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Author(s)	Dempwolff, Ulrich
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A NOTE ON SEMIFIELD PLANES ADMITTING IRREDUCIBLE PLANAR BAER COLLINEATIONS

ULRICH DEMPWOLFF

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Abstract

In this note we study finite semifield planes which admit an irreducible planar Baer collineation. This continues previous work of N. Johnson [5].

1. Introduction

In [5] N. Johnson investigates semifield planes of order q^4 , $q = p^f$, p a prime, which have rank 2 over the kernel and which admit a planar Baer collineation π of order r , where r is a p -primitive prime divisor of $q + 1$. He proves that such planes are obtained from semifield planes of order q^2 and rank 2 by an elegant construction due to Hiramane et al. [3] (and generalized by Johnson [6]). In this note we remove the restriction on the rank and weaken slightly the assumption on π by assuming that π is an irreducible Baer collineation, that is π acts irreducibly on $[X, \pi]$ for any fiber X being fixed by π . In Section 2 we show that these planes have usually a structure which is a natural generalization of the rank 2 case. However there is an additional possibility which we call the indecomposable case. In Section 3 we discuss a computer enumeration of semifield planes of order 2^8 and 5^4 which admit an irreducible Baer collineation. We find examples which are genuinely of rank 4, i.e. can not be obtained from a rank 2 example by the operations associated with the cubical array of a semifield [8]. In Section 4 we present three series of semifield planes genuinely of rank ≥ 4 admitting irreducible Baer collineations. While two series belong to known classes of semifield planes the third series generalizes some examples of Section 3 and it seems that this class has not been described in the literature before.

2. Irreducible planar Baer collineations on semifield planes

Set $V = K^n$, $K = \text{GF}(p)$, p a prime and let $\Sigma \subseteq \text{GL}(V) \cup 0$ be a spread set of a (pre-)semifield, i.e. Σ is an additive group. Let $\psi: V \rightarrow \Sigma$ be an arbitrary group isomorphism. Then we can associate with Σ a pre-semifield $S = S(\Sigma)$: the additive group is $(V, +)$ and the semifield multiplication is defined by $x * y = x\psi(y)$.

Set $W = V^2$ and define as usual by $\mathcal{S} = \mathcal{S}_\Sigma = \{V(\infty), V(\sigma) \mid \sigma \in \Sigma\}$ the associated spread. Here $V(\infty) = 0 \times V$ and $V(\sigma) = \{(v, v\sigma) \mid v \in V\}$, $\sigma \in \Sigma$ (the notation agrees with [9]). Finally, we denote by $\mathbf{P} = \mathbf{P}(W, \mathcal{S}) = \mathbf{P}_\Sigma$ the translation plane defined by \mathcal{S} .

Let $\pi \in \text{GL}(W)$ induce a planar Baer collineation, i.e. $n = 2m$, $\dim W_0 = \dim W_1 = n$ where $W_0 = C_W(\pi) = \ker(\pi - 1)$, $W_1 = [W, \pi] = \text{Im}(\pi - 1)$ and π fixes $p^m + 1$ fibers. Let Y be any fiber which is fixed by π . We call π an *irreducible* planar Baer collineation if π as a $\text{GF}(p)$ -linear Operator is irreducible on $Y \cap W_1$; i.e. $Y = (Y \cap W_1) \oplus (Y \cap W_0)$. We choose our notation such that $V(\infty)$ and $V(0)$ are fixed by π . Following Johnson [5] we choose bases of these spaces according to the decompositions $V(0) = (V(0) \cap W_1) \oplus (V(0) \cap W_0)$ and $V(\infty) = (V(\infty) \cap W_0) \oplus (V(\infty) \cap W_1)$. Hence (the notion π -morphism stands for a homomorphism of $\text{GF}(p)\langle\pi\rangle$ -modules):

Lemma 2.1. *With the assumptions from above one has:*

- (a) *With respect to the decomposition $W = V(0) \oplus V(\infty)$ the collineation π has a matrix $\text{diag}(\mathcal{X}, \mathcal{Y})$, $\mathcal{X}, \mathcal{Y} \in \text{GL}(n, p)$, with $\mathcal{X} = \text{diag}(P, 1)$, $\mathcal{Y} = \text{diag}(1, Q)$, $P, Q \in \text{GL}(m, p)$ and $|\pi| = |P| = |Q|$.*
- (b) *The matrix representation $T: \Sigma \rightarrow K^{n \times n}$ has the form*

$$T(\sigma) = \begin{pmatrix} T_{11}(\sigma) & T_{12}(\sigma) \\ T_{21}(\sigma) & T_{22}(\sigma) \end{pmatrix},$$

with quadratic blocks of size m . π acts on $T(\Sigma)$ by $T(\sigma^\pi) = \mathcal{X}^{-1}T(\sigma)\mathcal{Y}$. The maps $T_{ij}: \Sigma \rightarrow K^{m \times m}$ are π -morphisms with respect to the actions $T_{11}(\sigma^\pi) = P^{-1}T_{11}(\sigma)$, $T_{12}(\sigma^\pi) = P^{-1}T_{12}(\sigma)Q$, $T_{21}(\sigma^\pi) = T_{21}(\sigma)$, and $T_{22}(\sigma^\pi) = T_{22}(\sigma)Q$.

The following result generalizes Section 2 of [5].

Proposition 2.2. *We use the assumptions and the notations of the lemma:*

- (a) *$m = 2k$ and $|\pi|$ divides $p^k + 1$.*
- (b) *Set $\Sigma_0 = C_\Sigma(\pi)$ and $\Sigma_1 = [\Sigma, \pi]$. Then $\Sigma = \Sigma_0 \oplus \Sigma_1$ and $|\Sigma_0| = |\Sigma_1| = p^m$.*
- (c) *Choosing the basis of W in a suitable way one has $P = Q$. Moreover $L = K[Q]$ is a subring of $K^{m \times m}$ which is isomorphic to $\text{GF}(p^m)$.*
- (d) *There exists a semifield spread set $\bar{\Sigma} \subseteq K^{m \times m}$ and an additive bijection $\alpha: L \rightarrow \bar{\Sigma}$ with:*

$$T(\Sigma_0) = \left\{ \left(\begin{array}{cc} 0 & u \\ \alpha(u) & 0 \end{array} \right) \mid u \in L \right\}$$

- (e) *We have a π -morphism $\beta: L \rightarrow T_{12}(\Sigma_1)$ such that*

$$T(\Sigma_1) = \left\{ \left(\begin{array}{cc} u & \beta(u) \\ 0 & u^{p^k} \end{array} \right) \mid u \in L \right\}.$$

Moreover there exists a matrix $B \in K^{m \times m}$ such that $\beta(u) = \sum a_i Q^{-i} B Q^i$ where u has the form $u = f(Q)$, $f \in K[X]$, $f = \sum a_i X^i$.

(f) Let $|\pi| = p^k + 1$. Then $\beta = 0$ (i.e. $B = 0$) for $p > 2$. For $p = 2$ let π act via conjugation with Q on $K^{m \times m}$. There exists a π -subspace U of $K^{m \times m}$ of order 2^{3m} with $B \in U$.

Proof. By our assumptions π is a p' -element and $\Sigma = \Sigma_0 \oplus \Sigma_1$ by the theorem of Maschke.

Let $0 \neq \sigma \in \Sigma_0$. Then $T_{11}(\sigma) = P^{-1}T_{11}(\sigma)$, $T_{12}(\sigma) = P^{-1}T_{12}(\sigma)Q$ and $T_{22}(\sigma) = T_{22}(\sigma)Q$. This implies $T_{11}(\sigma) = T_{22}(\sigma) = 0$ (as Q (P^{-1}) acts fixed-point-freely on $K^{m \times m}$ by right (left) multiplication) and $T_{12}(\sigma), T_{21}(\sigma) \in GL(m, p)$. Moreover there exist $\lambda, \mu \in GF(p^m)$ having the order of $|\pi|$, such that $\lambda, \lambda^p, \dots, \lambda^{p^{m-1}}$ are the eigenvalues of P and $\mu, \mu^p, \dots, \mu^{p^{m-1}}$ are the eigenvalues of Q . Since both operators are irreducible the eigenvalues in either case are pairwise different. Act with π on $K^{m \times m}$ via $X^\pi = P^{-1}XQ$. Then $T_{12}(\sigma)$ is fixed under this action. As π has on $K^{m \times m}$ the eigenvalues $\lambda^{-p^i} \mu^{p^j}$, $0 \leq i, j \leq m-1$ we must have $\lambda^{p^i} = \mu^{p^j}$ with i, j suitable chosen. Then P and Q have the same minimal polynomial over K and are therefore conjugate in $GL(m, p)$. By choosing an appropriate basis of $V(0) \cap W_1$ we can assume $P = Q$. Again as Q is irreducible $L = K[Q] \simeq GF(p^m)$ and $C_{K^{m \times m}}(Q) = L$. Thus $T_{12}(\sigma) \in L$. (b), (c) and (d) follow.

Assume now $0 \neq \sigma \in \Sigma_1$. Since $\Sigma_1 = [\Sigma_1, \pi]$ we see $T_{21}(\sigma) = 0$. Then there exist $A, C \in GL(m, p)$ and $B \in K^{m \times m}$ with

$$T(\sigma) = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}.$$

The transformation $\text{diag}(1, A, C^{-1}, 1) \in GL(W)$ commutes with π . Considering the associated basis transformation we may assume $A = C = 1$. As $\Sigma_1 = \langle \sigma^{\pi^i} \mid i = 0, 1, 2, \dots \rangle$ we see that β has the form described in (e).

Suppose Q and Q^{-1} have different minimal polynomials over K . Then we have a $f = \sum_i a_i X^i \in K[X]$ with $f(Q^{-1}) = 0 \neq f(Q)$ and

$$T\left(\sum_i a_i \sigma^{\pi^i}\right) = \begin{pmatrix} 0 & * \\ 0 & f(Q) \end{pmatrix} \in \Sigma_1,$$

a contradiction. Hence there exists a k with $\lambda^{-1} = \lambda^{p^k}$, i.e. $\lambda^{p^k+1} = 1$ respectively. Thus $|\pi| = |Q|$ is a divisor of $(p^{2k} - 1, p^m - 1) = p^t - 1$, where $t = (2k, m)$. Irreducibility implies $m = 2k$ and $|\pi|$ divides $p^k + 1$. (a) and (e) follow.

Assume finally $|\pi| = p^k + 1$ and consider first the case $p > 2$. Pick $0 \neq \sigma \in \Sigma_1$ as above. Then $Q_0 = Q^{(p^k+1)/2} = -1$ and $-2^{-1}[T(\sigma), \pi^{(p^k+1)/2}] = 1 \in \Sigma_1$ which implies $B = 0$.

Now consider the case $p = 2$ and assume $B \neq 0$. Then the mappings T_{11}, T_{12}, T_{22} are all π -monomorphisms into $K^{m \times m}$. Hence $T_{11}(\Sigma_1) \simeq T_{12}(\Sigma_1) \simeq T_{22}(\Sigma_1)$ as π -modules. The lemma shows that $T_{11}(\Sigma_1)$ and $T_{22}(\Sigma_1)$ are isomorphic to K^m where $D: \langle \pi \rangle \rightarrow \text{GL}(K^m)$ is the natural action on this space via multiplication with Q . On the other hand $T_{12}(\Sigma_1)$ is a π -submodule of $\Delta = K^{m \times m}$ with the action $X^\pi = Q^{-1}XQ$. Choose $\Phi \in \text{GL}(m, 2)$ such that $Q^\Phi = Q^2$ (see [4], Kapitel II, 7.3 Satz, p.187). Then

$$\Delta = \bigoplus_{j=0}^{m-1} \Phi^j L,$$

is a decomposition into π -modules. Obviously, the module $\Phi^j L$ induces the representation $D^{1-2^j} \sim D^{2^j-1}$ and π has on this module the eigenvalues $\lambda^{2^j-1}, (\lambda^{2^j-1})^2, \dots, (\lambda^{2^j-1})^{2^{m-1}}$. Assume that B projects nontrivially into $\Phi^j L$. Then $\lambda^{2^j-1} \in \{\lambda, \lambda^2, \dots, \lambda^{2^{m-1}}\}$, i.e. $\lambda^{2^l-2^j+1} = 1$ with a suitably chosen l . We conclude

$$2^l - 2^j + 1 \equiv 0 \pmod{2^k + 1}.$$

We claim that solutions only occur for $(l, j) = (0, 1), (k - 1, 2k - 1), (k + 1, k)$ in that case. Then assertion (f) will follow.

In order to prove the claim we distinguish 4 cases according as to whether or not $j(l) \leq k$ or $j(l) > k$. Assume first $j, l \leq k$. Then $|2^l - 2^j + 1| \leq 2^k$ and thus $2^l - 2^j + 1 = 0$. This forces $j = 1, l = 0$. Assume next $j, l > k$. As $2^k \equiv -1 \pmod{2^k + 1}$ we have $-2^{l-k} + 2^{j-k} + 1 \equiv 0 \pmod{2^k + 1}$ and hence $j = k, l = k + 1$ by the previous case. But this contradicts $j > k$. The case $l \leq k < j$ leads to $j = 2k - 1, l = k - 1$ in a similar manner. The case $j \leq k < l$ implies $j = k, l = k + 1$. □

REMARKS. (a) Use the notation of the proposition. If $\beta = 0$ then the group $\langle \pi \rangle$ can be extended in $\text{Aut}(\mathbf{P}_\Sigma)$ to a cyclic group $\langle \pi^* \rangle$ of order $p^k + 1$ of planar collineations ($\pi^* = \text{diag}(Q^*, 1, 1, Q^*)$ with $Q^* \in L$ of order $p^k + 1$).

(b) If \mathbf{P}_Σ has the kernel $F \simeq \text{GF}(q), q = p^m$, and if π is a F -linear map we see that $T_{12}(\sigma^\pi) = T_{12}(\sigma)$ for $\sigma \in \Sigma$. Hence $\beta = 0$. These assumptions are satisfied in the situation of Johnson [5] and thus the results of Section 2 of [5] are a consequence of Proposition 2.2.

DEFINITION. Use the notation of the proposition. We call Σ *decomposable* if $T_{12}(\Sigma_1) = 0$ (i.e. $\beta = 0$) and *indecomposable* if $T_{12}(\Sigma_1) \neq 0$ (i.e. $\beta \neq 0$). Let $\text{MinRk}(\Sigma)$ be the minimum of the dimensions of the associated pre-semifield over the seminuclei (left, right, and middle nucleus).

REMARK. Consider the cubical array associated with Σ (see Knuth [8]). Clearly, any member Σ' of the cubical array admits a planar irreducible Baer-collineation too.

On the other hand the kernel of Σ and of Σ' can be different (see [1]); indeed the operations associated with a cubical array permute the roles of the left, right, and middle nuclei by the natural action of the group $\text{Sym}(3)$ (recall that the left nucleus is isomorphic to the kernel of \mathbf{P}_Σ as we are using the conventions of [9], p.24). In order to obtain examples which are *genuinely* different from the examples provided by [5] we are interested in the following questions.

- Are there indecomposable examples?
- Are there examples Σ with $\text{MinRk}(\Sigma) > 2$?

The next result concerns the computation of the seminuclei.

Denote by K_l, K_m, K_r the left, middle and right nucleus of the pre-semifield $S = S(\Sigma)$. The multiplicative groups K_l^*, K_m^*, K_r^* are isomorphic to the groups of $((0, 0), L_\infty)$ -homologies, $((0), V(\infty))$ -homologies, and of $((\infty), V(0))$ -homologies, [9], p.24. Using coordinates we therefore obtain

$$\begin{aligned}
 K_l &\simeq k_l = \{(X, Y) \in (\text{GL}(n, p) \cup 0)^2 \mid XA = AY, A \in T(\Sigma)\}, \\
 K_m &\simeq k_m = \{X \in \text{GL}(n, p) \cup 0 \mid XT(\Sigma) \subseteq T(\Sigma)\}, \\
 K_r &\simeq k_r = \{X \in \text{GL}(n, p) \cup 0 \mid T(\Sigma)X \subseteq T(\Sigma)\}.
 \end{aligned}$$

The planar collineation acts on k_m by conjugation with \mathcal{X} (notation of Lemma 2.1), on k_r by conjugation with \mathcal{Y} , and on k_l by conjugation with $(\mathcal{X}, \mathcal{Y})$. Finally, for $u = f(Q) \in L, f \in K[X]$, we denote the elements of Σ_0 and Σ_1 corresponding to u by

$$s_0(u) = \begin{pmatrix} 0 & u \\ \alpha(u) & 0 \end{pmatrix}, \quad s_1(u) = \begin{pmatrix} u & \beta(u) \\ 0 & \bar{u} \end{pmatrix}$$

where $\bar{u} = u^{p^k}$.

Lemma 2.3. *Use the notation from above.*

- (a) k_l is the field of pairs $(\text{diag}(u, u), \text{diag}(u, u))$, $u \in L$ with $\alpha(v)u = u\alpha(v)$ and $\beta(v)u = u\beta(v)$ for all $v \in L$.
- (b) k_m is the field of matrices $\text{diag}(u, \bar{u})$, $u \in L$ with $\alpha(uv) = \bar{u}\alpha(v)$ and $\beta(uv) = u\beta(v)$ for all $v \in L$.
- (c) k_r is the field of matrices $\text{diag}(u, \bar{u})$, $u \in L$ with $\alpha(vu) = \alpha(v)\bar{u}$ and $\beta(vu) = \beta(v)\bar{u}$ for all $v \in L$.

Proof. (b) Suppose $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in k_m - C_{k_m}(\pi)$. Then $0 \neq B = A^\mathcal{X} - A \in k_m$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix}$ with $\det B_{12} \neq 0 \neq \det B_{21}$. Let M be the additive group generated by $B, B^\mathcal{X}, B^{\mathcal{X}^2}, \dots$. Then $|M| \geq p^m$: The row space (column space) K^m is under the natural action $Q: v \mapsto vQ$ ($Q: v^t \mapsto Qv^t$) an irreducible $\text{GF}(p)\langle Q \rangle$ -module. Hence

$B_{12}, B_{12}Q, B_{12}Q^2, \dots$ generate (as an additive group) a group of order $\geq p^m$. As $B^2 \in k_m - M$ we see that even $|k_m| \geq p^{m+1}$ holds. On the other hand Σ is a vector space over k_m . This implies $|k_m| = p^n$. Hence Σ is Desarguesian. However a Desarguesian spread does not admit an irreducible, planar Baer collineation, a contradiction. Hence π centralizes k_m . This shows that $0 \neq A \in k_m$ has the form $A = \text{diag}(A_1, A_2)$ with $A_i \in \text{GL}(m, p)$ and $A\Sigma_i = \Sigma_i$, $i = 0, 1$. From $As_1(1) \in \Sigma_1$ we deduce $A_1 = u \in L$ and $A_2 = \bar{u}$. Finally $As_1(v) = s_1(uv)$ implies $u\beta(v) = \beta(uv)$ for all $v \in L$. Similarly one obtains $\bar{u}\alpha(v) = \alpha(uv)$ for all $v \in L$.

(c) follows by symmetry.

(a) By considering the action of π on k_l one observes as before that π centralizes k_l . This shows that the elements in k_l have the form $(\text{diag}(A_1, A_2), \text{diag}(B_1, B_2))$. From $\text{diag}(A_1, A_2)s_1(1) = s_1(1)\text{diag}(B_1, B_2)$ we deduce $A_i = B_i \in L$ and as $\text{diag}(A_1, A_2)s_0(1) = s_0(1)\text{diag}(A_1, A_2)$ we see $A_1 = A_2 = u \in L$. Finally we get $u\alpha(v) = \alpha(v)u$ and $u\beta(v) = \beta(v)u$ from $\text{diag}(u, u)s_i(v) = s_i(v)\text{diag}(u, u)$, $i = 0, 1$. □

3. Small orders

Semifields of order 2^4 and 3^4 are known [2]. For order 2^4 the example with an irreducible planar Baer collineation has dimension 2 over the kernel. For order 3^4 all 8 examples with such collineations have $\text{MinRk}(\Sigma) = 2$. By a straightforward computer enumeration we determined the semifield planes of order 2^8 and 5^4 with this property. We summarize the results; more details are displayed on my home page: www.mathematik.uni-kl.de/~dempw/dempw_IrrCol_semi.html.

ORDER 2^8 . There are 14 semifield planes which admit an irreducible planar Baer collineation. They are all decomposable and for 13 of them we have $\text{MinRk}(\Sigma) = 2$. For the remaining spread set Σ we have $\text{MinRk}(\Sigma) = 4$. A multiplication of an associated semifield $S(\Sigma)$ (which is identified as a $\text{GF}(2)$ -space with $\text{GF}(16)^2$) is given by

$$(u, v) * (x, y) = (ux + v(z^{12}x + z^8\bar{x}) + \bar{v}(x + z^3\bar{x}), uy + v\bar{x}).$$

Here z is a generator of $\text{GF}(16)^*$ with $z^4 + z + 1 = 0$ and $\bar{x} = x^4$.

ORDER 5^4 . There are 36 semifield planes which admit an irreducible planar Baer collineation. For 21 spread sets we have $\text{MinRk}(\Sigma) = 2$. For remaining the 15 semifield planes $\text{MinRk}(\Sigma) = 4$ holds. Moreover 9 of these semifield planes are decomposable and 6 indecomposable. We now describe the multiplication rules of the associated semifields in the case $\text{MinRk}(\Sigma) = 4$. For this purpose we identify $S(\Sigma)$ with $\text{GF}(25)^2$ and denote by z generator of $\text{GF}(25)^*$ with $z^2 - z + 2 = 0$. The 9 decomposable spread sets are partitioned in 3 cubical arrays each of them having 3 members. We present the multiplication rule for representatives from each cubical array. It has the form

$$(u, v) * (x, y) = (ux + v(ay + b\bar{y}) + \bar{v}(cy + d\bar{y}), uy + v\bar{x})$$

with $(a, b, c, d) = (z^{a'}, z^{b'}, z^{c'}, z^{d'})$ and (a', b', c', d') is one of the following quadruples

$$(13, 14, 14, 22), \quad (14, 15, 5, 6), \quad (20, 20, 12, 19),$$

and $\bar{x} = x^5$. The 6 indecomposable spread sets represent 6 cubical arrays with one member. The multiplication has the form

$$(u, v) * (x, y) = (ux + bv\bar{y}, uy + a\bar{u}x + v\bar{x})$$

with $(a, b) \in \{(z^5, z^7), (z^{13}, z^9), (z, 1), (z^9, z^8), (z, z), (z^5, z)\}$.

In the indecomposable case it is not difficult to see that a semifield with the opposite multiplication (i.e. $(x, y) \circ (u, v) = (u, v) * (x, y)$) is isotopic to a semifield of type II in the notation of Knuth [8], p.215.

4. Series with $\text{MinRk}(\Sigma) \geq 4$

We present three series of semifield planes \mathbf{P}_Σ admitting irreducible, planar Baer collineations and with $\text{MinRk}(\Sigma) \geq 4$. Two of these series are described for instance by Knuth in [8] while the third series generalizes examples of the previous section.

In this section we will use Oyama’s [10] description of vectors and matrices which for convenience we sketch briefly. Let $F = \text{GF}(q)$ and $E = \text{GF}(q^m)$. The vector space F^m is identified with the F -space ${}^0F^m$ of vectors of the form $((a)) = (a, a^q, \dots, a^{q^{m-1}}) \in E^m, a \in E$. The F -endomorphisms of ${}^0F^m$ form the F -space ${}^0F^{m \times m}$ of matrices $(a_{ij}) \in E^{m \times m}$ with the property $a_{i+1, j+1} = a_{ij}^q, 0 \leq i, j < m$ (indices are read modulo m). Such a matrix is determined by its first column and thus we define $[a_0, \dots, a_{m-1}]^t := (a_{ij})$ if $(a_0, \dots, a_{m-1}) = (a_{00}, \dots, a_{m-1,0})$. Set

$$T_k(a) = \sum_{i=0}^{m-1} a^{q^i} E_{k+i, i}.$$

Then $[a_0, \dots, a_{m-1}]^t = \sum_{i=0}^{m-1} T_i(a_i)$. We have the multiplication rules

$$T_j(u)T_k(v) = T_{j+k}(u^{q^k} v), \quad T_k(a)^{-1} = T_{m-k}(a^{-q^{m-k}}), \quad a \neq 0.$$

The main advantage of Oyama’s notation is that the cyclic Singer group $\{T_0(u) \mid u \in E - \{0\}\}$ of order $q^m - 1$ is a group of diagonal matrices.

We apply these notations to the notions of Section 2. We have $W = V \times V$ with $V = {}^0F^N \times {}^0F^N$ where $m = N \cdot r$ and $q = p^r, p$ a prime. We write (u, v) for a typical element in V instead of $((u)), ((v))$. The matrices of a spread set will have the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in {}^0\text{GL}(N, F) \cup 0$$

where ${}^0\text{GL}(N, F)$ is the group of invertible elements in ${}^0F^{N \times N}$. The matrices $T_0(u)$, $u \in E$, form a field L of matrices isomorphic to E (see Proposition 2.2). The planar collineation π has the form $\text{diag}(T_0(u), 1, 1, T_0(u))$ where $u \in E^*$ has order $p^k + 1$ in the decomposable case (has an order dividing $p^k + 1$ in the indecomposable case).

EXAMPLE 4.1. Set $K = F = \text{GF}(p)$, p a prime and $E = \text{GF}(p^m)$, $m = 2k$. Choose $a, b \in \{0, \dots, m - 1\}$ such that $E \neq E^{p^{b+1}}E^{p^{a-b+1}}E^{p^{b+k}-1}$ and pick $g \in E - E^{p^{b+1}}E^{p^{a-b+1}}E^{p^{b+k}-1}$. The exponents of p are always read modulo $m = 2k$ in this example. Then

$$(u, v) * (x, y) = (ux + gv^{p^a}y^{p^b}, uy + vx^{p^k})$$

defines a semifield multiplication on V . It can easily be seen that no zero divisors occur. In fact it is also not hard to see that this semifield is isotopic to a semifield defined in [8] on p.215 by (7.16). Let Σ be the spread set associated with this semifield and $T(\Sigma)$ its coordinatization. It has the form $T(\Sigma) = \{s(x, y) \mid x, y \in E\}$ where

$$s(x, y) = \begin{pmatrix} T_0(x) & T_0(y) \\ T_a(gy^{p^b}) & T_0(x^{p^k}) \end{pmatrix}.$$

Then Σ is invariant under π and the mappings α, β of Proposition 2.2 have the form $\alpha(T_0(y)) = T_a(gy^{p^b})$ and $\beta(T_0(x)) = 0$. In particular Σ is decomposable. We have:

- (1) $k_l \simeq \text{GF}(p^{(a,m)})$, $k_m \simeq \text{GF}(p^{(a+k-b,m)})$, $k_r \simeq \text{GF}(p^{(k-b,m)})$:

By Lemma 2.3 the element $s(u, 0)$ lies in k_m iff for all y

$$T_a(g uy^{p^b}) = \alpha(T_0(uy)) = T_0(\bar{u})\alpha(T_0(y)) = T_0(u^{p^k})T_a(gy^{p^b}) = T_a(gu^{p^{a+k}}y^{p^b}).$$

This shows $u^{p^b} = u^{p^{a+k}}$ and thus $k_m \simeq \text{GF}(p^{(a+k-b,m)})$. The other assertions follow similarly.

- (2) For any choice of a, b, g we have $\text{MinRk}(\Sigma) < n = 2m$. If $p > 2$ or if $p = 2$ and k is odd one can choose a, b, g such that $\text{MinRk}(\Sigma) = m$:

$\text{MinRk}(\Sigma) = n$ and $m = 2k$ implies $(a + k - b, 2k) = (k - b, 2k) = (a, 2k) = 1$. But then $a, k - b$ are odd and 2 divides $(a + k - b, 2k)$, a contradiction.

Assume first $p > 2$ and let g be a nonsquare in E . Then $a = 2, b = 1$, imply $\text{MinRk}(\Sigma) = m$.

Assume now $p = 2$. The condition for the existence of a semifield multiplication of the desired type is equivalent to $(2^{2k} - 1, 2^{b+k} - 1, 2^{|a-b|} + 1, 2^b + 1) > 1$. This in turn is equivalent to

$$a_2 > b_2 = k_2$$

where x_2 denotes the 2-part of a positive integer x : Recall that $(2^x + 1, 2^y - 1) = 2^{(x,y)} + 1$ iff $x/(x, y) \equiv 1, y/(x, y) \equiv 0 \pmod{2}$ (and $= 1$ otherwise) and $(2^x + 1, 2^y + 1) = 2^{(x,y)} + 1$

iff $x/(x, y) \equiv y/(x, y) \equiv 1 \pmod{2}$ (and $= 1$ otherwise). Then $(2^{2k} - 1, 2^b + 1) > 1$ and $(2^{b+k} - 1, 2^b + 1) > 1$ imply $b_2 = k_2$ and $(2^{|a-b|} + 1, 2^b + 1) > 1$ implies $a_2 > b_2$. On the other hand if $a_2 > b_2 = k_2$ we observe $(2^{2k} - 1, 2^{b+k} - 1, 2^{|a-b|} + 1, 2^b + 1) = 2^{(a,b,k)} + 1 > 1$. If k is odd we can take $a = 2, b = 1$ as before. Now (2) follows.

CONCLUSION. The preceding examples show that in any characteristic there exist decomposable semifield planes \mathbf{P}_Σ with arbitrary large $\text{MinRk}(\Sigma)$ and which admit irreducible planar Baer collineations.

For the next two examples we have $F = \text{GF}(q), E = \text{GF}(q^2)$, with $q = p^k, p$ a prime. We write \bar{x} for x^q .

EXAMPLE 4.2. First we generalize the indecomposable examples of order 5^4 from Section 3. Let q be an odd prime power. Choose $a, b \in E$ such that $y^{q+1} + ay - b \neq 0$ for $y \in E$. Then the multiplication

$$(u, v) * (x, y) = (ux + bv\bar{y}, uy + a\bar{u}x + v\bar{x})$$

is a semifield multiplication of a semifield which is isotopic to an opposite semifield of Knuth type II (see [8], (7.17.II)). The semifield spread set Σ has the coordinatization $T(\Sigma) = S(a, b) = \{s(x, y) \mid x, y \in E\} \subseteq {}^0F^{4 \times 4}$ where $s(x, y)$ is given by

$$s(x, y) = \begin{pmatrix} T_0(x) & T_0(y) + T_1(ax) \\ T_0(b\bar{y}) & T_0(\bar{x}) \end{pmatrix}.$$

Note that $\det s(x, y) \neq 0$ for any nontrivial $(x, y) \in E \times E$ by our choice of a and b . The mappings α, β of Proposition 2.2 have the form $\alpha(T_0(y)) = T_0(b\bar{y})$ and $\beta(T_0(x)) = T_1(ax)$. In particular we are in the indecomposable case. Assume $\pi = \text{diag}(T_0(\delta), 1, 1, T_0(\delta))$, is a planar Baer collineation.

(1) π has order 3 and divides $q + 1$. The collineation is irreducible iff $q = p$ is a prime $\equiv -1 \pmod{3}$:

The first row of $s(x, y)^\pi$ is $(T_0(x\delta^{-1}), T_0(y) + T_1(ax\delta^{1-q}))$. We deduce $\delta^{-1} = \delta^{1-q}$, i.e. $|\pi| = 3$ as $|\pi|$ divides $p^k + 1 = q + 1$. In order to be an irreducible Baer collineation $T_0(\delta)$ must be irreducible as a K -linear operator on ${}^0F^2$ (note that this is a stronger requirement than assuming merely the irreducibility as a F -linear operator). This forces $k = 1$ and $p \equiv -1 \pmod{3}$.

(2) $\text{MinRk}(\Sigma) = 4$:

By 2.3 an element in k_m has the form $\text{diag}(T_0(w), T_0(\bar{w}))$, $w \in E$. To compute k_m we must determine these $w \in E$ with

$$T_1(a\bar{w}x) = T_0(w)T_1(ax) = T_0(w)\beta(T_0(x)) = \beta(T_0(wx)) = T_1(awx)$$

for all $x \in E$. This shows $w \in F$ and thus $k_m \simeq F$. A similar computation shows $k_r \simeq F$. To determine k_l we inspect the equation

$$T_1(a\bar{w}x) = T_0(w)\beta(T_0(x)) = \beta(T_0(x))T_0(w) = T_1(awx)$$

which again forces $w \in F$. Hence $k_l \simeq F$ and $\text{MinRk}(\Sigma) = 4$ follows.

CONCLUSION. For any odd prime $p \equiv -1 \pmod{3}$ there exist indecomposable semifield planes \mathbf{P}_Σ of order p^4 with $\text{MinRk}(\Sigma) = 4$ which admit irreducible planar Baer collineations of order 3.

EXAMPLE 4.3. Now we generalize the decomposable rank 4 examples of orders 4^4 and 5^4 . We consider the additive group $S(a, b, c, d) = \{s(x, y) \mid x, y \in E\} \subseteq {}^0F^{4 \times 4}$ where $s(x, y)$ is defined by

$$s(x, y) = \begin{pmatrix} T_0(x) & T_0(y) \\ T_0(ay + b\bar{y}) + T_1(cy + d\bar{y}) & T_0(\bar{x}) \end{pmatrix}.$$

Denote by $n: E \rightarrow F$ the norm and by $\text{tr}: E \rightarrow F$ the trace. A computation shows

$$\begin{aligned} \det s(x, y) &= n(x)^2 + n(y)^2(n(a) + n(b) - n(c) - n(d)) - n(x)n(y) \text{tr}(b) \\ &\quad - n(x) \text{tr}(ay^2) + n(y) \text{tr}((a\bar{b} - c\bar{d})y^2). \end{aligned}$$

Suppose that we have chosen the parameters a, b, c, d such that $T(\Sigma) = S(a, b, c, d)$ is the coordinatization of a (decomposable) spread set Σ . Then we deduce from Proposition 2.2 that \mathbf{P}_Σ admits an irreducible planar Baer collineation of order $q + 1$. Clearly, every seminucleus has a subfield isomorphic to F . Similar computations as in Examples 4.1 and 4.2 show $k_l \simeq E$ iff $c = d = 0$, $k_m \simeq E$ iff $a = d = 0$, and $k_r \simeq E$ iff $a = c = 0$. Therefore we have $\text{MinRk}(\Sigma) = 4$ if at least two of the parameters a, c, d are nontrivial. The semifield multiplication has the form

$$(u, v) * (x, y) = (ux + v(ay + b\bar{y}) + \bar{v}(cy + d\bar{y}), uy + v\bar{x}).$$

The following lemma (and for $q = 5$ by Section 3) shows that the parameters a, b, c, d always can be chosen such that $a, c, d \neq 0$ and that $S(a, b, c, d)$ is a spread set.

Lemma 4.4. *Use the notations of Example 4.3 and assume $q > 3$ and $q \neq 5$. There exist $a, b, c, d \in E$ such that $S(a, b, c, d)$ is a spread set. In addition one can choose a, c, d to be not 0.*

Proof. We first show that one can choose non-zero $u, v \in E$ such that the mapping $d_{u,v}: E \times E \rightarrow E$ defined by

$$d_{u,v}(x, y) = n(x) + un(y) + vy^2$$

has zero only for $(x, y) = (0, 0)$. Then we choose $a, b, c, d \in E$ such that $\det s(x, y) = n(d_{u,v}(x, y))$ for $(x, y) \in E \times E$ and that in addition $a, c, d \neq 0$. Then $\det s(x, y) \neq 0$ for any nontrivial $(x, y) \in E \times E$, and the assertions of the lemma follow.

Let $E = F[\alpha]$. If q is odd we can assume $\alpha^2 = t \in F - F^2$ and if q is even we can assume $\alpha^2 = t\alpha + 1, t \in F$, chosen suitably.

STEP 1. Write elements $z \in E$ as $z = z_1 + \alpha z_2, z_1, z_2 \in F$. Choose $0 \neq u \in E, 0 \neq v_2 \in F$ such that $v_2 \pm u_2 \neq 0$ and if $\text{char } F = 2$ in addition $u_2 \neq 0$ (but otherwise arbitrary).

Assume first that q is odd. Then $n(x) = x_1^2 - tx_2^2, y^2 = y_1^2 + ty_2^2 + 2\alpha y_1 y_2$. This shows

$$d_{u,v}(x, y) = n(x) + u_1 n(y) + v_1(y_1^2 + ty_2^2) + 2tv_2 y_1 y_2 + \alpha((u_2 + v_2)y_1^2 + (v_2 - u_2)ty_2^2 + 2v_1 y_1 y_2).$$

We now choose v_1 such that

$$Q(X, Y) = (u_2 + v_2)X^2 + 2v_1 XY + (v_2 - u_2)tY^2$$

is a anisotropic quadratic form, i.e. the discriminant $D = 4(v_1^2 - (v_2^2 - u_2^2)t)$ is a non-square. For the existence of such a v_1 recall that a nondegenerate quadratic form over F in two variables represents all elements of F . Therefore for $a \neq 0$ the set $F^2 + a(F^*)^2$ contains a nonsquare. This implies that $F^2 + a$ contains a nonsquare too. Taking $a = (v_2^2 - u_2^2)t$ the claim about v_1 follows.

This shows $d_{u,v}(x, y) \in E - F$ for $(x, y) \in E \times E^*$. Hence $d_{u,v}(x, y) \neq 0$ for $(x, y) \neq (0, 0)$.

Assume now that q is even. Then $\text{tr}(\alpha) = t, \bar{\alpha} = \alpha^{-1}$ and $n(x) = x_1^2 + x_2^2 + tx_1 x_2, y^2 = y_1^2 + y_2^2 + \alpha t y_2^2$. Hence

$$v y^2 = v_1(y_1^2 + y_2^2) + tv_2 y_2^2 + \alpha(v_1 t y_2^2 + v_2 y_1^2 + v_2 y_2^2 + t^2 v_2 y_2^2).$$

This implies

$$d_{u,v}(x, y) = R + \alpha((u_2 + tv_1 + v_2 + t^2 v_2)y_2^2 + (u_2 + v_2)y_1^2 + tu_2 y_1 y_2)$$

with $R \in F$. Choose $v_1 \in F$ such that

$$Q(X, Y) = (u_2 + v_1 t + v_2 + t^2 v_2)Y^2 + (v_2 + u_2)X^2 + tu_2 XY$$

is a anisotropic quadratic form. The existence of such a v_1 follows from the fact that for $0 \neq \alpha_0 \in F$ at least one of the polynomials $f_\beta(X) = X^2 + \alpha_0 X + \beta, \beta \in F$, must be irreducible. As before $d_{u,v}(x, y) \neq 0$ for $(x, y) \neq (0, 0)$.

STEP 2. First we observe that

$$n(d_{u,v}(x, y)) = n(x)^2 + n(x)n(y) \text{tr}(u) + n(y)^2(n(u) + n(v)) + n(x) \text{tr}(v y^2) + n(y) \text{tr}(\bar{u} v y^2).$$

This together with the condition that $\det s(x, y) = n(d_{u,v}(x, y))$ for any $(x, y) \in E \times E$ shows that the parameters a, \dots, d must satisfy the equations

$$(1.1) \quad n(u) + n(v) = n(a) + n(b) - n(c) - n(d),$$

$$(1.2) \quad \text{tr}(u) = -\text{tr}(b),$$

$$(1.3) \quad v = -a,$$

$$(1.4) \quad \bar{u}v = a\bar{b} - c\bar{d}.$$

Eliminating a we have

$$(2.1) \quad n(u) = n(b) - n(c) - n(d),$$

$$(2.2) \quad \text{tr}(u) = -\text{tr}(b),$$

$$(2.3) \quad c\bar{d} = -v(\bar{b} + \bar{u}).$$

Assume first that q is odd. Then $b_1 = -u_1$ by (2.2) and by (2.3) $c = v\bar{d}^{-1}(b_2 + u_2)\alpha$ and therefore $n(c) = n(v)n(d)^{-1}(-t)(b_2 + u_2)^2$. Thus $n(d)$ must be a solution of the equation

$$X^2 + t(b_2^2 - u_2^2)X - t(b_2 + u_2)^2n(v) = 0$$

whose discriminant is $D = (b_2 + u_2)^2(t^2(b_2 - u_2)^2 + 4tn(v))$. We claim that one can choose b_2 such that D is a square and that $b_2 + u_2 \neq 0$, i.e. $a, c, d \neq 0$. This is implied by the following observation (where we choose $A = t^2$, $B = 4tn(v)$, $X = b_2 - u_2$, and $Y = 1$):

Claim. *Let $Q(X, Y) = AX^2 + BY^2$ be a nondegenerate quadratic form over F . Then there exist at least 2 elements $w_1, w_2 \in F^*$ such that the value of Q at $(X, Y) = (w_i, 1)$, $i = 1, 2$, is a nontrivial square.*

One knows that Q has every element in F^* precisely $q + 1$ times as a value if Q is elliptic and $q - 1$ times as a value if Q is hyperbolic. Consider pairs in $\mathcal{L} = F^* \times F^*$ which has the partition

$$\mathcal{L} = \bigcup_{f \in F^*} F^*(f, 1).$$

The values of Q on the elements of a class $F^*(f, 1)$ differ only by squares. The set $F^* \times \{0\} \cup \{0\} \times F^*$ produces at most $2(q - 1)$ nontrivial squares. Thus Q has on \mathcal{L} at least $(q - 1)^2/2 - 2(q - 1) = (q - 1)(q - 5)/2$ times as a value a nontrivial square. As $q > 5$, there is at least one class whose values are nontrivial squares, say a class of $F^*(f, 1)$. Then the values of two classes, that of $F^*(f, 1)$ and that of $F^*(f, -1)$ are nontrivial squares. The claim follows.

We now assume that q is even. Equations (2.1)–(2.3) lead to $u_2 = b_2$ and

$$(3.1) \quad (b_1 + u_1)^2 + tu_2(b_1 + u_1) = n(c) + n(d),$$

$$(3.2) \quad c\bar{d} = v(\bar{b} + \bar{u}).$$

Then $c = v\bar{d}^{-1}(b_1 + u_1)$ and therefore $n(d)$ must be a solution of the equation

$$X^2 + ((b_1 + u_1)^2 + tu_2(b_1 + u_1))X + (b_1 + u_1)^2n(v) = 0.$$

Choose $b_1 = u_1 + tu_2$ and d such that $n(d) = tu_2\sqrt{n(v)}$. Then the equation holds and Step 2 is done and $a, c, d \neq 0$ if we take $u_2 \neq 0$ in Step 1. □

CONCLUSION. For any prime power $q \geq 4$ there exist decomposable semifield planes \mathbf{P}_Σ which admit irreducible planar Baer collineations of order $q + 1$ and with $\text{MinRk}(\Sigma) = 4$.

REMARKS. We keep the notations of this section.

(a) To verify the existence of the examples of Example 4.3 we use a particular construction in Lemma 4.4; there may be more ways to obtain such examples. However by a rough estimate we see that this special method already produces at least $q(q - 1)(q^2 - 1)$ examples of order q^4 .

(b) Our investigation raises more questions than they answer. The following problems deserve further attention:

- Find decomposable examples of order p^{4n} , p a prime, with $\text{MinRk}(\Sigma) = 4n$. So far only for p^4 , $p \geq 5$, and $\text{MinRk}(\Sigma) = 4$ the series of Example 4.3 provide such examples.
- Find more indecomposable examples, in particular examples in characteristic 2 and/or examples with a large order of π .

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FB Mathematik, Universität
67653 Kaiserslautern
Germany
e-mail: dempwolff@mathematik.uni-kl.de