



Title	Equivariant isotopies of semifree G-manifolds
Author(s)	Komiya, Katsuhiro
Citation	Osaka Journal of Mathematics. 1983, 20(2), p. 383-397
Version Type	VoR
URL	https://doi.org/10.18910/6543
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

EQUIVARIANT ISOTOPIES OF SEMIFREE G -MANIFOLDS

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

KATSUHIRO KOMIYA

(Received June 18, 1981)

1. Introduction

In the previous paper [3] we studied the set of equivariant isotopy classes of equivariant smooth embeddings of a sphere with semifree linear action into a euclidean representation space. In this paper we will study more general case, i.e., the set of equivariant isotopy classes of equivariant smooth embeddings of a manifold into another manifold, where the manifolds in question have a smooth semifree action.

Let G be a compact Lie group, and M, N smooth G -manifolds. Two smooth G -embeddings f and g of M into N are called G -isotopic, if there is a smooth G -map

$$H: M \times [0, 1] \rightarrow N$$

such that, for any $t \in [0, 1]$, $H_t = H|_{M \times \{t\}}$ is a smooth G -embedding, and that $H_0 = f$, $H_1 = g$. Such H is called a *smooth G -isotopy* between f and g . The *G -isotopy class* $[f]$ is the set of all smooth G -embeddings G -isotopic to f . Denote by $\text{Iso}^G(M, N)$ the set of all G -isotopy classes of smooth G -embeddings of M into N . Fix a smooth G -embedding f of M into N , and denote by $\text{Iso}_f^G(M, N)$ the set of all G -isotopy classes of smooth G -embeddings G -homotopic to f . If N is a euclidean representation space of G , then N is G -contractible, and then

$$\text{Iso}_f^G(M, N) = \text{Iso}^G(M, N)$$

for any smooth G -embedding f of M into N .

For $x \in M$ denote by G_x the isotropy subgroup of G at x . An action of G on M is called *semifree* if, for any $x \in M$, G_x is either trivial or is all of G . If, moreover, the fixed point set

$$M^G = \{x \in M \mid G_x = G\}$$

is neither empty nor is all of M , the action is called *properly semifree*. For $x \in M^G$ denote by M_x^G the connected component of M^G containing x . Choose

a point from each connected component of M^G , and let $C(M^G)$ be the set of these points. Then M^G is the disjoint union of M_x^G for all $x \in C(M^G)$.

Let M, N be smooth properly semifree G -manifolds, and f a smooth G -embedding of M into N . This paper will proceed as follows. In section 2 we define $\Gamma_f(M_x^G)$ as the set of homotopy classes of cross sections of a fibre bundle over M_x^G , and give a definition of a transformation

$$\Phi: \text{Iso}_f^G(M, N) \rightarrow \prod_{x \in C(M^G)} \Gamma_f(M_x^G).$$

Under dimensional conditions we prove the surjectivity of Φ in section 3, and prove the injectivity of Φ in section 4. Finally in section 5 we analyze $\Gamma_f(M_x^G)$ by using obstruction theory.

REMARK. If the G -action on M is properly semifree, a normal representation of G at a fixed point has no fixed point except the origin. Any compact Lie group G does not always admit a fixed point free (outside the origin) representation. Finite groups which admit fixed point free representations are classified by Wolf [5]. If G is positive dimensional, then there are only three possibilities: $G \cong S^3$, S^1 , and its normalizer $N(S^1)$ in S^3 (e.g. as shown in Bredon [2; 8.5]). Thus the groups considered in this paper are finite groups, S^1 , $N(S^1)$, and S^3 .

2. Transformation Φ

Let M, N be smooth properly semifree G -manifolds, and f a smooth G -embedding of M into N . Choose once and for all a set $C(M^G)$ such that M^G is the disjoint union of M_x^G for all $x \in C(M^G)$. For any $x \in C(M^G)$, let

$$\nu(M_x^G) = (\tau(M)|M_x^G)/\tau(M_x^G)$$

be the normal bundle of M_x^G in M . Denote by $\nu_y(M_x^G)$ the fibre over $y \in M_x^G$. This is a representation of G which has no fixed point outside the origin. Denote by

$$\text{Mon}^G(\nu_y(M_x^G), \nu_{f(y)}(N_{f(x)}^G))$$

the set of all G -monomorphisms from $\nu_y(M_x^G)$ to $\nu_{f(y)}(N_{f(x)}^G)$, and define

$$\text{Mon}_f^G(\nu(M_x^G), \nu(N_{f(x)}^G)) = \bigcup_{y \in M_x^G} \text{Mon}^G(\nu_y(M_x^G), \nu_{f(y)}(N_{f(x)}^G)).$$

By the standard manner this becomes a smooth fibre bundle over M_x^G . The set of continuous (resp. smooth) cross sections of this bundle is in bijective correspondence with the set of continuous (resp. smooth) G -vector bundle monomorphisms from $\nu(M_x^G)$ to $\nu(N_{f(x)}^G)$ which cover

$$f_x^G = f|_{M_x^G}: M_x^G \rightarrow N_{f(x)}^G.$$

Denote by $\Gamma_f(M_x^G)$ the set of homotopy classes of continuous cross sections of $\text{Mon}_f^G(\nu(M_x^G), \nu(N_{f(x)}^G))$. Note that we may take smooth ones as representatives of classes in $\Gamma_f(M_x^G)$ by the differentiable approximation theorem [4; 6.7].

Let $g: M \rightarrow N$ be a smooth G -embedding G -homotopic to f . Note that $N_{g(x)}^G = N_{f(x)}^G$ for any $x \in C(M^G)$. Then two maps

$$g_x^G, f_x^G: M_x^G \rightarrow N_{f(x)}^G$$

are homotopic, i.e., there is a homotopy

$$H: M_x^G \times [0, 1] \rightarrow N_{f(x)}^G$$

with $H_0 = g_x^G$ and $H_1 = f_x^G$. By Bierstone [1] we may lift H to a G -homotopy of G -vector bundle monomorphism

$$\tilde{H}: \nu(M_x^G) \times [0, 1] \rightarrow \nu(N_{f(x)}^G)$$

with

$$\tilde{H}_0 = \tilde{d}_x g: \nu(M_x^G) \rightarrow \nu(N_{f(x)}^G),$$

where $\tilde{d}_x g$ is the G -vector bundle monomorphism induced from the differential $dg: \tau(M) \rightarrow \tau(N)$ of g . Then \tilde{H}_1 is a G -vector bundle monomorphism which covers f_x^G . Let

$$\Phi_x(g): M_x^G \rightarrow \text{Mon}_f^G(\nu(M_x^G), \nu(N_{f(x)}^G))$$

be a cross section corresponding to \tilde{H}_1 . $\Phi_x(g)$ is determined dependently on H and its lifting \tilde{H} . But, if $N_{f(x)}^G$ is $(\dim M_x^G + 1)$ -connected, the homotopy class of $\Phi_x(g)$ does not depend on H and \tilde{H} . More precisely we show

Lemma 1. *Let $g, h: M \rightarrow N$ be smooth G -embeddings G -homotopic to f . If g and h are G -isotopic, and if $N_{f(x)}^G$ is $(\dim M_x^G + 1)$ -connected, then $\Phi_x(g)$ and $\Phi_x(h)$ are homotopic as cross section.*

Proof. Let

$$\tilde{H}^{(i)}: \nu(M_x^G) \times [0, 1] \rightarrow \nu(N_{f(x)}^G), \quad i = 0, 1,$$

be G -homotopies of G -vector bundle monomorphism which cover G -homotopies

$$H^{(i)}: M_x^G \times [0, 1] \rightarrow N_{f(x)}^G, \quad i = 0, 1,$$

such that

- (1) $H_0^{(0)} = f, H_1^{(0)} = g, H_0^{(1)} = h, H_1^{(1)} = f,$
- (2) $\tilde{H}_0^{(0)} = \tilde{d}_x g, \tilde{H}_1^{(0)} = \tilde{d}_x h,$
- (3) $\tilde{H}_0^{(0)}$ and $\tilde{H}_1^{(1)}$ correspond to $\Phi_x(g)$ and $\Phi_x(h)$, respectively.

Let $K: M \times [0, 1] \rightarrow N$ be a smooth G -isotopy with $K_0 = g$ and $K_1 = h$. Since $N_{f(x)}^G$ is $(\dim M_x^G + 1)$ -connected, there is a homotopy

$$E: M_x^G \times [0, 3] \times [0, 1] \rightarrow N_{f(x)}^G$$

such that, for any $(y, t, s) \in M_x^G \times (\{0, 3\} \times [0, 1] \cup [0, 3] \times \{1\})$,

$$E(y, t, s) = f(y),$$

and for any $(y, t, 0) \in M_x^G \times [0, 3] \times \{0\}$,

$$E(y, t, 0) = \begin{cases} H^{(0)}(y, t) & \text{if } 0 \leq t \leq 1 \\ K(y, t-1) & \text{if } 1 \leq t \leq 2 \\ H^{(1)}(y, t-2) & \text{if } 2 \leq t \leq 3. \end{cases}$$

Define

$$k: \nu(M_x^G) \times [0, 3] \rightarrow \nu(N_{f(x)}^G)$$

as, for any $(v, t) \in \nu(M_x^G) \times [0, 3]$,

$$k(v, t) = \begin{cases} \tilde{H}^{(0)}(v, t) & \text{if } 0 \leq t \leq 1 \\ \tilde{d}_x K(v, t-1) & \text{if } 1 \leq t \leq 2 \\ \tilde{H}^{(1)}(v, t-2) & \text{if } 2 \leq t \leq 3. \end{cases}$$

Then k is a G -vector bundle monomorphism, and covers $E|_{M_x^G \times [0, 3] \times \{0\}}$. By Bierstone [1] we obtain a G -homotopy of G -vector bundle monomorphism

$$\tilde{E}: \nu(M_x^G) \times [0, 3] \times [0, 1] \rightarrow \nu(N_{f(x)}^G)$$

such that $\tilde{E}_0 = k$ and that \tilde{E} covers E . Then

$$\tilde{E}|_{\nu(M_x^G) \times (\{0, 3\} \times [0, 1] \cup [0, 3] \times \{1\})}$$

covers f_x^G on each level M_x^G , and

$$\begin{aligned} \tilde{E}|_{\nu(M_x^G) \times \{0\} \times \{0\}} &= \tilde{H}_0^{(0)}, \\ \tilde{E}|_{\nu(M_x^G) \times \{3\} \times \{0\}} &= \tilde{H}_1^{(1)}. \end{aligned}$$

Thus we see that $\Phi_x(g)$ and $\Phi_x(h)$ are homotopic as cross section. Q.E.D.

If $N_{f(x)}^G$ is $(\dim M_x^G + 1)$ -connected for all $x \in C(M^G)$, then, by Lemma 1, we may define a transformation

$$\Phi: \text{Iso}_f^G(M, N) \rightarrow \prod_{x \in C(M^G)} \Gamma_f(M_x^G)$$

as

$$\Phi([g]) = \prod_{x \in C(M^G)} [\Phi_x(g)]$$

for any $[g] \in \text{Iso}_f^G(M, N)$. If N is a euclidean representation space of G , then N^G is contractible and Φ is always defined.

Define

$$\dim N^G = \max \{ \dim N_x^G \mid x \in C(N^G) \} .$$

We obtain

Theorem 2. *Let M, N be smooth properly semifree G -manifolds without boundary, M compact, and f a smooth G -embedding of M into N . Assume that $N_{f(x)}^G$ is $(\dim M_x^G + 1)$ -connected for any $x \in C(M^G)$. Then the transformation*

$$\Phi: \text{Iso}_f^G(M, N) \rightarrow \prod_{x \in C(M^G)} \Gamma_f(M_x^G)$$

satisfies that

(a) if

$$\dim M + \max \{ \dim M, \dim N^G \} < \dim N + \dim G ,$$

then Φ is surjective,

(b) if

$$2 \dim M_x^G + 1 < \dim N_{f(x)}^G \quad \text{for any } x \in C(M^G) ,$$

and if

$$\dim M + \max \{ \dim M, \dim N^G \} + 1 < \dim N + \dim G ,$$

then Φ is bijective.

The surjectivity of Φ will be proven in the next section 3, and the injectivity of Φ in section 4.

3. Surjectivity of Φ

First we provide a lemma for the proof of surjectivity of Φ .

Lemma 3. *Let $\alpha: X \rightarrow Y$ be a map. Let $\xi \rightarrow X$ and $\zeta \rightarrow Y$ be a - and b -dimensional G -sphere bundles over X and Y , respectively. Here G acts trivially on both X and Y , and freely on ξ . Assume that X is a finite connected complex, and that A is a subcomplex of X . Let $\varphi: \xi|_A \rightarrow \zeta$ be a fibre preserving G -map which covers $\alpha|_A$. If*

$$\dim X + a \leq b + \dim G ,$$

then φ is extended to a fibre preserving G -map from ξ to ζ which covers α .

Proof. Denote by $\text{Map}^G(\xi_x, \zeta_{\alpha(x)})$ the set of G -maps from the fibre ξ_x of ξ over $x \in X$ to the fibre $\zeta_{\alpha(x)}$ of ζ over $\alpha(x) \in Y$. Give the compact-open topology to the set. Define

$$\text{Map}_a^G(\xi, \zeta) = \bigcup_{x \in X} \text{Map}^G(\xi_x, \zeta_{\alpha(x)}).$$

By the standard manner this becomes a fibre bundle over X with fibre $\text{Map}^G(\xi_x, \zeta_{\alpha(x)})$. The set of cross sections of $\text{Map}_a^G(\xi, \zeta) \rightarrow X$ is in bijective correspondence with the set of fibre preserving G -maps from ξ to ζ which cover α . Let

$$s(\varphi): A \rightarrow \text{Map}_a^G(\xi, \zeta) | A$$

be the cross section corresponding to φ . To prove the lemma we extend $s(\varphi)$ over X . For this it suffices to see that the fibre $\text{Map}^G(\xi_x, \zeta_{\alpha(x)})$ is $(\dim X - 1)$ -connected. For any i with $0 \leq i \leq \dim X - 1$, let D^{i+1} be the canonical $(i+1)$ -dimensional disc with trivial G -action, S^i its boundary, and

$$\beta: S^i \rightarrow \text{Map}^G(\xi_x, \zeta_{\alpha(x)})$$

any map. We should like to extend β over D^{i+1} . By the exponential law β gives a G -map

$$\tilde{\beta}: S^i \times \xi_x \rightarrow \zeta_{\alpha(x)}.$$

From the hypothesis,

$$\dim D^{i+1} \times \xi_x / G \leq b$$

and $\zeta_{\alpha(x)}$ is $(b-1)$ -connected. Then, as in the proof of Lemma 5 in [3], we may extend $\tilde{\beta}$ to a G -map on $D^{i+1} \times \xi_x$. Thus we may also extend β over D^{i+1} . Q.E.D.

From Lemma 3 we obtain

Corollary 4. *Let $\xi \rightarrow X$ and $\zeta \rightarrow Y$ be a - and b -dimensional G -vector bundles over X and Y , respectively. Here G acts trivially on both X and Y , and freely on both ξ and ζ outside the zero sections. Assume X is a finite complex. Let*

$$\varphi, \psi: \xi \rightarrow \zeta$$

be G -vector bundle monomorphisms which cover a map $\alpha: X \rightarrow Y$. If

$$\dim X + a < b + \dim G,$$

then there exists a fibre preserving G -homotopy

$$H: \xi \times [0, 1] \rightarrow \zeta$$

such that

- (1) $H_0 = \varphi, H_1 = \psi$,
- (2) H_t covers α for any $t \in [0, 1]$, (H_t is not necessarily linear on fibres of ξ .)

(3) $H((\xi - X) \times [0, 1]) \subset \zeta - Y$, where X and Y are regarded as the zero sections of ξ and ζ , respectively.

Proof. Let $S(\xi)$ and $S(\zeta)$ be associated G -sphere bundles of ξ and ζ , respectively. Since φ and ψ are monic on each fibre of ξ ,

$$\begin{aligned}\varphi(S(\xi)) &\subset \zeta - Y, \quad \text{and} \\ \psi(S(\xi)) &\subset \zeta - Y.\end{aligned}$$

Let $r: \zeta - Y \rightarrow S(\zeta)$ be the radial retraction. Apply Lemma 3 to

$$r \circ \varphi \cup r \circ \psi: S(\xi) \times \{0, 1\} \rightarrow S(\zeta). \quad \text{Q.E.D.}$$

We now begin the proof of surjectivity of Φ under the assumption (a) of Theorem 2. Let

$$\alpha = \prod_{x \in \mathcal{O}(M^G)} [s_x] \in \prod_{x \in \mathcal{O}(M^G)} \Gamma_f(M_x^G)$$

be any element. We will construct a smooth G -embedding g of M into N with $\Phi([g]) = \alpha$. Let

$$t_x: \nu(M_x^G) \rightarrow \nu(N_{f(x)}^G)$$

be a G -vector bundle monomorphism covering f_x^G which corresponds to s_x . Without loss of generality we may assume t_x is smooth. From the assumption (a) and Corollary 4 we obtain a fibre preserving G -homotopy

$$H^{(1)}: \nu(M_x^G) \times [0, 1] \rightarrow \nu(N_{f(x)}^G)$$

such that

- (1) $H_0^{(1)} = \tilde{d}_x f$, $H_1^{(1)} = t_x$,
- (2) $H_t^{(1)}$ covers f_x^G for any $t \in [0, 1]$,
- (3) $H^{(1)}((\nu(M_x^G) - M_x^G) \times [0, 1]) \subset \nu(N_{f(x)}^G) - N_{f(x)}^G$.

Define

$$t = \bigcup_{x \in \mathcal{O}(M^G)} t_x: \nu(M^G) = \bigcup_{x \in \mathcal{O}(M^G)} \nu(M_x^G) \rightarrow \nu(N^G).$$

Making use of exponential maps as in the proof of Lemma 6 of [3], from t we obtain a G -homotopy

$$H^{(2)}: T_{3\varepsilon}(M^G) \times [0, 1] \rightarrow N$$

such that

- (1) $H_0^{(2)} = f|T_{3\varepsilon}(M^G)$,
- (2) $H_1^{(2)}$ is a smooth G -embedding with $\tilde{d}H_1^{(2)} = t$,
- (3) $H^{(2)}((T_{3\varepsilon}(M^G) - M^G) \times [0, 1]) \subset N - N^G$, where $T_{3\varepsilon}(M^G)$ is a G -equivariant closed tubular neighborhood of M^G in M with radius $3\varepsilon > 0$. Using $H^{(2)}$

and f , we may construct a smooth G -map

$$g^{(1)}: M \rightarrow N$$

such that

- (1) $g^{(1)}$ and f are G -homotopic,
- (2) for some $\delta, \gamma > 0$ with $\gamma < 3\varepsilon$

$$(g^{(1)})^{-1}(T_\delta(N^G)) \subset \text{Int } T_\gamma(M^G),$$

- (3) $g^{(1)} = H_1^{(2)}$ on $T_\gamma(M^G)$, hence $g^{(1)} = f$ on M^G .

In fact, $g^{(1)}$ can be constructed as follows. First define a G -map

$$h: M \rightarrow N$$

as the followings:

$$h(x) = H_1^{(2)}(x) \quad \text{for } x \in T_\varepsilon(M^G),$$

$$h(x) = H^{(2)}\left(\frac{\varepsilon x}{\|x\|}, 2 - \frac{\|x\|}{\varepsilon}\right) \quad \text{for } x \in T_{2\varepsilon}(M^G) - \text{Int } T_\varepsilon(M^G), \text{ where } \|x\|$$

denotes the length of x in $T_{3\varepsilon}(M^G)$,

$$h(x) = f\left(\left(2 - \frac{3\varepsilon}{\|x\|}\right)x\right) \quad \text{for } x \in T_{3\varepsilon}(M^G) - \text{Int } T_{2\varepsilon}(M^G), \text{ and}$$

$$h(x) = f(x) \quad \text{for } x \in M - \text{Int } T_{3\varepsilon}(M^G).$$

Next, smooth h to obtain the desired $g^{(1)}$.

Define

$$K = M - \text{Int } (g^{(1)})^{-1}(T_\delta(N^G)), \quad \text{and}$$

$$L = N - \text{Int } T_\delta(N^G).$$

These are smooth free G -manifolds with boundary. Since $g^{(1)}(K) \subset L$, we obtain a smooth G -map

$$g^{(1)}|K: K \rightarrow L.$$

Passing to orbit spaces, we also obtain a smooth map

$$g^{(2)} = (g^{(1)}|K)/G: K/G \rightarrow L/G,$$

which is an embedding on a neighborhood of $\partial K/G$ in K/G . From the assumption (a),

$$2 \dim K/G < \dim L/G.$$

Thus $g^{(2)}$ is homotoped to a smooth embedding, precisely there is a smooth homotopy

$$H^{(3)}: K/G \times [0, 1] \rightarrow L/G$$

such that

- (1) $H_0^{(3)} = g^{(2)}$,
- (2) $H_1^{(3)}$ is a smooth embedding, and
- (3) $H^{(3)}$ is a constant homotopy on a neighborhood of $\partial K/G$.

Since the natural projections $K \rightarrow K/G$ and $L \rightarrow L/G$ are smooth G -fibre bundles, then by Bierstone [1] we obtain a smooth G -homotopy

$$H^{(4)}: K \times [0, 1] \rightarrow L$$

such that $H_0^{(4)} = g^{(1)}|_K$, and that $H_1^{(4)}$ is a smooth G -embedding. Moreover, we can choose $H^{(4)}$ so that it is a constant homotopy on a neighborhood of ∂K in K , hence that $H_1^{(4)} = g^{(1)}$ on the neighborhood. Then, from $g^{(1)}$ and $H_1^{(4)}$, we obtain a smooth G -embedding

$$g^{(3)}: M \rightarrow N$$

such that

- (1) $g^{(3)}$ is G -homotopic to f , and
- (2) $g^{(3)} = g^{(1)} = H_1^{(2)}$ on a neighborhood of M^G in M .

Thus

$$\tilde{d}g^{(3)} = \tilde{d}H_1^{(2)} = t: \nu(M^G) \rightarrow \nu(N^G),$$

and

$$\Phi([g^{(3)}]) = \prod_{x \in \mathcal{O}(M^G)} [s_x].$$

This completes the proof for the surjectivity of Φ under the assumption (a) of Theorem 2.

4. Injectivity of Φ

In this section we will show the injectivity of Φ under the assumption (b) of Theorem 2. Let

$$\Phi([g]) = \Phi([h]) \quad \text{in} \quad \prod_{x \in \mathcal{O}(M^G)} \Gamma_f(M_x^G)$$

for $[g], [h] \in \text{Iso}_f^G(M, N)$. We will construct a smooth G -isotopy between g and h .

First, since g and h are G -homotopic, there is a G -homotopy

$$H^{(1)}: M \times [0, 1] \rightarrow N$$

with $H_0^{(1)} = g$ and $H_1^{(1)} = h$. By the assumption

$$2 \dim M_x^G + 1 < \dim N_{f(x)}^G \quad \text{for all } x \in C(M^G),$$

we see

$$f^G, g^G, h^G: M^G \rightarrow N^G$$

are isotopic each other. From this and $\Phi([g]) = \Phi([h])$ we obtain a smooth G -homotopy of G -vector bundle monomorphism

$$H^{(2)}: \nu(M^G) \times [0, 1] \rightarrow \nu(N^G)$$

such that

- (1) $H_0^{(2)} = \tilde{d}g, H_1^{(2)} = \tilde{d}h$, and
- (2) $H^{(2)}$ covers a smooth isotopy: $M^G \times [0, 1] \rightarrow N^G$.

Making use of exponential maps as in the proof of Lemma 6 of [3], from $H^{(2)}$ we obtain, for an appropriate $\varepsilon > 0$, a smooth G -isotopy

$$H^{(3)}: T_{4\varepsilon}(M^G) \times [0, 1] \rightarrow N$$

with $H_0^{(3)} = g|T_{4\varepsilon}(M^G)$ and with $H_1^{(3)} = h|T_{4\varepsilon}(M^G)$. Since $N_{f(x)}^G$ is $(\dim M_x^G + 1)$ -connected for any $x \in C(M^G)$, we may obtain a homotopy

$$H^{(4)}: (M^G \times [0, 1]) \times [0, 1] \rightarrow N^G$$

such that

- (1) $H_0^{(4)} = H^{(3)}|M^G \times [0, 1]$,
- (2) $H_1^{(4)} = H^{(1)}|M^G \times [0, 1]$,
- (3) $H_t^{(4)}|M^G \times \{0\} = g^G$ for any $t \in [0, 1]$, and
- (4) $H_t^{(4)}|M^G \times \{1\} = h^G$ for any $t \in [0, 1]$.

Define a G -homotopy

$$H^{(5)}: M \times [0, 1] \rightarrow N$$

as follows: for any $(x, t) \in M \times [0, 1]$,

$$H^{(5)}(x, t) = H^{(3)}(x, t) \quad \text{if } x \in T_\varepsilon(M^G),$$

$$H^{(5)}(x, t) = H^{(3)}\left(\left(\frac{2\varepsilon}{\|x\|} - 1\right)x, t\right) \quad \text{if } x \in T_{2\varepsilon}(M^G) - \text{Int } T_\varepsilon(M^G),$$

$$H^{(5)}(x, t) = H^{(4)}\left(\pi(x), t, \frac{\|x\|}{\varepsilon} - 2\right) \quad \text{if } x \in T_{3\varepsilon}(M^G) - \text{Int } T_{2\varepsilon}(M^G),$$

where $\pi: T_{3\varepsilon}(M^G) \rightarrow M^G$ is the canonical projection,

$$H^{(5)}(x, t) = H^{(1)}\left(4\left(1 - \frac{3\varepsilon}{\|x\|}\right)x, t\right) \quad \text{if } x \in T_{4\varepsilon}(M^G) - \text{Int } T_{3\varepsilon}(M^G),$$

$$H^{(5)}(x, t) = H^{(1)}(x, t) \quad \text{if } x \in M - \text{Int } T_{4\varepsilon}(M^G).$$

Then $H_0^{(5)}$ and g are G -homotopic, and its homotopy can be so chosen as to be constant on $T_\varepsilon(M^G)$. Similarly for $H_1^{(5)}$ and h . From these homotopies we obtain a G -homotopy

$$H^{(6)}: M \times [0, 1] \rightarrow N$$

such that $H_0^{(6)} = g$, $H_1^{(6)} = h$, and that $H^{(6)}$ is a smooth G -isotopy on $T_*(M^G)$.

Define

$$L = (M - \text{Int } T_*(M^G)) \times [0, 1].$$

Note the G -action on L is free. Let G act diagonally on $L \times N$. Passing a G -map

$$id \times H^{(6)}: L \rightarrow L \times N$$

to orbit spaces, we obtain a map

$$\alpha^{(1)} = id \times H^{(6)}/G: L/G \rightarrow (L \times N)/G.$$

Consider a submanifold

$$(L \times N^G)/G = L/G \times N^G$$

of $(L \times N)/G$. Then

$$\alpha^{(1)}(\partial L/G) \cap L/G \times N^G = \emptyset.$$

From the assumption (b),

$$\dim L/G < \dim (L \times N)/G - \dim L/G \times N^G.$$

Thus $\alpha^{(1)}$ can be so homotoped that its image does not intersect $L/G \times N^G$, i.e., there is a map

$$\alpha^{(2)}: L/G \rightarrow (L \times N)/G$$

which is homotopic to $\alpha^{(1)}$ relative to $\partial L/G$, and whose image does not intersect $L/G \times N^G$. From this we obtain a G -map

$$\alpha^{(3)}: L \rightarrow N$$

which is G -homotopic to $H^{(6)}|_L$ relative to ∂L , and whose image does not intersect N^G . Define

$$H^{(7)}: M \times [0, 1] \rightarrow N$$

as

$$\begin{aligned} H^{(7)} &= H^{(6)} && \text{on } T_*(M^G) \times [0, 1], \text{ and} \\ H^{(7)} &= \alpha^{(3)} && \text{on } L. \end{aligned}$$

Then $H^{(7)}$ is a G -homotopy between g and h , and a smooth G -isotopy particularly on $T_*(M^G)$. We see

$$M^G \times [0, 1] = (H^{(7)})^{-1}(N^G).$$

At this point it only remains to deform $H^{(7)}$ outside a neighborhood of M^G to a smooth G -isotopy. It can be done similarly to the proof in [3]. So we will merely give an outline. Since $M \times [0, 1]$ is compact, for small $\delta > 0$,

$$\text{Int } T_{\varepsilon/2}(M^G) \times [0, 1] \supset (H^{(7)})^{-1}(T_\delta(N^G)).$$

Let η be a level preserving G -diffeomorphism of $M \times [0, 1]$ such that

$$\begin{aligned} \eta(T_\varepsilon(M^G) \times [0, 1]) &= T_\varepsilon(M^G) \times [0, 1], \text{ and} \\ \eta(T_{\varepsilon/2}(M^G) \times [0, 1]) &= (H^{(7)})^{-1}(T_\delta(N^G)). \end{aligned}$$

Define

$$\begin{aligned} P &= M - \text{Int } T_{\varepsilon/2}(M^G), \text{ and} \\ Q &= N - \text{Int } T_\delta(N^G). \end{aligned}$$

Consider a G -homotopy

$$H^{(7)} \circ \eta: P \times [0, 1] \rightarrow Q,$$

which is a smooth G -isotopy on a neighborhood of ∂P . From the assumption (b),

$$2 \dim P + 1 < \dim Q + \dim G.$$

Then $H^{(7)} \circ \eta$ may be deformed to a smooth G -isotopy

$$H^{(8)}: P \times [0, 1] \rightarrow Q$$

such that

- (1) $H_0^{(8)} = g \circ \eta_0|_P$,
- (2) $H_1^{(8)} = h \circ \eta_1|_P$,
- (3) $H^{(8)} = H^{(7)} \circ \eta$ on (n.b.d of ∂P) $\times [0, 1]$.

From $H^{(7)}$ and $H^{(8)}$ we obtain a smooth G -isotopy between g and h . This completes the proof for the injectivity of Φ under the assumption (b) of Theorem 2.

5. Analysis of $\Gamma_r(M_x^G)$

In this section we will analyze $\Gamma_r(M_x^G)$.

Let $\{V_j | j \in J(G)\}$ be a complete set of fixed point free (outside the origin), nonisomorphic, irreducible, real representations of G . For any $j \in J(G)$ denote by \mathbf{F}_j the set of G -endomorphisms of V_j , $\text{Hom}^G(V_j, V_j)$, which is the field of real numbers \mathbf{R} , complex numbers \mathbf{C} , or quaternions \mathbf{Q} . V_j is the real restriction of a complex representation if $\mathbf{F}_j = \mathbf{C}$, and of a quaternionic representation if $\mathbf{F}_j = \mathbf{Q}$.

For any $y \in M_x^G$, $\nu_y(M_x^G)$ and $\nu_{f(y)}(N_{f(x)}^G)$ are fixed point free (outside the origin) representations of G . Let

$$\begin{aligned}\nu_y(M_x^G) &\cong \bigoplus_{j \in J(G)} m_{x,j} V_j, \quad \text{and} \\ \nu_{f(y)}(N_{f(x)}^G) &\cong \bigoplus_{j \in J(G)} n_{f(x),j} V_j\end{aligned}$$

be the decompositions into irreducible representations, where all $m_{x,j}$ and all $n_{f(x),j}$ are nonnegative integers independent of $y \in M_x^G$, and where mV_j denotes the direct sum of m copies of V_j . Since $\tilde{d}_x f$ embeds $\nu_y(M_x^G)$ into $\nu_{f(y)}(N_{f(x)}^G)$, we see

$$m_{x,j} \leq n_{f(x),j}$$

for any $j \in J(G)$. As seen in § 1 of [3], $\text{Mon}^G(m_{x,j}V_j, n_{f(x),j}V_j)$ is identified with $V(m_{x,j}, n_{f(x),j}; \mathbf{F}_j)$, where $V(m, n; \mathbf{F}_j)$ is the Stiefel manifold of m -frames (not necessarily orthonormal) in the n -dimensional vector space $n\mathbf{F}_j$ over \mathbf{F}_j .

We may split the normal bundle $\nu(M_x^G)$ into Whitney sum

$$\bigoplus_{j \in J(G)} \nu(M_x^G)_j.$$

Here each $\nu(M_x^G)_j$ is a G -vector bundle over M_x^G whose fibre is $m_{x,j}V_j$, and as whose structure group we may take $\Lambda(m_{x,j}; \mathbf{F}_j)$, where $\Lambda(m; \mathbf{F}_j)$ denotes the orthogonal group $O(m)$ if $\mathbf{F}_j = \mathbf{R}$, the unitary group $U(m)$ if $\mathbf{F}_j = \mathbf{C}$, and the symplectic group $Sp(m)$ if $\mathbf{F}_j = \mathbf{Q}$. Similarly for the normal bundle $\nu(N_{f(x)}^G)$. Thus we may split the fibre bundle

$$\text{Mon}_f^G(\nu(M_x^G), \nu(N_{f(x)}^G))$$

into Whitney sum

$$\bigoplus_{j \in J(G)} B_j.$$

Here each B_j is a fibre bundle over M_x^G whose fibre is $V(m_{x,j}, n_{f(x),j}; \mathbf{F}_j)$, and whose structure group is $\Lambda(m_{x,j}; \mathbf{F}_j) \times \Lambda(n_{f(x),j}; \mathbf{F}_j)$.

We easily obtain

Theorem 5. *If both $\nu(M_x^G)$ and $\nu(N_{f(x)}^G)$ are product bundles, then there is a bijective correspondence*

$$\Gamma_f(M_x^G) \approx \prod_{j \in J(G)} [M_x^G, V(m_{x,j}, n_{f(x),j}; \mathbf{F}_j)],$$

where $[,]$ denotes the homotopy set.

The Stiefel manifolds are q -simple for any $q \geq 0$. According to [4; 30.2], denote by $B_j(\pi_q)$ the bundle of q -th homotopy groups associated with B_j . Define

$$d_j = \dim_{\mathbf{R}} \mathbf{F}_j, \quad \text{and} \\ q_j = d_j(n_{f(x),j} - m_{x,j} + 1) - 1,$$

then $V(m_{x,j}, n_{f(x),j}; \mathbf{F}_j)$ is $(q_j - 1)$ -connected and its q_j -th homotopy group is nonzero. So from (37.2) and (37.5) of [4] we obtain

Theorem 6. (a) *If*

$$\dim M_x^G \leq q_j + 1$$

for any j with $m_{x,j} \neq 0$, then there is a surjective correspondence

$$\Gamma_f(M_x^G) \rightarrow \prod_{j \in J(G)} H^{q_j}(M_x^G; B_j(\pi_{q_j})).$$

(b) *If*

$$\dim M_x^G \leq q_j$$

for any j with $m_{x,j} \neq 0$, then there is a bijective correspondence

$$\Gamma_f(M_x^G) \approx \prod_{j \in J(G)} H^{q_j}(M_x^G; B_j(\pi_{q_j})).$$

For many cases $B_j(\pi_q)$ becomes a product bundle. In fact we will see this for the cases (i)~(iv) in the next Proposition. So for these cases we may replace $H^{q_j}(M_x^G; B_j(\pi_{q_j}))$, in Theorem 6, by the ordinary cohomology groups $H^{q_j}(M_x^G; \pi_{q_j}(V(m_{x,j}, n_{f(x),j}; \mathbf{F}_j)))$.

Proposition 7. $B_j(\pi_q)$ is a product bundle for each case of the followings

(i)~(iv):

- (i) G is not of order 2 (including infinite groups),
- (ii) both $\nu(M_x^G)$ and $\nu(N_{f(x)}^G)$ are orientable,
- (iii) G is of order 2, $m_{x,j} \geq 2$, and $q = n_{f(x),j} - m_{x,j}$ is odd,
- (iv) M_x^G is simply connected.

Proof.

$$G_j = \Lambda(m_{x,j}; \mathbf{F}_j) \times \Lambda(n_{f(x),j}; \mathbf{F}_j)$$

is the structure group of B_j . The action of G_j on the fibre $V(m_{x,j}, n_{f(x),j}; \mathbf{F}_j)$ induces automorphisms of $\pi_q = \pi_q(V(m_{x,j}, n_{f(x),j}; \mathbf{F}_j))$. Let H_j be the subgroup which acts as the identity in π_q . Then G_j/H_j is the structure group of $B_j(\pi_q)$.

(i) From the table in [5; p. 208], we see that $\mathbf{F}_j = \mathbf{C}$ or \mathbf{Q} if G is not of order 2. Thus G_j is connected, and $G_j = H_j$. So the structure group of $B_j(\pi_q)$ is trivial, and the bundle is a product bundle.

(ii) The structure group of $B_j(\pi_q)$ may be reduced to a connected group. Thus, as seen above, $B_j(\pi_q)$ is a product bundle.

(iii) For this case we see

$$\pi_q(V(m_{x,j}, n_{f(x),j}; \mathbf{F}_j)) = \mathbf{Z}_2,$$

and the identity is the only automorphism of \mathbf{Z}_2 . Thus $B_j(\pi_q)$ is a product bundle.

(iv) Clear since the fibre of $B_j(\pi_q)$ is discrete. Q.E.D.

References

- [1] E. Bierstone: *The equivariant covering homotopy property for differentiable G-fibre bundles*, J. Differential Geom. **8** (1973), 615–622.
- [2] G.E. Bredon: *Introduction to compact transformation groups*, Academic Press, New York and London, 1972.
- [3] K. Komiya: *Equivariant embeddings and isotopies of a sphere in a representation*, J. Math. Soc. Japan **34** (1982), 425–444.
- [4] N. Steenrod: *The topology of fibre bundles*, Princeton University Press, Princeton, 1951.
- [5] J.A. Wolf: *Spaces of constant curvature* (4-th edition), Publish or Perish Inc., Berkeley, 1977.

Department of Mathematics
Faculty of Science
Yamaguchi University
Yoshida, Yamaguchi 753
Japan

