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ON THE BOUNDARY BEHAVIOR OF THE DIRICHLET SOLUTIONS AT AN IRREGULAR BOUNDARY POINT

Dedicated to Professor Makoto Ohtsuka on his 60th birthday

TERUO IKEGAMI

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Introduction. In the classical potential theory, O. Frostman [2] investigated the boundary behavior of the Dirichlet solution H_f^U for continuous boundary data f at an irregular boundary point x of a bounded domain U of R^n. And it was revealed that the cluster set of H_f^U at x is a segment with a possible exception. In other words, the cluster set of harmonic measures at x has two extreme points —the Dirac measure ε_x and the balayaged measure $\varepsilon_x^{\complement U}$. A generalization of this result was given by Constantinescu-Cornea [1] in an axiomatic setting in a more comprehensive context. Recently, J. Lukeš–J. Malý [6] considered this problem in a relatively compact open subset of a harmonic space. The present paper is a contribution to this problem under a resolutive compactification.

Let X be a $\mathcal{P}$-harmonic space with countable base in the sense of Constantinescu–Cornea [1] and X^* be a resolutive compactification. Let U be an open set of X. The closure $\bar{U}$ of U in X^* is a resolutive compactification of U. Suppose that $\partial U=(\bar{U}\setminus U)\cap X\neq\emptyset$. For a sequence $\{b_k\}$ converging to $x\in\partial U$ and satisfying $\varepsilon_{b_k}^{\complement U}\to\varepsilon_x^{\complement U}$, the harmonic measure of U at b_k converges to a measure λ_x. If x is irergular for $\bar{U}$, λ_x enjoyes remarkable properties stated in Theorem 7, which has a counterpart with the results of Lukeš-Malý [6] and Hyvönen [3], and is connected with a version of maximal sequences considered by Smyrnélis [7]. In view of the work of Lukeš-Malý, we can decide the structure of the cluster set $\mathcal{N}_x^U$ of harmonic measures and reveal that the type of $\mathcal{N}_x^U$ is a local property. We can also conclude the same result for the cluster set of the normalized Dirichlet solutions.

1. Preliminaries

Let X be a $\mathcal{P}$-harmonic space with countable base in the sense of Constantinescu-Cornea [1] and X^* be a resolutive compactification of X. We assume that there exists a function s_0 which is bounded superharmonic on X and $\inf s_0>0$. We write $\Delta=X^*\setminus X$.

For an open subset U of X, we set $\Delta U=\partial U\cup(\Delta\cap\bar{U})$, where $\partial U=(\bar{U}\setminus U)\cap X$ and $\bar{U}$ is the closure of U in X^*.

For a numerical function f on ∂U (resp. ΔU, resp. Δ) we define

$$\bar{H}_f^{U,X}(a)=\inf\left\{v(a)\,;\begin{array}{l}\text{hyperharmonic on } U\text{, lower bounded,}\\ v\geq 0 \text{ outside a compact subset of } X,\\ \underline{\lim}\, v\geq f \text{ on } \partial U\end{array}\right\}$$

(resp.

$$\bar{H}_f^{U}(a)=\inf\left\{v(a)\,;\begin{array}{l}\text{hyperharmonic on } U\text{, lower bounded,}\\ \underline{\lim}\, v\geq f \text{ on } \Delta U\end{array}\right\},$$

resp.

$$\bar{H}_f(a)=\inf\left\{v(a)\,;\begin{array}{l}\text{hyperharmonic on } X\text{, lower bounded,}\\ \underline{\lim}\, v\geq f \text{ on } \Delta\end{array}\right\}),$$

and $\underline{H}_f^{U,X}=-\bar{H}_{(-f)}^{U,X}$ (resp. $\underline{H}_f^U=-\bar{H}_{(-f)}^U$, resp. $\underline{H}_f=-\bar{H}_{(-f)}$).
When $\bar{H}_f^{U,X}=\underline{H}_f^{U,X}$ and harmonic we write it $H_f^{U,X}$. Similarly we define H_f^U and H_f.

In the following, we denote by $\cdot\,|A$, the restriction on A.

Proposition 1. *The closure $\bar{U}$ of U in X^* is a resolutive compactification. For $f\in \boldsymbol{C}(\Delta U)$, let f^* be a finite continuous extension of f onto X^* and let $u=H_{f^*|\Delta}$. Then we have*

$$(1.1)\qquad H_f^U=H_{f-u}^{U,X}+u\,.$$

The proposition is proved quite in the same way as in [5], Prop. 1.

In the sequel, we denote by λ_a the harmonic measure of $\bar{U}$ at a, i.e., $\lambda_a(f)=H_f^U(a)$ for every $f\in \boldsymbol{C}(\Delta U)$, and stands ε_x^A for the balayaged measure of the Dirac measure ε_x on A [1].

Corollary 2. *$x\in\partial U$ is regular for $\bar{U}$ if and only if $\varepsilon_x^{CU}=\varepsilon_x$.*

Proof. If g is continuous on ∂U and has a compact support, then g can be extended continuously to be 0 on Δ, thus $H_g^U=H_g^{U,X}$. This proves that if x is regular for $\bar{U}$ then we have $\varepsilon_x^{CU}(g)=g(x)$ for all continuous functions g on ∂U with compact support, since $\varepsilon_x^{CU}(g)=\lim_k \varepsilon_{b_k}^{CU}(g)=\lim_k H_g^{U,X}(b_k)=\lim_k H_g^U(b_k)=g(x)$ for some $\{b_k\}$ converging to x, i.e., $\varepsilon_x^{CU}=\varepsilon_x$. The converse is also true, since $\varepsilon_x^{CU}=\varepsilon_x$ means that $\{\varepsilon_{a_k}^{CU}\}$ converges to ε_x for every $\{a_k\}$ tending to x and the following Corollary 3 deduces the result.

Corollary 3. *Let $x\in\partial U$ and $\{a_k\}$ be a sequence of points of U tending to x. If $\{\varepsilon_{a_k}^{CU}\}$ converges for μ, then $\{\lambda_{a_k}\}$ converges vaguely.*

Proof. Using the same notation as in Proposition 1,

$$H_f^U=H_{f-u}^{U,X}+u$$

and $\lim_k \lambda_{a_k}(f)=\lim_k H_f^U(a_k)=\lim_k H_{f-u}^{U,X}(a_k)+u(x)$. By [1] Cor. 7.2.6, $\lim_k H_{f-u}^{U,X}(a_k)=\lim_k \varepsilon_{a_k}^{\complement U}(f-u)=\mu(f-u)$, since $|f^*-u|\leq p$ for a potential p on X.

Corollary 4. *The regularity of $x\in\partial U$ is a local property, that is, x is regular for $\bar{U}$ if and only if it is regular for $\overline{U\cap V}$ for every neighborhood V of x.*

Proof. The regularity of x for U is equivalent to $\varepsilon_x^{\complement U}=\varepsilon_x$ and the latter is equivalent to the fact that $X\setminus U$ is not thin at x. This is also equivalent to the fact that $X\setminus(U\cap V)$ is not thin at x since $X\setminus V$ is thin at x.

2. The definition of λ_x

Lemma 5. *Let $x\in\partial U$, $f\in C\ (\Delta U)$, f^* be a continuous extension of f on X^* and let $u=H_{f^*|\Delta}$. Then $u(x)-\varepsilon_x^{\complement U}(u)$ depends only on f.*

Proof. Consider the sequence $\{b_k\}$ such that $b_k\to x$ and $\varepsilon_{b_k}^{\complement U}\to\varepsilon_x^{\complement U}$. By Corollary 3, the sequence $\{\lambda_{b_k}\}$ of harmonic measures with respect to $\bar{U}$ converges and $\lim_k \lambda_{b_k}(f)=\lim_k H_f^U(b_k)=\lim_k H_{f-u}^{U,X}(b_k)+u(x)=\varepsilon_x^{\complement U}\ (f-u)+u(x)$. If we denote this limit by λ, then $\varepsilon_x(u)-\varepsilon_x^{\complement U}(u)=\lambda(f)-\varepsilon_x^{\complement U}(f)$, and the last expression shows that $\varepsilon_x(u)-\varepsilon_x^{\complement U}(u)$ depends only on f and independent of f^*.

We can see also that if $\varepsilon_{a_k}^{\complement U}\to\varepsilon_x$ then $\lambda_{a_k}\to\varepsilon_x$.

We shall denote by λ_x the vague limit of $\{\lambda_{b_k}\}$ corresponding to the sequence $\{b_k\}$ satisfying $\varepsilon_{b_k}^{\complement U}\to\varepsilon_x^{\complement U}$, and call $\{b_k\}$ to be maximal at x in $\bar{U}$. Thus using the above notation we have

(2.1) $\lambda_x(f)=\varepsilon_x^{\complement U}\ (f)+\varepsilon_x(u)-\varepsilon_x^{\complement U}\ (u)$ for every $f\in C(\Delta U)$.

If U is a relatively compact open set of X, then λ_x is just $\varepsilon_x^{\complement U}$. We can see $\lambda_x\neq\varepsilon_x^{\complement U}$ in general. In fact, we have

Proposition 6. *If $X\setminus U$ is compact, then $\lambda_x|\partial U=\varepsilon_x^{\complement U}$.*

Proof. For $f\in C(\Delta U)$ with $f=0$ on $\bar{U}\cap\Delta$
$\lambda_x(f)=\lim_k \lambda_{b_k}(f)=\lim_k H_f^U(b_k)=\lim_k H_f^{U,X}(b_k)=\lim_k \varepsilon_{b_k}^{\complement U}(f)=\varepsilon_x^{\complement U}(f)$.

3. The properties of λ_x

We denote by $\Gamma(U)$ the harmonic boundary of $\bar{U}$, i.e.,

$$\Gamma(U)=\overline{\bigcup_{a\in U}\operatorname{Supp}.\ \lambda_a}$$

We define, for $x\in\partial U$

$\mathfrak{N}_x^U=\{\lambda;\ \exists\ \{a_k\}\subset U,\ a_k\to x,\ \lambda_{a_k}\to\lambda \text{ vaguely}\}$.

Theorem 7. *Let $x\in\partial U$ be irregular for $\bar{U}$. Then we have:*

(1) $\lambda_x\neq\varepsilon_x$ *and* $\mathfrak{N}_x^U\subset\{t\ \varepsilon_x+(1-t)\lambda_x;\ 0\leq t\leq 1\}$,

(2) $\lambda_x(s)=\underline{\lim}_x H_s^U$ *for every* s *continuous on* $\bar{U}$ *and superharmonic on* U,

(3) *if* $x\notin\Gamma(U)$ *then* $\mathfrak{M}_x^U=\{\lambda_x\}$, *thus for every* $f\in\boldsymbol{C}(\Delta U)$ H_f^U *is extendable continuously at* x *to the value* $\lambda_x(f)$.

(4) *for* $f\in\boldsymbol{C}(\Delta U)$, *non-negative and* $f=0$ *in a neighborhood of* x *we have* $\lambda_x(f)=\overline{\lim}_x H_f^U$,

(5) $\varepsilon_x\in\mathfrak{M}_x^U$ *if and only if* $x\in\Gamma(U)$,

(6) *let* $x\in\Gamma(U)$ *and* $f\in\boldsymbol{C}(\Delta U)$ *such that* $f(x)\neq\lambda_x(f)$, *then only one of the following cases occurs*:

(i) $\underline{\lim}_x H_f^U=f(x)<\lambda_x(f)=\overline{\lim}_x H_f^U$,

(ii) $\underline{\lim}_x H_f^U=\lambda_x(f)<f(x)=\overline{\lim}_x H_f^U$.

Proof. To prove (1), we note that the convergence of $\{\lambda_{a_k}\}$ is equivalent to the convergence of $\{\varepsilon_{a_k}^{\complement U}\}$. Suppose that there exists $\{a_k\}$ auch thst $a_k\to x$ and $\lambda_{a_k}\to\lambda$, then we may find $t\in[0,1]$ so that $\varepsilon_{a_k}^{\complement U}\to t\varepsilon_x+(1-t)\varepsilon_x^{\complement U}$. For $f\in\boldsymbol{C}(\Delta U)$ we have $\lambda(f)=\lim_k\lambda_{a_k}(f)=\lim_k H_f^U(a_k)=\lim_k H_{f-u}^{U,X}(a_k)+u(x)=\lim_k \varepsilon_{a_k}^{\complement U}(f-u)+u(x)=t\ \varepsilon_x(f-u)+(1-t)\ \varepsilon_x^{\complement U}(f-u)+u(x)=t\ \varepsilon_x(f)+(1-t)\ [\varepsilon_x(u)-\varepsilon_x^{\complement U}(u)+\varepsilon_x^{\complement U}(f)]=t\varepsilon_x(f)+(1-t)\lambda_x(f)$.

(2): let $\{b_k\}$ be maximal at x in $\bar{U}$ and fix a function s.

$$\underline{\lim}_x H_s^U\leq\lim_k H_s^U(b_k)=\lambda_x(s)\leq s(x).$$

Then there exists $\lambda\in\mathfrak{M}_x^U$ so that $\lambda(s)=\underline{\lim}_x H_s^U$, for there is a sequence $\{a_k\}$ satisfying $a_k\to x$ and $\lim_k\lambda_{a_k}(s)=\underline{\lim}_x H_s^U$, and therefore there is a subsequence of $\{\lambda_{a_k}\}$ converging to λ vaguely. We can not conclude $\lambda_x(s)>\underline{\lim}_x H_s^U$ since, by (1), $\lambda=t\ \varepsilon_x+(1-t)\lambda_x$ for some $t\in[0,1]$.

(3): suppose that there is a function $f\in\boldsymbol{C}(\Delta U)$ such that $\underline{\lim}_x H_f^U<\overline{\lim}_x H_f^U$. Then as in the proof (2) there exist λ', $\lambda''\in\mathfrak{M}_x^U$ satisfying

$$(2.2)\quad \begin{aligned}&\underline{\lim}_x H_f^U=\lambda'(f),\ \lambda'=t'\ \varepsilon_x+(1-t')\ \lambda_x\ 0\leq t'\leq 1\\ &\overline{\lim}_x H_f^U=\lambda''(f),\ \lambda''=t''\varepsilon_x+(1-t'')\ \lambda_x\ 0\leq t''\leq 1.\end{aligned}$$

Hence, $(t''-t')\ [f(x)-\lambda_x(f)]>0$. This is impossible, since the support of λ_x is contained in $\Gamma(U)$ and $x\notin\Gamma(U)$.

(4): let $f\in\boldsymbol{C}(\Delta U)$, $f\geq 0$, $f=0$ on a neighborhood of x and let $\{b_k\}$ be maximal at x in $\bar{U}$. As above, we have $\lambda\in\mathfrak{M}_x^U$ such that $\lambda(f)=\overline{\lim}_x H_f^U$ and $\lambda=t\varepsilon_x+(1-t)\lambda_x$ for $t\in[0,1]$. We claim that $t=0$; in fact, $\overline{\lim}_x H_f^U=\lambda(f)=tf(x)+(1-t)\lambda_x(f)=(1-t)\ \lambda_x(f)=(1-t)\lim_k H_f^U(b_k)\leq\lim_k H_f^U(b_k)\leq\overline{\lim}_x H_f^U$, which implies that $t=0$.

(5), (6): if $x\notin\Gamma(U)$ then by (2), $\mathfrak{M}_x^U=\{\lambda_x\}$ and $\lambda_x\neq\varepsilon_x$ thus $\varepsilon_x\notin\mathfrak{M}_x^U$, i.e., $\varepsilon_x\in\mathfrak{M}_x^U$ implies that $x\in\Gamma(U)$. To prove the converse suppose that $x\in\Gamma(U)$. Then, by (1), we have $\lambda_x\in\mathfrak{M}_x^U$ $\lambda_x\neq\varepsilon_x$. Letting $f\in\boldsymbol{C}(\Delta U)$ with $f(x)\neq\lambda_x(f)$, by virtue of the definition of the harmonic boundary $\Gamma(U)$ and the fact that for a continuous extension f^* of f on $\bar{U}$ there is a potential q on U such that $|H_f^U-f^*|\leq q$, we obtain

$$\underline{\lim}_x H_f^U \leq f(x) \leq \overline{\lim}_x H_f^U.$$

However, the only possible cases are (i) $\underline{\lim}_x H_f^U = f(x)$ or (ii) $\overline{\lim}_x H_f^U = f(x)$; for if $\underline{\lim}_x H_f^U < f(x) < \overline{\lim}_x H_f^U$ then as in the proof of (3), there are $t', t'' \in [0, 1]$ so that

$$t'f(x)+(1-t')\lambda_x(f) < f(x) < t''f(x)+(1-t'')\lambda_x(f),$$

i.e., $(1-t')[f(x)-\lambda_x(f)]>0$ and $(1-t'')[f(x)-\lambda_x(f)]<0$, which is absurd. We shall consider the case (i). We may find also $\lambda', \lambda'' \in \mathfrak{N}_x^U$ such that

$$\lambda'(f) = \underline{\lim}_x H_f^U,\ \lambda' = t'\varepsilon_x + (1-t')\lambda_x,\ 0 \leq t' \leq 1,$$
$$\lambda''(f) = \overline{\lim}_x H_f^U,\ \lambda'' = t''\varepsilon_x + (1-t'')\lambda_x,\ 0 \leq t'' \leq 1.$$

The equality $t'f(x)+(1-t')\lambda_x(f)=f(x)$ means that $t'=1$ and $\varepsilon_x=\lambda'\in\mathfrak{N}_x^U$. On the other hand, the inequality $f(x)<t''f(x)+(1-t'')\lambda_x(f)$ means that $t''<1$ and $\lambda_x(f)>f(x)$. This implies $t''=0$, since if $t''>0$ then we are led to the contradiction that $\lambda''(f)=\overline{\lim}_x H_f^U = t''f(x)+(1-t'')\lambda_x(f) < \lambda_x(f) \leq \overline{\lim}_x H_f^U$. Similarly in the case (ii), we have $\varepsilon_x \in \mathfrak{N}_x^U$ and $\lambda_x(f)=\underline{\lim}_x H_f^U$.

REMARKS. In the Theorem 7, (1) was proved by O. Frostman [2] in the classical potential theory. The fundamental contribution to the behavior of normalized solution $H_f^{U,X}$ in the axiomatic potential theory is due to Constantinescu–Cornea [1], and when U is relatively compact open set the precise investigation was given by Lukeš–Malý [6].

(3) has a counterpart in a result of J. Hyvönen [3] (Cor. 1.6).

(4) is considered to be a refined variant of a theorem of Smyrnélis [7] (Cor. 2) and the maximal sequence $\{b_k\}$ corresponds to "une suite maximal".

It is plausible that λ_x is the vague limit of $\{\varepsilon_x^{\complement U_n}\}$, where $U_n = U \cap X_n$ with a compact exhaustion $\{X_n\}$ of X.

4. The structure of $\mathfrak{N}_x^U$

In [6] Lukeš–Malý proved that in the case where U is a relatively compact open set $\mathfrak{N}_x^U$ has only four types: (1) $\mathfrak{N}_x^U=\{\varepsilon_x\}$, (2) $\mathfrak{N}_x^U=\{\varepsilon_x^{\complement U}\}$, (3) $\mathfrak{N}_x^U=\{\varepsilon_x, \varepsilon_x^{\complement U}\}$, (4) $\mathfrak{N}_x^U=\{t\varepsilon_x+(1-t)\varepsilon_x^{\complement U};\ 0\leq t\leq 1\}$. The situation is quite the same in our consideration. That is, in the same argument as in [6], we can prove

Theorem 8. *Let $x\in\partial U$. $\mathfrak{N}_x^U$ has the following four types:*

(1) $\mathfrak{N}_x^U=\{\varepsilon_x\}$, *i.e., x is regular for $\bar{U}$,*

(2) $\mathfrak{N}_x^U=\{\lambda_x\}$, *i.e., x is semi-regular for $\bar{U}$,*

(3) $\mathfrak{N}_x^U=\{\varepsilon_x, \lambda_x\}$, *i.e., x is weak-irregular,*

(4) $\mathfrak{N}_x^U=\{t\,\varepsilon_x+(1-t)\lambda_x{}'\ 0\leq t\leq 1\}$.

The type of $\mathfrak{N}_x^U$ is a local property, i.e., the above types are unaltered if we con-

sider $\overline{U \cap V}$ in stead of $\bar{U}$, where V is a neighborhood of x. Therefore when X is elliptic no boundary point $x \in \partial U$ is weak-irregular.

To prove the theorem we need some lemmas.

Lemma 9. *In a resolutive compactification, let x be a point of the harmonic boundary Γ. If f is bounded, resolutive and continuous in a neighborhood of x, then we have*

$$\underline{\lim}_x H_f \leq f(x) \leq \overline{\lim}_x H_f.$$

Proof. Suppose that $|f| \leq M$. Let V_1, V_2 be neighborhoods of x such that $\bar{V}_1 \subset V_2$, and f is continuous on $V_2 \cap \Delta$. And let φ_1, φ_2 be continuous on Δ such that

$$\varphi_1 = \begin{cases} f \text{ on } \bar{V}_1 \\ M \text{ on } \Delta \backslash V_2 \end{cases} \quad \text{and} \quad \varphi_2 = \begin{cases} f \text{ on } \bar{V}_1 \\ -M \text{ on } \Delta \backslash V_2. \end{cases}$$

And finally let $f_1 = \max(f, \varphi_1)$ and $f_2 = \min(f, \varphi_2)$. Then

$$\underline{\lim}_x H_f \leq \underline{\lim}_x H_{f_1} \leq f_1(x) = f(x) = f_2(x) \leq \overline{\lim}_x H_{f_2} \leq \overline{\lim}_x H_f.$$

Lemma 10. *$x \in \Gamma$ (U) if and only if $x \in \Gamma$ $(U \cap V)$ for every neighborhood V of x.*

Proof. Suppose that $x \in \Gamma$ $(U \cap V)$ for some V. Then by [6] Cor. 19, there is a sequence $\{a_k\}$ tending to x and $\lim_k H_\varphi^{U \cap V}(a_k) = \varphi(x)$ for every resolutive and bounded function φ which is continuous at x. Since $H_f^U = H_\varphi^{U \cap V}$ for every $f \in \boldsymbol{C}$ (ΔU), where $\varphi = f$ on $\partial U \cap \bar{V}$ and $\varphi = H_f^U$ on $\partial V \cap U$, we conclude that $\lambda_{a_k} \to \varepsilon_x$ and $x \in \Gamma(U)$.

Next, suppose that $x \notin \Gamma(U \cap V)$ then x is irregular for $\overline{U \cap V}$, and, by Corollary 4, x is irregular for $\bar{U}$. On the other hand, every bounded harmonic function on $U \cap V$ is extended continuously at x, [3]. Cor. 1.6. Therefore every bounded harmonic function on U, in particular H_f^U, is extended continuously at x, which implies that $x \notin \Gamma$ (U).

Proof of Theorem 8. Suppose that $x \in \Gamma(U)$ and x is irregular, then $\lambda_x \neq \varepsilon_x$ and $\{\varepsilon_x, \lambda_x\} \subset \mathfrak{N}_x^U$. If the case (4) does not occur, then, in view of Theorem 7 (1), there exist $t \in (0, 1)$ and $f \in \boldsymbol{C}$ (ΔU) such that $t\varepsilon_x + (1-t)\lambda_x \notin \mathfrak{N}_x^U$ and $f(x) \neq \lambda_x(f)$. We may assume that $f(x) > \lambda_x(f)$. Then there is a neighborhood V of x such that $u = H_f^U \neq \alpha$ on $U \cap V$, where $\alpha = tf(x) + (1-t)\lambda_x(f)$. Denoting by

$$V_1 = \{a \in U \cap V;\ u(a) > \alpha\},$$
$$V_2 = \{a \in U \cap V;\ u(a) < \alpha\},$$

we have $V_i \neq \emptyset$ $(i=1, 2)$ and $V_1 \cup V_2 = U \cap V$.

Let

$$\varphi_i = \begin{cases} f & \text{on } \partial U \cap \bar{V}_i \\ u & \text{on } \partial V_i \cap U. \end{cases} \qquad (i=1, 2)$$

The functions φ_i are bounded, resolutive and $u = H_{\varphi_i}^{V_i}$ on V_i.

Since

$$\overline{\lim}_x H_{\varphi_2}^{V_2} = \overline{\lim_{\substack{a \to x \\ a \in V_2}}} H_f^U(a) \leq \alpha < f(x) = \varphi_2(x)$$

we have $x \notin \Gamma(V_2)$ by Lemma 9. Then there exists $\lim_{\substack{a \to x \\ a \in V_2}} H_{\varphi_2}^{V_2}(a)$ ([3], Cor. 1.6), which means that $\lim_{\substack{a \to x \\ a \in V_2}} \lambda_a = \lambda_x$.

The fact that $x \in \partial V_1 \cap \partial V_2$ and x is irregular for $\bar{V}_2$ implies that $X \setminus V_2$ is thin at x and V_2 is not thin at x; Further $X \setminus V_1$ is not thin at x, for $V_2 \subset X \setminus V_1$, which means that x is regular for $\bar{V}_1$. Thus there exists $\lim_{\substack{a \to x \\ a \in V_1}} H_{\varphi_1}^{V_1}(a)$ ([6], Cor. 19) and we can conclude that $\lim_{\substack{a \to x \\ a \in V_1}} \lambda_a = \varepsilon_x$.

The remaining part of the theorem is proved by the above consideration and Lemma 10, since regularity is a local property.

Finally, we shall remark on the cluster set $\mathcal{M}_x^U$ of balayaged measures $\varepsilon_a^{\complement U}$ at x.

If f is continuous on ∂U and has a compact carrier then f is extended continuously on ΔU to be 0 on $\Delta \cap \bar{U}$, then we have

$$H_f^U = H_f^{U,X}$$

and this shows

Theorem 11. *Let $x \in \partial U$. We have*

(1) $\mathcal{M}_x^U = \{\varepsilon_x\}$ *if x is regular for $\bar{U}$,*

(2) $\mathcal{M}_x^U = \{\varepsilon_x^{\complement U}\}$ *if x is semi-regular,*

(3) $\mathcal{M}_x^U = \{\varepsilon_x, \varepsilon_x^{\complement U}\}$ *if x is weak-irregular,*

(4) *otherwise,* $\mathcal{M}_x^U = \{t\varepsilon_x + (1-t)\varepsilon_x^{\complement U};\ 0 \leq t \leq 1\}$.

References

[1] C. Constantinescu-A. Cornea: Potential theory on harmonic spaces, Berlin-Heidelberg-New York, Springer, 1972.

[2] O. Frostman: *Les points irrĕguliers dans la théorie du potentiel et la critĕre de Wiener,* Medd. Lunds Univ. Mat. Sem. **4** (1939), 1–10.

[3] J. Hyvönen: *On the harmonic continuation of bounded harmonic functions,* Math. Ann. **245** (1979), 151–157.

[4] T. Ikegami: *On the regularity of boundary points in a resolutive compactification of a harmonic space,* Osaka J. Math. **14** (1977), 271–289.

[5] T. Ikegami: *Remarks on the regularity of boundary points in a resolutive compactification,* Osaka J. Math. **17** (1980), 177–186.

[6] J. Lukše-J-Malý: *On the boundary behavior of the Perron generalized solution,* Math. Ann. **257** (1981), 355–366.

[7] E.P. Smyrnélis: *Sur les limites fine des fonctions harmoniques et les suites maximales,* Bull. Sci. Math. **97** (1973), 161–175.

Department of Mathematics
Osaka City University
Sugimoto-cho, Sumiyoshi-ku
Osaka 558, Japan