamental relations (A.17), \dots , (A.21).

Let C be the quotient of $R \langle x_{ij} | i, j=1, \dots, n \rangle$ by the two-sided ideal generated by

$$\begin{array}{rcl} x_{ij}x_{is} - qx_{is}x_{ij} & (j < s), \\ x_{ij}x_{rj} - qx_{rj}x_{ij} & (i < r), \\ x_{ij}x_{rs} - x_{rs}x_{ij} & (i < r, j > s), \\ x_{ij}x_{rs} - x_{rs}x_{ij} - (q - q^{-1})x_{is}x_{rj} & (i < r, j < s), \end{array}$$

where $R\langle x_{ij} | i, j=1, \dots, n \rangle$ is the tensor algebra of the free *R*-module with basis $\{x_{ij} | i, j=1, \dots, n\}$.

Lemma A.2. (i) C is a free R-module with basis

$$\{x_{11}^{a_{11}}x_{12}^{a_{12}}\cdots x_{1n}^{a_{1n}}x_{21}^{a_{21}}\cdots x_{2n}^{a_{2n}}\cdots x_{nn}^{a_{nn}}|a_{ij} \in \mathbb{Z}_{\geq 0}\}.$$

(ii) C is an integral domain; i.e., if f, g are elements of C satisfying fg=0, we have f=0 or g=0.

Proof. (i) It is easily checked that the R-module C is generated by the elements

$$x_{11}^{a_{11}}x_{12}^{a_{12}}\cdots x_{1n}^{a_{1n}}x_{21}^{a_{21}}\cdots x_{2n}^{a_{2n}}\cdots x_{nn}^{a_{nn}} \qquad (a_{ij} \in \mathbb{Z}_{\geq 0}).$$

Considering the natural algebra homomorphism $C \to U_{\hbar}(\mathfrak{gl}_n)^{\circ} (x_{ij} \mapsto \hat{\rho}_{ij})$, it is enough to show that the elements

$$\hat{\rho}_{11}^{a_{11}}\hat{\rho}_{12}^{a_{12}}\cdots\hat{\rho}_{1n}^{a_{1n}}\hat{\rho}_{21}^{a_{21}}\cdots\hat{\rho}_{2n}^{a_{2n}}\cdots\hat{\rho}_{nn}^{a_{nn}} \in U_{\hbar}(\mathfrak{gl}_{n})^{*} \qquad (a_{ij} \in \mathbb{Z}_{\geq 0})$$

are linearly independent over R. This follows from the facts that $U_{\hbar}(\mathfrak{gl}_n)^*$ is a torsion free R-module and that the elements

$$\hat{\rho}_{11}^{a_{11}} \hat{\rho}_{12}^{a_{12}} \cdots \hat{\rho}_{1n}^{a_{1n}} \hat{\rho}_{21}^{a_{21}} \cdots \hat{\rho}_{2n}^{a_{2n}} \cdots \hat{\rho}_{nn}^{a_{nn}} \mod \hbar \qquad (a_{ij} \in \mathbb{Z}_{\ge 0})$$

of $U_{\hbar}(\mathfrak{gl}_n)^*/\hbar U_{\hbar}(\mathfrak{gl}_n)^* = U(\mathfrak{gl}_n)^*$ are linearly independent over k.

(ii) This follows from (i) and the fact that $C/\hbar C$ is an integral domain.

Set

$$\varphi = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{l(\sigma)} x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{n\sigma(n)} - 1 \in C ,$$

$$\varphi^0 = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{l(\sigma)} y_{1\sigma(1)} y_{2\sigma(2)} \cdots y_{n\sigma(n)} - 1 \in k[y_{ij}] ,$$

where $k[y_{ij}]$ is the polynomial ring with variables y_{ij} $(i, j=1, \dots, n)$. We have natural identifications $k[SL_n]=k[y_{ij}]/(\varphi^0)$ and $k\otimes_R C=k[y_{ij}]$. Let C_p (resp. $A_{\hbar}[SL_n]_p$, resp. $k[y_{ij}]_p$, resp. $k[SL_n]_p$) be the linear span in C (resp. $A_{\hbar}[SL_n]$, resp $k[y_{ij}]$, resp. $k[SL_n]$) of the mononials in x_{ij} (resp. ρ_{ij} , resp. y_{ij} , resp. y_{ij} mod φ^0) of degree $\leq p$.

Lemma A.3. (i) C_p is a free *R*-module of rank=dim $k[y_{ij}]_p$. (ii) $C_{p-n}\varphi$ is a free *R*-module of rank=dim $k[y_{ij}]_{p-n}$. (iii) dim $(k \otimes_R (C_p/C_{p-n}\varphi))$ =dim $k[y_{ij}]_p$ -dim $k[y_{ij}]_{p-n}$. (iv) A [SL] is a free *R*-databased of rank dim $k[y_{ij}]_p$ -dim $k[y_{ij}]_p$.

(iv) $A_{\hbar}[SL_n]_p$ is a free *R*-dodule of rank $\geq \dim k[y_{ij}]_p - \dim k[y_{ij}]_{p-n}$.

Proof. The statements (i), (ii), (iii) are clear from Lemma A.2. Let us show (iv). Since $A_{\hbar}[SL_n]_{\rho}$ is a finitely generated *R*-submodule of the free *R*-module $A_{\hbar}[SL_n]$, it is a free *R*-module of finite rank. Hence the surjectivity of the *k*-linear map

$$A_{\hbar}[SL_n]_{\rho}/\hbar A_{\hbar}[SL_n]_{\rho} \to A_{\hbar}[SL_n]_{\rho}/(A_{\hbar}[SL_n]_{\rho} \cap \hbar A_{\hbar}[SL_n]) \simeq k[SL_n]_{\rho}$$

implies that

$$\operatorname{rank} A_{\hbar}[SL_{n}]_{p} = \dim (A_{\hbar}[SL_{n}]_{p})/\hbar A_{\hbar}[SL_{n}]_{p})$$

$$\geq \dim k[SL_{n}]_{p}$$

$$= \dim k[y_{ij}]_{p} - \dim k[y_{ij}]_{p-n} \square$$

Proof of Proposition A.1. We have to show that the natural algebra homomorphism $C/C \varphi C \rightarrow A_{\hbar}/[SL_n] (x_{ij} \mapsto \rho_{ij})$ is an isomorphism. Since this is surjective by Lemma 2.3.1, it is sufficient to show that the *R*-homomorphism $C_p/(C \varphi C \cap C_p) \rightarrow A_{\hbar}[SL_n]$ is injective for each p. Therefore it suffices to prove that the surjective *R*-homomorphism $C_p/C_{p-n}\varphi \rightarrow A_{\hbar}[SL_n]_p$ is an isomorphism. This follows from Lemma A.3. \square

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Depatrment of Mathematics Osaka University Toyonaka 560, Japan