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# ON A CONSTRUCTION OF NULL ELECTROMAGNETIC FIELDS

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**Introduction.** In this paper we consider Maxwell's equations with a certain nonlinear condition and give an elementary method of constructing the solutions of these.

After the work of Penrose [5], complex manifold techniques have been used for representing the solutions of Maxwell's equations. It is now known that the solutions are represented in terms of cohomology classes on an open complex manifold with coefficients in a certain holomorphic vector bundles (cf. Penrose [5], Wells [7]). But it is not always easy to have the solutions in the explicit form using this representation. The purpose of this paper is to give a direct method of constructing the solutions. Our approach is based on the work of Robinson [6]. In [6] he studied a particular class of the solutions, so-called null electromagnetic fields and found the connection between these fields and the special families of null lines.

We give a brief summary of the results of [6]. The solutions of Maxwell's equations, namely, electromagnetic fields are represented by means of the differential 2-forms on Minkowski space. The differential 2-forms induce the linear mappings from the tangent space to the contangent space by contraction. The intersection $N$ of the kernels of the transformations induced by $F$ and $*F$ plays an essential role, where $F$ is a differential 2-form and $*F$ is the Hodge dual of $F$. If $F$ is a null electromagnetic field, $N$ has dimension 1 and is null. Therefore we have a family of null lines (null rays associated with a null electromagnetic field). This family satisfies some nonlinear equations which are called shear-free equations. We say that a family of null lines is a shear-free null congruence if it satisfies shear-free equations. Null electromagnetic fields are constructed from shear-free null congruences.

In the process of carrying out Robinson's method of constructing null electromagnetic fields we must solve an overdetermined system of differential equations (3.11) which has coefficients related to a shear-free null congruence. In the present paper we solve Eqs. (3.11) exactly and construct null electromagnetic fields. At this stage the theorem of Kerr which asserts that every analytic shear-free null congruence is obtained from a complex analytic homogeneous function with four variables has important place. We find that the shear-free nature is the integrability condition of Eqs. (3.11). This condition is equivalent to the existence of the analytic function associated with a shear-free null congruence. This fact allows us to construct the solutions of Eqs. (3.11) using the shear-free null congruence and the associated analytic function.

The contents of the paper are as follows. In the first section Maxwell's equations are presented in a form which is invariant with respect to Lorentz transformations and the definition of null electromagnetic fields is given. In §2 using the spinor components of the differential 2-forms, we rewrite Maxwell's euqations in $SL(2, C)$ invariant form. In §3 we review the relation between null electromagnetic fields and shear-free null congruences given in [6]. We remark that the spinor language adopted here simplifies the proof of [6]. The main results are in §4 and §5: we prove in §4 the existence of the solutions of Eqs. (3.11) by showing its compatibility and in §5 give a method of constructing all of its solutions using the Kerr theorem.

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## 1. Maxwell's equations

Maxwell's equations, which describe the time evolution of electric fields $E=(E_1, E_2, E_3)$ and magnetic fields $B=(B_1, B_2, B_3)$ in affine 3-space $R^3=\{(x, y, z); x, y, z\in R\}$, classically take the form

$$\begin{aligned} &\frac{d}{dt}E-\mathrm{rot}\,B=0\,, \quad \mathrm{div}\,E=0\,, \\ &\frac{d}{dt}B+\mathrm{rot}\,E=0\,, \quad \mathrm{div}\,B=0\,. \end{aligned} \tag{1.1}$$

We want to rewrite Eqs. (1.1) in a form which is invariant with respect to Lorentz transformations. Let $(M, g)$ be the Minkowksi space, namely, $M$ is affine 4-space $R^4$ with Cartesian coordinates $(x^0, x^1, x^2, x^3)$; $x^0=t$, $x^1=x$, $x^2=y$, $x^3=z$ and $g$ is the metric form on $M$ defined by $g=(dx^0)^2-(dx^1)^2-(dx^2)^2-(dx^3)^2$. We define a 2-form $F$ as follows:

$$F=F_{ij}dx^i\wedge dx^j$$

where

$$[F_{ij}]=\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{1.2}$$

REMARK. We suppress the summation sign every time that the summation has to be done on an index which appears twice in the term.

We denote by $\wedge^2 T^*M$ the space of all differential 2-forms on $M$. The metric $g$ induces a Hodge $*$-operator on $\wedge^2 T^*M$:

$$*\colon \wedge^2 T^*M \to \wedge^2 T^*M .$$

We recall that if

$$F = F_{i_1 i_2} dx^{i_1} \wedge dx^{i_2} ,$$

then

$$(*F)_{j_1 j_2} = (sgn\ \sigma) F^{i_1 i_2}$$

where

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ i_1 & i_2 & j_1 & j_2 \end{pmatrix} ,$$

$$F^{i_1 i_2} = g^{i_1 k_1} g^{i_2 k_2} F_{k_1 k_2}$$

and

$$(g^{ij}) = diag(1, -1, -1, -1) .$$

Then Eqs. (1.1) are equivalent to

$$dF = 0 , \qquad d{*F} = 0 . \tag{1.3}$$

Hodge $*$-operator is linear and satisfies $*^2 = -1$. Therefore $*$ has eigenvalues $+i$, $-i$. Let $\wedge^2 T^*M \otimes C$ be the complexification of $\wedge^2 T^*M$, then we have $\wedge^2 T^*M \otimes C = \wedge_+^2 \oplus \wedge_-^2$ where $\wedge_+^2$ and $\wedge_-^2$ denote the $+i$, $-i$ eigenspaces. So $F$ has the decomposition $F = F_+ + F_-$, $F_+ \in \wedge_+^2$, $F_- \in \wedge_-^2$ and Maxwell's equations for real forms become

$$dF_+ = 0$$

or equivalently

$$dF_- = 0 .$$

We next give the definition of null electromagnetic fields which are the main objects of our study. The tangent space at any point of $M$ is equipped with the inner product $g$. This induces naturally an inner product on $\wedge^2 T^*M$, which we also denote by $g$.

DEFINITION. We say that a differential 2-form $F$ is null if $g(F, F) = 0$ and $g(F, *F) = 0$. In particular null solutions of Maxwell's equations are called null electromagnetic fields.

## 2. The spinor form of Maxwell's equations

In the previous section Maxwell's equations are given in a Lorentz invariant form. We rewrite these in $SL(2, C)$ invariant form. The formalism is based on the isomorphism between the group $SL(2, C)$ and the universal covering of the connected component of the Lorentz group. We need the following notations;

$$(\varepsilon_{AB}) = (\varepsilon^{AB}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$(\varepsilon_{A'B'}) = (\varepsilon^{A'B'}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$(\sigma_0{}^{AA'}) = (1/2)^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$(\sigma_1{}^{AA'}) = (1/2)^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$(\sigma_2{}^{AA'}) = (1/2)^{1/2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$(\sigma_3{}^{AA'}) = (1/2)^{1/2} \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

$$(g_{ij}) = (g^{ij}) = diag(1, -1, -1, -1).$$

To raise or lower indices, we use the formulas

$$\xi_B = \xi^A \varepsilon_{AB} \quad \text{and} \quad \xi^A = \varepsilon^{AB} \xi_B .$$

Then we have the identities

$$g_{ij} = \sigma_i{}^{I_1 I_2'} \sigma_j{}^{J_1 J_2'} \varepsilon_{I_1 J_1} \varepsilon_{I_2' J_2'} \tag{2.1}$$

for $i, j = 0, 1, 2, 3$.

For a differential 2-form $F = F_{ij} dx^i \wedge dx^j$ we define $F_{AA'BB'}$ for $A, B = 0, 1$ and $A', B' = 0', 1'$ by

$$F_{AA'BB'} = F_{ij} \sigma^i{}_{AA'} \sigma^j{}_{BB'}$$

where

$$\sigma^i{}_{AA'} = g^{ik} \varepsilon_{AC} \varepsilon_{A'C'} \sigma_k{}^{CC'} \quad \text{and} \quad \sigma^j{}_{BB'} = g^{jl} \varepsilon_{BD} \varepsilon_{B'D'} \sigma_l{}^{DD'} .$$

We call $F_{AA'BB'}$ the spinor components of $F$. It follows from (2.1) that

$$F_{ij} F^{ij} = F_{AA'BB'} F^{AA'BB'} . \tag{2.2}$$

Here we give properties of the spinor components of $F$. The details are referred to Penrose [3].

(i) Let $\phi_{AB}=(1/2)F_{BM'A}{}^{M'}$ and $\psi_{A'B'}=(1/2)F_{MA'}{}^{M}{}_{B'}$. Then we have

$$F_{AA'BB'} = \phi_{AB}\varepsilon_{A'B'}+\psi_{A'B'}\varepsilon_{AB} \tag{2.3}$$

and

$$\phi_{AB} = \phi_{BA}\,, \quad \psi_{A'B'} = \psi_{B'A'}\,. \tag{2.4}$$

(ii) Let $F_{AA'BB'}=\phi_{AB}\varepsilon_{A'B'}+\psi_{A'B'}\varepsilon_{AB}$. Then $F$ is a real form if and only if $\overline{\phi_{AB}}=\psi_{A'B'}$.

(iii) The spinor components of $F_-$ and $F_+$ are $\phi_{AB}\varepsilon_{A'B'}$ and $\psi_{A'B'}\varepsilon_{AB}$ respectively and we have

$$*F_{AA'BB'} = -i\phi_{AB}\varepsilon_{A'B'}+i\psi_{A'B'}\varepsilon_{AB}\,. \tag{2.5}$$

We next rewrite Maxwell's equations using the spinor components of $F$. Introducing new variables $x_{AA'}$ for $A=0,1$ and $A'=0', 1'$ by $x_{AA'}=\sigma^{i}{}_{AA'}x_i$, we define differential operators $\nabla^{AA'}$ by $\nabla^{AA'}=\dfrac{\partial}{\partial x_{AA'}}$, for example $\nabla^{01'}=-(1/2)^{1/2}\left(\dfrac{\partial}{\partial x^2}+i\dfrac{\partial}{\partial x^3}\right)$. Next proposition gives the spinor form of Maxwell's equations. Its proof is also referred to Penrose [3].

**Proposition 2.1.** *If $F_{AA'BB'}=\phi_{AB}\varepsilon_{A'B'}+\psi_{A'B'}\varepsilon_{AB}$ is the spinor components of a differential 2-form $F$, then Maxwell's equations for $F$ take the form*

$$\nabla^{AA'}\phi_{AB} = 0 \;\; \textit{and} \;\; \phi_{AB} = \phi_{BA}\,. \tag{3.6}$$

Hereafter we investigate Maxwell's equations in this spinor form.

## 3. Null electromagnetic fields and shear-free null congruences

The relation between null electromagnetic fields and shear-free null congruences is discussed. We first characterize spinor fields which represent null differential 2-forms.

**Proposition 3.1.** *Let $F_{AA'BB'}=\phi_{AB}\varepsilon_{A'B'}+\psi_{A'B'}\varepsilon_{AB}$ be the spinor components of a differential 2-form $F$. Then $F$ is null if and only if $\phi_{AB}\phi^{AB}=0$ and $\psi_{A'B'}\psi^{A'B'}=0$ hold.*

Proof. Assume that $F=F_{ij}dx^i\wedge dx^j$ is null: $F_{ij}F^{ij}=0$ and $F_{ij}*F^{ij}=0$. Then we have by (2.2)

$$F_{AA'BB'}F^{AA'BB'} = F_{ij}F^{ij} = 0\,.$$

Therefore,

$$\phi_{AB}\phi^{AB}\varepsilon_{A'B'}\varepsilon^{A'B'}+\psi_{A'B'}\psi^{A'B'}\varepsilon_{AB}\varepsilon^{AB}$$
$$+\phi_{AB}\varepsilon^{AB}\psi^{A'B'}\varepsilon_{A'B'}+\psi_{A'B'}\varepsilon^{A'B'}\phi^{AB}\varepsilon_{AB}=0\,.$$

Symmetric nature of $\phi_{AB}$ and $\psi_{A'B'}$ implies

$$\phi_{AB}\varepsilon^{AB}=0 \quad\text{and}\quad \psi_{A'B'}\varepsilon^{A'B'}=0\,.$$

Hence it follows that

$$\phi_{AB}\phi^{AB}+\psi_{A'B'}\psi^{A'B'}=0\,. \tag{3.1}$$

We recall that

$$F_{AA'BB'}{*F}^{AA'BB'}=F_{ij}{*F}^{ij}=0\,,$$
$${*F}^{AA'BB'}=-i\phi^{AB}\varepsilon^{A'B'}+i\psi^{A'B'}\varepsilon^{AB}\,.$$

Similarly as above, we have

$$\phi_{AB}\phi^{AB}-\psi_{A'B'}\psi^{A'B'}=0\,. \tag{3.2}$$

By (3.1) and (3.2), we find that

$$\phi_{AB}\phi^{AB}=0 \quad\text{and}\quad \psi_{A'B'}\psi^{A'B'}=0\,. \tag{3.3}$$

Conversely we can verify that (3.3) imply that $F$ is null.

Q.E.D.

Consider a spinor field $\phi_{AB}$ which represents a null differential 2-form. From Proposition 3.1 it follows that $\phi_{AB}\phi^{AB}=0$. But direct calculation shows $\phi_{AB}\phi^{AB}=2det[\phi_{AB}]$. Therefore $\phi_{AB}$, identified with an element of $M(2,C)$, has an eigenvalue 0. Hence we can choose a non-zero spinor field $n^A$ such that $\phi_{AB}n^B=0$.

**Proposition 3.2.** *If a non-zero symmetric spinor field* $\phi_{AB}$ *satisfies* $\nabla^{AA'}\phi_{AB}=0$, *then* $n_B n_J\nabla^{JJ'}n^B=0$.

Proof. From the definition $n_A$ is a spinor field such that

$$\phi_{AB}n^B=0\,. \tag{3.4}$$

Here (3.4) implies that $\phi_{BJ}$ and $n_J$ are linearly dependent, whence

$$\phi_{JB}n_C-\phi_{BC}n_J=0\,. \tag{3.5}$$

Differentiating (3.4) and (3.5), we find that

$$(n^J\nabla_{JJ'}\phi_{AB})n^B+\phi_{AB}n^J\nabla_{JJ'}n^B=0 \tag{3.6}$$

and

$$(\nabla^{JJ'}\phi_{JB})n_C+\phi_{JB}\nabla^{JJ'}n_C-(\nabla^{JJ'}\phi_{BC})n_J-\phi_{BC}\nabla^{JJ'}n_J=0\,.$$

By our assumption on $\phi_{AB}$ it follows that

$$n^J\nabla_{JJ'}\phi_{BC}=\phi^J{}_B\nabla_{JJ'}n_C-\phi_{BC}\nabla_{JJ'}n^J\,. \tag{3.7}$$

Multiply (3.7) by $n^B$ and use $\phi_{JB}n^B=0$. Then we have

$$n^Bn^J\nabla_{JJ'}\phi_{BC}=0\,.$$

Hence from (3.6) we find that

$$\phi_{AB}n^J\nabla_{JJ'}n^B=0\,.$$

Therefore $n^J\nabla_{JJ'}n^B$ are 0 eigenvectors and that they are parallel to $n^B$. This implies

$$n_Bn^J\nabla_{JJ'}n^B=0\,.$$

Q.E.D.

DEFINITION. We say that a spinor field $n_A$ is a shear-free null congruence if $n_A$ satisfies

$$n_Bn_J\nabla^{JJ'}n^B=0 \qquad \text{for } J'=0',\,1'\,.$$

It is now known from Proposition 3.2 that a spinor field $n^A$ such that $\phi_{AB}n^B=0$ is a shear-free null congruence if $\phi_{AB}$ is a null electromagnetic field. Conversely we can construct null electromagnetic fields from shear-free null congruences as explained in the following.

**Proposition 3.3.** *Let $n_A$ be a shear-free null congruence. If a symmetric spinor field $\phi_{AB}$ satisfies $\phi_{AB}n^B=0$, then $(\nabla^{AA'}\phi_{AB})n^B=0$.*

Proof. The assumption $\phi_{BC}n^B=0$ implies

$$(n_A\nabla^{AA'}\phi_{BC})n^B+\phi_{BC}n_A\nabla^{AA'}n^B=0\,. \tag{3.8}$$

Since $\phi_{AB}n^B=0$, we have

$$\phi_{AB}n_C-\phi_{BC}n_A=0\,. \tag{3.9}$$

Differentiating (3.9) and using $\phi_{AB}n^B=0$, we find that

$$n^B(\nabla^{AA'}\phi_{AB})n_C=(\nabla^{AA'}\phi_{BC})n_An^B\,.$$

It follows from (3.8) that

$$n^B(\nabla^{AA'}\phi_{AB})n_C=-\phi_{BC}n_A\nabla^{AA'}n^B\,. \tag{3.10}$$

Shear-free nature of $n_A$ implies the existence of the spinor field $\zeta^{A'}$ so that $n_A\nabla^{AA'}n^B=\zeta^{A'}n^B$. Therefore (3.10) become

$$n^B(\nabla^{AA'}\phi_{AB})n_C=-\phi_{BC}n^B\zeta^{A'}=0\,.$$

Hence we have

$$(\nabla^{AA'}\phi_{AB})n^B = 0\,.$$

Q.E.D.

Let $\phi_{AB}$ be the same as above and $\eta_A$ be the spinor field such that $\phi_{AB}=\eta_A n_B$. Symmetric nature of $\phi_{AB}$ implies $\eta_A n^A=0$. So there exists the scalar function $\kappa$ so that $\eta_A=\kappa n_A$. We have then $\phi_{AB}=\kappa n_A n_B$. Since $(\nabla^{AA'}\phi_{AB})n^B=0$, there exists the spinor field $\xi^{A'}$ so that $\nabla^{AA}\,\phi_{AB}=\xi^{A'}n_B$. Then $\xi^{A'}$ can be represented by means of $\zeta^{A'}$ introduced in the proof of Proposition 3.3, $\kappa$ and $n_A$ as follows.

Differentiating $\phi_{AB}=\kappa n_A n_B$, we find that

$$\begin{aligned}\nabla^{AA'}\phi_{AB} &= (n_A\nabla^{AA'}\kappa)n_B+\kappa(n_A\nabla^{AA'}n_B+n_B\nabla^{AA'}n_A)\\ &= \{n_B\nabla^{AA'}\kappa+(\nabla^{AA'}n_A+\zeta^{A'})\kappa\}n_B\,.\end{aligned}$$

By the definition of $\xi^{A'}$ we have

$$\xi^{A'} = n_A\nabla^{AA'}\kappa+(\nabla^{AA'}n_A+\zeta^{A'})\kappa\,.$$

If the function $\kappa$ satisfies

$$n_A\nabla^{AA'}\kappa+(\nabla^{AA'}n_A+\zeta^{A'})\kappa = 0\,,$$

then we have $\xi^{A'}=0$, namely, $\nabla^{AA'}\phi_{AB}=0$. Hence $\phi_{AB}=\kappa n_A n_B$ is a null electromagnetic field. Summarizing these, we have the next theorem.

**Theorem 3.4.** *Let $n_A$ be a shear-free null congruence and $\zeta^{A'}$ be the spinor field defined by $n_A\nabla^{AA'}n_B=\zeta^{A'}n_B$. If a function $\kappa$ satisfies*

(3.11) $$n_A\nabla^{AA'}\kappa+(\nabla^{AA'}n_A+\zeta^{A'})\kappa = 0\,,$$

*then $\phi_{AB}=\kappa n_A n_B$ is a null electromagnetic field.*

Eqs. (3.11) is an overdetermined system and so the existence of the solutions is not always obvious. In the next section the existence of the solutions will be proved in analytic case.

## 4. Existence theorem

We will dsicuss in complex analytic category.

Eqs. (3.11) are equivalent to homogeneous equations (cf. Courant and Hilbert [1], 31–32)

(4.1) $$n_A\nabla^{AA'}f-(\nabla^{AA'}n_A+\zeta^{A'})\frac{\partial}{\partial\kappa}f = 0\,.$$

We define vector fields $X^{A'}$ on $C^5$ by

$$X^{A'} = n_A\nabla^{AA'} - (\nabla^{AA'}n_A + \zeta^{A'})\frac{\partial}{\partial\kappa} \qquad \text{for } A' = 0', 1'.$$

We need the next definition before we explain a property of $X^{A'}$.

DEFINITION. Let $X_i (i=1,2,\cdots,m)$ be vector fields on $C^n$ where $m<n$. We say that the set of vector fields $\{X_i\}_{i=1}^m$ is a complete system if there eixst functions $\lambda_{ijk}$ for $i,j,k=1,2,\cdots,m$ so that

$$[X_i, X_j] = \sum_{k=1}^m \lambda_{ijk} X_k$$

where

$$[X_i, X_j] = X_iX_j - X_jX_i .$$

For any set of vector fields $\{X_i\}_{i=1}^m$ we define a system of differential equations as follows:

$$X_i f = 0 \qquad \text{for } i = 1, 2, \cdots, m$$

where $f$ is an unknown function. The following theorem for the complete systems is well known (cf. Eisenhart [2]).

**Theorem 4.1.** *Let $X_i$ for $i=1,2,\cdots,m$ be independent vector fields on $C^n$ where $m<n$. If $\{X_i\}_{i=1}^m$ is a complete system, then the system of equations*

$$X_i f = 0 \qquad \textit{for } i = 1, 2, \cdots, m$$

*has $n-m$ independent solutions.*

The system of vector fields $\{X^{A'}\}_{A'=0',1'}$ is the complete system.

**Theorem 4.2.** *Let $n_A$ be a shear-free null congruence and $\zeta^{A'}$ be the spinor field such that $n_A\nabla^{AA'}n_B = \zeta^{A'}n_B$. Then the system of vector fields $\{X^{A'}\}_{A'=0',1'}$ defined by*

$$X^{A'} = n_A\nabla^{AA'} - (\nabla^{AA'}n_A + \zeta^{A'})\frac{\partial}{\partial\kappa}$$

*is the complete system.*

Proof. By direct calculation we have

$$[X^{0'}, X^{1'}] = -n_A\zeta_{A'}\nabla^{AA'} - (n_A\nabla^{AA'}\nabla^B{}_{A'}n_B + n_A\nabla^{AA'}\zeta_{A'})\frac{\partial}{\partial\kappa}. \tag{4.2}$$

We recall that

$$n_A\nabla^{AA'}n_B = \zeta^{A'}n_B .$$

Differentiating we find that

$$n_A\nabla^B{}_{A'}\nabla^{AA'}n_B+n_A\nabla^{AA'}\zeta_{A'}=\zeta^{A'}\nabla^B{}_{A'}n_B-(\nabla^B{}_{A'}n_A)(\nabla^{AA'}n_B).$$

We can easily verify

$$(\nabla^B{}_{A'}n_A)(\nabla^{AA'}n_B)=0$$

and we have

$$n_A\nabla^B{}_{A'}\nabla^{AA'}n_B+n_A\nabla^{AA'}\zeta_{A'}=\zeta^{A'}\nabla^B{}_{A'}n_B. \tag{4.3}$$

Substituting (4.3) into (4.2), we find that

$$[X^{0'},X^{1'}]=-\zeta_{0'}X^{0'}-\zeta_{1'}X^{1'}.$$

Hence $\{X^{A'}\}_{A'=0',1'}$ is the complete system.

Q.E.D.

## 5. Kerr theorem and its application to the construction of null electromagnetic fields

Further we study Eqs. (3.11). Let $n_A$ be a shear-free null congruence. Then Kerr theorem asserts that shear-free nature of $n_A$, which is the integrability condition of Eqs. (3.11), is equivalent to the existence of a certain complex analytic homogeneous function related to $n_A$. In §5 we consider local solutions.

**Theorem 5.1** (Kerr). *An analytic spinor $n_A$ is shear-free if and only if there exists a homogeneous function $f(Z^0,Z^1,Z^2,Z^3)$ which defines a surface in $P^3(C)$ and satisfies*

$$f(n^A,-in^Ax_{AA'})=0.$$

REMARK. The term 'homogeneous' means that

$$f(\lambda Z^0,\lambda Z^1,\lambda Z^2,\lambda Z^3)=\lambda^m f(Z^0,Z^1,Z^2,Z^3)$$

holds for any $\lambda\in C$, where $m$ is a fixed integer. In this case $f$ is called $m$-homogeneous.

Its proof is found in Penrose [4]. Here we give a simple example.

EXAMPLE 5.1. Put $f=Z^0-Z^1+iZ^2+iZ^3$. Then

$$n_0=-1-x_{00'}-x_{01'}\quad\text{and}\quad n_1=1-x_{10'}-x_{11'}$$

satisfy

$$f(n^A,-in^Ax_{AA'})=0.$$

Hence $n_A$ is a shear-free null congruence.

We want to represent the solutions of Eqs. (3.11) using $n_A$ and $f$. For this purpose we need some lemmas.

Consider a shear-free null congruence $n_A$ and a homogeneous function $f(Z^0, Z^1, Z^2, Z^3)$ such that $f(n^A, -in^A x_{AA'})=0$. Put $Z^0=\omega^0$, $Z^1=\omega^1$, $Z^2=\pi_{0'}$, $Z^3=\pi_{1'}$.

**Lemma 5.2.** *There exists the scalar function $\kappa$ such that*

$$\kappa\Big(\frac{\partial f}{\partial\omega^A}-ix_{AA'}\frac{\partial f}{\partial\pi_{A'}}\Big)=n_A \qquad \textit{for } A=0,1\,.$$

Proof. From Euler identity we have

$$\Big(\frac{\partial f}{\partial\omega^A}-ix_{AA'}\frac{\partial f}{\partial\pi_{A'}}\Big)n^A=0\,.$$

Hence we can find the function $\kappa$ such that

$$\kappa\Big(\frac{\partial f}{\partial\omega^A}-ix_{AA'}\frac{\partial f}{\partial\pi_{A'}}\Big)=n_A\,.$$

Q.E.D.

**Lemma 5.3.** *The function $\kappa$ of Lemma 5.2 satisfies*

$$i\kappa\frac{\partial f}{\partial\pi_{B'}}n^B=n_A\nabla^{BB'}n^A\,.$$

Proof. Differentiating $f(n^A, -in^A x_{AA'})=0$, we find that

$$\kappa\Big(\frac{\partial f}{\partial\omega^A}-ix_{AA'}\frac{\partial f}{\partial\pi_{A'}}\Big)\nabla^{BB'}n^A-i\kappa\frac{\partial f}{\partial\pi_{B'}}n^B=0\,.$$

Using Lemma 5.2, we have

$$n_A\nabla^{BB'}n^A-i\kappa\frac{\partial f}{\partial\pi_{B'}}n^B=0\,.$$

Q.E.D.

We can easily verify the following.

**Lemma 5.4.** *For any spinor $n_A$*

$$n_A\nabla^{AA'}n^B-n_A\nabla^{BA'}n^A=n^B\nabla^{AA'}n_A$$

*hold for $A'=0', 1'$ and $B=0, 1$.*

Now we can prove the next theorem which is one of the main results in this paper.

**Theorem 5.5.** *Let $f$ be an $m$-homogeneous function which defines a surface in $P^3(C)$ and $n_A$ be the shear-free null congruence such that $f(n^A, -in^A x_{AA'})=0$.*

*Then the function* $\kappa$ *defined by*

$$\kappa\Big(\frac{\partial f}{\partial\omega^A}-ix_{AA'}\frac{\partial f}{\partial\pi_{A'}}\Big)=n_A \qquad \textit{for} \quad A=0,1$$

*satisfies*

$$(n_A\nabla^{AA'}\kappa)n_B+(m-4)\kappa n_A\nabla^{AA'}n_B+\kappa\nabla^{AA'}(n_An_B)=0\,. \tag{5.1}$$

Proof. Differentiating $\kappa\Big(\frac{\partial f}{\partial\omega^B}-ix_{BB'}\frac{\partial f}{\partial\pi_{B'}}\Big)=n_B$, we find that

$$\begin{aligned} &n_A\nabla^{AA'}n_B \\ &=(n_A\nabla^{AA'}\kappa)\Big(\frac{\partial f}{\partial\omega^B}-ix_{BB'}\frac{\partial f}{\partial\pi_{B'}}\Big)+\kappa n_A\nabla^{AA'}\Big(\frac{\partial f}{\partial\omega^B}-ix_{BB'}\frac{\partial f}{\partial\pi_{B'}}\Big)\,. \end{aligned} \tag{5.2}$$

Here

$$\begin{aligned} &\nabla^{AA'}\Big(\frac{\partial f}{\partial\omega^B}-ix_{BB'}\frac{\partial f}{\partial\pi_{B'}}\Big) \\ &=\Big(\frac{\partial^2 f}{\partial\omega^C\partial\omega^B}-ix_{CC'}\frac{\partial^2 f}{\partial\pi_{C'}\partial\omega^B}\Big)\nabla^{AA'}n^C-i\frac{\partial^2 f}{\partial\pi_{A'}\partial\omega^B}n^A-i\delta^A{}_B\frac{\partial f}{\partial\pi_{A'}} \\ &\quad -ix_{BB'}\Big(\frac{\partial^2 f}{\partial\omega^C\partial\pi_{B'}}-ix_{CC'}\frac{\partial^2 f}{\partial\pi_{C'}\partial\pi_{B'}}\Big)\nabla^{AA'}n^C-x_{BB'}\frac{\partial^2 f}{\partial\pi_{A'}\partial\pi_{B'}}n^A \end{aligned}$$

and that we have

$$\begin{aligned} &n_A\nabla^{AA'}\Big(\frac{\partial f}{\partial\omega^B}-ix_{BB'}\frac{\partial f}{\partial\pi_{B'}}\Big) \\ &=\Big(\frac{\partial^2 f}{\partial\omega^C\partial\omega^B}-ix_{CC'}\frac{\partial^2 f}{\partial\pi_{C'}\partial\omega^B}\Big)n_A\nabla^{AA'}n^C-i\frac{\partial f}{\partial\pi_{A'}}n_B \\ &\quad -ix_{BB'}\Big(\frac{\partial^2 f}{\partial\omega^C\partial\pi_{B'}}-ix_{CC'}\frac{\partial^2 f}{\partial\pi_{C'}\partial\pi_{B'}}\Big)n_A\nabla^{AA'}n^C\,. \end{aligned}$$

We recall that

$$n_A\nabla^{BB'}n^A=i\kappa\frac{\partial f}{\partial\pi_{B'}}n^B$$

and there exists the spinor field $\zeta^{A'}$ so that

$$n_A\nabla^{AA'}n^C=n^C\zeta^{A'}\,.$$

It follows that

$$\begin{aligned} &\kappa n_A\nabla^{AA'}\Big(\frac{\partial f}{\partial\omega^B}-ix_{BB'}\frac{\partial f}{\partial\pi_{B'}}\Big) \\ &=\kappa\Big(\frac{\partial^2 f}{\partial\omega^C\partial\omega^B}-ix_{CC'}\frac{\partial^2 f}{\partial\pi_{C'}\partial\omega^B}\Big)n^C\zeta^{A'}-n_A\nabla_B{}^{A'}n^A \end{aligned}$$

$$-i\kappa x_{BB'}\Big(\frac{\partial^2 f}{\partial\omega^C\partial\pi_{B'}}-ix_{CC'}\frac{\partial^2 f}{\partial\pi_{C'}\partial\pi_{B'}}\Big)n^C\zeta^{A'}.$$

Using Euler identity, we have

$$\begin{aligned}&\kappa n_{A'}\nabla^{AA'}\Big(\frac{\partial f}{\partial\omega^B}-ix_{BB'}\frac{\partial f}{\partial\pi_{B'}}\Big)\\ &=(m-1)\kappa\Big(\frac{\partial f}{\partial\omega^B}-ix_{BB'}\frac{\partial f}{\partial\pi_{B'}}\Big)\zeta^{A'}-n_A\nabla_B{}^{A'}n^A\\ &=(m-1)n_A\nabla^{AA'}n_B-n_A\nabla_B{}^{A'}n^A.\end{aligned}$$

Substituting these into (5.2) and using Lemma 5.4, we find that

$$(n_A\nabla^{AA'}\kappa)n_B+(m-4)\kappa n_A\nabla^{AA'}n_B+\kappa\nabla^{AA'}(n_An_B)=0.$$

Q.E.D.

**Corollary 5.6.** *If $m=4$, $\kappa$ is a solution of Eqs.* (3.11).

Proof. Since

$$(n_A\nabla^{AA'}\kappa)n_B+\kappa\nabla^{AA'}(n_An_B)=0,$$

we have

$$\{n_A\nabla^{AA'}\kappa+(\nabla^{AA'}n_A+\zeta^{A'})\kappa\}n_B=0$$

where $\zeta^{A'}$ is the spinor such that $n_A\nabla^{AA'}n_B=\zeta^{A'}n_B$.
Hence we have

$$n_A\nabla^{AA'}\kappa+(\nabla^{AA'}n_A+\zeta^{A'})\kappa=0.$$

Q.E.D.

It is shown that $\kappa$ is a solution of Eqs. (3.11) if it is obtained from 4-homogeneous analytic function. For an arbitrary $m$ we can construct the solutions of Eqs. (3.11) from $\kappa$ as follows.

**Lemma 5.7.** *Let $n_A$ be a shear-free null congruence and $g(Z^0,Z^1,Z^2,Z^3)$ be an arbitrary $m$-homogeneous analytic function. Then $\lambda=g(n^A, -in^Ax_{AA'})$ satisfies*

$$(n_J\nabla^{JJ'}\lambda)n_A=m(n_J\nabla^{JJ'}n_A)\lambda.$$

Proof. By direct calculation we have

$$\begin{aligned}&\{n_J\nabla^{JJ'}g(n_A, -in^Ax_{AA'})\}n^B\\ &=\Big(\frac{\partial g}{\partial\omega_A}-ix_{AA'}\frac{\partial g}{\partial\pi_{A'}}\Big)(n_J\nabla^{JJ'}n^A)n^B\\ &=\Big(\frac{\partial g}{\partial\omega_A}-ix_{AA'}\frac{\partial g}{\partial\pi_{A'}}\Big)n^A\zeta^{J'}n^B\end{aligned}$$

$$= mg(n^A, -in^A x_{AA'})\zeta^{J'} n^B$$

$$= mg(n^A, -in^A x_{AA'}) n_J \nabla^{JJ'} n^B .$$

Q.E.D.

**Theorem 5.8.** *Let $f$ be an m-homogeneous function which defines a surface in $P^3(C)$ and $n_A$ be the shear-free null congruence such that $f(n^A, -in^A x_{AA'})=0$. Consider the scalar function $\kappa$ defined by*

$$\kappa\left(\frac{\partial f}{\partial \omega^A} - i x_{AA'} \frac{\partial f}{\partial \pi_{A'}}\right) = n_A$$

*and the function $\lambda = g(n^A, -in^A x_{AA'})$ where $g$ is an arbitrary $(m-4)$-homogeneous analytic function. Then $\hat{\kappa} = \lambda\kappa$ is a solution of Eqs.* (3.11). *Conversely every analytic solution of Eqs.* (3.11) *is obtained locally in this way.*

Using Theorem 3.4 and Theorem 5.8, we have the following.

**Corollary 5.9.** *Put $\phi_{AB} = \hat{\kappa} n_A n_B$. Then $\phi_{AB}$ is a null electromagnetic field.*

Proof of Theorem 5.8. The first statement follows at once from Theorem 5.5 and Lemma 5.7. In the following we prove the second statement.

Consider a system of equations for an unknown $\chi$:

$$n_A \nabla^{AA'} \chi = 0 \qquad \text{for } A' = 0', 1' . \tag{5.3}$$

**Lemma 5.10.** *If $\kappa_1$ and $\kappa_2$ be the solutions of Eqs.* (3.11), *then there exists a solution $\chi$ of Eqs.* (5.3) *so that $\kappa_2 = \chi\kappa_1$.*

Proof. The assumption on $\kappa_1$ and $\kappa_2$ implies

$$(n_A \nabla^{AA'} \kappa_1) n_B + \nabla^{AA'}(n_A n_B)\kappa_1 = 0$$

and

$$(n_A \nabla^{AA'} \kappa_2) n_B + \nabla^{AA'}(n_A n_B)\kappa_2 = 0 .$$

Setting $\chi = \kappa_2/\kappa_1$, we have

$$n_A \nabla^{AA'} \chi = 0 .$$

Q.E.D.

By virtue of Lemma 5.10 if we have all solutions of Eqs. (5.3), then we can obtain all solutions of Eqs. (3.11).

Set $X = n_1/n_0$. Then Eqs. (5.3) are equivalent to

$$\nabla^{0A'} \chi + X \nabla^{1A'} \chi = 0 \qquad \text{for } A' = 0', 1' . \tag{5.4}$$

Here we note that shear-free nature of $n_A$ is conformally invariant and therefore

$(1, X)$ is also shear-free. Hence we have

$$\partial_u X + X \partial_{\bar{\zeta}} X = 0\,, \qquad \partial_\zeta X + X \partial_v X = 0 \tag{5.5}$$

where

$$(x_{AA'}) = \begin{bmatrix} u & \zeta \\ \bar{\zeta} & v \end{bmatrix}$$

and $\partial_u = \dfrac{\partial}{\partial u}$, $\partial_{\bar{\zeta}} = \dfrac{\partial}{\partial \bar{\zeta}}$ etc.

Using the same notations as above, Eqs. (5.4) become

$$\partial_u \chi + X \partial_{\bar{\zeta}} \chi = 0\,, \qquad \partial_\zeta \chi + X \partial_v \chi = 0\,. \tag{5.6}$$

Eqs. (5.6) are solved easily. In fact two independent solutions are obtained.

**Lemma 5.11.** *We have two independent solutions* $\chi_1 = v - \zeta X$ *and* $\chi_2 = \bar{\zeta} - uX$ *of Eqs.* (5.6).

Proof. By (5.5) we find that

$$\partial_u \chi_1 + X \partial_{\bar{\zeta}} \chi_1 = -\zeta(\partial_u X + X \partial_{\bar{\zeta}} X) = 0$$

and

$$\partial_\zeta \chi_1 + X \partial_v \chi_1 = -\zeta(\partial_\zeta X + X \partial_v X) = 0\,.$$

Also we have

$$\partial_u \chi_2 + X \partial_{\bar{\zeta}} \chi_2 = 0\,, \qquad \partial_\zeta \chi_2 + X \partial_v \chi_2 = 0\,.$$

Q.E.D.

Therefore every solution of Eqs. (5.6) is given by

$$\chi = h(\bar{\zeta} - uX, v - \zeta X)$$

where $h$ is an arbitrary complex analytic function. Now we return to spinor representation. We define a 0-homogeneous complex analytic function $\hat{h}(Z^1, Z^2, Z^3)$ by

$$\hat{h}(Z^1, Z^2, Z^3) = h\left(i \frac{Z^2}{Z^1}, i \frac{Z^3}{Z^1}\right)$$

Then we have

$$\hat{h}(n^1, -in^A x_{AA'}) = h\left(\frac{1}{n^1} n^A x_{AA'}\right)$$
$$= h(\xi - uX, v - \zeta X) .$$

Hence every solution of Eqs. (3.11) is represented in the form which is given in the first statement.

Q.E.D.

EXAMPLE 5.1′. In the case of Example 5.1, we find $\kappa=-1$. Put $g=-1/(Z^1)^3$. Then we have $\lambda=-1/(1+x_{00'}+x_{01'})^3$. Hence we have a null electromagnetic field $\phi_{AB}$:

$$\phi_{00} = 1/(1+x_{00'}+x_{01'}) ,$$
$$\phi_{01} = (x_{10'}+x_{11'}-1)/(1+x_{00'}+x_{01'})^2 ,$$
$$\phi_{11} = (x_{10'}+x_{11'}-1)^2/(1+x_{00'}+x_{01'})^3 .$$

EXAMPLE 5.2. Put $f=Z^0-iZ^1$. Then $(n_A)=(-1, i)$ is a shear-free null congruence such that $f(n^A, -in^A x_{AA'})=0$. In this case we have $\kappa=-1$. Therefore we have

$$(\kappa n_A n_B) = \begin{bmatrix} -1 & i \\ i & 1 \end{bmatrix}$$

Put $g=-\frac{1}{2} \sin \left\{\frac{iZ^2-Z^3}{(2)^{1/2}Z^0}\right\} \left\{\frac{1}{(Z^0)^3}\right\}$. Then we have

$$(\phi_{AB}) = \begin{bmatrix} \frac{i}{2} \sin (x^0-x^2) & \frac{1}{2} \sin (x^0-x^3) \\ \frac{1}{2} \sin (x^0-x^3) & -\frac{i}{2} \sin (x^0-x^3) \end{bmatrix} .$$

In tensor form we have

$$(E_1, E_2, E_3) = (\sin (z-t), 0, 0) ,$$
$$(B_1, B_2, B_3) = (0, \sin (z-t), 0) .$$

This electromagnetic field represents light wave in charge free vacuum space.

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