Kernel Functions of Diffusion Equations (*I*)

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Let D be an open bounded set in a d-dimensional Euclidean space $\mathcal{E}$.

By Δ we understand the Laplacian with respect to given coordinates. Consider the diffusion equation

$$(1) \qquad \frac{\partial U}{\partial t} = \Delta U$$

on D. By a *kernel function* $K(x, y; t)$ we understand a function on $\mathcal{E} \times \mathcal{E} \times [0, \infty)$ satisfying following properties:

(i) $K(x, y; t) = 0$ when either x or y is on the boundary of D, if $K(x, y; t)$ is continuous on boundary for a fixed t, or the boundary ∂D is smooth.

(ii) For a fixed y

$$(2) \qquad \frac{\partial}{\partial t} K(x, y; t) = \Delta_x K(x, y; t)$$

where Δ_x is understood as the Laplacian on the variable x.

The purpose of this paper is to give a new way of constructing the kernel function on D which coincides with the Green's function when ∂D, the boundary of D is smooth.

In preparations we shall define some notations. Coordinates of points $x, y, \cdots$ on $\mathcal{E}$ will be written x^i; $1 \leq i \leq d$, y^j; $1 \leq j \leq d$, etc. The euclidean distance between two points x and y is denoted by

$$(3) \qquad |x-y| = (\sum_i^d (x^i - y^i)^2)^{1/2} .$$

Let

$$(4) \qquad E_t(x, y) = (2\sqrt{\pi t})^{-d} \exp\left(-(4t)^{-1}|x-y|^2\right) .$$

Lemma 1. *Take a point x on D. Let $S(h)$ be a solid sphere around*

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(1), see (15) and (8)

(2), see (8)

x with its radius $(4h)^{1/6}$ *where h is sufficiently small that* $S(h)$ *is within*
D.   Then

(5) $$1 \geqq \underset{S(h)}{\int} E_h(x, y)dy \geqq 1 - 2^d \exp\left(-\frac{1}{4}(h)^{-2/3}\right).$$

Proof.   Consider polar coordinates ρ, θ^i, $1 \leqq i \leqq d-1$, around x where
θ^i's denote angular coordinates.   Then

(6) $$\underset{D-S(h)}{\int} E_h(x, y)dy \leqq \overset{\infty}{\underset{(4h)^{1/3}}{\int}} d\rho \underset{\theta}{\int} \rho^{d-1}(2\sqrt{\pi h})^{-d} \exp\left(-(4h)^{-1}\rho^2\right)d\theta \, .$$

Notice that for any y outside of $S(h)$

(7) $$|x-y| \geqq \frac{1}{2}(4h)^{1/6} + \frac{1}{2}|x-y|$$

and consequently

(8) $$|x-y|^2 \geqq \frac{1}{4}\left((4h)^{1/3} + |x-y|^2\right)$$

Hence

(9) $$\underset{D-S(h)}{\int} E_h(x, y)dy \leqq (2\sqrt{\pi h})^{-d} \underset{\mathcal{E}-S(h)}{\int} \exp\left[-\frac{1}{4}(4h)^{-2/3} - \frac{1}{4}|x-y|^2/4h\right]dy$$

$$\leqq 2^d \exp\left(-\frac{1}{4}(4h)^{-2/3}\right)(2\sqrt{\pi 4h})^{-d} \int \exp\left(-|x-y|^2/16h\right)dy$$

$$\leqq 2^d \exp\left(-\frac{1}{4}(4h)^{-2/3}\right).$$

However it is well-known that

(10) $$\underset{S(h)}{\int} E_h(x, y)dy \leqq 1 \, ,$$

for any h.

Hence we have

$$1 \geqq \int E_h(x, y)dy \geqq 1 - 2^d \exp\left(-\frac{1}{4}(h)^{-2/3}\right),$$

which proves the lemma.

Given two functions ϕ and ψ of two variables, we define a convolution

(11) $$(\phi * \psi)(x, y) = \underset{D}{\int} \phi(x, z)\psi(z, y)dz \, .$$

Then

Lemma 2.

(12) $(E_t * E_h)(x, y)$

$$\leqq E_{t+h}(x, y)\left[\left(2\sqrt{\pi}\sqrt{\frac{th}{t+h}}\right)^{-d} \underset{D}{\int} \exp\left(-\frac{t+h}{4th}\left|z - \frac{tx+hy}{t+h}\right|^2\right)dz\right.$$

Proof.   By direct computations

(13)

$E_t(x, z)E_h(z, y)$

$= (2\sqrt{\pi t})^{-d}(2\sqrt{\pi h})^{-d} \exp\left(-\frac{1}{4t}|x-z|^2 - \frac{1}{4h}|z-y|^2\right)$

$= (2\sqrt{\pi(t+h)})^{-d}\left(2\sqrt{\pi}\sqrt{\frac{th}{t+h}}\right)^{-d} \exp\left(-|x-y|^2/4(t+h) - \frac{t+h}{4th}\left|z - \frac{tx+hy}{t+h}\right|^2\right).$

Hence we have (12) by integrating (13) with respect to z over D. By iterating these processes m times for $E_{t/n}(x, y)$ we can define

(14) $\overbrace{(E_{t/n} * E_{t/n} *^m \cdots * E_{t/n})}(x, y) = (E_{t/n}*)^m(x, y)$.

Lemma 3. *Suppose that x and y be on a compact convex set Q contained in D.   Then for small t,*

(15) $(E_{tl(n)}*)^{2^n}(x, y) = E_t(x, y)(1+0(\exp - t^{-1/2}))$

where $l(n) = 2^{-n}$.

Proof.   Let t be so small that around any x in Q, $S(t)$, i.e. the solid sphere with its radius $\sqrt[q]{4t}$ is contained in D.
It is easy to see that

(16) $(E_{i/n}*)^2(x, y) \leqq E_{2/tn}(x, y)$,

$(E_{t/n}*)^3(x, y) \leqq (E_{2/tn}*E_{t/n})(x, y)$,

$\cdots\cdots$,

By making convolutions sucessively

(16′) $(E_{t/n}*)^n(x, y) \leqq E_t(x, y)$.

From (16′), by replacing t by t/m we have

$(E_{t/mn}*)^{mn}(x, y) \leqq (E_{t/m}*)^m(x, y)$.

This means that if we introduce a partial order $<$ into positive integers such that $m < n$ means n is divided by m, then $(E_{t/m}*)^m$ is decreasing when m is increasing.   Hence there exists the limit in the sense of Moore Smith for integers 2^n because these are linearly ordered.
Now let Q_k be the set of points whose distance from Q is less than

$$\sum_{j=1}^{k} \sqrt[6]{4tl(j)}$$

where $l(j)=2^{-j}$.

Notice that the distance between the two sets Q_{k+1} and Q_k is equal to $\sqrt[4]{4tl(k+1)}$. Therefore for x and y in Q_k

(17)
$$\int_D E_{tl(k)}(x,\ z)\ E_{tl(k)}(x,\ y)dz$$

$$\geq \int_{Q_{k+1}} E_{tl(k)}(x,\ z)\ E_{tl(k)}(z,\ y)dz$$

$$\geq E_{2tl(k)}(x,\ y) \int_{Q_{k+1}} E_{tl(k)/2}\Big(z,\ \frac{1}{2}(x+y)\Big)dz$$

$$= E_{tl(k-1)}(x,\ y)\Big(1-2^d \exp\Big(-\frac{1}{4}(tl(k+1))^{-2/3}\Big)\Big).$$

Repeating these processes

(18) $E_{tl(n)}*)^{2^n}(x,\ y)$

$$\geq (E_{tl(n-1)}*)^{2^{n-1}}(x,\ y)\Big(1-2^d \exp\Big(-\frac{1}{4}(tl(n+1))\Big)^{-2/3}\Big)^{l(n-1)}$$

$$\geq,\ \cdots,\ E_t(x,\ y)\sum_{j=1}^n \Big(1-2^d \exp\Big(-\frac{1}{4}(tl(n-j+1))\Big)^{-2/3}\Big)^{l(j-1)}$$

$$\geq E_t(x,\ y)\Big(1-\sum_{j=1}^n 2^d 2^{n-j+1} \exp\Big(-\frac{1}{4}(tl(n-j+1))^{-2/3}\Big)\Big)$$

$$\geq E_t(x,\ y)(1-2^d \int_1^\infty \exp\Big(-\frac{1}{4}(t^{-2/3}\xi^{2/3})d\xi(1-0(t))\Big)$$

$$\geq E_t(x,\ y)(1-0(\exp(-t^{-1/2})))$$

uniformly in n when t is small. This proves the lemma.

REMARK : Since $E_t(x,\ y)$ is uniformly continuous in t over D and larger than a constant, $1/n$ can be approximated with dyadic number $pl(m)$ from below in such a way that

$$E_{t/n}(x,\ y) \geq E_{tpl(m)}(x,\ y)e^{-\delta/n}$$

for a preassigned δ. Then

$$(E_{t/n}*)^n(x,\ y) \geq (E_{tpl(m)}*)^n(x,\ y)e^{-\delta}$$
$$\geq E_{tpnl(m)}(x,\ y)(1-0(\exp(-t^{-1/2})))\ .$$

New let m go to infinity. Then δ goes to zero and $pl(m)$ to $1/n$. Hence we have

(19) $(E_{t/n}*)^n(x,\ y) \geq E_t(x,\ y)(1-0(\exp(-t^{-1/2})))\ .$

These (16') and (19) prove the lemma.

Lemma 4. *When both x and y are in Q,*

(20) $$\lim_{n \to \infty} (E_{t/n} *)^n(x, y) = K(x, y; t) \quad exists.$$

Proof. Set $t = h/m$ in (19). Then

$$(E_{hm/n} *)^{mn}(x, y) \geq (E_{h/m} *)^m(x, y)(1 - 0(\exp(-(h/m)^{-1/2}))^m$$
$$= (E_{t/m} *)^m(x, y)(1 - 0(1/m)).$$

Evidently, however

(21) $$(E_{t/mn} *)^{mn}(x, y) \leq (E_{t/m} *)^m(x, y)$$

and $(E_{t/m} *)^m(x, y)$ is uniformly bounded above by $E_t(x, y)$. Hence

(22) $$|(E_{t/mn} *)^{mn}(x, y) - (E_{t/m} *)^m(x, y)| \leq 0(1/m)E_t(x, y).$$

If both m and n be larger than m_0, then from (23) it follows that

(23) $$|(E_{t/m} *)^m(x, y) - (E_{t/n} *)^n(x, y)| \leq |(E_{t/m} *)^m(x, y) - (E_{t/mn} *)^{mn}(x \ y)|$$
$$+ |(E_{t/mn} *)^{mn}(x, y) - (E_{t/n} *)^n(x, y)|$$

$$\leq (0(1/m) + 0(1/n))E_t(x, y) = 0(1/m_0)E_t(x, y).$$

This proves the existence of the limit (20).

Lemma 5. *The diffusion equation* (1) *on D has a unique solution up to the initial function under the condition of* 0–*boundary-value when* $t \neq 0$.

Proof. We have only to prove that if U be a solution of (1) with 0–boundary-value and $U(x ; t) = 0$ everywhere when $t = 0$, then $U(x, t) = 0$ for any t and for any x.
Now

(24) $$\frac{d}{dt} \int_D (U(x, t))^2 dx = 2 \int_D \frac{\partial}{\partial t} U(x, t) U(x, t) dx = \int_D (\Delta U) U dx,$$

and by the virtue of Stoke's theorem,

$$= -2 \int_D \sum_{i=1}^a \left(\frac{\partial}{\partial x^i} U(x, t)\right)^2 dx \leq 0.$$

Hence $\int (U(x, 0))^2 dx = 0$ implies $\int (U(x, t))^2 dx = 0$, i.e. $U(x, t) = 0$. This prove the lemma.

Lemma 6. *Let D′ be an open convex domain, and p be a point at a distance less than* $ch^{1/2}$ *from D′. Then there exists a constant* $\varepsilon > 0$

depending only on c so that when h goes to zero

(25) $$\int_{D'} E_h(p,\ y)dy \geqq \varepsilon$$

holds uniformly in h.

Proof. Consider the transformation of coordinates $x^i \to \xi^i = (2\sqrt{h})^{-1}$ $(x^i - p^j)$ with its origin p. In ξ's, D' is mapped to a convex open set $D'_1(h)$ near to origin, and similar to D'. These $D'_1(h)$ approaches a convex open domain D'_1 when h goes to 0.

However, for a small positive ε,

(26) $$\int_{D'} E_h(p,\ y)dy = \int_{D'_1(h)} E_1(0,\ \eta)d\eta \geqq \varepsilon\ .$$

where $\eta^i = (2\sqrt{h})^{-1}(v^i - p^i)$.

An open set O is called regularly open if it coincides with the open kernel of its closure.

Lemma 7. *Suppose that the boundary ∂D of a regularly open D be a rectilinear simplicial complex. If either x or y be on ∂D, then*

$$\lim_n (E_{t/n} *)^n(x,\ y) = 0\ .$$

Proof. Let x be on ∂D. It is easy to construct a convex open D' outside of D whose boundary contains x.

Set
$$m = [n/2]\ .$$

Clearly

$$(E_{t/n} *)^m(x,\ y) \leqq E_{mt/n}(x,\ y)\ .$$

Take the c defined in Lemma 6 sufficiently large that for any h less than t

(27) $$\int_{|y| \geqq ct^{1/2}} E_h(0,\ v)dy$$

$$\leqq \int_{|\eta| \geqq c/2} E_1(0,\ \eta)d\eta \leqq \frac{1}{2}\varepsilon\ .$$

By T_n we denote the set of points $\{z\ ;\ |x - z| \leqq c(t/n)^{1/2}\}$. The for y in T_n

$$(E_{t/n} *)^{m+1}(x,\ y) \leqq \int_D E_{m/tn}(z,\ y)dz$$

(28) $$= E_{(m+1)t/n}(x,\ y) \int_D E_{mt/(m+1)n}\left(z,\ \frac{x + my}{m+1}\right) dz$$

$$\leqq E_{(m+1)t/n}(x,\ y)\left(\int_{D \cap Tn} + \int_{\mathcal{E} - Tn}\right) E_{m/t(m+1)n}\left(z,\ \frac{x + my}{m+1}\right) dz$$

$$\leq E_{(m+1)t/n}(x,\ y)\left(1-\varepsilon+\frac{1}{2}\varepsilon\right)$$

$$= E_{(m+1)t/n}(x,\ y)\left(1-\frac{1}{2}\varepsilon\right).$$

because of (27) and of the previous lemma.

By iterating these processes, we have

$$(E_{t/n}*)^{m+i}(x,\ y)\leq E_{(m+i)t/n}(x,\ y)\left(1-\frac{1}{2}\varepsilon\right)^{i}$$

and in particular

(29)
$$(E_{t/n}*)^{n-1}(x,\ y)\leq E_{(n-1)t/n}\left(1-\frac{1}{2}\varepsilon\right)^{n-m-1}.$$

From this it follows that

$$(E_{t/n}*)^{n}(x,\ y)\leq \underset{D\cap T_n}{\int}(E_{t/n}*)^{n-1}(x,\ z)E_{t/n}(z,\ y)dz$$

$$+ \underset{D-T_n\cap D}{\int}E_{t/n}(x,\ z)(E_{t/n}*)^{n-1}(z,\ y)dz$$

$$\leq \left(1-\frac{1}{2}\varepsilon\right)^{n-m-1}\underset{\mathcal{E}}{\int}E_{(n-1)t/n}(x,\ z)E_{t/n}(z,\ y)dz$$

$$+ \underset{|\zeta|\geq c}{\int}E_1(0,\ \zeta)d\zeta A$$

$$= \left(1-\frac{1}{2}\varepsilon\right)^{n-m-1}E_t(x,\ y)+\underset{|\zeta|\geq c}{\int}E_1(0,\ \zeta)d\zeta A$$

where

$$A = \sup_{z\in D,\ \frac{1}{2}\leq h\leq t}E_h(z,\ y).$$

Now let c vary and go to infinity slow enough as n tends to infinity, that ε depending on c by (29) satisfies

(30)
$$\lim_n\left(1-\frac{1}{2}\varepsilon\right)^{n-m}=0.$$

For instance if we take c's in such a way that $\varepsilon=\varepsilon(n)=n^{-1/2}$, (30) holds.

Under these modifications we can commute

(31)
$$\lim_n(E_{t/n}*)^{n}(x,\ y)\leq\lim_n\left(1-\frac{1}{2}\varepsilon\right)^{n-m}E_t(x,\ y)$$

$$+\lim_n\underset{|\zeta|\geq c}{\int}E_1(0,\ \zeta)d\zeta A = 0$$

which prove the lemma.

REMARK. Lemma 7 holds for a regularly open D with a smooth boundary and the same kind of proof works for it. Moreover we have

$$(32) \qquad \lim_n (E_{t/n} *)^n (x, \ y) = 0$$

when either x or y is outside of D.

Lemma 8. *Let* $\phi \ (y)$ *be a continuous function over the closure of* D, *and* x *be a point on* D. *Then*

$$(33) \qquad \lim_{h \to 0} \int_D \overline{\lim}_n (E_{h/n} *)^n (x, \ y) \phi(y) dy = \phi(x) \ .$$

Proof. Let $S(h)$ be the solid sphere of radius $\sqrt[3]{4h}$ around x. Set

$$(34) \qquad A(h) = \sup_{y \in S(h)} \phi(y)$$

and

$$B(h) = \inf_{y \in S(h)} \phi(y) \ .$$

Then from (5), considering $S(h)$ as Q,

$$(35) \qquad \begin{aligned} &\int_D \overline{\lim}_n (E_{h/n} *)^n (x, \ y) \phi(y) dy \\ &\leq A(h) \int_{S(h)} E_h(x, \ y) dy + \int_{D - S(h)} E_h(x, \ y) \phi(y) dy \\ &\leq A(h) + 0 \left(2^d \exp\left(-\frac{1}{4} (h)^{-2/3} \right) \right) . \end{aligned}$$

However from (15),

$$(36) \qquad \begin{aligned} &\int_D \overline{\lim}_n (E_{h/n} *)^n (x, \ y) \phi(y) dy \\ &\geq B(h) (1 - 0 (\exp(-h^{-1/2}) \int_D E_h(x, \ y) dy \end{aligned}$$

Since the right hand sides of both (35) and (36) approaches $\phi(x)$ when h goes to zero, we have

$$\lim_{h \to 0} \int_D \overline{\lim}_n (E_{h/n} *)^n (x, \ y) \phi(y) dy = \phi(x) \ .$$

Corollary.

$$(37) \qquad \lim_{h \to 0} \int_D \underline{\lim}_n (E_{h/n} *)^n (x, \ y) \phi(y) dy = \phi(x) \ .$$

The proof is obtained by changing $\overline{\lim}$ to $\underline{\lim}$ in Lemma 8.

Lemma 9. *Suppose that* $\phi(y)$ *be a* C^2 *function over the closure of* D. *Then,*

(38)　　　　$\lim_{h \to 0}(1/h)(\int_D \overline{\lim}(E_{h/n}*)^n(x, y)\phi(y)dy - \phi(x)) = \Delta\phi$

Proof.

(39)　　　　$\int_{S(h)} \overline{\lim}(E_{h/n}*)^n(x, y)\phi(y)dy - \phi(x) = \int_{S(h)} E_h(x, y)\phi(y)dy - \phi(x)$

　　　　　　　$+ 0(\exp(-h^{-1/2})) \int_{S(h)} \phi(y)dy$.

On the other hand by (5)

(40)　　　　$\int_{D-S(h)} \overline{\lim}(E_{h/n}*)(x, y)\phi(y)dy \leq \sup_{y \in D}|\phi(y)| 2^d \exp\left(-\frac{1}{4}(4h)^{-2/3}\right)$.

Hence

(41)　　　　$(1/h)\left[\int_D \overline{\lim}(E_{h/}*)^n(x, y)\phi(y)dy - \phi(x)\right]$

　　　　　　　$= (1/h)(\int_{S(h)} E_h(x, y)\phi(y)dy - \phi(x)) + 0(h^{-1} \exp(-h^{-1/2}))$.

It is well known that

(42)　　　　$\lim_{h \to 0}(1/h)(\int_{S(h)} E_h(x, y)\phi(y)dy - \phi(x)) = (\Delta\phi)_x$.

Hence we have

　　　　　　　$\lim_{h \to 0}(1/h)\left[\int_D \overline{\lim}_n(E_{h/n}*)^n(x, y)\phi(y)dy - \phi(x)\right] = (\Delta\phi)_x$

Corollary.

(43)　　　　$\lim_{h \to 0}(1/h)\left[\int_D \underline{\lim}_n(E_{h/n}*)^n(x, y)\phi(y)dy - \phi(x)\right] = (\Delta\phi)_x$.

Lemma 10.

(44)　　　　$\overline{\lim}_n(E_{t/n}*)^n(x, y) = \underline{\lim}_n(E_{t/n}*)^n(x, y)$ *exists.*

We denote this by $K(x, y; t)$.

Proof. From (16) it follows that

　　　　　　　$(E_{t/mn}*)^m(x, y) \leq E_{t/n}(x, y)$

and hence

　　　　　　　$(E_{t/mn}*)^{mn}(x, y) \leq (E_{t/n}*)^n(x, y)$.

Therefore the limit in the sense of Moore-Smith existis for integers 2^m's when we introduce an partial order in such a way that $m < p$ when m is a divisor of p.

In general, first we approximate $1/m$ with $pl(n)$. Then $E_{t/m}(x, y)$ is uniformly approximated in such a way that for a preassigned δ

$$E_{t/m}(x, y) \geqq E_{tpI(n)}(x, y)e^{-\delta/m}.$$

Then

$$(E_{t/m}*)^m(x, y) \geqq (E_{tpI(n)}*)^m(x, y)e^{-\delta} \geqq \lim_j (E_{t/pI(n+j)}*)^{m2^j}(x, y)e^{-\delta}.$$

Since this holds for any δ,

$$(E_{t/m}*)^m(x, y) \geqq \lim_j (E_{tI(j)/m}*)^{2^j_m}(x, y) = \lim_j (E_{tI(j)}*)^{2^j}(x, y).$$

Conversely, when m is large, we have

$$E_{tp/m}(x, y) \leqq E_{tI(n)}(x, y)e^{\delta I(n)}$$

with a suitable integer p.

Hence it follows that

$$\overline{\lim}(E_{t/m}*)^m(x, y) \leqq \overline{\lim}(E_{pt/m}*)^{2^n}(x, y) \leqq \lim_n (E_{tI(n)}*)^{2^n}(x, y)e^{\delta}$$

which hold for any δ. Hence

$$\lim(E_{t/m}*)^m(x, y) = \lim_n (E_{tI(n)}*)^{2^n}(x, y)$$

has been proved. This proves the lemma.

Introduce a one parameter family of operators K_t by

(45)
$$(K_t\phi)(x) = \int_D K(x, y; t)\phi(y)dy.$$

Lemma 11. $\{K_t\}$ *forms a one parameter semi-group of operators.*

Proof. For two positive reals t and s

(46)
$$(K_t(K_s\phi))(x) = \int_D K(x, y; t) \int_D K(y, z; s)\phi(z)dzdy$$

$$= \int_D (\int_D K(x, y; t) K(y, z; s)dy)\phi(z)dz$$

$$= \int_D \lim_n (E_{(t+s)/n}*)^n(x, z)\phi(z)dz$$

$$= (K_{t+s}\phi)(x)$$

which proves the lemma.

Lemma 12. *Suppose that for a C^2 continuous $\phi(x)$,*

$$\lim_{h \to 0}(1/h)((K_h\phi)(x) - \phi(x))$$

exists everywhere. Then the limit is equal to $(\Delta\phi)_x$.

Proof. Take a point x and denote by $S(h)$ the solid sphere of radius $\sqrt[p]{4h}$ around x. Then from (19)

$$\int_D K(x, y; h)\phi(y)dy = \int_{S(h)} K(x, y; h)\phi(y)dy + 0\,(\exp(-h^{-1/2}))$$

(46)
$$= \int_{S(h)} E_h(x, y)\phi(y)dy + 0\,(\exp(-h^{-1/2}))$$

$$= \int_{S(h)} \int_{\varepsilon}^{h} \frac{\partial}{\partial t} E_t(x, y)dt\phi(y)dy + \int_{S(h)} E_\varepsilon(x, y)\phi(y)dy + 0\,(\exp(-h^{-1/2}))$$

$$= \Delta_x \int_{S(h)} \int_{\varepsilon}^{h} E_t(x, y)\phi(y)dy\, dt + \int_{S(h)} E_\varepsilon(x, y)\phi(y)dy + 0\,(\exp(-h^{-1/2})),$$

for some small ε.

Hence

(47)
$$1/h((E_h\phi)(x) - \phi(x)) = \Delta_x \int_{S(h)} (1/h) \int_{\varepsilon}^{h} E_t(x, y)\phi(y)dydt$$

$$+ (1/h)(\int_{S(h)} E_\varepsilon(x, y)\phi(y)dy - \phi(x) + 0\,(h^{-1}\exp(-h^{-1/2})).$$

When ε goes to zero,

(48)
$$(1/h)(K_h\phi)(x) - \phi(x)) = \Delta_x \int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dtdy$$

$$+ 0\,(h^{-1}\exp(-h^{-1/2})).$$

and therefore

(49)
$$\lim_{h\to 0}(1/h)((K_h\phi)(x) - \phi(x)) = \Delta_x \lim_h \int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dtdy$$

which proves that $(\Delta\phi)_x$ exists and is equal to $\frac{\partial}{\partial h}(K_h\phi)(x)$. Here we use the fact that $\int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dtdy$ approaches $\phi(x)$ uniformly together with its second derivatives with the order of $h^{-1}\exp(-h^{-1/2})$.

Lemma 13. *Let $\phi(x)$ be a C^2 function over the closure of D. Then for any x in D and for any t*

(50)
$$\left[\frac{\partial}{\partial h}(K_{t+h}\phi)\right]_{h=0}(x) = \Delta_x(K_t\phi)$$

exists.

Proof. By Lemma 9,

(51)
$$\lim_h (1/h)(K_h\phi - \phi) = \Delta_x\phi.$$

However

(52)
$$K_{t+h}\phi - K_t\phi = K_t(K_h\phi - \phi).$$

Hence

$$\left(\frac{\partial}{\partial h}K_{t+h}\phi\right)(x) = [K_t \lim \frac{1}{h}(K_h\phi - \phi)](x) = (K_t\Delta\phi)(x).$$

By the previous Lemma

$$(K_t \Delta \phi)(x) = (\Delta K_t \phi)(x) = \left(\frac{\partial}{\partial h} K_{t+h} \phi \right)(x)$$

because $\frac{\partial}{\partial h} K_{t+h} \phi = \lim_{h \to 0} (1/h)(K_h K_t \phi - K_t \phi)$ exists. This completes the proof.

Corollary. $\int_D K(x, y ; t)\phi(y)dy$ *is a solution of a differential equation* $\frac{\partial}{\partial t} U = \Delta U,$ *for any continuous function* $\phi(x)$.

Proof. When $\phi(x)$ is C^2, then U is a solution of $\partial U/\partial t = \Delta U$. Now let $\phi_n(x)$'s converge to $\phi(x)$ where all ϕ_n's are C^2. Then the corresponding U_n's converge to a weak solution which, by a theorem by Nirenberg (1), is a genuine solution. This completes the proof.

Lemma 14. $K(x, y ; t)$ *is C^2 both in x and in y.*

Proof. From the previous corollary it is evident that $\int K(x, y; t)\phi(y)dy$ is C^2 in x for any $t > 0$. Hence for an $h < t$

(53) $K(x, y ; t) = \int_D K(x, z ; h) \int_D K(z, y ; t-h)dz$

is C^2 in x. By the construction of $K(x, y ; t)$

(54) $K(x, y ; t) = K(y, x ; t)$.

Therefore $K(x, y ; t)$ is C^2 in y.

Theorem 1. *Suppose that D is a regularly open set with either smooth or rectilinear boundary. Then $K(x, y ; t)$ defined in Lemma 10 is the kernel function of the differential equation*

(55) $\frac{\partial}{\partial t} U = \Delta U$

over D.

Proof. By Lemma 7

(56) $K(x, y ; t) = 0$

if either x or y be on ∂D. From its construction it follows directly that $K(x, y ; t) > 0$ for $x \neq y$. Lemma 14 says $K(x, y ; t)$ is C^2 both in x and in y.

Therefore the result in Lemma 12, i.e.

(57) $$\frac{\partial}{\partial t} \int_D K(x, y; t)\phi(y)dy = \Delta_x \int K(x, y; t)\phi(y)dy$$

implies $\frac{\partial}{\partial t} K(x, y; t-h) = \Delta_x K(x, y; t-h)$, which proves the theorem.

NOTICE. Theorem 1 holds when any point on ∂D is on a boundary of an open convex set D' disjoint with D.

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