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Kernel Functions of Diffusion Equations (I)

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Let D be an open bounded set in a d-dimensional Euclidean space $\mathcal{E}$.

By Δ we understand the Laplacian with respect to given coordinates. Consider the diffusion equation

$$\frac{\partial U}{\partial t} = \Delta U \tag{1}$$

on D. By a *kernel function* $K(x, y; t)$ we understand a function on $\mathcal{E}\times\mathcal{E}\times[0, \infty)$ satisfying following properties:

(i) $K(x, y; t)=0$ when either x or y is on the boundary of D, if $K(x, y; t)$ is continuous on boundary for a fixed t, or the boundary ∂D is smooth.

(ii) For a fixed y

$$\frac{\partial}{\partial t} K(x, y; t) = \Delta_x K(x, y; t) \tag{2}$$

where Δ_x is understood as the Laplacian on the variable x.

The purpose of this paper is to give a new way of constructing the kernel function on D which coincides with the Green's function when ∂D, the boundary of D is smooth.

In preparations we shall define some notations. Coordinates of points $x, y, \cdots$ on $\mathcal{E}$ will be written x^i; $1\leq i\leq d$, y^j; $1\leq j\leq d$, etc. The euclidean distance between two points x and y is denoted by

$$|x-y| = (\sum_i^d (x^i-y^i)^2)^{1/2}. \tag{3}$$

Let

$$E_t(x, y) = (2\sqrt{\pi t})^{-d} \exp(-(4t)^{-1}|x-y|^2). \tag{4}$$

Lemma 1. *Take a point x on D. Let $S(h)$ be a solid sphere around*

(1), see (15) and (8)
(2), see (8)

x with its radius $(4h)^{1/6}$ *where h is sufficiently small that* $S(h)$ *is within D. Then*

$$1 \geqq \int_{S(h)} E_h(x, y)dy \geqq 1 - 2^d \exp\left(-\frac{1}{4}(h)^{-2/3}\right). \tag{5}$$

Proof. Consider polar coordinates ρ, θ^i, $1 \leqq i \leqq d-1$, around x where θ^i's denote angular coordinates. Then

$$\int_{D-S(h)} E_h(x, y)dy \leqq \int_{(4h)^{1/3}}^{\infty} d\rho \int_{\theta} \rho^{d-1}(2\sqrt{\pi h})^{-d} \exp(-(4h)^{-1}\rho^2)d\theta. \tag{6}$$

Notice that for any y outside of $S(h)$

$$|x-y| \geqq \frac{1}{2}(4h)^{1/6} + \frac{1}{2}|x-y| \tag{7}$$

and consequently

$$|x-y|^2 \geqq \frac{1}{4}\left((4h)^{1/3} + |x-y|^2\right) \tag{8}$$

Hence

$$\begin{aligned} \int_{D-S(h)} E_h(x, y)dy &\leqq (2\sqrt{\pi h})^{-d} \int_{\mathcal{E}-S(h)} \exp\left[-\frac{1}{4}(4h)^{-2/3} - \frac{1}{4}|x-y|^2/4h\right] dy \\ &\leqq 2^d \exp\left(-\frac{1}{4}(4h)^{-2/3}\right)(2\sqrt{\pi 4h})^{-d} \int \exp(-|x-y|^2/16h)dy \\ &\leqq 2^d \exp\left(-\frac{1}{4}(4h)^{-2/3}\right). \end{aligned} \tag{9}$$

However it is well-known that

$$\int_{S(h)} E_h(x, y)dy \leqq 1, \tag{10}$$

for any h.

Hence we have

$$1 \geqq \int E_h(x, y)dy \geqq 1 - 2^d \exp\left(-\frac{1}{4}(h)^{-2/3}\right),$$

which proves the lemma.

Given two functions ϕ and ψ of two variables, we define a convolution

$$(\phi * \psi)(x, y) = \int_D \phi(x, z)\psi(z, y)dz. \tag{11}$$

Then

Lemma 2.

$$\begin{aligned} &(E_t * E_h)(x, y) \\ &\leqq E_{t+h}(x, y)\left[\left(2\sqrt{\pi}\sqrt{\frac{th}{t+h}}\right)^{-d} \int_D \exp\left(-\frac{t+h}{4th}\left|z - \frac{tx+hy}{t+h}\right|^2\right)dz\right. \end{aligned} \tag{12}$$

Proof. By direct computations

(13)

$$
\begin{aligned}
&E_t(x, z)E_h(z, y)\\
&=(2\sqrt{\pi t})^{-d}(2\sqrt{\pi h})^{-d}\exp\left(-\frac{1}{4t}|x-z|^2-\frac{1}{4h}|z-y|^2\right)\\
&=(2\sqrt{\pi(t+h)})^{-d}\left(2\sqrt{\pi}\sqrt{\frac{th}{t+h}}\right)^{-d}\exp\left(-|x-y|^2/4(t+h)-\frac{t+h}{4th}\left|z-\frac{tx+hy}{t+h}\right|^2\right).
\end{aligned}
$$

Hence we have (12) by integrating (13) with respect to z over D. By iterating these processes m times for $E_{t/n}(x, y)$ we can define

$$(\overbrace{E_{t/n} * E_{t/n} * \cdots * E_{t/n}}^{m})(x, y) = (E_{t/n} *)^m(x, y)\,. \tag{14}$$

Lemma 3. *Suppose that x and y be on a compact convex set Q contained in D. Then for small t,*

$$(E_{tl(n)} *)^{2^n}(x, y) = E_t(x, y)\,(1+0(\exp-t^{-1/2})) \tag{15}$$

where $l(n)=2^{-n}$.

Proof. Let t be so small that around any x in Q, $S(t)$, i.e. the solid sphere with its radius $\sqrt[6]{4t}$ is contained in D.

It is easy to see that

$$
\begin{aligned}
&(E_{i/n} *)^2(x, y) \leqq E_{2/tn}(x, y)\,,\\
&(E_{t/n} *)^3(x, y) \leqq (E_{2/tn} * E_{t/n})(x, y)\,,\\
&\cdots\cdots,
\end{aligned} \tag{16}
$$

By making convolutions sucessively

$$(E_{t/n} *)^n(x, y) \leqq E_t(x, y)\,. \tag{16'}$$

From (16′), by replacing t by t/m we have

$$(E_{t/mn} *)^{mn}(x, y) \leqq (E_{t/m} *)^m(x, y)\,.$$

This means that if we introduce a partial order $<$ into positive integers such that $m < n$ means n is divided by m, then $(E_{t/m} *)^m$ is decreasing when m is increasing. Hence there exists the limit in the sense of Moore Smith for integers 2^n because these are linearly ordered.

Now let Q_k be the set of points whose distance from Q is less than

$$\sum_{j=1}^{k} \sqrt[6]{4tl(j)}$$

where $l(j)=2^{-j}$.

Notice that the distance between the two sets Q_{k+1} and Q_k is equal to $\sqrt[6]{4tl(k+1)}$. Therefore for x and y in Q_k

$$
\begin{aligned}
(17)\qquad & \int_D E_{tl(k)}(x, z)\, E_{tl(k)}(x, y)dz \\
& \geqq \int_{Q_{k+1}} E_{tl(k)}(x, z)\, E_{tl(k)}(z, y)dz \\
& \geqq E_{2tl(k)}(x, y) \int_{Q_{k+1}} E_{tl(k)/2}\left(z, \frac{1}{2}(x+y)\right)dz \\
& = E_{tl(k-1)}(x, y)\left(1-2^d \exp\left(-\frac{1}{4}(tl(k+1)\right)^{-2/3}\right).
\end{aligned}
$$

Repeating these processes

$$
\begin{aligned}
(18)\qquad & E_{tl(n)}*)^{2^n}(x, y) \\
& \geqq (E_{tl(n-1)}*)^{2^{n-1}}(x, y)\left(1-2^d \exp\left(-\frac{1}{4}(tl(n+1))\right)^{-2/3}\right)^{l(n-1)} \\
& \geqq, \cdots, E_t(x, y) \sum_{j=1}^{n}\left(1-2^d \exp\left(-\frac{1}{4}(tl(n-j+1))\right)^{-2/3}\right)^{l(j-1)} \\
& \geqq E_t(x, y)\left(1-\sum_{j=1}^{n} 2^d 2^{n-j+1} \exp\left(-\frac{1}{4}(tl(n-j+1))^{-2/3}\right)\right) \\
& \geqq E_t(x, y)(1-2^d \int_1^\infty \exp\left(-\frac{1}{4}(t^{-2/3}\xi^{2/3})d\xi(1-0(t))\right) \\
& \geqq E_t(x, y)(1-0(\exp(-t^{-1/2})))
\end{aligned}
$$

uniformly in n when t is small. This proves the lemma.

REMARK: Since $E_t(x, y)$ is uniformly continuous in t over D and larger than a constant, $1/n$ can be approximated with dyadic number $pl(m)$ from below in such a way that

$$E_{t/n}(x, y) \geqq E_{tpl(m)}(x, y)e^{-\delta/n}$$

for a preassigned δ. Then

$$
\begin{aligned}
(E_{t/n}*)^n(x, y) & \geqq (E_{tpl(m)}*)^n(x, y)e^{-\delta} \\
& \geqq E_{tpnl(m)}(x, y)(1-0(\exp(-t^{-1/2}))).
\end{aligned}
$$

New let m go to infinity. Then δ goes to zero and $pl(m)$ to $1/n$. Hence we have

$$(19)\qquad (E_{t/n}*)^n(x, y) \geqq E_t(x, y)(1-0(\exp(-t^{-1/2}))).$$

These (16′) and (19) prove the lemma.

Lemma 4. *When both x and y are in Q,*

$$\lim_{n\to\infty} (E_{t/n}*)^n(x, y) = K(x, y; t) \quad \textit{exists.} \tag{20}$$

Proof. Set $t=h/m$ in (19). Then

$$\begin{aligned}(E_{hm/n}*)^{mn}(x, y) &\geqq (E_{h/m}*)^m(x, y)(1-0(\exp(-(h/m)^{-1/2}))^m \\ &= (E_{t/m}*)^m(x, y)(1-0(1/m)).\end{aligned}$$

Evidently, however

$$(E_{t/mn}*)^{mn}(x, y) \leqq (E_{t/m}*)^m(x, y) \tag{21}$$

and $(E_{t/m}*)^m(x, y)$ is uniformly bounded above by $E_t(x, y)$. Hence

$$|(E_{t/mn}*)^{mn}(x, y)-(E_{t/m}*)^m(x, y)| \leqq 0(1/m)E_t(x, y). \tag{22}$$

If both m and n be larger than m_0, then from (23) it follows that

$$\begin{aligned}|(E_{t/m}*)^m(x, y)-(E_{t/n}*)^n(x, y)| &\leqq |(E_{t/m}*)^m(x, y)-(E_{t/mn}*)^{mn}(x\ y)| \\ &\quad +|(E_{t/mn}*)^{mn}(x, y)-(E_{t/n}*)^n(x, y)| \\ &\leqq (0(1/m)+0(1/n))E_t(x, y) = 0(1/m_0)E_t(x, v).\end{aligned} \tag{23}$$

This proves the existence of the limit (20).

Lemma 5. *The diffusion equation* (1) *on D has a unique solution up to the initial function under the condition of* 0*–boundary-value when* $t \neq 0$.

Proof. We have only to prove that if U be a solution of (1) with 0–boundary-value and $U(x; t)=0$ everywhere when $t=0$, then $U(x, t)=0$ for any t and for any x.

Now

$$\frac{d}{dt}\int_D (U(x, t))^2 dx = 2\int_D \frac{\partial}{\partial t} U(x, t)U(x, t)dx = \int_D (\Delta U)U dx, \tag{24}$$

and by the virtue of Stoke's theorem,

$$= -2\int_D \sum_{i=1}^{d}\left(\frac{\partial}{\partial x^i} U(x, t)\right)^2 dx \leqq 0.$$

Hence $\int (U(x, 0))^2 dx = 0$ implies $\int (U(x, t))^2 dx = 0$, i.e. $U(x, t)=0$. This prove the lemma.

Lemma 6. *Let D' be an open convex domain, and p be a point at a distance less than $ch^{1/2}$ from D'. Then there exists a constant* $\varepsilon > 0$

depending only on c so that when h goes to zero

(25) $$\int_{D'} E_h(p, y)dy \geqq \varepsilon$$

holds uniformly in h.

Proof. Consider the transformation of coordinates $x^i \to \xi^i = (2\sqrt{h})^{-1}(x^i - p^j)$ with its origin p. In ξ's, D' is mapped to a convex open set $D'_1(h)$ near to origin, and similar to D'. These $D'_1(h)$ approaches a convex open domain D'_1 when h goes to 0.

However, for a small positive ε,

(26) $$\int_{D'} E_h(p, y)dy = \int_{D'_1(h)} E_1(0, \eta)d\eta \geqq \varepsilon .$$

where $\eta^i = (2\sqrt{h})^{-1}(y^i - p^i)$.

An open set O is called regularly open if it coincides with the open kernel of its closure.

Lemma 7. *Suppose that the boundary ∂D of a regularly open D be a rectilinear simplicial complex. If either x or y be on ∂D, then*

$$\lim_n (E_{t/n} *)^n(x, y) = 0 .$$

Proof. Let x be on ∂D. It is easy to construct a convex open D' outside of D whose boundary contains x.

Set

$$m = [n/2] .$$

Clearly

$$(E_{t/n} *)^m(x, y) \leqq E_{mt/n}(x, y) .$$

Take the c defined in Lemma 6 sufficiently large that for any h less than t

(27) $$\int_{|y| \geqq ct^{1/2}} E_h(0, y)dy \leqq \int_{|\eta| \geqq c/2} E_1(0, \eta)d\eta \leqq \frac{1}{2}\varepsilon .$$

By T_n we denote the set of points $\{z ; |x-z| \leqq c(t/n)^{1/2}\}$. The for y in T_n

(28) $$\begin{aligned}(E_{t/n} *)^{m+1}(x, y) &\leqq \int_D E_{m/tn}(z, y)dz \\ &= E_{(m+1)t/n}(x, y) \int_D E_{mt/(m+1)n}\left(z, \frac{x+my}{m+1}\right) dz \\ &\leqq E_{(m+1)t/n}(x, y)\Big(\int_{D \cap T_n} + \int_{\mathcal{E} - T_n}\Big) E_{m/t(m+1)n}\left(z, \frac{x+my}{m+1}\right) dz\end{aligned}$$

$$\leqq E_{(m+1)t/n}(x, y)\left(1-\varepsilon+\frac{1}{2}\varepsilon\right)$$
$$= E_{(m+1)t/n}(x, y)\left(1-\frac{1}{2}\varepsilon\right).$$

because of (27) and of the previous lemma.

By iterating these processes, we have

$$(E_{t/n}*)^{m+i}(x, y) \leqq E_{(m+i)t/n}(x, y)\left(1-\frac{1}{2}\varepsilon\right)^{i}$$

and in particular

$$(E_{t/n}*)^{n-1}(x, y) \leqq E_{(n-1)t/n}\left(1-\frac{1}{2}\varepsilon\right)^{n-m-1}. \tag{29}$$

From this it follows that

$$(E_{t/n}*)^{n}(x, y) \leqq \int_{D\cap T_n}(E_{t/n}*)^{n-1}(x, z)E_{t/n}(z, y)dz$$
$$+\int_{D-T_n\cap D}E_{t/n}(x, z)(E_{t/n}*)^{n-1}(z, y)dz$$
$$\leqq\left(1-\frac{1}{2}\varepsilon\right)^{n-m-1}\int_{\mathcal{E}}E_{(n-1)t/n}(x, z)E_{t/n}(z, y)dz$$
$$+\int_{|\zeta|\geqq c}E_1(0, \zeta)d\zeta A$$
$$=\left(1-\frac{1}{2}\varepsilon\right)^{n-m-1}E_t(x, y)+\int_{|\zeta|\geqq c}E_1(0, \zeta)d\zeta A$$

where

$$A=\sup_{z\in D,\ \frac{1}{2}\leqq h\leqq t}E_h(z, y).$$

Now let c vary and go to infinity slow enough as n tends to infinity, that ε depending on c by (29) satisfies

$$\lim_n\left(1-\frac{1}{2}\varepsilon\right)^{n-m}=0. \tag{30}$$

For instance if we take c's in such a way that $\varepsilon=\varepsilon(n)=n^{-1/2}$, (30) holds.

Under these modifications we can commute

$$\lim_n(E_{t/n}*)^n(x, y)\leqq\lim_n\left(1-\frac{1}{2}\varepsilon\right)^{n-m}E_t(x, y) \tag{31}$$
$$+\lim_n\int_{|\zeta|\geqq c}E_1(0, \zeta)d\zeta A=0$$

which prove the lemma.

REMARK. Lemma 7 holds for a regularly open D with a smooth boundary and the same kind of proof works for it. Moreover we have

$$\lim_n (E_{t/n} *)^n(x, y) = 0 \tag{32}$$

when either x or y is outside of D.

Lemma 8. *Let ϕ (y) be a continuous function over the closure of D, and x be a point on D. Then*

$$\lim_{h\to 0} \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y)\phi(y)dy = \phi(x) . \tag{33}$$

Proof. Let $S(h)$ be the solid sphere of radius $\sqrt[6]{4h}$ around x. Set

$$A(h) = \sup_{y\in S(h)}\phi(y) \tag{34}$$

and

$$B(h) = \inf_{y\in S(h)}\phi(y) .$$

Then from (5), considering $S(h)$ as Q,

$$\begin{aligned} &\int_D \overline{\lim}_n (E_{h/n} *)^n(x, y)\phi(y)dy \\ &\leqq A(h) \int_{S(h)} E_h(x, y)dy + \int_{D-S(h)} E_h(x, y)\phi(y)dy \\ &\leqq A(h) + 0\,(2^d \exp\left(-\frac{1}{4}(h)^{-2/3}\right) . \end{aligned} \tag{35}$$

However from (15),

$$\begin{aligned} &\int_D \overline{\lim}_n (E_{h/n} *)^n(x, y)\phi(y)dy \\ &\geqq B(h)(1-0\,(\exp(-h^{-1/2}) \int_D E_h(x, y)dy \end{aligned} \tag{36}$$

Since the right hand sides of both (35) and (36) approaches $\phi(x)$ when h goes to zero, we have

$$\lim_{h\to 0} \int_D \overline{\lim}_n(E_{h/n} *)^n(x, y)\phi(y)dy = \phi(x) .$$

Corollary.

$$\lim_{h\to 0} \int_D \underline{\lim}_n(E_{h/n} *)^n(x, y)\phi(y)dy = \phi(x) . \tag{37}$$

The proof is obtained by changing $\overline{\lim}$ to $\underline{\lim}$ in Lemma 8.

Lemma 9. *Suppose that $\phi(y)$ be a C^2 function over the closure of D. Then,*

(38) $$\lim_{h\to 0}(1/h)(\int_D \overline{\lim}(E_{h/n}*)^n(x,\ y)\phi(y)dy-\phi(x))=\Delta\phi$$

Proof.

(39) $$\int_{S(h)} \overline{\lim}(E_{h/n}*)^n(x,\ y)\phi(y)dy-\phi(x)=\int_{S(h)} E_h(x,\ y)\phi(y)dy-\phi(x)$$
$$+0(\exp(-h^{-1/2}))\int_{S(h)}\phi(y)dy\,.$$

On the other hand by (5)

(40) $$\int_{D-S(h)} \overline{\lim}(E_{h/n}*)(x,\ y)\phi(y)dy\leqq\sup_{y\in D}|\phi(y)|2^d\exp\left(-\frac{1}{4}(4h)^{-2/3}\right).$$

Hence

(41) $$(1/h)[\int_D \overline{\lim}(E_{h/}*)^n(x,\ y)\phi(y)dy-\phi(x)]$$
$$=(1/h)(\int_{S(h)} E_h(x,\ y)\phi(y)dy-\phi(x))+0(h^{-1}\exp(-h^{-1/2}))\,.$$

It is well known that

(42) $$\lim_{h\to 0}(1/h)(\int_{S(h)} E_h(x,\ y)\phi(y)dy-\phi(x))=(\Delta\phi)_x\,.$$

Hence we have

$$\lim_{h\to 0}(1/h)[\int_D \overline{\lim}_n(E_{h/n}*)^n(x,\ y)\phi(y)dy-\phi(x)]=(\Delta\phi)_x$$

Corollary.

(43) $$\lim_{h\to 0}(1/h)[\int_D \underline{\lim}_n(E_{h/n}*)^n(x,\ y)\phi(y)dy-\phi(x)]=(\Delta\phi)_x\,.$$

Lemma 10.

(44) $\overline{\lim}_n(E_{t/n}*)^n(x,\ y)=\underline{\lim}_n(E_{t/n}*)^n(x,\ y)$ *exists.*

We denote this by $K(x,y;t)$.

Proof. From (16) it follows that

$$(E_{t/mn}*)^m(x,\ y)\leqq E_{t/n}(x,\ y)$$

and hence

$$(E_{t/mn}*)^{mn}(x,\ y)\leqq(E_{t/n}*)^n(x,\ y)\,.$$

Therefore the limit in the sense of Moore-Smith existis for integers 2^m's when we introduce an partial order in such a way that $m<p$ when m is a divisor of p.

In general, first we approximate $1/m$ with $pl(n)$. Then $E_{t/m}(x,y)$ is uniformly approximated in such a way that for a preassigned δ

$$E_{t/m}(x,\ y) \geqq E_{tpl(n)}(x,\ y)e^{-\delta/m}\,.$$

Then

$$(E_{t/m}*)^m(x,\ y) \geqq (E_{tpl(n)}*)^m(x,\ y)e^{-\delta} \geqq \lim_j (E_{t/pl(n+j)}*)^{m2^j}(x,\ y)e^{-\delta}\,.$$

Since this holds for any δ,

$$(E_{t/m}*)^m(x,\ y) \geqq \lim_j (E_{tl(j)/m}*)^{2^j_m}(x,\ y) = \lim_j (E_{tl(j)}*)^{2^j}(x,\ y)\,.$$

Conversely, when m is large, we have

$$E_{tp/m}(x,\ y) \leqq E_{tl(n)}(x,\ y)e^{\delta l(n)}$$

with a suitable integer p.

Hence it follows that

$$\overline{\lim}(E_{t/m}*)^m(x,\ y) \leqq \overline{\lim}(E_{pt/m}*)^{2^n}(x,\ y) \leqq \lim_n (E_{tl(n)}*)^{2^n}(x,\ y)e^{\delta}$$

which hold for any δ. Hence

$$\lim(E_{t/m}*)^m(x,\ y) = \lim_n (E_{tl(n)}*)^{2^n}(x,\ y)$$

has been proved. This proves the lemma.

Introduce a one parameter family of operators K_t by

$$(K_t\phi)(x) = \int_D K(x,\ y\,;\ t)\phi(y)dy\,. \tag{45}$$

Lemma 11. *$\{K_t\}$ forms a one parameter semi-group of operators.*

Proof. For two positive reals t and s

$$\begin{aligned}(K_t(K_s\phi))(x) &= \int_D K(x,\ y\,;\ t)\int_D K(y,\ z\,;\ s)\phi(z)dzdy \\ &= \int_D \Big(\int_D K(x, y\,;\ t)\,K(y,\ z\,;\ s)dy\Big)\phi(z)dz \\ &= \int_D \lim_n (E_{(t+s)/n}*)^n(x,\ z)\phi(z)dz \\ &= (K_{t+s}\phi)(x)\end{aligned} \tag{46}$$

which proves the lemma.

Lemma 12. *Suppose that for a C^2 continuous $\phi(x)$,*

$$\lim_{h\to 0}(1/h)((K_h\phi)(x) - \phi(x))$$

exists everywhere. Then the limit is equal to $(\Delta\phi)_x$.

Proof. Take a point x and denote by $S(h)$ the solid sphere of radius $\sqrt[9]{4h}$ around x. Then from (19)

$$
\begin{aligned}
&\int_D K(x, y\,;\, h)\phi(y)dy = \int_{S(h)} K(x, y\,;\, h)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
(46)\qquad &= \int_{S(h)} E_h(x, y)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
&= \int_{S(h)} \int_{\varepsilon}^{h} \frac{\partial}{\partial t} E_t(x, y)dt\phi(y)dy + \int_{S(h)} E_\varepsilon(x, y)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
&= \Delta_x \int_{S(h)} \int_{\varepsilon}^{h} E_t(x, y)\phi(y)dy\,dt + \int_{S(h)} E_\varepsilon(x, y)\phi(y)dy + 0(\exp(-h^{-1/2}))\,,
\end{aligned}
$$

for some small ε.

Hence

$$
\begin{aligned}
(47)\qquad & 1/h((E_h\phi)(x) - \phi(x)) = \Delta_x \int_{S(h)} (1/h) \int_{\varepsilon}^{h} E_t(x, y)\phi(y)dydt \\
&\quad + (1/h)\Big(\int_{S(h)} E_\varepsilon(x, y)\phi(y)dy - \phi(x) + 0(h^{-1}\exp(-h^{-1/2}))\,.
\end{aligned}
$$

When ε goes to zero,

$$
\begin{aligned}
(48)\qquad & (1/h)(K_h\phi)(x) - \phi(x)) = \Delta_x \int_{S(h)} (1/h) \int_{0}^{h} E_t(x, y)\phi(y)dtdy \\
&\quad + 0(h^{-1}\exp(-h^{-1/2}))\,.
\end{aligned}
$$

and therefore

$$
(49)\qquad \lim_{h\to 0}(1/h)((K_h\phi)(x) - \phi(x)) = \Delta_x \lim_h \int_{S(h)} (1/h) \int_{0}^{h} E_t(x, y)\phi(y)dtdy
$$

which proves that $(\Delta\phi)_x$ exists and is equal to $\frac{\partial}{\partial h}(K_h\phi)(x)$. Here we use the fact that $\int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dtdy$ approaches $\phi(x)$ uniformly together with its second derivatives with the order of $h^{-1}\exp(-h^{-1/2})$.

Lemma 13. *Let $\phi(x)$ be a C^2 function over the closure of D. Then for any x in D and for any t*

$$
(50)\qquad \left[\frac{\partial}{\partial h}(K_{t+h}\phi)\right]_{h=0}(x) = \Delta_x(K_t\phi)
$$

exists.

Proof. By Lemma 9,

$$
(51)\qquad \lim_h (1/h)(K_h\phi - \phi) = \Delta_x\phi\,.
$$

However

$$
(52)\qquad K_{t+h}\phi - K_t\phi = K_t(K_h\phi - \phi)\,.
$$

Hence

$$
\left(\frac{\partial}{\partial h} K_{t+h}\phi\right)(x) = [K_t \lim \frac{1}{h}(K_h\phi - \phi)](x) = (K_t\Delta\phi)(x)\,.
$$

By the previous Lemma

$$(K_t\Delta\phi)(x) = (\Delta K_t\phi)(x) = \left(\frac{\partial}{\partial h}K_{t+h}\phi\right)(x)$$

because $\frac{\partial}{\partial h}K_{t+h}\phi = \lim_{h\to 0}(1/h)(K_hK_t\phi - K_t\phi)$ exists. This completes the proof.

Corollary. *$\int_D K(x, y; t)\phi(y)dy$ is a solution of a differential equation $\frac{\partial}{\partial t}U = \Delta U$, for any continuous function $\phi(x)$.*

Proof. When $\phi(x)$ is C^2, then U is a solution of $\partial U/\partial t = \Delta U$. Now let $\phi_n(x)$'s converge to $\phi(x)$ where all ϕ_n's are C^2. Then the corresponding U_n's converge to a weak solution which, by a theorem by Nirenberg (1), is a genuine solution. This completes the proof.

Lemma 14. *$K(x, y; t)$ is C^2 both in x and in y.*

Proof. From the previous corollary it is evident that $\int K(x, y; t)\phi(y)dy$ is C^2 in x for any $t>0$. Hence for an $h<t$

$$K(x, y; t) = \int_D K(x, z; h) \int_D K(z, y; t-h)dz \tag{53}$$

is C^2 in x. By the construction of $K(x, y; t)$

$$K(x, y; t) = K(y, x; t)\,. \tag{54}$$

Therefore $K(x, y; t)$ is C^2 in y.

Theorem 1. *Suppose that D is a regularly open set with either smooth or rectilinear boundary. Then $K(x, y; t)$ defined in Lemma* 10 *is the kernel function of the differential equation*

$$\frac{\partial}{\partial t}U = \Delta U \tag{55}$$

over D.

Proof. By Lemma 7

$$K(x, y; t) = 0 \tag{56}$$

if either x or y be on ∂D. From its construction it follows directly that $K(x, y; t)>0$ for $x \neq y$. Lemma 14 says $K(x, y; t)$ is C^2 both in x and in y.

Therefore the result in Lemma 12, i.e.

$$\frac{\partial}{\partial t}\int_D K(x, y;\ t)\phi(y)dy = \Delta_x \int K(x, y;\ t)\phi(y)dy \tag{57}$$

implies $\frac{\partial}{\partial t}K(x, y;\ t-h) = \Delta_x K(x, y;\ t-h)$, which proves the theorem.

NOTICE. Theorem 1 holds when any point on ∂D is on a boundary of an open convex set D' disjoint with D.

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