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ON THE BOUNDARY BEHAVIOR OF THE DIRICHLET SOLUTIONS AT AN IRREGULAR BOUNDARY POINT

Dedicated to Professor Makoto Ohtsuka on his 60th birthday

TERUO IKEGAMI

(Received October 11, 1983)

Introduction. In the classical potential theory, O. Frostman [2] investigated the boundary behavior of the Dirichlet solution $H_f$ for continuous boundary data $f$ at an irregular boundary point $x$ of a bounded domain $U$ of $R^*$. And it was revealed that the cluster set of $H_f$ at $x$ is a segment with a possible exception. In other words, the cluster set of harmonic measures at $x$ has two extreme points—the Dirac measure $\delta_x$ and the balayaged measure $\beta_xU$. A generalization of this result was given by Constantinescu-Cornea [1] in an axiomatic setting in a more comprehensive context. Recently, J. Lukes-J. Malý [6] considered this problem in a relatively compact open subset of a harmonic space. The present paper is a contribution to this problem under a resolutive compactification.

Let $X$ be a $\mathcal{P}$-harmonic space with countable base in the sense of Constantinescu-Cornea [1] and $X^*$ be a resolutive compactification. Let $U$ be an open set of $X$. The closure $\bar{U}$ of $U$ in $X^*$ is a resolutive compactification of $U$. Suppose that $\partial U=(\bar{U}\setminus U)\cap X\neq\emptyset$. For a sequence $\{b_k\}$ converging to $x\in\partial U$ and satisfying $\delta_{b_k}\rightarrow\delta_xU$, the harmonic measure of $U$ at $b_k$ converges to a measure $\lambda_x$. If $x$ is irregular for $\bar{U}$, $\lambda_x$ enjoys remarkable properties stated in Theorem 7, which has a counterpart with the results of Lukes-Malý [6] and Hyvönen [3], and is connected with a version of maximal sequences considered by Smyrnélis [7]. In view of the work of Lukes-Malý, we can decide the structure of the cluster set $\mathcal{M}^U_x$ of harmonic measures and reveal that the type of $\mathcal{M}^U_x$ is a local property. We can also conclude the same result for the cluster set of the normalized Dirichlet solutions.

1. Preliminaries

Let $X$ be a $\mathcal{P}$-harmonic space with countable base in the sense of Constantinescu-Cornea [1] and $X^*$ be a resolutive compactification of $X$. We assume that there exists a function $s_x$ which is bounded superharmonic on $X$ and $\inf s_x>0$. We write $\Delta=X^*\setminus X$. 
For an open subset $U$ of $X$, we set $\Delta U = \partial U \cup (\Delta \cap \bar{U})$, where $\partial U = (\bar{U} \setminus U) \cap X$ and $\bar{U}$ is the closure of $U$ in $X^*$.

For a numerical function $f$ on $\partial U$ (resp. $\Delta U$, resp. $\Delta$) we define

$$H_f^{U,x}(a) = \inf \left\{ v(a) ; \begin{array}{l}
\text{hyperharmonic on } U, \text{ lower bounded,} \\
\lim v \geq f \text{ on } \partial U
\end{array} \right\}$$

(resp.

$$H_f^x(a) = \inf \left\{ v(a) ; \begin{array}{l}
\text{hyperharmonic on } U, \text{ lower bounded,} \\
\lim v \geq f \text{ on } \Delta U
\end{array} \right\},$$

and $H_f^{U,x} = -H_f^x$ (resp. $H_f^x = -H_f^x$), resp. $H_f = -H(\cdot -)$.

When $H_f^{U,x} = H_f^x$ and harmonic we write it $H_f^{U,x}$. Similarly we define $H_f^x$ and $H_f$.

In the following, we denote by $\cdot|A$, the restriction on $A$.

**Proposition 1.** The closure $\bar{U}$ of $U$ in $X^*$ is a resolutive compactification. For $f \in C(\partial U)$, let $f^*$ be a finite continuous extension of $f$ onto $X^*$ and let $u = H_f^*|_A$. Then we have

$$(1.1) \quad H_f^y = H_f^x + u.$$  

The proposition is proved quite in the same way as in [5], Prop. 1.

In the sequel, we denote by $\lambda_a$ the harmonic measure of $\bar{U}$ at $a$, i.e., $\lambda_a(f) = H_f^a$ (a) for every $f \in C(\Delta U)$, and stands $\varepsilon \delta$ for the balayaged measure of the Dirac measure $\delta_a$ on $A$ [1].

**Corollary 2.** $x \in \partial U$ is regular for $U$ if and only if $\varepsilon \delta U = \varepsilon_x$.

Proof. If $g$ is continuous on $\partial U$ and has a compact support, then $g$ can be extended continuously to be 0 on $\Delta$, thus $H_f^y = H_f^{U,x}$. This proves that if $x$ is regular for $\bar{U}$ then we have $\varepsilon \delta U = g(x)$ for all continuous functions $g$ on $\partial U$ with compact support, since $\varepsilon \delta U = \lim \varepsilon \delta \delta U = \lim H_f^{U,x}(b) = \lim H_f(b) = g(x)$ for some $\{b\}$ converging to $x$, i.e., $\varepsilon \delta \delta U = \varepsilon_x$. The converse is also true, since $\varepsilon \delta U = \varepsilon_x$ means that $\{\varepsilon \delta U\}$ converges to $\varepsilon_x$ for every $\{a\}$ tending to $x$ and the following Corollary 3 deduces the result.

**Corollary 3.** Let $x \in \partial U$ and $\{a\}$ be a sequence of points of $U$ tending to $x$. If $\{\varepsilon \delta U\}$ converges for $\mu$, then $\{\lambda_{a}\} \text{ converges vaguely.}$

Proof. Using the same notation as in Proposition 1,

$$H_f^y = H_f^{U,x} + u$$
and \( \lim_k \lambda_{x_k}(f) = \lim_k H^U_{x_k}(a_k) = \lim_k H^U_{x_k}(a_k) + u(x) \). By [1] Cor. 7.2.6, \( \lim_k H^U_{x_k}(a_k) = \lim_k \varepsilon^U_x(f - u) = \mu(f - u) \), since \( |f^* - u| \leq \rho \) for a potential \( \rho \) on \( X \).

**Corollary 4.** The regularity of \( x \in \partial U \) is a local property, that is, \( x \) is regular for \( \bar{U} \) if and only if it is regular for \( \bar{U} \cap \bar{V} \) for every neighborhood \( V \) of \( x \).

**Proof.** The regularity of \( x \) for \( U \) is equivalent to \( \varepsilon^U_x = \varepsilon^x \) and the latter is equivalent to the fact that \( X \setminus U \) is not thin at \( x \). This is also equivalent to the fact that \( X \setminus (U \cap V) \) is not thin at \( x \) since \( X \setminus V \) is thin at \( x \).

2. The definition of \( \lambda_x \)

**Lemma 5.** Let \( x \in \partial U, f \in C(\Delta U), f^* \) be a continuous extension of \( f \) on \( X^* \) and let \( u = H^U_{f^*x} \). Then \( u(x) = \varepsilon^U_x(u) \) depends only on \( f \) and \( \lambda_x \).

**Proof.** Consider the sequence \( \{b_k\} \) such that \( b_k \to x \) and \( \varepsilon^U_x \to \varepsilon^U_x \). By Corollary 3, the sequence \( \{\lambda_{b_k}\} \) of harmonic measures with respect to \( U \) converges and \( \lim_k \lambda_{b_k}(f) = \lim_k H^U_{b_k}(b_k) = \lim_k H^U_{b_k}(b_k) + u(x) = \varepsilon^U_x(f - u) + u(x) \).

If we denote this limit by \( \lambda \), then \( \varepsilon_x(u) - \varepsilon^U_x(u) = \lambda(f) - \varepsilon^U_x(f) \), and the last expression shows that \( \varepsilon_x(u) - \varepsilon^U_x(u) \) depends only on \( f \) and independent of \( f^* \).

We can see also that if \( \varepsilon^U_x \to \varepsilon^x \) then \( \lambda_{b_k} \to \lambda_x \).

We shall denote by \( \lambda_x \) the vague limit of \( \{\lambda_{b_k}\} \) corresponding to the sequence \( \{b_k\} \) satisfying \( \varepsilon^U_x \to \varepsilon^U_x \), and call \( \{b_k\} \) to be maximal at \( x \) in \( U \). Thus using the above notation we have

\[
(2.1) \quad \lambda_x(f) = \varepsilon^U_x(f) + \varepsilon_x(u) - \varepsilon^U_x(u) \quad \text{for every } f \in C(\Delta U).
\]

If \( U \) is a relatively compact open set of \( X \), then \( \lambda_x \) is just \( \varepsilon^U_x \). We can see \( \lambda_x \) is \( \varepsilon^U_x \) in general. In fact, we have

**Proposition 6.** If \( X \setminus U \) is compact, then \( \lambda_x \mid \partial U = \varepsilon^U_x \).

**Proof.** For \( f \in C(\Delta U) \) with \( f = 0 \) on \( \bar{U} \cap \Delta \)

\[
\lambda_x(f) = \lim_k \lambda_{b_k}(f) = \lim_k H^U_{b_k}(b_k) + \lim_k H^U_{b_k}(b_k) = \lim_k \varepsilon^U_x(f) = \varepsilon^U_x(f).
\]

3. The properties of \( \lambda_x \)

We denote by \( \Gamma(U) \) the harmonic boundary of \( U \), i.e.,

\[
\Gamma(U) = \bigcup_{x \in \partial U} \text{Supp} \cdot \lambda_x
\]

We define, for \( x \in \partial U \)

\[
\mathcal{H}_x = \{ \lambda_x \} \text{ s.t. } \exists \{a_k\} \subset U, a_k \to x, x \to \lambda_x \text{ vaguely}.
\]

**Theorem 7.** Let \( x \in \partial U \) be irregular for \( \bar{U} \). Then we have:

1. \( \lambda_x \neq \varepsilon_x \) and \( \mathcal{C}_x \subset \{ t \varepsilon_x + (1-t) \lambda_x ; 0 \leq t \leq 1 \} \).
(2) $\lambda_s(s) = \lim_s H^U_s$ for every $s$ continuous on $\bar{U}$ and superharmonic on $U$.
(3) if $x \in \Gamma(U)$ then $\mathcal{N}^U_x = \{\lambda_x\}$, thus for every $f \in \mathcal{C}(\Delta U)$, $H^U_y$ is extendable continuously at $x$ to the value $\lambda_x(f)$.
(4) for $f \in \mathcal{C}(\Delta U)$, non-negative and $f=0$ in a neighborhood of $x$ we have $\lambda_x(f) = \lim_s H^U_s$.
(5) $\varepsilon_x \in \mathcal{N}^U_x$ if and only if $x \in \Gamma(U)$.
(6) let $x \in \Gamma(U)$ and $f \in \mathcal{C}(\Delta U)$ such that $f(x) \neq \lambda_x(f)$, then only one of the following cases occurs:
   (i) $\lim_s H^U_y = f(x) < \lambda_x(f) = \lim_s H^U_s$,
   (ii) $\lim_s H^U_y = \lambda_x(f) < f(x) = \lim_s H^U_s$.

Proof. To prove (1), we note that the convergence of $\{\lambda_a\}$ is equivalent to the convergence of $\{\varepsilon_x^U\}$. Suppose that there exists $\{a_k\}$ such that $a_k \to x$ and $\lambda_{a_k} \to \lambda$, then we may find $t \in [0, 1]$ so that $\varepsilon_x^U \to t \varepsilon_x + (1-t) \varepsilon_x^U$. For $f \in \mathcal{C}(\Delta U)$ we have $\lambda(f) = \lim_s \lambda_{a_k}(f) = \lim_s H^U_y(a_k) = \lim_s H^U_y(x) + u(x) = \lim_s \varepsilon_x^U(f-u)+u(x) = t \varepsilon_x^U(f-u)+u(x) = \lim_s \varepsilon_x^U(f-u) = \lim_s \varepsilon_x^U(f)$.

(2): let $\{b_k\}$ be maximal at $x$ in $\bar{U}$ and fix a function $s$.

Then there exists $\lambda \in \mathcal{N}^U_x$ so that $\lambda(s) = \lim_s H^U_s$, for there is a sequence $\{a_k\}$ satisfying $a_k \to x$ and $\lim_s \lambda_{a_k}(s) = \lim_s H^U_s$, and therefore there is a subsequence of $\{\lambda_{a_k}\}$ converging to $\lambda$ vaguely. We can not conclude $\lambda(s) > \lim_s H^U_s$ since, by (1), $\lambda = t \varepsilon_x + (1-t) \lambda_x$ for some $t \in [0, 1]$.

(3): suppose that there is a function $f \in \mathcal{C}(\Delta U)$ such that $\lim_s H^U_y < \lim_s H^U_s$. Then as in the proof (2) there exist $\lambda', \lambda'' \in \mathcal{N}^U_x$ satisfying

$$\lim_s H^U_y = \lambda'(f), \lambda' = t' \varepsilon_x + (1-t') \lambda_x, 0 \leq t' \leq 1$$
$$\lim_s H^U_y = \lambda''(f), \lambda'' = t'' \varepsilon_x + (1-t'') \lambda_x, 0 \leq t'' \leq 1.$$ 

Hence, $(t'-t') [f(x) - \lambda_x(f)] > 0$. This is impossible, since the support of $\lambda_x$ is contained in $\Gamma(U)$ and $x \in \Gamma(U)$.

(4): let $f \in \mathcal{C}(\Delta U)$, $f \geq 0$, $f=0$ on a neighborhood of $x$ and let $\{b_k\}$ be maximal at $x$ in $\bar{U}$. As above, we have $\lambda \in \mathcal{N}^U_x$ such that $\lambda(f) = \lim_s H^U_s$ and $\lambda = t \varepsilon_x + (1-t) \lambda_x$ for $t \in [0, 1]$. We claim that $t=0$; in fact, $\lim_s H^U_y = \lambda(f) = t \varepsilon_x + (1-t) \lambda_x$ implies that $t=0$.

(5), (6): if $x \in \Gamma(U)$ then by (2), $\mathcal{N}^U_x = \{\lambda_x\}$ and $\lambda_x \neq \varepsilon_x$ thus $\varepsilon_x \notin \mathcal{N}^U_x$, i.e., $\varepsilon_x \notin \mathcal{N}^U_x$ implies that $x \notin \Gamma(U)$. To prove the converse suppose that $x \notin \Gamma(U)$. Then, by (1), we have $\lambda_x \in \mathcal{N}^U_x$. Letting $f \in \mathcal{C}(\Delta U)$ with $f(x) \neq \lambda_x(f)$, by virtue of the definition of the harmonic boundary $\Gamma(U)$ and the fact that for a continuous extension $f^*$ of $f$ on $\bar{U}$ there is a potential $q$ on $U$ such that $|H^U_y - f^*| \leq q$, we obtain
\[ \lim_{\lambda} H^x f \leq f(x) \leq \lim_{\lambda} H^x f. \]

However, the only possible cases are (i) \( \lim_{\lambda} H^x f = f(x) \) or (ii) \( \lim_{\lambda} H^x f = f(x) \); for if \( \lim_{\lambda} H^x f < f(x) < \lim_{\lambda} H^x f \) then as in the proof of (3), there are \( t', t'' \in [0, 1] \) so that
\[ t'f(x) + (1 - t') \lambda_x f < f(x) < t''f(x) + (1 - t'') \lambda_x f, \]
i.e., \( (1 - t') [f(x) - \lambda_x f] > 0 \) and \( (1 - t'') [f(x) - \lambda_x f] < 0 \), which is absurd. We shall consider the case (i). We may find also \( \lambda', \lambda'' \in \mathcal{I}^U_x \) such that
\[ \lambda'(x) = \lim_{\lambda} H^x f, \lambda' = t' \mathcal{E} + (1 - t') \lambda_x, 0 \leq t' \leq 1, \]
\[ \lambda''(x) = \lim_{\lambda} H^x f, \lambda'' = t'' \mathcal{E} + (1 - t'') \lambda_x, 0 \leq t'' \leq 1. \]

The equality \( t'f(x) + (1 - t') \lambda_x f = f(x) \) means that \( t' = 1 \) and \( \mathcal{E} = \lambda' \in \mathcal{I}^U_x \).

On the other hand, the inequality \( f(x) < t''f(x) + (1 - t'') \lambda_x f \) means that \( t'' < 1 \) and \( \lambda_x (f) < f(x) \). This implies \( t'' = 0 \), since if \( t'' > 0 \) then we are led to the contradiction that \( \lambda''(x) = \lim_{\lambda} H^x f = t''f(x) + (1 - t'') \lambda_x f \leq \lim_{\lambda} H^x f \).

Similarly in the case (ii), we have \( \mathcal{E} = \lambda'' \in \mathcal{I}^U_x \) and \( \lambda_x (f) = \lim_{\lambda} H^x f \).

**Remarks.** In the Theorem 7, (1) was proved by O. Frostman [2] in the classical potential theory. The fundamental contribution to the behavior of normalized solution \( H^x f \) in the axiomatic potential theory is due to Constantinescu-Cornea [1], and when \( U \) is relatively compact open set the precise investigation was given by Lukes-Maly [6].

(3) has a counterpart in a result of J. Hyvönen [3] (Cor. 1.6).

(4) is considered to be a refined variant of a theorem of Smyrnélis [7] (Cor. 2) and the maximal sequence \( \{ \mathcal{E}_x \} \) corresponds to "une suite maximal".

It is plausible that \( \lambda_x \) is the vague limit of \( \{ \mathcal{E}_x U \} \), where \( U_n = U \cap X_n \) with a compact exhaustion \( \{ X_n \} \) of \( X \).

### 4. The structure of \( \mathcal{I}^U_x \)

In [6] Lukes-Malý proved that in the case where \( U \) is a relatively compact open set \( \mathcal{I}^U_x \) has only four types: (1) \( \mathcal{I}^U_x = \{ \mathcal{E}_x \} \), (2) \( \mathcal{I}^U_x = \{ \mathcal{E}_x U \} \), (3) \( \mathcal{I}^U_x = \{ \mathcal{E}_x, \mathcal{E}_x U \} \), (4) \( \mathcal{I}^U_x = \{ t \mathcal{E}_x + (1 - t) \mathcal{E}_x U; 0 \leq t \leq 1 \} \). The situation is quite the same in our consideration. That is, in the same argument as in [6], we can prove

**Theorem 8.** Let \( x \in \partial U \). \( \mathcal{I}^U_x \) has the following four types:

1. \( \mathcal{I}^U_x = \{ \mathcal{E}_x \} \), \( x \) is regular for \( \bar{U} \),
2. \( \mathcal{I}^U_x = \{ \lambda_x \} \), \( x \) is semi-regular for \( \bar{U} \),
3. \( \mathcal{I}^U_x = \{ \mathcal{E}_x, \lambda_x \} \), \( x \) is weak-irregular,
4. \( \mathcal{I}^U_x = \{ t \mathcal{E}_x + (1 - t) \lambda_x; 0 \leq t \leq 1 \} \).

The type of \( \mathcal{I}^U_x \) is a local property, i.e., the above types are unaltered if we con-
sider $U \cap V$ in stead of $U$, where $V$ is a neighborhood of $x$. Therefore when $X$ is elliptic no boundary point $x \in \partial U$ is weak-irregular.

To prove the theorem we need some lemmas.

**Lemma 9.** In a resolutive compactification, let $x$ be a point of the harmonic boundary $\Gamma$. If $f$ is bounded, resolutive and continuous in a neighborhood of $x$, then we have

$$\lim_{x} H_{f} \leq f(x) \leq \lim_{x} H_{f}.$$

Proof. Suppose that $|f| \leq M$. Let $V_{1}, V_{2}$ be neighborhoods of $x$ such that $V_{1} \subset V_{2}$, and $f$ is continuous on $V_{2} \cap \Delta$. And let $\varphi_{1}, \varphi_{2}$ be continuous on $\Delta$ such that

$$\varphi_{1} = \begin{cases} f & \text{on } V_{1} \\ M & \text{on } \Delta \setminus V_{2} \end{cases} \quad \text{and} \quad \varphi_{2} = \begin{cases} f & \text{on } V_{1} \\ -M & \text{on } \Delta \setminus V_{2} \end{cases}.$$

And finally let $f_{1} = \max(f, \varphi_{1})$ and $f_{2} = \min(f, \varphi_{2})$. Then

$$\lim_{x} H_{f} \leq \lim_{x} H_{f_{1}} \leq f_{1}(x) = f(x) = f_{2}(x) \leq \lim_{x} H_{f_{2}} \leq \lim_{x} H_{f}.$$

**Lemma 10.** $x \in \Gamma(U)$ if and only if $x \in \Gamma(U \cap V)$ for every neighborhood $V$ of $x$.

Proof. Suppose that $x \in \Gamma(U \cap V)$ for some $V$. Then by [6] Cor. 19, there is a sequence $\{a_{k}\}$ tending to $x$ and $\lim_{x} H_{f}^{U \cap V}(a_{k}) = \varphi(x)$ for every resolutive and bounded function $\varphi$ which is continuous at $x$. Since $H_{f}^{U} = H_{f}^{U \cap V}$ for every $f \in C(\Delta U)$, where $\varphi = f$ on $\partial U \cap V$ and $\varphi = H_{f}^{U}$ on $\partial V \cap U$, we conclude that $\lambda_{\varphi} \rightarrow \lambda_{x}$ and $x \in \Gamma(U)$.

Next, suppose that $x \notin \Gamma(U \cap V)$ then $x$ is irregular for $U \cap \overline{V}$, and, by Corollary 4, $x$ is irregular for $U$. On the other hand, every bounded harmonic function on $U \cap V$ is extended continuously at $x$, [3]. Cor. 1.6. Therefore every bounded harmonic function on $U$, in particular $H_{f}^{U}$, is extended continuously at $x$, which implies that $x \notin \Gamma(U)$.

Proof of Theorem 8. Suppose that $x \in \Gamma(U)$ and $x$ is irregular, then $\lambda_{x} \in E_{x}$ and $\{E_{x}, \lambda_{x}\} \subset \mathcal{M}_{x}^{U}$. If the case (4) does not occur, then, in view of Theorem 7 (1), there exist $t \in (0, 1)$ and $f \in C(\Delta U)$ such that $t \varepsilon_{x} + (1-t)\lambda_{x} \in \mathcal{M}_{x}^{U}$ and $f(x) \neq \lambda_{x}(f)$. We may assume that $f(x) > \lambda_{x}(f)$. Then there is a neighborhood $V$ of $x$ such that $u = H_{f}^{U} + \alpha$ on $U \cap V$, where $\alpha = tf(x) + (1-t)\lambda_{x}(f)$. Denoting by

$$V_{1} = \{a \in U \cap V; u(a) > \alpha\},$$

$$V_{2} = \{a \in U \cap V; u(a) < \alpha\},$$
we have \( V_i \neq \emptyset \) \((i = 1, 2)\) and \( V_1 \cup V_2 = U \cap V \).

Let

\[
\varphi_i = \begin{cases} 
  f & \text{on } \partial U \cap V_i \\
  u & \text{on } \partial V_i \cap U.
\end{cases} \quad (i = 1, 2)
\]

The functions \( \varphi_i \) are bounded, resolutive and \( u = H_{\varphi_i}^V \) on \( V_i \).

Since

\[
\lim_{a \to x} H_{\varphi_i}^V = \lim_{a \to x} H_f^V (a) = \alpha < f(x) = \varphi_i(x)
\]

we have \( x \in \Gamma(V) \) by Lemma 9. Then there exists \( \lim_{a \to x} H_{\varphi_i}^V(a) \) \([3, \text{Cor. } 1.6]\), which means that \( \lim \lambda_a = \lambda_x \).

The fact that \( x \in \partial V_1 \cap \partial V_2 \) and \( x \) is irregular for \( V_2 \) implies that \( X \setminus V_2 \) is thin at \( x \) and \( V_2 \) is not thin at \( x \); Further \( X \setminus V_1 \) is not thin at \( x \), for \( V_2 \subset X \setminus V_1 \), which means that \( x \) is regular for \( V_1 \). Thus there exists \( \lim_{a \to x} H_{\varphi_i}^V(a) \) \([6, \text{Cor. } 19]\) and we can conclude that \( \lim \lambda_a = \varepsilon_x \).

The remaining part of the theorem is proved by the above consideration and Lemma 10, since regularity is a local property.

Finally, we shall remark on the cluster set \( \mathcal{M}_x^U \) of balayaged measures \( \varepsilon_x^U \) at \( x \).

If \( f \) is continuous on \( \partial U \) and has a compact carrier then \( f \) is extended continuously on \( \Delta U \) to be 0 on \( \Delta \cap \overline{U} \), then we have

\[
H_f^V = H_f^{U,x}
\]

and this shows

**Theorem 11.** Let \( x \in \partial U \). We have

1. \( \mathcal{M}_x^U = \{\varepsilon_x\} \) if \( x \) is regular for \( U \),
2. \( \mathcal{M}_x^U = \{\varepsilon_x^U\} \) if \( x \) is semi-regular,
3. \( \mathcal{M}_x^U = \{\varepsilon_x, \varepsilon_x^U\} \) if \( x \) is weak-irregular,
4. otherwise, \( \mathcal{M}_x^U = \{\varepsilon_x + (1-t)\varepsilon_x^U \mid 0 \leq t \leq 1\} \).
References


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