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Kernel Functions of Diffusion Equations (I)

By Hidehiko YAMABE

Let D be an open bounded set in a d -dimensional Euclidean space \mathcal{E} .

By Δ we understand the Laplacian with respect to given coordinates. Consider the diffusion equation

$$(1) \quad \frac{\partial U}{\partial t} = \Delta U$$

on D . By a *kernel function* $K(x, y; t)$ we understand a function on $\mathcal{E} \times \mathcal{E} \times [0, \infty)$ satisfying following properties:

(i) $K(x, y; t) = 0$ when either x or y is on the boundary of D , if $K(x, y; t)$ is continuous on boundary for a fixed t , or the boundary ∂D is smooth.

(ii) For a fixed y

$$(2) \quad \frac{\partial}{\partial t} K(x, y; t) = \Delta_x K(x, y; t)$$

where Δ_x is understood as the Laplacian on the variable x .

The purpose of this paper is to give a new way of constructing the kernel function on D which coincides with the Green's function when ∂D , the boundary of D is smooth.

In preparations we shall define some notations. Coordinates of points x, y, \dots on \mathcal{E} will be written $x^i; 1 \leq i \leq d, y^j; 1 \leq j \leq d$, etc. The euclidean distance between two points x and y is denoted by

$$(3) \quad |x - y| = \left(\sum_i^d (x^i - y^i)^2 \right)^{1/2}.$$

Let

$$(4) \quad E_t(x, y) = (2\sqrt{\pi t})^{-d} \exp(- (4t)^{-1} |x - y|^2).$$

Lemma 1. *Take a point x on D . Let $S(h)$ be a solid sphere around*

(1), see (15) and (8)

(2), see (8)

x with its radius $(4h)^{1/6}$ where h is sufficiently small that $S(h)$ is within D . Then

$$(5) \quad 1 \geq \int_{S(h)} E_h(x, y) dy \geq 1 - 2^d \exp\left(-\frac{1}{4}(h)^{-2/3}\right).$$

Proof. Consider polar coordinates $\rho, \theta^i, 1 \leq i \leq d-1$, around x where θ^i 's denote angular coordinates. Then

$$(6) \quad \int_{D-S(h)} E_h(x, y) dy \leq \int_{(4h)^{1/3}}^{\infty} d\rho \int_{\theta} \rho^{d-1} (2\sqrt{\pi h})^{-d} \exp(- (4h)^{-1} \rho^2) d\theta.$$

Notice that for any y outside of $S(h)$

$$(7) \quad |x-y| \geq \frac{1}{2}(4h)^{1/6} + \frac{1}{2}|x-y|$$

and consequently

$$(8) \quad |x-y|^2 \geq \frac{1}{4} \left((4h)^{1/3} + |x-y|^2 \right)$$

Hence

$$(9) \quad \begin{aligned} \int_{D-S(h)} E_h(x, y) dy &\leq (2\sqrt{\pi h})^{-d} \int_{\mathcal{E}-S(h)} \exp\left[-\frac{1}{4}(4h)^{-2/3} - \frac{1}{4}|x-y|^2/4h\right] dy \\ &\leq 2^d \exp\left(-\frac{1}{4}(4h)^{-2/3}\right) (2\sqrt{\pi 4h})^{-d} \int \exp(-|x-y|^2/16h) dy \\ &\leq 2^d \exp\left(-\frac{1}{4}(4h)^{-2/3}\right). \end{aligned}$$

However it is well-known that

$$(10) \quad \int_{S(h)} E_h(x, y) dy \leq 1,$$

for any h .

Hence we have

$$1 \geq \int E_h(x, y) dy \geq 1 - 2^d \exp\left(-\frac{1}{4}(h)^{-2/3}\right),$$

which proves the lemma.

Given two functions ϕ and ψ of two variables, we define a convolution

$$(11) \quad (\phi * \psi)(x, y) = \int_D \phi(x, z) \psi(z, y) dz.$$

Then

Lemma 2.

$$(12) \quad \begin{aligned} (E_t * E_h)(x, y) \\ \leq E_{t+h}(x, y) \left[\left(2\sqrt{\pi} \sqrt{\frac{th}{t+h}} \right)^{-d} \int_D \exp\left(-\frac{t+h}{4th} \left| z - \frac{tx+hy}{t+h} \right|^2\right) dz \right] \end{aligned}$$

Proof. By direct computations

$$\begin{aligned}
 (13) \quad & E_t(x, z)E_h(z, y) \\
 &= (2\sqrt{\pi t})^{-d}(2\sqrt{\pi h})^{-d} \exp\left(-\frac{1}{4t}|x-z|^2-\frac{1}{4h}|z-y|^2\right) \\
 &= (2\sqrt{\pi(t+h)})^{-d}\left(2\sqrt{\pi}\sqrt{\frac{th}{t+h}}\right)^{-d} \exp\left(-|x-y|^2/4(t+h)-\frac{t+h}{4th}\left|z-\frac{tx+hy}{t+h}\right|^2\right).
 \end{aligned}$$

Hence we have (12) by integrating (13) with respect to z over D . By iterating these processes m times for $E_{t/n}(x, y)$ we can define

$$(14) \quad \overbrace{(E_{t/n} * E_{t/n} * \dots * E_{t/n})}^m(x, y) = (E_{t/n} *)^m(x, y).$$

Lemma 3. *Suppose that x and y be on a compact convex set Q contained in D . Then for small t ,*

$$(15) \quad (E_{t/l(n)} *)^{2^n}(x, y) = E_t(x, y) (1 + o(\exp - t^{-1/2}))$$

where $l(n) = 2^{-n}$.

Proof. Let t be so small that around any x in Q , $S(t)$, i.e. the solid sphere with its radius $\sqrt[3]{4t}$ is contained in D .

It is easy to see that

$$\begin{aligned}
 (16) \quad & (E_{t/n} *)^2(x, y) \leq E_{2/tn}(x, y), \\
 & (E_{t/n} *)^3(x, y) \leq (E_{2/tn} * E_{t/n})(x, y), \\
 & \dots,
 \end{aligned}$$

By making convolutions successively

$$(16') \quad (E_{t/n} *)^n(x, y) \leq E_t(x, y).$$

From (16'), by replacing t by t/m we have

$$(E_{t/mn} *)^{mn}(x, y) \leq (E_{t/m} *)^m(x, y).$$

This means that if we introduce a partial order $<$ into positive integers such that $m < n$ means n is divided by m , then $(E_{t/m} *)^m$ is decreasing when m is increasing. Hence there exists the limit in the sense of Moore Smith for integers 2^n because these are linearly ordered.

Now let Q_k be the set of points whose distance from Q is less than

$$\sum_{j=1}^k \sqrt[3]{4tl(j)}$$

where $l(j) = 2^{-j}$.

Notice that the distance between the two sets Q_{k+1} and Q_k is equal to $\sqrt[4]{4tl(k+1)}$. Therefore for x and y in Q_k

$$\begin{aligned}
 (17) \quad & \int_D E_{t l(k)}(x, z) E_{t l(k)}(x, y) dz \\
 & \geq \int_{Q_{k+1}} E_{t l(k)}(x, z) E_{t l(k)}(z, y) dz \\
 & \geq E_{2tl(k)}(x, y) \int_{Q_{k+1}} E_{t l(k)/2}\left(z, \frac{1}{2}(x+y)\right) dz \\
 & = E_{t l(k-1)}(x, y) \left(1 - 2^d \exp\left(-\frac{1}{4}(tl(k+1))\right)^{-2/3}\right).
 \end{aligned}$$

Repeating these processes

$$\begin{aligned}
 (18) \quad & E_{t l(n)} *)^{2^n}(x, y) \\
 & \geq (E_{t l(n-1)} *)^{2^{n-1}}(x, y) \left(1 - 2^d \exp\left(-\frac{1}{4}(tl(n+1))\right)^{-2/3}\right)^{l(n-1)} \\
 & \geq, \dots, E_t(x, y) \sum_{j=1}^n \left(1 - 2^d \exp\left(-\frac{1}{4}(tl(n-j+1))\right)^{-2/3}\right)^{l(j-1)} \\
 & \geq E_t(x, y) \left(1 - \sum_{j=1}^n 2^d 2^{n-j+1} \exp\left(-\frac{1}{4}(tl(n-j+1))\right)^{-2/3}\right) \\
 & \geq E_t(x, y) (1 - 2^d \int_1^\infty \exp\left(-\frac{1}{4}(t^{-2/3} \xi^{2/3})\right) d\xi (1 - 0(t))) \\
 & \geq E_t(x, y) (1 - 0(\exp(-t^{-1/2})))
 \end{aligned}$$

uniformly in n when t is small. This proves the lemma.

REMARK: Since $E_t(x, y)$ is uniformly continuous in t over D and larger than a constant, $1/n$ can be approximated with dyadic number $pl(m)$ from below in such a way that

$$E_{t/n}(x, y) \geq E_{t pl(m)}(x, y) e^{-\delta/n}$$

for a preassigned δ . Then

$$\begin{aligned}
 (E_{t/n} *)^n(x, y) & \geq (E_{t pl(m)} *)^n(x, y) e^{-\delta} \\
 & \geq E_{t pl(m)}(x, y) (1 - 0(\exp(-t^{-1/2}))).
 \end{aligned}$$

New let m go to infinity. Then δ goes to zero and $pl(m)$ to $1/n$. Hence we have

$$(19) \quad (E_{t/n} *)^n(x, y) \geq E_t(x, y) (1 - 0(\exp(-t^{-1/2}))).$$

These (16') and (19) prove the lemma.

Lemma 4. *When both x and y are in Q ,*

$$(20) \quad \lim_{n \rightarrow \infty} (E_{t/n} *)^n(x, y) = K(x, y; t) \text{ exists.}$$

Proof. Set $t = h/m$ in (19). Then

$$\begin{aligned} (E_{hm/n} *)^{mn}(x, y) &\geq (E_{h/m} *)^m(x, y)(1 - 0(\exp(-h/m)^{-1/2}))^m \\ &= (E_{t/m} *)^m(x, y)(1 - 0(1/m)). \end{aligned}$$

Evidently, however

$$(21) \quad (E_{t/mn} *)^{mn}(x, y) \leq (E_{t/m} *)^m(x, y)$$

and $(E_{t/m} *)^m(x, y)$ is uniformly bounded above by $E_t(x, y)$. Hence

$$(22) \quad |(E_{t/mn} *)^{mn}(x, y) - (E_{t/m} *)^m(x, y)| \leq 0(1/m)E_t(x, y).$$

If both m and n be larger than m_0 , then from (23) it follows that

$$(23) \quad \begin{aligned} |(E_{t/m} *)^m(x, y) - (E_{t/n} *)^n(x, y)| &\leq |(E_{t/m} *)^m(x, y) - (E_{t/mn} *)^{mn}(x, y)| \\ &\quad + |(E_{t/mn} *)^{mn}(x, y) - (E_{t/n} *)^n(x, y)| \\ &\leq 0(1/m) + 0(1/n)E_t(x, y) = 0(1/m_0)E_t(x, y). \end{aligned}$$

This proves the existence of the limit (20).

Lemma 5. *The diffusion equation (1) on D has a unique solution up to the initial function under the condition of 0-boundary-value when $t \neq 0$.*

Proof. We have only to prove that if U be a solution of (1) with 0-boundary-value and $U(x; t) = 0$ everywhere when $t = 0$, then $U(x, t) = 0$ for any t and for any x .

Now

$$(24) \quad \frac{d}{dt} \int_D (U(x, t))^2 dx = 2 \int_D \frac{\partial}{\partial t} U(x, t) U(x, t) dx = \int_D (\Delta U) U dx,$$

and by the virtue of Stoke's theorem,

$$= -2 \int_D \sum_{i=1}^a \left(\frac{\partial}{\partial x^i} U(x, t) \right)^2 dx \leq 0.$$

Hence $\int (U(x, 0))^2 dx = 0$ implies $\int (U(x, t))^2 dx = 0$, i.e. $U(x, t) = 0$.

This prove the lemma.

Lemma 6. *Let D' be an open convex domain, and p be a point at a distance less than $ch^{1/2}$ from D' . Then there exists a constant $\varepsilon > 0$*

depending only on c so that when h goes to zero

$$(25) \quad \int_{D'} E_h(p, y) dy \geq \varepsilon$$

holds uniformly in h .

Proof. Consider the transformation of coordinates $x^i \rightarrow \xi^i = (2\sqrt{h})^{-1}(x^i - p^i)$ with its origin p . In ξ 's, D' is mapped to a convex open set $D'_1(h)$ near to origin, and similar to D' . These $D'_1(h)$ approaches a convex open domain D'_1 when h goes to 0.

However, for a small positive ε ,

$$(26) \quad \int_{D'} E_h(p, y) dy = \int_{D'_1(h)} E_1(0, \eta) d\eta \geq \varepsilon.$$

where $\eta^i = (2\sqrt{h})^{-1}(y^i - p^i)$.

An open set O is called regularly open if it coincides with the open kernel of its closure.

Lemma 7. *Suppose that the boundary ∂D of a regularly open D be a rectilinear simplicial complex. If either x or y be on ∂D , then*

$$\lim_n (E_{t/n} *)^n(x, y) = 0.$$

Proof. Let x be on ∂D . It is easy to construct a convex open D' outside of D whose boundary contains x .

Set

$$m = [n/2].$$

Clearly

$$(E_{t/n} *)^m(x, y) \leq E_{mt/n}(x, y).$$

Take the c defined in Lemma 6 sufficiently large that for any h less than t

$$(27) \quad \int_{|y| \geq ct^{1/2}} E_h(0, y) dy \leq \int_{|\eta| \geq c/2} E_1(0, \eta) d\eta \leq \frac{1}{2} \varepsilon.$$

By T_n we denote the set of points $\{z; |x-z| \leq c(t/n)^{1/2}\}$. The for y in T_n

$$(28) \quad \begin{aligned} (E_{t/n} *)^{m+1}(x, y) &\leq \int_D E_{m+1/n}(z, y) dz \\ &= E_{(m+1)t/n}(x, y) \int_D E_{mt/(m+1)n} \left(z, \frac{x+my}{m+1} \right) dz \\ &\leq E_{(m+1)t/n}(x, y) \left(\int_{D \cap T_n} + \int_{\mathcal{E} - T_n} \right) E_{m+1/n} \left(z, \frac{x+my}{m+1} \right) dz \end{aligned}$$

$$\begin{aligned} &\leq E_{(m+1)t/n}(x, y) \left(1 - \varepsilon + \frac{1}{2} \varepsilon\right) \\ &= E_{(m+1)t/n}(x, y) \left(1 - \frac{1}{2} \varepsilon\right). \end{aligned}$$

because of (27) and of the previous lemma.

By iterating these processes, we have

$$(E_{t/n} *)^{m+i}(x, y) \leq E_{(m+i)t/n}(x, y) \left(1 - \frac{1}{2} \varepsilon\right)^i$$

and in particular

$$(29) \quad (E_{t/n} *)^{n-1}(x, y) \leq E_{(n-1)t/n} \left(1 - \frac{1}{2} \varepsilon\right)^{n-m-1}.$$

From this it follows that

$$\begin{aligned} (E_{t/n} *)^n(x, y) &\leq \int_{D \cap T_n} (E_{t/n} *)^{n-1}(x, z) E_{t/n}(z, y) dz \\ &\quad + \int_{D - T_n \cap D} E_{t/n}(x, z) (E_{t/n} *)^{n-1}(z, y) dz \\ &\leq \left(1 - \frac{1}{2} \varepsilon\right)^{n-m-1} \int_{\mathcal{E}} E_{(n-1)t/n}(x, z) E_{t/n}(z, y) dz \\ &\quad + \int_{|\xi| \geq c} E_1(0, \xi) d\xi A \\ &= \left(1 - \frac{1}{2} \varepsilon\right)^{n-m-1} E_t(x, y) + \int_{|\xi| \geq c} E_1(0, \xi) d\xi A \end{aligned}$$

where

$$A = \sup_{z \in D, \frac{1}{2} \leq h \leq t} E_h(z, y).$$

Now let c vary and go to infinity slow enough as n tends to infinity, that ε depending on c by (29) satisfies

$$(30) \quad \lim_n \left(1 - \frac{1}{2} \varepsilon\right)^{n-m} = 0.$$

For instance if we take c 's in such a way that $\varepsilon = \varepsilon(n) = n^{-1/2}$, (30) holds.

Under these modifications we can commute

$$(31) \quad \begin{aligned} \lim_n (E_{t/n} *)^n(x, y) &\leq \lim_n \left(1 - \frac{1}{2} \varepsilon\right)^{n-m} E_t(x, y) \\ &\quad + \lim_n \int_{|\xi| \geq c} E_1(0, \xi) d\xi A = 0 \end{aligned}$$

which prove the lemma.

REMARK. Lemma 7 holds for a regularly open D with a smooth boundary and the same kind of proof works for it. Moreover we have

$$(32) \quad \lim_n (E_{r/n} *)^n(x, y) = 0$$

when either x or y is outside of D .

Lemma 8. *Let $\phi(y)$ be a continuous function over the closure of D , and x be a point on D . Then*

$$(33) \quad \lim_{h \rightarrow 0} \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy = \phi(x).$$

Proof. Let $S(h)$ be the solid sphere of radius $\sqrt[3]{4h}$ around x . Set

$$(34) \quad A(h) = \sup_{y \in S(h)} \phi(y)$$

and

$$B(h) = \inf_{y \in S(h)} \phi(y).$$

Then from (5), considering $S(h)$ as Q ,

$$(35) \quad \begin{aligned} & \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy \\ & \leq A(h) \int_{S(h)} E_h(x, y) dy + \int_{D-S(h)} E_h(x, y) \phi(y) dy \\ & \leq A(h) + 0 \left(2^d \exp \left(-\frac{1}{4} (h)^{-2/3} \right) \right). \end{aligned}$$

However from (15),

$$(36) \quad \begin{aligned} & \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy \\ & \geq B(h) (1 - 0 \left(\exp(-h^{-1/2}) \int_D E_h(x, y) dy \right)) \end{aligned}$$

Since the right hand sides of both (35) and (36) approaches $\phi(x)$ when h goes to zero, we have

$$\lim_{h \rightarrow 0} \int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy = \phi(x).$$

Corollary.

$$(37) \quad \lim_{h \rightarrow 0} \int_D \underline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy = \phi(x).$$

The proof is obtained by changing $\overline{\lim}$ to $\underline{\lim}$ in Lemma 8.

Lemma 9. *Suppose that $\phi(y)$ be a C^2 function over the closure of D . Then,*

$$(38) \quad \lim_{h \rightarrow 0} (1/h) \left(\int_D \overline{\lim} (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) \right) = \Delta \phi$$

Proof.

$$(39) \quad \int_{S(h)} \overline{\lim} (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) = \int_{S(h)} E_h(x, y) \phi(y) dy - \phi(x) + 0 (\exp(-h^{-1/2})) \int_{S(h)} \phi(y) dy.$$

On the other hand by (5)

$$(40) \quad \int_{D-S(h)} \overline{\lim} (E_{h/n} *)^n(x, y) \phi(y) dy \leq \sup_{y \in D} |\phi(y)| 2^d \exp\left(-\frac{1}{4} (4h)^{-2/3}\right).$$

Hence

$$(41) \quad (1/h) \left[\int_D \overline{\lim} (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) \right] = (1/h) \left(\int_{S(h)} E_h(x, y) \phi(y) dy - \phi(x) \right) + 0(h^{-1} \exp(-h^{-1/2})).$$

It is well known that

$$(42) \quad \lim_{h \rightarrow 0} (1/h) \left(\int_{S(h)} E_h(x, y) \phi(y) dy - \phi(x) \right) = (\Delta \phi)_x.$$

Hence we have

$$\lim_{h \rightarrow 0} (1/h) \left[\int_D \overline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) \right] = (\Delta \phi)_x$$

Corollary.

$$(43) \quad \lim_{h \rightarrow 0} (1/h) \left[\int_D \underline{\lim}_n (E_{h/n} *)^n(x, y) \phi(y) dy - \phi(x) \right] = (\Delta \phi)_x.$$

Lemma 10.

$$(44) \quad \overline{\lim}_n (E_{t/n} *)^n(x, y) = \underline{\lim}_n (E_{t/n} *)^n(x, y) \text{ exists.}$$

We denote this by $K(x, y; t)$.

Proof. From (16) it follows that

$$(E_{t/mn} *)^m(x, y) \leq E_{t/n}(x, y)$$

and hence

$$(E_{t/mn} *)^{mn}(x, y) \leq (E_{t/n} *)^n(x, y).$$

Therefore the limit in the sense of Moore-Smith exists for integers 2^n 's when we introduce an partial order in such a way that $m < p$ when m is a divisor of p .

In general, first we approximate $1/m$ with p/n . Then $E_{t/m}(x, y)$ is uniformly approximated in such a way that for a preassigned δ

$$E_{t/m}(x, y) \geq E_{t_{pI(n)}}(x, y)e^{-\delta/m}.$$

Then

$$(E_{t/m} *)^m(x, y) \geq (E_{t_{pI(n)}} *)^m(x, y)e^{-\delta} \geq \lim_j (E_{t_{pI(n+j)}} *)^{m_2^j}(x, y)e^{-\delta}.$$

Since this holds for any δ ,

$$(E_{t/m} *)^m(x, y) \geq \lim_j (E_{t_{I(j)/m}} *)^{2_m^j}(x, y) = \lim_j (E_{t_{I(j)}} *)^{2^j}(x, y).$$

Conversely, when m is large, we have

$$E_{t_{p/m}}(x, y) \leq E_{t_{I(n)}}(x, y)e^{\delta I(n)}$$

with a suitable integer p .

Hence it follows that

$$\overline{\lim} (E_{t/m} *)^m(x, y) \leq \overline{\lim} (E_{pI(m)} *)^{2^n}(x, y) \leq \lim_n (E_{t_{I(n)}} *)^{2^n}(x, y)e^\delta$$

which hold for any δ . Hence

$$\lim (E_{t/m} *)^m(x, y) = \lim_n (E_{t_{I(n)}} *)^{2^n}(x, y)$$

has been proved. This proves the lemma.

Introduce a one parameter family of operators K_t by

$$(45) \quad (K_t \phi)(x) = \int_D K(x, y; t) \phi(y) dy.$$

Lemma 11. $\{K_t\}$ forms a one parameter semi-group of operators.

Proof. For two positive reals t and s

$$(46) \quad \begin{aligned} (K_t(K_s \phi))(x) &= \int_D K(x, y; t) \int_D K(y, z; s) \phi(z) dz dy \\ &= \int_D \left(\int_D K(x, y; t) K(y, z; s) dy \right) \phi(z) dz \\ &= \int_D \lim_n (E_{(t+s)/n} *)^n(x, z) \phi(z) dz \\ &= (K_{t+s} \phi)(x) \end{aligned}$$

which proves the lemma.

Lemma 12. Suppose that for a C^2 continuous $\phi(x)$,

$$\lim_{h \rightarrow 0} (1/h) ((K_h \phi)(x) - \phi(x))$$

exists everywhere. Then the limit is equal to $(\Delta \phi)_x$.

Proof. Take a point x and denote by $S(h)$ the solid sphere of radius $\sqrt[3]{4h}$ around x . Then from (19)

$$\begin{aligned}
 \int_D K(x, y; h)\phi(y)dy &= \int_{S(h)} K(x, y; h)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
 (46) \quad &= \int_{S(h)} E_h(x, y)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
 &= \int_{S(h)} \int_{\varepsilon}^h \frac{\partial}{\partial t} E_t(x, y)dt\phi(y)dy + \int_{S(h)} E_{\varepsilon}(x, y)\phi(y)dy + 0(\exp(-h^{-1/2})) \\
 &= \Delta_x \int_{S(h)} \int_{\varepsilon}^h E_t(x, y)\phi(y)dy dt + \int_{S(h)} E_{\varepsilon}(x, y)\phi(y)dy + 0(\exp(-h^{-1/2})),
 \end{aligned}$$

for some small ε .

Hence

$$\begin{aligned}
 (47) \quad 1/h((E_h\phi)(x) - \phi(x)) &= \Delta_x \int_{S(h)} (1/h) \int_{\varepsilon}^h E_t(x, y)\phi(y)dydt \\
 &\quad + (1/h) \left(\int_{S(h)} E_{\varepsilon}(x, y)\phi(y)dy - \phi(x) + 0(h^{-1} \exp(-h^{-1/2})) \right).
 \end{aligned}$$

When ε goes to zero,

$$\begin{aligned}
 (48) \quad (1/h)(K_h\phi)(x) - \phi(x) &= \Delta_x \int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dt dy \\
 &\quad + 0(h^{-1} \exp(-h^{-1/2})).
 \end{aligned}$$

and therefore

$$(49) \quad \lim_{h \rightarrow 0} (1/h)((K_h\phi)(x) - \phi(x)) = \Delta_x \lim_h \int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dt dy$$

which proves that $(\Delta\phi)_x$ exists and is equal to $\frac{\partial}{\partial h}(K_h\phi)(x)$. Here we use the fact that $\int_{S(h)} (1/h) \int_0^h E_t(x, y)\phi(y)dt dy$ approaches $\phi(x)$ uniformly together with its second derivatives with the order of $h^{-1} \exp(-h^{-1/2})$.

Lemma 13. *Let $\phi(x)$ be a C^2 function over the closure of D . Then for any x in D and for any t*

$$(50) \quad \left[\frac{\partial}{\partial h} (K_{t+h}\phi) \right]_{h=0} (x) = \Delta_x (K_t\phi)$$

exists.

Proof. By Lemma 9,

$$(51) \quad \lim_h (1/h)(K_h\phi - \phi) = \Delta_x \phi.$$

However

$$(52) \quad K_{t+h}\phi - K_t\phi = K_t(K_h\phi - \phi).$$

Hence

$$\left(\frac{\partial}{\partial h} K_{t+h}\phi \right) (x) = [K_t \lim \frac{1}{h} (K_h\phi - \phi)](x) = (K_t \Delta\phi)(x).$$

By the previous Lemma

$$(K_t \Delta \phi)(x) = (\Delta K_t \phi)(x) = \left(\frac{\partial}{\partial h} K_{t+h} \phi \right)(x)$$

because $\frac{\partial}{\partial h} K_{t+h} \phi = \lim_{h \rightarrow 0} (1/h)(K_{t+h} \phi - K_t \phi)$ exists. This completes the proof.

Corollary. $\int_D K(x, y; t) \phi(y) dy$ is a solution of a differential equation $\frac{\partial}{\partial t} U = \Delta U$, for any continuous function $\phi(x)$.

Proof. When $\phi(x)$ is C^2 , then U is a solution of $\partial U / \partial t = \Delta U$. Now let $\phi_n(x)$'s converge to $\phi(x)$ where all ϕ_n 's are C^2 . Then the corresponding U_n 's converge to a weak solution which, by a theorem by Nirenberg (1), is a genuine solution. This completes the proof.

Lemma 14. $K(x, y; t)$ is C^2 both in x and in y .

Proof. From the previous corollary it is evident that $\int K(x, y; t) \phi(y) dy$ is C^2 in x for any $t > 0$. Hence for an $h < t$

$$(53) \quad K(x, y; t) = \int_D K(x, z; h) \int_D K(z, y; t-h) dz$$

is C^2 in x . By the construction of $K(x, y; t)$

$$(54) \quad K(x, y; t) = K(y, x; t).$$

Therefore $K(x, y; t)$ is C^2 in y .

Theorem 1. Suppose that D is a regularly open set with either smooth or rectilinear boundary. Then $K(x, y; t)$ defined in Lemma 10 is the kernel function of the differential equation

$$(55) \quad \frac{\partial}{\partial t} U = \Delta U$$

over D .

Proof. By Lemma 7

$$(56) \quad K(x, y; t) = 0$$

if either x or y be on ∂D . From its construction it follows directly that $K(x, y; t) > 0$ for $x \neq y$. Lemma 14 says $K(x, y; t)$ is C^2 both in x and in y .

Therefore the result in Lemma 12, i.e.

$$(57) \quad \frac{\partial}{\partial t} \int_D K(x, y; t) \phi(y) dy = \Delta_x \int K(x, y; t) \phi(y) dy$$

implies $\frac{\partial}{\partial t} K(x, y; t-h) = \Delta_x K(x, y; t-h)$, which proves the theorem.

NOTICE. Theorem 1 holds when any point on ∂D is on a boundary of an open convex set D' disjoint with D .

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