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EQUIVARIANT ISOTOPIES OF SEMIFREE G-MANIFOLDS

Dedicated to Professor Minoru Nakaoka on his sixtieth birthday

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1. Introduction

In the previous paper [3] we studied the set of equivariant isotopy classes of equivariant smooth embeddings of a sphere with semifree linear action into a euclidean representation space. In this paper we will study more general case, i.e., the set of equivariant isotopy classes of equivariant smooth embeddings of a manifold into another manifold, where the manifolds in question have a smooth semifree action.

Let G be a compact Lie group, and M, N smooth G-manifolds. Two smooth G-embeddings f and g of M into N are called *G-isotopic*, if there is a smooth G-map

$$H: M\times[0, 1] \rightarrow N$$

such that, for any $t\in[0, 1]$, $H_t=H\,|\,M\times\{t\}$ is a smooth G-embedding, and that $H_0=f$, $H_1=g$. Such H is called a *smooth G-isotopy* between f and g. The *G-isotopy class* $[f]$ is the set of all smooth G-embeddings G-isotopic to f. Denote by $\mathrm{Iso}^G(M, N)$ the set of all G-isotopy classes of smooth G-embeddings of M into N. Fix a smooth G-embedding f of M into N, and denote by $\mathrm{Iso}^G_f(M, N)$ the set of all G-isotopy classes of smooth G-embeddings G-homotopic to f. If N is a euclidean representation space of G, then N is G-contractible, and then

$$\mathrm{Iso}^G_f(M, N) = \mathrm{Iso}^G(M, N)$$

for any smooth G-embedding f of M into N.

For $x\in M$ denote by G_x the isotropy subgroup of G at x. An action of G on M is called *semifree* if, for any $x\in M$, G_x is either trivial or is all of G. If, moreover, the fixed point set

$$M^G = \{x\in M\,|\,G_x = G\}$$

is neither empty nor is all of M, the action is called *properly semifree*. For $x\in M^G$ denote by M^G_x the connected component of M^G containing x. Choose a point from each connected component of M^G, and let $C(M^G)$ be the set of these points. Then M^G is the disjoint union of M_x^G for all $x \in C(M^G)$.

Let M, N be smooth properly semifree G-manifolds, and f a smooth G-embedding of M into N. This paper will proceed as follows. In section 2 we define $\Gamma_f(M_x^G)$ as the set of homotopy classes of cross sections of a fibre bundle over M_x^G, and give a definition of a transformation

$$\Phi\colon \mathrm{Iso}_f^G(M, N) \to \prod_{x \in C(M^G)} \Gamma_f(M_x^G)\,.$$

Under dimensional conditions we prove the surjectivity of Φ in section 3, and prove the injectivity of Φ in section 4. Finally in section 5 we analyze $\Gamma_f(M_x^G)$ by using obstruction theory.

REMARK. If the G-action on M is properly semifree, a normal representation of G at a fixed point has no fixed point except the origin. Any compact Lie group G does not always admit a fixed point free (outside the origin) representation. Finite groups which admit fixed point free representations are classified by Wolf [5]. If G is positive dimensional, then there are only three posibilities: $G \cong S^3$, S^1, and its normalizer $N(S^1)$ in S^3 (e.g. as shown in Bredon [2; 8.5]). Thus the groups considered in this paper are finite groups, S^1, $N(S^1)$, and S^3.

2. Transformation Φ

Let M, N be smooth properly semifree G-manifolds, and f a smooth G-embedding of M into N. Choose once and for all a set $C(M^G)$ such that M^G is the disjoint union of M_x^G for all $x \in C(M^G)$. For any $x \in C(M^G)$, let

$$\nu(M_x^G) = (\tau(M)\,|\,M_x^G)/\tau(M_x^G)$$

be the normal bundle of M_x^G in M. Denote by $\nu_y(M_x^G)$ the fibre over $y \in M_x^G$. This is a representation of G which has no fixed point outside the origin. Denote by

$$\mathrm{Mon}^G(\nu_y(M_x^G),\ \nu_{f(y)}(N_{f(x)}^G))$$

the set of all G-monomorphisms from $\nu_y(M_x^G)$ to $\nu_{f(y)}(N_{f(x)}^G)$, and define

$$\mathrm{Mon}_f^G(\nu(M_x^G),\ \nu(N_{f(x)}^G)) = \bigcup_{y \in M_x^G} \mathrm{Mon}^G(\nu_y(M_x^G),\ \nu_{f(y)}(N_{f(x)}^G))\,.$$

By the standard manner this becomes a smooth fibre bundle over M_x^G. The set of continuous (resp. smooth) cross sections of this bundle is in bijective correspondence with the set of continuous (resp. smooth) G-vector bundle monomorphisms from $\nu(M_x^G)$ to $\nu(N_{f(x)}^G)$ which cover

$$f_x^G = f\,|\,M_x^G\colon M_x^G \to N_{f(x)}^G\,.$$

Denote by $\Gamma_f(M_x^G)$ the set of homotopy classes of continuous cross sections of $\mathrm{Mon}_f^G(\nu(M_x^G), \nu(N_{f(x)}^G))$. Note that we may take smooth ones as representatives of classes in $\Gamma_f(M_x^G)$ by the differentiable approximation theorem [4; 6.7].

Let $g: M \to N$ be a smooth G-embedding G-homotopic to f. Note that $N_{g(x)}^G = N_{f(x)}^G$ for any $x \in C(M^G)$. Then two maps

$$g_x^G, f_x^G: M_x^G \to N_{f(x)}^G$$

are homotopic, i.e., there is a homotopy

$$H: M_x^G \times [0, 1] \to N_{f(x)}^G$$

with $H_0 = g_x^G$ and $H_1 = f_x^G$. By Bierstone [1] we may lift H to a G-homotopy of G-vector bundle monomorphism

$$\tilde{H}: \nu(M_x^G) \times [0, 1] \to \nu(N_{f(x)}^G)$$

with

$$\tilde{H}_0 = \tilde{d}_x g: \nu(M_x^G) \to \nu(N_{f(x)}^G),$$

where $\tilde{d}_x g$ is the G-vector bundle monomorphism induced from the differential $dg: \tau(M) \to \tau(N)$ of g. Then $\tilde{H}_1$ is a G-vector bundle monomorphism which covers f_x^G. Let

$$\Phi_x(g): M_x^G \to \mathrm{Mon}_f^G(\nu(M_x^G), \nu(N_{f(x)}^G))$$

be a cross section corresponding to $\tilde{H}_1$. $\Phi_x(g)$ is determined dependently on H and its lifting $\tilde{H}$. But, if $N_{f(x)}^G$ is $(\dim M_x^G + 1)$-connected, the homotopy class of $\Phi_x(g)$ does not depend on H and $\tilde{H}$. More precisely we show

Lemma 1. *Let $g, h: M \to N$ be smooth G-embeddings G-homotopic to f. If g and h are G-isotopic, and if $N_{f(x)}^G$ is $(\dim M_x^G + 1)$-connected, then $\Phi_x(g)$ and $\Phi_x(h)$ are homotopic as cross section.*

Proof. Let

$$\tilde{H}^{(i)}: \nu(M_x^G) \times [0, 1] \to \nu(N_{f(x)}^G), \qquad i = 0, 1,$$

be G-homotopies of G-vector bundle monomorphism which cover G-homotopies

$$H^{(i)}: M_x^G \times [0, 1] \to N_{f(x)}^G, \qquad i = 0, 1,$$

such that

(1) $H_0^{(0)} = f$, $H_1^{(0)} = g$, $H_0^{(1)} = h$, $H_1^{(1)} = f$,
(2) $\tilde{H}_1^{(0)} = \tilde{d}_x g$, $\tilde{H}_0^{(1)} = \tilde{d}_x h$,
(3) $\tilde{H}_0^{(0)}$ and $\tilde{H}_1^{(1)}$ correspond to $\Phi_x(g)$ and $\Phi_x(h)$, respectively.

Let $K: M\times[0, 1]\to N$ be a smooth G-isotopy with $K_0=g$ and $K_1=h$. Since $N^G_{f(x)}$ is $(\dim M^G_x+1)$-connected, there is a homotopy

$$E: M^G_x\times[0, 3]\times[0, 1]\to N^G_{f(x)}$$

such that, for any $(y, t, s)\in M^G_x\times(\{0, 3\}\times[0, 1]\cup[0, 3]\times\{1\})$,

$$E(y, t, s)=f(y),$$

and for any $(y, t, 0)\in M^G_x\times[0, 3]\times\{0\}$,

$$E(y, t, 0)=\begin{cases} H^{(0)}(y, t) & \text{if } 0\leq t\leq 1 \\ K(y, t-1) & \text{if } 1\leq t\leq 2 \\ H^{(1)}(y, t-2) & \text{if } 2\leq t\leq 3\,. \end{cases}$$

Define

$$k: \nu(M^G_x)\times[0, 3]\to\nu(N^G_{f(x)})$$

as, for any $(v, t)\in\nu(M^G_x)\times[0, 3]$,

$$k(v, t)=\begin{cases} \tilde{H}^{(0)}(v, t) & \text{if } 0\leq t\leq 1 \\ \tilde{d}_xK(v, t-1) & \text{if } 1\leq t\leq 2 \\ \tilde{H}^{(1)}(v, t-2) & \text{if } 2\leq t\leq 3\,. \end{cases}$$

Then k is a G-vecotr bundle monomorphism, and covers $E\,|\,M^G_x\times[0, 3]\times\{0\}$. By Bierstone [1] we obtain a G-homotopy of G-vector bundle monomorphism

$$\tilde{E}: \nu(M^G_x)\times[0, 3]\times[0, 1]\to\nu(N^G_{f(x)})$$

such that $\tilde{E}_0=k$ and that $\tilde{E}$ covers E. Then

$$\tilde{E}\,|\,\nu(M^G_x)\times(\{0, 3\}\times[0, 1]\cup[0, 3]\times\{1\})$$

covers f^G_x on each level M^G_x, and

$$\tilde{E}\,|\,\nu(M^G_x)\times\{0\}\times\{0\}=\tilde{H}^{(0)}_0\,,$$
$$\tilde{E}\,|\,\nu(M^G_x)\times\{3\}\times\{0\}=\tilde{H}^{(1)}_1\,.$$

Thus we see that $\Phi_x(g)$ and $\Phi_x(h)$ are homotopic as cross section. Q.E.D.

If $N^G_{f(x)}$ is $(\dim M^G_x+1)$-connected for all $x\in C(M^G)$, then, by Lemma 1, we may define a transformation

$$\Phi: \mathrm{Iso}^G_f(M, N)\to\prod_{x\in C(M^G)}\Gamma_f(M^G_x)$$

as

$$\Phi([g])=\prod_{x\in C(M^G)}[\Phi_x(g)]$$

for any $[g]\in \mathrm{Iso}_f^G(M, N)$. If N is a euclidean representation space of G, then N^G is contractible and Φ is always defined.

Define

$$\dim N^G = \max\{\dim N_x^G \mid x\in C(N^G)\} .$$

We obtain

Theorem 2. *Let M, N be smooth properly semifree G-manifolds without boundary, M compact, and f a smooth G-embedding of M into N. Assume that $N_{f(x)}^G$ is $(\dim M_x^G+1)$-connected for any $x\in C(M^G)$. Then the transformation*

$$\Phi\colon \mathrm{Iso}_f^G(M, N) \to \prod_{x\in C(M^G)} \Gamma_f(M_x^G)$$

satisfies that

(a) *if*

$$\dim M+\max\{\dim M, \dim N^G\}<\dim N+\dim G ,$$

then Φ is surjective,

(b) *if*

$$2\dim M_x^G+1<\dim N_{f(x)}^G \qquad \text{for any } x\in C(M^G) ,$$

and if

$$\dim M+\max\{\dim M, \dim N^G\}+1<\dim N+\dim G ,$$

then Φ is bijective.

The surjectivity of Φ will be proven in the next section 3, and the injectivity of Φ in section 4.

3. Surjectivity of Φ

First we provide a lemma for the proof of surjectivity of Φ.

Lemma 3. *Let α: $X\to Y$ be a map. Let $\xi\to X$ and $\zeta\to Y$ be a- and b-dimensional G-sphere bundles over X and Y, respectively. Here G acts trivially on both X and Y, and freely on ξ. Assume that X is a finite connected complex, and that A is a subcomplex of X. Let φ: $\xi|A\to\zeta$ be a fibre preserving G-map which covers $\alpha|A$. If*

$$\dim X+a\leq b+\dim G ,$$

then φ is extended to a fibre preserving G-map from ξ to ζ which covers α.

Proof. Denote by $\mathrm{Map}^G(\xi_x, \zeta_{\alpha(x)})$ the set of G-maps from the fibre ξ_x of ξ over $x\in X$ to the fibre $\zeta_{\alpha(x)}$ of ζ over $\alpha(x)\in Y$. Give the compact-open topology to the set. Define

$$\mathrm{Map}^G_\alpha(\xi, \zeta) = \bigcup_{x\in X} \mathrm{Map}^G(\xi_x, \zeta_{\alpha(x)}) .$$

By the standard manner this becomes a fibre bundle over X with fibre $\mathrm{Map}^G(\xi_x, \zeta_{\alpha(x)})$. The set of cross sections of $\mathrm{Map}^G_\alpha(\xi, \zeta)\to X$ is in bijective correspondence with the set of fibre preserving G-maps from ξ to ζ which cover α. Let

$$s(\varphi)\colon A \to \mathrm{Map}^G_\alpha(\xi, \zeta)\,|\,A$$

be the cross section corresponding to φ. To prove the lemma we extend $s(\varphi)$ over X. For this it suffices to see that the fibre $\mathrm{Map}^G(\xi_x, \zeta_{\alpha(x)})$ is ($\dim X-1$)-connected. For any i with $0\leq i\leq \dim X-1$, let D^{i+1} be the canonical ($i+1$)-dimensional disc with trivial G-action, S^i its boundary, and

$$\beta\colon S^i \to \mathrm{Map}^G(\xi_x, \zeta_{\alpha(x)})$$

any map. We should like to extend β over D^{i+1}. By the exponential law β gives a G-map

$$\tilde\beta\colon S^i\times \xi_x \to \zeta_{\alpha(x)} .$$

From the hypothesis,

$$\dim D^{i+1}\times \xi_x/G\leq b$$

and $\zeta_{\alpha(x)}$ is ($b-1$)-connected. Then, as in the proof of Lemma 5 in [3], we may extend $\tilde\beta$ to a G-map on $D^{i+1}\times \xi_x$. Thus we may also extend β over D^{i+1}. Q.E.D.

From Lemma 3 we obtain

Corollary 4. *Let $\xi\to X$ and $\zeta\to Y$ be a- and b-dimensional G-vecotr bundles over X and Y, respectively. Here G acts trivially on both X and Y, and freely on both ξ and ζ outside the zero sections. Assume X is a finite complex. Let*

$$\varphi,\ \psi\colon \xi \to \zeta$$

be G-vector bundle monomorphisms which cover a map $\alpha\colon X\to Y$. If

$$\dim X+a<b+\dim G ,$$

then there exists a fibre preserving G-homotopy

$$H\colon \xi\times [0, 1] \to \zeta$$

such that

(1) $H_0=\varphi$, $H_1=\psi$,

(2) *H_t covers α for any $t\in[0, 1]$, (H_t is not necessarily linear on fibres of ξ.)*

(3) $H((\xi-X)\times[0, 1])\subset\zeta-Y$, *where* X *and* Y *are regarded as the zero sections of* ξ *and* ζ, *respectively.*

Proof. Let $S(\xi)$ and $S(\zeta)$ be associated G-sphere bundles of ξ and ζ, respectively. Since φ and ψ are monic on each fibre of ξ,

$$\varphi(S(\xi))\subset\zeta-Y, \quad \text{and}$$
$$\psi(S(\xi))\subset\zeta-Y.$$

Let r: $\zeta-Y\to S(\zeta)$ be the radial retraction. Apply Lemma 3 to

$$r\circ\varphi\cup r\circ\psi: S(\xi)\times\{0, 1\}\to S(\zeta). \qquad \text{Q.E.D.}$$

We now begin the proof of surjectivity of Φ under the assumption (a) of Theorem 2. Let

$$\alpha=\prod_{x\in C(M^G)}[s_x]\in\prod_{x\in C(M^G)}\Gamma_f(M_x^G)$$

be any element. We will construct a smooth G-embedding g of M into N with $\Phi([g])=\alpha$. Let

$$t_x: \nu(M_x^G)\to\nu(N_{f(x)}^G)$$

be a G-vector bundle monomorphism covering f_x^G which corresponds to s_x. Without loss of generality we may assume t_x is smooth. From the assumption (a) and Corollary 4 we obtain a fibre preserving G-homotopy

$$H^{(1)}: \nu(M_x^G)\times[0, 1]\to\nu(N_{f(x)}^G)$$

such that

(1) $H_0^{(1)}=\tilde{d}_xf$, $H_1^{(1)}=t_x$,

(2) $H_t^{(1)}$ covers f_x^G for any $t\in[0, 1]$,

(3) $H^{(1)}((\nu(M_x^G)-M_x^G)\times[0, 1])\subset\nu(N_{f(x)}^G)-N_{f(x)}^G$.

Define

$$t=\bigcup_{x\in C(M^G)}t_x: \nu(M^G)=\bigcup_{x\in C(M^G)}\nu(M_x^G)\to\nu(N^G).$$

Making use of exponential maps as in the proof of Lemma 6 of [3], from t we obtain a G-homotopy

$$H^{(2)}: T_{3\varepsilon}(M^G)\times[0, 1]\to N$$

such that

(1) $H_0^{(2)}=f\mid T_{3\varepsilon}(M^G)$,

(2) $H_1^{(2)}$ is a smooth G-embedding with $\tilde{d}H_1^{(2)}=t$,

(3) $H^{(2)}((T_{3\varepsilon}(M^G)-M^G)\times[0, 1])\subset N-N^G$, where $T_{3\varepsilon}(M^G)$ is a G-equivariant closed tubular neighborhood of M^G in M with radius $3\varepsilon>0$. Using $H^{(2)}$

and f, we may construct a smooth G-map

$$g^{(1)}: M \to N$$

such that

(1) $g^{(1)}$ and f are G-homotopic,

(2) for some δ, $\gamma>0$ with $\gamma<3\varepsilon$

$$(g^{(1)})^{-1}(T_\delta(N^G)) \subset \operatorname{Int} T_\gamma(M^G),$$

(3) $g^{(1)}=H_1^{(2)}$ on $T_\gamma(M^G)$, hence $g^{(1)}=f$ on M^G.

In fact, $g^{(1)}$ can be constructed as follows. First define a G-map

$$h: M \to N$$

as the followings:

$$h(x) = H_1^{(2)}(x) \qquad \text{for } x \in T_\varepsilon(M^G),$$

$$h(x) = H^{(2)}\left(\frac{\varepsilon x}{||x||}, 2-\frac{||x||}{\varepsilon}\right) \qquad \text{for } x \in T_{2\varepsilon}(M^G)-\operatorname{Int} T_\varepsilon(M^G), \text{ where } ||x||$$

denotes the length of x in $T_{3\varepsilon}(M^G)$,

$$h(x) = f\left(\left(2-\frac{3\varepsilon}{||x||}\right)x\right) \qquad \text{for } x \in T_{3\varepsilon}(M^G)-\operatorname{Int} T_{2\varepsilon}(M^G), \text{ and}$$

$$h(x) = f(x) \qquad \text{for } x \in M-\operatorname{Int} T_{3\varepsilon}(M^G).$$

Next, smooth h to obtain the desired $g^{(1)}$.

Define

$$K = M-\operatorname{Int}(g^{(1)})^{-1}(T_\delta(N^G)), \quad \text{and}$$
$$L = N-\operatorname{Int} T_\delta(N^G).$$

These are smooth free G-manifolds with boundary. Since $g^{(1)}(K) \subset L$, we obtain a smooth G-map

$$g^{(1)}|K: K \to L.$$

Passing to orbit spaces, we also obtain a smooth map

$$g^{(2)} = (g^{(1)}|K)/G: K/G \to L/G,$$

which is an embedding on a neighborhood of $\partial K/G$ in K/G. From the assumption (a),

$$2 \dim K/G < \dim L/G.$$

Thus $g^{(2)}$ is homotoped to a smooth embedding, precisely there is a smooth homotopy

$$H^{(3)}: K/G\times[0, 1]\rightarrow L/G$$

such that

(1) $H_0^{(3)}=g^{(2)}$,

(2) $H_1^{(3)}$ is a smooth embedding, and

(3) $H^{(3)}$ is a constant homotopy on a neighborhood of $\partial K/G$.

Since the natural projections $K\rightarrow K/G$ and $L\rightarrow L/G$ are smooth G-fibre bundles, then by Bierstone [1] we obtain a smooth G-homotopy

$$H^{(4)}: K\times[0, 1]\rightarrow L$$

such that $H_0^{(4)}=g^{(1)}|K$, and that $H_1^{(4)}$ is a smooth G-embedding. Moreover, we can choose $H^{(4)}$ so that it is a constant homotopy on a neighborhood of ∂K in K, hence that $H_1^{(4)}=g^{(1)}$ on the neighborhood. Then, from $g^{(1)}$ and $H_1^{(4)}$, we obtain a smooth G-embedding

$$g^{(3)}: M\rightarrow N$$

such that

(1) $g^{(3)}$ is G-homotopic to f, and

(2) $g^{(3)}=g^{(1)}=H_1^{(2)}$ on a neighborhood of M^G in M.

Thus

$$\tilde{d}g^{(3)}=\tilde{d}H_1^{(2)}=t: \nu(M^G)\rightarrow\nu(N^G),$$

and

$$\Phi([g^{(3)}])=\prod_{x\in C(M^G)}[s_x].$$

This completes the proof for the surjectivity of Φ under the assumption (a) of Theorem 2.

4. Injectivity of Φ

In this section we will show the injectivity of Φ under the assumption (b) of Theorem 2. Let

$$\Phi([g])=\Phi([h]) \quad \text{in} \quad \prod_{x\in C(M^G)}\Gamma_f(M_x^G)$$

for $[g]$, $[h]\in \mathrm{Iso}_f^G(M, N)$. We will construct a smooth G-isotopy between g and h.

First, since g and h are G-homotopic, there is a G-homotopy

$$H^{(1)}: M\times[0, 1]\rightarrow N$$

with $H_0^{(1)}=g$ and $H_1^{(1)}=h$. By the assumption

$$2\dim M_x^G+1<\dim N_{f(x)}^G \qquad \text{for all } x\in C(M^G),$$

we see

$$f^G,\ g^G,\ h^G\colon M^G \to N^G$$

are isotopic each other. From this and $\Phi([g])=\Phi([h])$ we obtain a smooth G-homotopy of G-vector bundle monomorphism

$$H^{(2)}\colon \nu(M^G)\times[0,1]\to\nu(N^G)$$

such that

(1) $H_0^{(2)}=\tilde{d}g$, $H_1^{(2)}=\tilde{d}h$, and

(2) $H^{(2)}$ covers a smooth isotopy: $M^G\times[0,1]\to N^G$.

Making use of exponential maps as in the proof of Lemma 6 of [3], from $H^{(2)}$ we obtain, for an appropriate $\varepsilon>0$, a smooth G-isotopy

$$H^{(3)}\colon T_{4\varepsilon}(M^G)\times[0,1]\to N$$

with $H_0^{(3)}=g\,|\,T_{4\varepsilon}(M^G)$ and with $H_1^{(3)}=h\,|\,T_{4\varepsilon}(M^G)$. Since $N^G_{f(x)}$ is $(\dim M^G_x+1)$-connected for any $x\in C(M^G)$, we may obtain a homotopy

$$H^{(4)}\colon (M^G\times[0,1])\times[0,1]\to N^G$$

such that

(1) $H_0^{(4)}=H^{(3)}\,|\,M^G\times[0,1]$,

(2) $H_1^{(4)}=H^{(1)}\,|\,M^G\times[0,1]$,

(3) $H_t^{(4)}\,|\,M^G\times\{0\}=g^G$ for any $t\in[0,1]$, and

(4) $H_t^{(4)}\,|\,M^G\times\{1\}=h^G$ for any $t\in[0,1]$.

Define a G-homotopy

$$H^{(5)}\colon M\times[0,1]\to N$$

as follows: for any $(x,t)\in M\times[0,1]$,

$$H^{(5)}(x,t)=H^{(3)}(x,t)\qquad\text{if}\ \ x\in T_\varepsilon(M^G),$$

$$H^{(5)}(x,t)=H^{(3)}\left(\left(\frac{2\varepsilon}{||x||}-1\right)x,\ t\right)\qquad\text{if}\ \ x\in T_{2\varepsilon}(M^G)-\operatorname{Int}T_\varepsilon(M^G),$$

$$H^{(5)}(x,t)=H^{(4)}\left(\pi(x),\ t,\ \frac{||x||}{\varepsilon}-2\right)\qquad\text{if}\ \ x\in T_{3\varepsilon}(M^G)-\operatorname{Int}T_{2\varepsilon}(M^G),$$

where $\pi\colon T_{3\varepsilon}(M^G)\to M^G$ is the canonical projection,

$$H^{(5)}(x,t)=H^{(1)}\left(4\left(1-\frac{3\varepsilon}{||x||}\right)x,\ t\right)\qquad\text{if}\ \ x\in T_{4\varepsilon}(M^G)-\operatorname{Int}T_{3\varepsilon}(M^G),$$

$$H^{(5)}(x,t)=H^{(1)}(x,t)\qquad\text{if}\ \ x\in M-\operatorname{Int}T_{4\varepsilon}(M^G).$$

Then $H_0^{(5)}$ and g are G-homotopic, and its homotopy can be so chosen as to be constant on $T_\varepsilon(M^G)$. Similarly for $H_1^{(5)}$ and h. From these homotopies we obtain a G-homotopy

$$H^{(6)}\colon M\times[0, 1]\to N$$

such that $H_0^{(6)}=g$, $H_1^{(6)}=h$, and that $H^{(6)}$ is a smooth G-isotopy on $T_\varepsilon(M^G)$.

Define

$$L=(M-\mathrm{Int}\, T_\varepsilon(M^G))\times[0, 1]\,.$$

Note the G-action on L is free. Let G act diagonally on $L\times N$. Passing a G-map

$$id\times H^{(6)}\colon L\to L\times N$$

to orbit spaces, we obtain a map

$$\alpha^{(1)}=id\times H^{(6)}/G\colon L/G\to(L\times N)/G\,.$$

Consider a submanifold

$$(L\times N^G)/G=L/G\times N^G$$

of $(L\times N)/G$. Then

$$\alpha^{(1)}(\partial L/G)\cap L/G\times N^G=\phi\,.$$

From the assumption (b),

$$\dim L/G<\dim(L\times N)/G-\dim L/G\times N^G\,.$$

Thus $\alpha^{(1)}$ can be so homotoped that its image does not intersect $L/G\times N^G$, i.e., there is a map

$$\alpha^{(2)}\colon L/G\to(L\times N)/G$$

which is homotopic to $\alpha^{(1)}$ relative to $\partial L/G$, and whose image does not intersect $L/G\times N^G$. From this we obtain a G-map

$$\alpha^{(3)}\colon L\to N$$

which is G-homotopic to $H^{(6)}|L$ relative to ∂L, and whose image does not intersect N^G. Define

$$H^{(7)}\colon M\times[0, 1]\to N$$

as

$$H^{(7)}=H^{(6)} \qquad \text{on } T_\varepsilon(M^G)\times[0, 1], \text{ and}$$
$$H^{(7)}=\alpha^{(3)} \qquad \text{on } L\,.$$

Then $H^{(7)}$ is a G-homotopy between g and h, and a smooth G-isotopy particularly on $T_\varepsilon(M^G)$. We see

$$M^G\times[0, 1]=(H^{(7)})^{-1}(N^G)\,.$$

At this point it only remains to deform $H^{(7)}$ outside a neighborhood of M^G to a smooth G-isotopy. It can be done similarly to the proof in [3]. So we will merely give an outline. Since $M\times[0, 1]$ is compact, for small $\delta>0$,

$$\text{Int } T_{\varepsilon/2}(M^G)\times[0, 1]\supset(H^{(7)})^{-1}(T_\delta(N^G)).$$

Let η be a level preserving G-diffeomorphism of $M\times[0, 1]$ such that

$$\eta(T_\varepsilon(M^G)\times[0, 1]) = T_\varepsilon(M^G)\times[0, 1], \text{ and}$$
$$\eta(T_{\varepsilon/2}(M^G)\times[0, 1]) = (H^{(7)})^{-1}(T_\delta(N^G)).$$

Define

$$P = M-\text{Int } T_{\varepsilon/2}(M^G), \text{ and}$$
$$Q = N-\text{Int } T_\delta(N^G).$$

Consider a G-homotopy

$$H^{(7)}\circ\eta\colon P\times[0, 1]\to Q,$$

which is a smooth G-isotopy on a neighborhood of ∂P. From the assumption (b),

$$2\dim P+1<\dim Q+\dim G.$$

Then $H^{(7)}\circ\eta$ may be deformed to a smooth G-isotopy

$$H^{(8)}\colon P\times[0, 1]\to Q$$

such that

(1) $H_0^{(8)}=g\circ\eta_0|P$,
(2) $H_1^{(8)}=h\circ\eta_1|P$,
(3) $H^{(8)}=H^{(7)}\circ\eta$ on (n.b.d of $\partial P)\times[0, 1]$.

From $H^{(7)}$ and $H^{(8)}$ we obtain a smooth G-isotopy between g and h. This completes the proof for the injectivity of Φ under the assumption (b) of Theorem 2.

5. Analysis of $\Gamma_f(M_x^G)$

In this section we will analyze $\Gamma_f(M_x^G)$.

Let $\{V_j | j\in J(G)\}$ be a complete set of fixed point free (outside the origin), nonisomorphic, irreducible, real representations of G. For any $j\in J(G)$ denote by $\boldsymbol{F}_j$ the set of G-endomorphisms of V_j, $\text{Hom}^G(V_j, V_j)$, which is the field of real numbers $\boldsymbol{R}$, complex numbers $\boldsymbol{C}$, or quaternions $\boldsymbol{Q}$. V_j is the real restriction of a complex representation if $\boldsymbol{F}_j=\boldsymbol{C}$, and of a quaternionic representation if $\boldsymbol{F}_j=\boldsymbol{Q}$.

For any $y \in M_x^G$, $\nu_y(M_x^G)$ and $\nu_{f(y)}(N_{f(x)}^G)$ are fixed point free (outside the origin) representations of G. Let

$$\nu_y(M_x^G) \cong \bigoplus_{j \in J(G)} m_{x,j} V_j \,, \quad \text{and}$$
$$\nu_{f(y)}(N_{f(x)}^G) \cong \bigoplus_{j \in J(G)} n_{f(x),j} V_j$$

be the decompositions into irreducible representations, where all $m_{x,j}$ and all $n_{f(x),j}$ are nonnegative integers independent of $y \in M_x^G$, and where mV_j denotes the direct sum of m copies of V_j. Since $\tilde{d}_x f$ embedds $\nu_y(M_x^G)$ into $\nu_{f(y)}(N_{f(x)}^G)$, we see

$$m_{x,j} \leq n_{f(x),j}$$

for any $j \in J(G)$. As seen in § 1 of [3], $\mathrm{Mon}^G(m_{x,j} V_j, n_{f(x),j} V_j)$ is identified with $V(m_{x,j}, n_{f(x),j}; \boldsymbol{F}_j)$, where $V(m, n; \boldsymbol{F}_j)$ is the Stiefel manifold of m-frames (not necessarily orthonormal) in the n-dimensional vector space $n\boldsymbol{F}_j$ over $\boldsymbol{F}_j$.

We may split the normal bundle $\nu(M_x^G)$ into Whitney sum

$$\bigoplus_{j \in J(G)} \nu(M_x^G)_j \,.$$

Here each $\nu(M_x^G)_j$ is a G-vector bundle over M_x^G whose fibre is $m_{x,j} V_j$, and as whose structure group we may take $\Lambda(m_{x,j}; \boldsymbol{F}_j)$, where $\Lambda(m; \boldsymbol{F}_j)$ denotes the orthogonal group $O(m)$ if $\boldsymbol{F}_j = \boldsymbol{R}$, the unitary group $U(m)$ if $\boldsymbol{F}_j = \boldsymbol{C}$, and the symplectic group $Sp(m)$ if $\boldsymbol{F}_j = \boldsymbol{Q}$. Similarly for the normal bundle $\nu(N_{f(x)}^G)$. Thus we may split the fibre bundle

$$\mathrm{Mon}_f^G(\nu(M_x^G), \nu(N_{f(x)}^G))$$

into Whitney sum

$$\bigoplus_{j \in J(G)} B_j \,.$$

Here each B_j is a fibre bundle over M_x^G whose fibre is $V(m_{x,j}, n_{f(x),j}; \boldsymbol{F}_j)$, and whose structure group is $\Lambda(m_{x,j}; \boldsymbol{F}_j) \times \Lambda(n_{f(x),j}; \boldsymbol{F}_j)$.

We easily obtain

Theorem 5. *If both $\nu(M_x^G)$ and $\nu(N_{f(x)}^G)$ are product bundles, then there is a bijective correspondence*

$$\Gamma_f(M_x^G) \approx \prod_{j \in J(G)} [M_x^G, V(m_{x,j}, n_{f(x),j}; \boldsymbol{F}_j)] \,,$$

where [,] *denotes the homotopy set.*

The Stiefel manifolds are q-simple for any $q \geq 0$. According to [4; 30.2], denote by $B_j(\pi_q)$ the bundle of q-th homotopy groups associated with B_j. Define

$$d_j = \dim_{\mathbf{R}} \mathbf{F}_j\,, \quad \text{and}$$
$$q_j = d_j(n_{f(x),j} - m_{x,j} + 1) - 1\,,$$

then $V(m_{x,j},\ n_{f(x),j};\ \mathbf{F}_j)$ is (q_j-1)-connected and its q_j-th homotopy group is nonzero. So from (37.2) and (37.5) of [4] we obtain

Theorem 6. (a) *If*

$$\dim M_x^G \leq q_j + 1$$

for any j with $m_{x,j} \neq 0$, *then there is a surjective correspondence*

$$\Gamma_f(M_x^G) \to \prod_{j\in J(G)} H^{q_j}(M_x^G;\ B_j(\pi_{q_j}))\,.$$

(b) *If*

$$\dim M_x^G \leq q_j$$

for any j with $m_{x,j} \neq 0$, *then there is a bijective correspondence*

$$\Gamma_f(M_x^G) \approx \prod_{j\in J(G)} H^{q_j}(M_x^G;\ B_j(\pi_{q_j}))\,.$$

For many cases $B_j(\pi_q)$ becomes a product bundle. In fact we will see this for the cases (i)~(iv) in the next Proposition. So for these cases we may replace $H^{q_j}(M_x^G;\ B_j(\pi_{q_j}))$, in Theorem 6, by the ordinary cohomology groups $H^{q_j}(M_x^G;\ \pi_{q_j}(V(m_{x,j},\ n_{f(x),j};\ \mathbf{F}_j)))$.

Proposition 7. $B_j(\pi_q)$ *is a product bundle for each case of the followings* (i)~(iv):

(i) *G is not of order* 2 (*including infinite groups*),

(ii) *both* $\nu(M_x^G)$ *and* $\nu(N_{f(x)}^G)$ *are orientable,*

(iii) *G is of order* 2, $m_{x,j} \geq 2$, *and* $q = n_{f(x),j} - m_{x,j}$ *is odd,*

(iv) M_x^G *is simply connected.*

Proof.

$$G_j = \Lambda(m_{x,j};\ \mathbf{F}_j) \times \Lambda(n_{f(x),j};\ \mathbf{F}_j)$$

is the structure group of B_j. The action of G_j on the fibre $V(m_{x,j},\ n_{f(x),j};\ \mathbf{F}_j)$ induces automorphisms of $\pi_q = \pi_q(V(m_{x,j},\ n_{f(x),j};\ \mathbf{F}_j))$. Let H_j be the subgroup which acts as the identity in π_q. Then G_j/H_j is the structure group of $B_j(\pi_q)$.

(i) From the table in [5; p. 208], we see that $\mathbf{F}_j = \mathbf{C}$ or $\mathbf{Q}$ if G is not of order 2. Thus G_j is connected, and $G_j = H_j$. So the structure group of $B_j(\pi_q)$ is trivial, and the bundle is a product bundle.

(ii) The structure group of $B_j(\pi_q)$ may be reduced to a connected group. Thus, as seen above, $B_j(\pi_q)$ is a product bundle.

(iii) For this case we see

$$\pi_q(V(m_{x,j}, n_{f(x),j}; \boldsymbol{F}_j)) = \boldsymbol{Z}_2,$$

and the identity is the only automorphism of $\boldsymbol{Z}_2$. Thus $B_j(\pi_q)$ is a product bundle.

(iv) Clear since the fibre of $B_j(\pi_q)$ is discrete. Q.E.D.

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