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SOME PROPERTIES OF INVERTIBLE SUBSTITUTIONS OF rank d , AND HIGHER DIMENSIONAL SUBSTITUTIONS

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0. Introduction

We denote by \mathcal{A}_d^* (resp., F_d) the free monoid (resp., the free group), with the empty word as unit, generated by an alphabet $\mathcal{A}_d := \{1, 2, \dots, d\}$ consisting of d letters. We consider an endomorphism σ on F_d , i.e., a group homomorphism from F_d to itself. An endomorphism σ will be referred to as a *substitution* if we can take a nonempty word $\sigma(i) \in \mathcal{A}_d^*$ for all $i \in \mathcal{A}_d$, cf. the first paragraph of Section 1. When is a substitution σ invertible as an endomorphism on F_d ? An answer to this question is known when $d = 2$, cf. Proposition 1. Our objective is to generalize Proposition 1 for arbitrary $d \geq 2$. We introduce a geometrical method in [2]; and we use a general method given in [6], where the so called *higher dimensional substitutions* $E_k(\sigma)$ ($0 \leq k \leq d$) are established for a given substitution σ on F_d .

Throughout the paper, we denote by \mathbf{Z} (resp., N , \mathbf{R}) the set of integers (resp., positive integers, real numbers), and by $\text{End}(F_d)$ (resp., $\text{Sub}(F_d)$, $\text{Aut}(F_d)$, $\text{IS}(F_d)$) the set of endomorphisms (resp., substitutions, automorphisms, invertible substitutions) on F_d .

Let $d \geq 2$ be an integer. We mean by $(\mathbf{x}, i_1 \wedge \dots \wedge i_k)$ the positively oriented unit cube of dimension k translated by \mathbf{x} in the Euclidean space \mathbf{R}^d :

$$(\mathbf{x}, i_1 \wedge \dots \wedge i_k) := \{\mathbf{x} + t_1 \mathbf{e}_{i_1} + \dots + t_k \mathbf{e}_{i_k} \mid 0 \leq t_n \leq 1, 1 \leq n \leq k\},$$

$$\mathbf{x} \in \mathbf{Z}^d, 0 \leq k \leq d, 1 \leq i_1 < \dots < i_k \leq d,$$

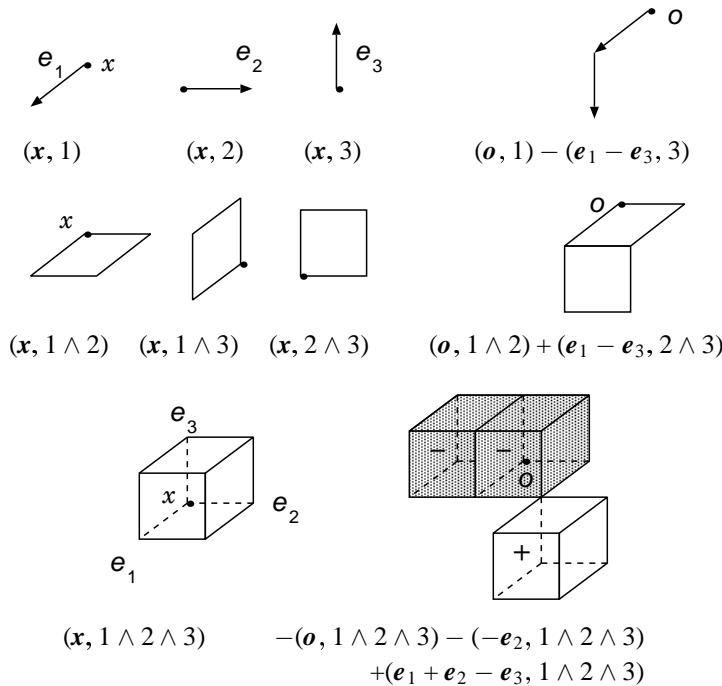
where $\{\mathbf{e}_i\}_{i=1,\dots,d}$ is the canonical basis of \mathbf{R}^d . In particular, for $k = 0$, the k dimensional unit cube $(\mathbf{x}, i_1 \wedge \dots \wedge i_k)$, which will be denoted by (\mathbf{x}, \bullet) , is considered to turn out a point \mathbf{x} . In general, for $\{i_1, i_2, \dots, i_k\}$ with $1 \leq i_m \leq d$, $1 \leq m \leq k$, we define

$$(\mathbf{x}, i_1 \wedge \dots \wedge i_k) := 0, \text{ if } i_n = i_m \text{ for some } n \neq m,$$

$$(\mathbf{x}, i_1 \wedge \dots \wedge i_k) := \epsilon(\tau)(\mathbf{x}, i_{\tau(1)} \wedge \dots \wedge i_{\tau(k)}) \quad (1 \leq i_{\tau(1)} < \dots < i_{\tau(k)} \leq d), \quad \text{otherwise,}$$

where τ is a permutation on $\{1, \dots, k\}$, and $\epsilon(\tau)$ is the signature of τ , which designates the orientation. We put

$$\Lambda_0 := \mathbf{Z}^d \times \{\bullet\},$$

Fig. 1. elements $\sum_{\lambda \in \Lambda_k} n_{\lambda} \lambda \in \mathcal{G}_k$, $k = 1, 2, 3$

$$\Lambda_k := \mathbf{Z}^d \times \{i_1 \wedge i_2 \wedge \cdots \wedge i_k \mid 1 \leq i_1 < \cdots < i_k \leq d\} \quad (1 \leq k \leq d).$$

We denote by \mathcal{G}_k the free \mathbf{Z} -module generated by the elements of Λ_k :

$$\mathcal{G}_k := \left\{ \sum_{\lambda \in \Lambda_k} n_{\lambda} \lambda \mid n_{\lambda} \in \mathbf{Z}, \#\{\lambda \in \Lambda_k \mid n_{\lambda} \neq 0\} < \infty \right\} \quad (0 \leq k \leq d).$$

We can identify the element $\sum_{\lambda \in \Lambda_k} n_{\lambda} \lambda \in \mathcal{G}_k$ with the union of oriented d dimensional unit cubes with their multiplicity, cf. Fig. 1.

For a word $P \in \mathcal{A}_d^*$, $|P|$ denotes the length of the word P . For $\sigma \in \text{Sub}(F_d)$, $i \in \mathcal{A}_d$ and $0 \leq k \leq l_i := |\sigma(i)|$, we define a word $P_k^{(i)} \in \mathcal{A}_d^*$ to be a prefix

$$P_k^{(i)} := w_1^{(i)} \cdots w_{k-1}^{(i)}$$

of $\sigma(i) = w_1^{(i)} \cdots w_k^{(i)} \cdots w_{l_i}^{(i)}$ ($w_j^{(i)} \in \mathcal{A}_d$ ($1 \leq j \leq l_i$)). A higher dimensional substitution $E_d(\sigma) : \mathcal{G}_d \rightarrow \mathcal{G}_d$ is a \mathbf{Z} -linear map (an endomorphism on a free \mathbf{Z} -module)

defined by

$$E_d(\sigma)(\mathbf{x}, 1 \wedge \cdots \wedge d) := \sum_{n_1=1}^{|\sigma(1)|} \cdots \sum_{n_d=1}^{|\sigma(d)|} (A_\sigma(\mathbf{x}) + \mathbf{f}(P_{n_1}^{(1)}) + \cdots + \mathbf{f}(P_{n_d}^{(d)}), w_{n_1}^{(1)} \wedge \cdots \wedge w_{n_d}^{(d)}),$$

where A_σ is the linear representation (or the so called characteristic matrix; see the beginning of Section 1.) of σ , so that it is of size $d \times d$ with integer entries; and where $\mathbf{f}(W) := {}^t(x_1, \dots, x_d) \in \mathbf{Z}^d$, $x_i = x_i(W)$ is the number of the occurrence of a letter i appearing in a word $W \in \mathcal{A}_d^*$. Now, we can state a result:

Proposition 1 ([2]). *Let $\sigma \in \text{Sub}(F_2)$ be a substitution with 2 letters. Then σ is invertible iff there exists $\mathbf{x} = \mathbf{x}_\sigma \in \mathbf{Z}^2$ such that*

$$E_2(\sigma)(\mathbf{o}, 1 \wedge 2) = \det(A_\sigma)(\mathbf{x}, 1 \wedge 2).$$

Related to generators of the group $\text{Aut}(F_d)$, the following result is well known.

Proposition 2 ([3]). *$\sigma \in \text{Aut}(F_d)$ iff σ is decomposed into the following three kinds of automorphisms:*

$$\alpha_{ij} : \begin{cases} i \rightarrow j \\ j \rightarrow i \\ k \rightarrow k \end{cases} \quad (i \neq j), \quad \beta_{ij} : \begin{cases} j \rightarrow ij \\ k \rightarrow k \end{cases} \quad (i \neq j), \quad \gamma_j : \begin{cases} j \rightarrow j^{-1} \\ k \rightarrow k \end{cases} \quad \text{for all } k \neq j.$$

α_{ij} , β_{ij} , γ_j are called *Nielsen's generators*. We shall use Proposition 2 for the proof of our main results (the following theorems). Noting that γ_i is not a substitution, we define $E_k(\sigma)$ ($0 \leq k \leq d$) not only for substitutions σ but also for endomorphisms σ . The map $E_k(\sigma)$ ($\sigma \in \text{End}(F_d)$) plays an important role in this paper.

In Section 1, we define $E_k(\sigma)$ ($0 \leq k \leq d$) for $\sigma \in \text{End}(F_d)$; and we prove

Theorem 1. *Let $\sigma \in \text{End}(F_d)$. If σ is invertible, then there exists $\mathbf{x} = \mathbf{x}_\sigma \in \mathbf{Z}^d$ such that*

$$E_d(\sigma)(\mathbf{o}, 1 \wedge 2 \wedge \cdots \wedge d) = \det(A_\sigma)(\mathbf{x}, 1 \wedge 2 \wedge \cdots \wedge d).$$

Roughly speaking, Theorem 1 says that the unit cube $(\mathbf{o}, 1 \wedge \cdots \wedge d)$ of dimension d is mapped to a unit cube of dimension d by $E_d(\sigma)$ if $\sigma \in \text{Aut}(F_d)$.

In Section 2, we consider the dual map $E_k^*(\sigma)$ of $E_k(\sigma)$. $E_k^*(\sigma)$ acts on a union of oriented $(d-k)$ dimensional unit cubes with their multiplicity; and by the map φ_{d-k} ($0 \leq k \leq d$), we can consider $\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1}$ as a map on \mathcal{G}_k . We apply

$E_k^*(\sigma)$ to getting the following theorem, which describes the relation between $E_k(\overline{\sigma^{-1}})$ and the dual map $E_{d-k}^*(\sigma)$, where \overline{W} is the mirror image of a word W and $\overline{\sigma}(i) := \overline{\sigma(i)}$, $i \in \mathcal{A}_d$.

Theorem 2. *Let σ be an automorphism on the free group F_d . Then there exists $\mathbf{x} \in \mathbf{Z}^d$ such that*

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} = \det(A_\sigma) \circ T(\mathbf{x}) \circ E_k(\overline{\sigma^{-1}}) \quad (0 \leq k \leq d),$$

where the map $T(\mathbf{x})$ is a translation by \mathbf{x} .

In the case of $k = 1$, Theorem 2 says that we can construct σ^{-1} by the figure of $E_{d-1}^*(\sigma)$.

When we study invertible substitutions $\sigma \in \text{Sub}(F_d)$ with $d \geq 3$, we encounter phenomena which do not occur in the case of $d = 2$. Accordingly, some results for $d = 2$ can not be extended for the case of $d \geq 3$. In Section 3, we see the gap between the cases of $d = 2$ and of $d \geq 3$ through some examples.

1. Map $E_k(\sigma)$ for an endomorphism σ on F_d

We put $\widehat{\mathcal{A}}_d := \{1^{\pm 1}, 2^{\pm 1}, \dots, d^{\pm 1}\}$, which is an alphabet consisting of $2d$ letters. We say that a word $W \in \widehat{\mathcal{A}}_d^*$ is a *reduced word* if W is the empty word, or $W = w_1 \cdots w_n$ ($w_i \in \widehat{\mathcal{A}}_d$) such that we can not find a number $1 \leq i \leq n-1$ satisfying $w_i = s^\rho$, $w_{i+1} = s^{-\rho}$, $s \in \mathcal{A}_d$ and $\rho \in \{-1, 1\}$. We write $W \doteq W'$ for two words $W, W' \in \widehat{\mathcal{A}}_d^*$ satisfying

$$W = W'; \quad W = UV \text{ and } W' = Us^\rho s^{-\rho}V; \quad \text{or } W = Us^\rho s^{-\rho}V \text{ and } W' = UV,$$

with $s \in \mathcal{A}_d$, $\rho \in \{-1, 1\}$. Two words $W, V \in \widehat{\mathcal{A}}_d^*$ are referred to be equivalent, and written as $W \approx V$, if there exist words $U_1, \dots, U_n \in \widehat{\mathcal{A}}_d^*$ such that $W \doteq U_1, U_i \doteq U_{i+1}$ ($1 \leq i \leq n-1$), $U_n \doteq V$. The relation \approx is an equivalence one, and $F_d = \widehat{\mathcal{A}}_d^*/\approx$ holds by the definition of free groups. For a given word $W \in \widehat{\mathcal{A}}_d^*$, $[W]$ denotes the element of F_d determined by $[W] \ni W$. Note that $\sigma \in \text{End}(F_d)$ is a substitution iff there exists a nonempty word $W(i) \in \mathcal{A}_d^*$ such that $W(i) \in [\sigma(i)]$ for each $1 \leq i \leq d$. In what follows, a word $W \in \widehat{\mathcal{A}}_d^*$ will be identified with the element $[W]$, cf. the definition of substitutions given in Section 0.

For $\sigma \in \text{End}(F_d)$, we can set

$$\sigma(i) = w_1^{(i)} \cdots w_k^{(i)} \cdots w_{l_i}^{(i)} \in \widehat{\mathcal{A}}_d^* \quad (w_k^{(i)} \in \widehat{\mathcal{A}}_d)$$

such that the word on the right-hand side is reduced one in $\widehat{\mathcal{A}}_d^*$ for each $i \in \mathcal{A}_d$. We define $P_k^{(i)}, S_k^{(i)} \in \widehat{\mathcal{A}}_d^*$ by

$$P_k^{(i)} = w_1^{(i)} \cdots w_{k-1}^{(i)}, \quad S_k^{(i)} = w_{k+1}^{(i)} \cdots w_{l_i}^{(i)}.$$

$P_k^{(i)}$ (resp., $S_k^{(i)}$) will be referred to as the k -prefix (resp., the k -suffix) of $\sigma(i)$. Note that $P_1^{(i)}$ is the empty word for any $i \in \mathcal{A}_d$. A canonical homomorphism $\mathbf{f}: F_d \rightarrow \mathbf{Z}^d$ is defined by $\mathbf{f}(i^{\pm 1}) = \pm \mathbf{e}_i$ ($i \in \mathcal{A}_d$). Then there exists a unique linear representation A_σ on \mathbf{Z}^d associated with σ such that the following diagram becomes commutative:

$$\begin{array}{ccc} F_d & \xrightarrow{\sigma} & F_d \\ \mathbf{f} \downarrow & & \downarrow \mathbf{f} \\ \mathbf{Z}^d & \xrightarrow{A_\sigma} & \mathbf{Z}^d \end{array}$$

We introduce $\widehat{\Lambda}_k$ ($0 \leq k \leq d$) formally defined by:

$$\begin{aligned} \widehat{\Lambda}_0 &:= \Lambda_0 = \mathbf{Z}^d \times \{\bullet\}, \\ \widehat{\Lambda}_k &:= \mathbf{Z}^d \times \{i_1 \wedge i_2 \wedge \cdots \wedge i_k \mid i_n \in \widehat{\mathcal{A}}_d\} \quad (1 \leq k \leq d). \end{aligned}$$

We denote by $\widehat{\mathcal{G}}_k$ the free \mathbf{Z} -module generated by the elements of $\widehat{\Lambda}_k$:

$$\widehat{\mathcal{G}}_k := \left\{ \sum_{\lambda' \in \widehat{\Lambda}_k} n_{\lambda'} \lambda' \mid n_{\lambda'} \in \mathbf{Z}, \#\{\lambda' \in \widehat{\Lambda}_k \mid n_{\lambda'} \neq 0\} < \infty \right\} \quad (0 \leq k \leq d).$$

DEFINITION 1. We denote by $\iota: \widehat{\mathcal{G}}_k \rightarrow \mathcal{G}_k$ the \mathbf{Z} -homomorphism (the \mathbf{Z} -linear map) defined by

$$\iota(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) := \begin{cases} 0 & \text{if } \|i_n\| = \|i_m\| \text{ for some } n \neq m, \\ \operatorname{sgn}(i_1) \cdots \operatorname{sgn}(i_k) \epsilon(\tau)(\mathbf{x} + \sum_{j=1}^k \chi(i_j), \|i_{\tau(1)}\| \wedge \cdots \wedge \|i_{\tau(k)}\|) & (1 \leq \|i_{\tau(1)}\| < \cdots < \|i_{\tau(k)}\| \leq d), \text{ otherwise,} \end{cases}$$

where τ is a permutation on $\{1, \dots, k\}$, $\epsilon(\tau)$ is the signature of τ , $\operatorname{sgn}(i^a)$ and $\|i^a\|$ means $\operatorname{sgn}(i^a) := a$, $\|i^a\| := i$, and

$$\chi(i^a) := \begin{cases} \mathbf{o} & \text{if } a = 1 \\ \mathbf{f}(i^a) & \text{if } a = -1 \end{cases} \quad (a \in \{-1, 1\}, i \in \mathcal{A}_d).$$

For two elements $g_1, g_2 \in \widehat{\mathcal{G}}_k$, we write $g_1 \sim g_2$ if $\iota(g_1) = \iota(g_2)$. It is easy to see that \sim is an equivalence relation. For example,

$$(\mathbf{o}, 2^{-1} \wedge 1) \sim \operatorname{sgn}(2^{-1}) \operatorname{sgn}(1) (\chi(2^{-1}) + \chi(1), 2 \wedge 1) \sim -(-\mathbf{e}_2, 2 \wedge 1) \sim (-\mathbf{e}_2, 1 \wedge 2).$$

Then, \mathcal{G}_k can be identified with a complete set of representatives of $\widehat{\mathcal{G}}_k / \sim$.

The geometrical meaning of the elements of \mathcal{G}_k ($0 \leq k \leq d$), we have already mentioned, leads us the following definition of a map $\delta_k: \mathcal{G}_k \rightarrow \mathcal{G}_{k-1}$, which is considered to be a boundary map.

DEFINITION 2. Boundary maps $\delta_k: \widehat{\mathcal{G}}_k \rightarrow \widehat{\mathcal{G}}_{k-1}$ ($1 \leq k \leq d$) are \mathbf{Z} -homomorphisms defined by

$$\begin{aligned} \delta_k(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) \\ := \sum_{n=1}^k (-1)^n \{(\mathbf{x}, i_1 \wedge \cdots \wedge \widehat{i_n} \wedge \cdots \wedge i_k) - (\mathbf{x} + \mathbf{f}(i_n), i_1 \wedge \cdots \wedge \widehat{i_n} \wedge \cdots \wedge i_k)\}. \end{aligned}$$

We note that $\delta_{k-1} \circ \delta_k = 0$ ($1 \leq k \leq d$) holds.

REMARK 1. The value of the map δ_k is independent of the choice of a representative, i.e., $g_1 \sim g_2$ ($g_1, g_2 \in \widehat{\mathcal{G}}_k$) implies $\delta_k(g_1) \sim \delta_k(g_2)$.

Let V and V' be \mathbf{Z} -modules. We mean by $\text{Hom}_{\mathbf{Z}}(V, V')$ (resp., $\text{End}_{\mathbf{Z}}(V)$) the set of \mathbf{Z} -linear maps from V to V' (resp., from V to itself). Now, we can define a map $E_k(\sigma) \in \text{End}_{\mathbf{Z}}(\mathcal{G}_k)$ for an endomorphism σ on F_d .

DEFINITION 3. Let $\sigma \in \text{End}(F_d)$. $E_k(\sigma): \mathcal{G}_k \rightarrow \mathcal{G}_k$ ($0 \leq k \leq d$) are \mathbf{Z} -linear maps defined by

$$\begin{aligned} E_0(\sigma)(\mathbf{x}, \bullet) &:= (A_\sigma(\mathbf{x}), \bullet), \\ E_k(\sigma)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) &:= \\ &\sum_{n_1=1}^{|\sigma(i_1)|} \cdots \sum_{n_k=1}^{|\sigma(i_k)|} (A_\sigma(\mathbf{x}) + \mathbf{f}(P_{n_1}^{(i_1)}) + \cdots + \mathbf{f}(P_{n_k}^{(i_k)}), w_{n_1}^{(i_1)} \wedge \cdots \wedge w_{n_k}^{(i_k)}) \quad (1 \leq k \leq d). \end{aligned}$$

We call $E_k(\sigma)$ a substitution of dimension k with respect to $\sigma \in \text{End}(F_d)$. We remark that in a certain sense, our definition is compatible with the boundary map δ_k :

Proposition 3. For $\sigma \in \text{End}(F_d)$ and for each $1 \leq k \leq d$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G}_k & \xrightarrow{E_k(\sigma)} & \mathcal{G}_k \\ \delta_k \downarrow & & \downarrow \delta_k \\ \mathcal{G}_{k-1} & \xrightarrow{E_{k-1}(\sigma)} & \mathcal{G}_{k-1} \end{array}$$

See Theorem 2.1 in [6], where the commutative diagram with $\sigma \in \text{Sub}(F_d)$ is given. The following theorem is one of our main results.

Theorem 1. *Let $\sigma \in \text{End}(F_d)$. If σ is invertible, then there exists $\mathbf{x} = \mathbf{x}_\sigma \in \mathbf{Z}^d$ such that*

$$E_d(\sigma)(\mathbf{o}, 1 \wedge 2 \wedge \cdots \wedge d) = \det(A_\sigma)(\mathbf{x}, 1 \wedge 2 \wedge \cdots \wedge d).$$

We need a lemma for the proof of Theorem 1.

Lemma 1. *Let an automorphism σ be decomposed as $\sigma = \sigma' \circ \sigma''$. Put*

$$\begin{aligned} \sigma(i) &= P_n^{(i)} w_n^{(i)} S_n^{(i)}, \\ \sigma'(i) &= P'_n^{(i)} w'_n^{(i)} S'_n^{(i)}, \quad \sigma''(i) = P''_n^{(i)} w''_n^{(i)} S''_n^{(i)}. \end{aligned}$$

Then the following statements are valid:

- (i) $A_\sigma = A_{\sigma'} A_{\sigma''}$.
- (ii) *For any pair (i, k) ($i \in \mathcal{A}_d$, $1 \leq k \leq |\sigma(i)|$), there exists a unique pair (m, n) such that*

$$P_k^{(i)} = \sigma'(P''_m^{(i)}) \quad P'_n^{(w''_m^{(i)})}, \quad 0 \leq m \leq |\sigma''(i)|, \quad 0 \leq n \leq |\sigma'(w''_m^{(i)})|.$$

- (iii) $E_k(\sigma) = E_k(\sigma') \circ E_k(\sigma'')$ ($0 \leq k \leq d$).

Proof. The assertions (i), (ii) can be easily seen. We show the equality

$$E_k(\sigma)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) = E_k(\sigma') \circ E_k(\sigma'')(\mathbf{x}, i_1 \wedge \cdots \wedge i_k).$$

For simplicity, we put

$$\sum := \sum_{m_1=1}^{|\sigma''(i_1)|} \cdots \sum_{m_k=1}^{|\sigma''(i_k)|} \sum_{n_1=1}^{|\sigma'(w''_m^{(i_1)})|} \cdots \sum_{n_k=1}^{|\sigma'(w''_m^{(i_k)})|}.$$

Using (i), (ii) in this lemma, we get

$$\begin{aligned} &E_k(\sigma)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) \\ &= \sum_{p_1=1}^{|\sigma(i_1)|} \cdots \sum_{p_k=1}^{|\sigma(i_k)|} \left(A_\sigma(\mathbf{x}) + \mathbf{f}(P_{p_1}^{(i_1)}) + \cdots + \mathbf{f}(P_{p_k}^{(i_k)}), w_{p_1}^{(i_1)} \wedge \cdots \wedge w_{p_k}^{(i_k)} \right) \\ &= \sum \left(A_{\sigma'} A_{\sigma''}(\mathbf{x}) + \mathbf{f}(\sigma'(P''_m^{(i_1)}) - P'_n^{(w''_m^{(i_1)})}) + \cdots + \mathbf{f}(\sigma'(P''_m^{(i_k)}) - P'_n^{(w''_m^{(i_k)})}), \right. \\ &\quad \left. w_{n_1}^{(w''_m^{(i_1)})} \wedge \cdots \wedge w_{n_k}^{(w''_m^{(i_k)})} \right) \\ &= \sum \left(A_{\sigma'} \left(A_{\sigma''}(\mathbf{x}) + \mathbf{f}(P''_m^{(i_1)}) + \cdots + \mathbf{f}(P''_m^{(i_k)}) \right) + \mathbf{f}(P'_n^{(w''_m^{(i_1)})}) + \cdots + \mathbf{f}(P'_n^{(w''_m^{(i_k)})}), \right. \\ &\quad \left. w_{n_1}^{(w''_m^{(i_1)})} \wedge \cdots \wedge w_{n_k}^{(w''_m^{(i_k)})} \right) \end{aligned}$$

$$\begin{aligned}
&= E_k(\sigma') \left\{ \sum_{m_1=1}^{|\sigma''(i_1)|} \cdots \sum_{m_k=1}^{|\sigma''(i_k)|} \left(A_{\sigma''}(\mathbf{x}) + \mathbf{f}(P''_{m_1}^{(i_1)}) + \cdots + \mathbf{f}(P''_{m_k}^{(i_k)}), w''_{m_1}^{(i_1)} \wedge \cdots \wedge w''_{m_k}^{(i_k)} \right) \right\} \\
&= E_k(\sigma') \circ E_k(\sigma'')(\mathbf{x}, i_1 \wedge \cdots \wedge i_k). \quad \square
\end{aligned}$$

Proof of Theorem 1. It suffices to show Theorem 1 only when $\sigma = \alpha_{ij}$, β_{ij} , γ_j . By easy calculation, we have

$$\begin{aligned}
E_d(\alpha_{ij})(\mathbf{o}, 1 \wedge \cdots \wedge i \cdots \wedge j \cdots \wedge d) &= (\mathbf{o}, 1 \wedge \cdots \wedge j \cdots \wedge i \cdots \wedge d) \\
&= -(\mathbf{o}, 1 \wedge 2 \wedge \cdots \wedge d), \\
E_d(\beta_{ij})(\mathbf{o}, 1 \wedge \cdots \wedge j \wedge \cdots \wedge d) &= (\mathbf{e}_i, 1 \wedge \cdots \wedge j \wedge \cdots \wedge d), \\
E_d(\gamma_j)(\mathbf{o}, 1 \wedge \cdots \wedge j \wedge \cdots \wedge d) &= (\mathbf{o}, 1 \wedge \cdots \wedge j^{-1} \wedge \cdots \wedge d) \\
&= -(-\mathbf{e}_j, 1 \wedge \cdots \wedge j \wedge \cdots \wedge d).
\end{aligned}$$

In view of the assertion (iii) in Lemma 1, we get Theorem 1, cf. Remark 6 in Section 2. \square

REMARK 2. It is not clear whether what the unit cube $(\mathbf{o}, 1 \wedge \cdots \wedge d)$ of dimension d is mapped to one unit cube of dimension d by $E_d(\sigma)$ implies that σ is an automorphism.

EXAMPLE 1. Let σ_R be the so called *Rauzy substitution*:

$$\sigma_R: \begin{cases} 1 \rightarrow 12 \\ 2 \rightarrow 13 \\ 3 \rightarrow 1 \end{cases} \quad \left(A_{\sigma_R} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right).$$

Then

$$E_3(\sigma_R)(\mathbf{o}, 1 \wedge 2 \wedge 3) = (2\mathbf{e}_1, 1 \wedge 2 \wedge 3).$$

By Remark 2, this doesn't imply that σ_R is invertible. But for the substitution σ_R , we have the inverse

$$\sigma_R^{-1}: \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 3^{-1}1 \\ 3 \rightarrow 3^{-1}2 \end{cases}.$$

We define another substitution σ_C given by

$$\sigma_C: \begin{cases} 1 \rightarrow 123 \\ 2 \rightarrow 112 \\ 3 \rightarrow 2333 \end{cases} \quad \left(A_{\sigma_C} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} \right).$$

Then

$$\begin{aligned} E_3(\sigma_C)(\mathbf{o}, 1 \wedge 2 \wedge 3) &= -(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, 1 \wedge 2 \wedge 3) - (\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3, 1 \wedge 2 \wedge 3) \\ &\quad + (2\mathbf{e}_1 + \mathbf{e}_2, 1 \wedge 2 \wedge 3). \end{aligned}$$

This together with Theorem 1 implies that σ_C is not invertible.

We consider all of substitutions satisfying $A_\sigma = A_{\sigma_C}$, whose cardinal number is 72. Then, we see that there does not exist a substitution σ satisfying $A_\sigma = A_{\sigma_C}$ such that

$$E_3(\sigma)(\mathbf{o}, 1 \wedge 2 \wedge 3) = -(\mathbf{x}, 1 \wedge 2 \wedge 3)$$

for some $\mathbf{x} \in \mathbf{Z}^3$. By Theorem 1, it means there does not exist an invertible substitution satisfying $A_\sigma = A_{\sigma_C}$, cf. Section 3.1. The matrix $A_\sigma = A_{\sigma_C}$ can be found in [5] as an example which does not give an invertible substitution.

2. Dual map $E_k^*(\sigma)$ for an endomorphism σ on F_d

We can define in the obvious manner a dual space. Since we are in an infinite dimensional \mathbf{Z} -module, this defines a complicated space; and restrict ourself to the set of dual maps with finite support. We denote this set by \mathcal{G}_k^* . It has a natural basis, and we write $(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*)$ for the dual vector of $(\mathbf{x}, i_1 \wedge \cdots \wedge i_k)$.

In fact, we introduce Λ_k^* ($0 \leq k \leq d$) formally defined by

$$\begin{aligned} \Lambda_0^* &:= \mathbf{Z}^d \times \{\bullet^*\}, \\ \Lambda_k^* &:= \mathbf{Z}^d \times \{i_1^* \wedge i_2^* \wedge \cdots \wedge i_k^* \mid 1 \leq i_1 < \cdots < i_k \leq d\} \quad (1 \leq k \leq d). \end{aligned}$$

We denote by \mathcal{G}_k^* the free \mathbf{Z} -module generated by the elements of Λ_k^* as follows:

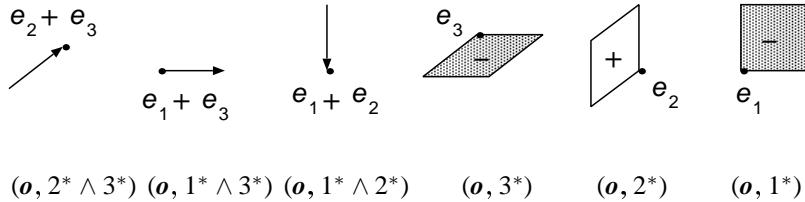
$$\mathcal{G}_k^* := \left\{ \sum_{\lambda^* \in \Lambda_k^*} n_{\lambda^*} \lambda^* \mid n_{\lambda^*} \in \mathbf{Z}, \#\{\lambda^* \in \Lambda_k^* \mid n_{\lambda^*} \neq 0\} < \infty \right\}.$$

REMARK 3. Following convention, we define a pairing by

$$\begin{aligned} &\langle (\mathbf{y}, j_1 \wedge \cdots \wedge j_k), (\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle \\ &:= \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{y} \text{ and } i_t = j_t \text{ for all } 1 \leq t \leq k \\ 0, & \text{otherwise} \end{cases}, \end{aligned}$$

$(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_k^*$, $(\mathbf{y}, j_1 \wedge \cdots \wedge j_k) \in \mathcal{G}_k$.

Thus \mathcal{G}_k^* can be considered as the set of dual maps with finite support.

Fig. 2. elements of \mathcal{G}_1^* , \mathcal{G}_2^* in the case of $d = 3$

We can define the \mathbf{Z} -linear isomorphism $\varphi_k: \mathcal{G}_k^* \rightarrow \mathcal{G}_{d-k}$ ($0 \leq k \leq d$) by

$$\begin{aligned} \varphi_0(\mathbf{x}, \bullet^*) &:= (\mathbf{x}, 1 \wedge \cdots \wedge d), \\ \varphi_k(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) &:= (-1)^{i_1 + \cdots + i_k} (\mathbf{x} + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_k}, j_1 \wedge \cdots \wedge j_{d-k}) \quad (1 \leq k \leq d), \end{aligned}$$

where $\{j_1, j_2, \dots, j_{d-k}\} = \mathcal{A}_d \setminus \{i_1, i_2, \dots, i_k\}$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{d-k}$. By virtue of the isomorphism φ_k , we can imagine an element $(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_k^*$ as the element $\varphi_k(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_{d-k}$ with geometrical meaning for each $0 \leq k \leq d$, see Fig. 2.

REMARK 4. In general, for $\Phi \in \text{Hom}_{\mathbf{Z}}(\mathcal{G}_s, \mathcal{G}_t)$ ($0 \leq s, t \leq d$), the dual map Φ^* of Φ is an element of $\text{Hom}_{\mathbf{Z}}(\mathcal{G}_t^*, \mathcal{G}_s^*)$ determined by $\langle F, \Phi^*(G^*) \rangle = \langle \Phi(F), G^* \rangle$ ($F \in \mathcal{G}_s, G^* \in \mathcal{G}_t^*$).

Proposition 4 ([6]). (i) *The dual boundary map $\delta_k^*: \mathcal{G}_{k-1}^* \rightarrow \mathcal{G}_k^*$ ($1 \leq k \leq d$) is given by*

$$\begin{aligned} \delta_1^*(\mathbf{x}, \bullet^*) &= \sum_{i=1}^d \{(\mathbf{x} - \mathbf{e}_i, i^*) - (\mathbf{x}, i^*)\}, \\ \delta_k^*(\mathbf{x}, i_1^* \wedge \cdots \wedge i_{k-1}^*) &= \sum_{n=1}^{d-k+1} (-1)^{j_n - n+1} \{(\mathbf{x}, i_1^* \wedge \cdots \wedge i_{j_n-n}^* \wedge j_n^* \wedge i_{j_n-n+1}^* \wedge \cdots \wedge i_{k-1}^*) \\ &\quad - (\mathbf{x} - \mathbf{e}_{j_n}, i_1^* \wedge \cdots \wedge i_{j_n-n}^* \wedge j_n^* \wedge i_{j_n-n+1}^* \wedge \cdots \wedge i_{k-1}^*)\}, \end{aligned}$$

where $\{j_1, j_2, \dots, j_{d-k+1}\} = \mathcal{A}_d \setminus \{i_1, i_2, \dots, i_{k-1}\}$ with $i_1 < \cdots < i_{k-1}$, $j_1 < \cdots < j_{d-k+1}$ and $i_{j_n-n} < j_n < i_{j_n-n+1}$ ($2 \leq k \leq d$).

(ii) *The following diagram commutes for each $1 \leq k \leq d$:*

$$\begin{array}{ccc} \mathcal{G}_{k-1}^* & \xrightarrow{\varphi_{k-1}} & \mathcal{G}_{d-k+1} \\ \delta_k^* \downarrow & & \downarrow \delta_{d-k+1} \\ \mathcal{G}_k^* & \xrightarrow{\varphi_k} & \mathcal{G}_{d-k} \end{array}$$

From the commutativity (ii) given above, we can see that δ_k^* is a boundary map with a geometrical sense.

By Remark 4, we can determine the dual map $E_k^*(\sigma)$ (on \mathcal{G}_k^*) of $E_k(\sigma)$ (on \mathcal{G}_k) for $\sigma \in \text{End}(F_d)$ under a minor condition on $\det(A_\sigma)$:

Proposition 5. (i) *Let σ be an endomorphism on the free group F_d satisfying $\det(A_\sigma) = \pm 1$. Then dual maps $E_k^*(\sigma): \mathcal{G}_k^* \rightarrow \mathcal{G}_k^*$ ($0 \leq k \leq d$) satisfies*

$$\begin{aligned} E_0^*(\sigma)(\mathbf{x}, \bullet^*) &= (A_\sigma^{-1}\mathbf{x}, \bullet^*), \\ E_k^*(\sigma)(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) &= \sum_{\tau \in S_k} \sum_{\|w_{n_1}^{(j_1)}\| = i_{\tau(1)}} \cdots \sum_{\|w_{n_k}^{(j_k)}\| = i_{\tau(k)}} \text{sgn}(w_{n_1}^{(j_1)}) \cdots \text{sgn}(w_{n_k}^{(j_k)}) \epsilon(\tau) \\ &\quad \left(A_\sigma^{-1} \left(\mathbf{x} - \sum_{m=1}^k \{\mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)})\} \right), j_1^* \wedge \cdots \wedge j_k^* \right) \quad (1 \leq j_1 < \cdots < j_k \leq d), \end{aligned}$$

where S_k is the symmetric group of rank k .

(ii) *The following diagram is commutative for each $1 \leq k \leq d$:*

$$\begin{array}{ccc} \mathcal{G}_{k-1}^* & \xrightarrow{E_{k-1}^*(\sigma)} & \mathcal{G}_{k-1}^* \\ \delta_k^* \downarrow & & \downarrow \delta_k^* \\ \mathcal{G}_k^* & \xrightarrow{E_k^*(\sigma)} & \mathcal{G}_k^* \end{array}$$

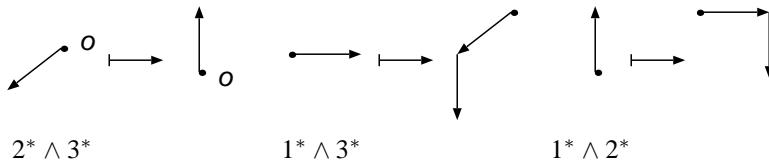
We remark that $\sigma \in \text{Aut}(F_d)$ implies $\det(A_\sigma) = \pm 1$.

Proof. We can prove the proposition in a similar fashion as that given in [6]. The dual map $E_k^*(\sigma) \in \text{End}_{\mathbb{Z}}(\mathcal{G}_k^*)$ of $E_k(\sigma)$ is given by the identity

$$\langle F, E_k^*(\sigma)(G^*) \rangle = \langle E_k(\sigma)(F), G^* \rangle$$

for $F \in \mathcal{G}_k$, $G^* \in \mathcal{G}_k^*$ by Remark 4. Therefore, we get

$$\begin{aligned} &\langle \mathbf{y}, j_1 \wedge \cdots \wedge j_k \rangle, E_k^*(\sigma)(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle \\ &= \langle E_k(\sigma)(\mathbf{y}, j_1 \wedge \cdots \wedge j_k), (\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle \\ &= \sum_{n_1=1}^{|\sigma(j_1)|} \cdots \sum_{n_k=1}^{|\sigma(j_k)|} \text{sgn}(w_{n_1}^{(j_1)}) \cdots \text{sgn}(w_{n_k}^{(j_k)}) \\ &\quad \left\langle \left(A_\sigma(\mathbf{y}) + \sum_{m=1}^k \{\mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)})\}, \|w_{n_1}^{(j_1)}\| \wedge \cdots \wedge \|w_{n_k}^{(j_k)}\| \right), (\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \right\rangle. \end{aligned}$$

Fig. 3. the map $E_2^*(\sigma_R)$

The pairing appearing in the summation is equal to

$$\operatorname{sgn}(w_{n_1}^{(j_1)}) \cdots \operatorname{sgn}(w_{n_k}^{(j_k)}) \epsilon(\tau)$$

if $A_\sigma(\mathbf{y}) + \sum_{m=1}^k \{\mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)})\} = \mathbf{x}$, and $\|w_{n_l}^{(j_l)}\| = i_{\tau(l)}$ ($1 \leq l \leq k$), $\tau \in S_k$; it is equal to 0, otherwise. Hence we get (i). The commutative diagram (ii) comes from Proposition 3. \square

We write $(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \simeq (\mathbf{y}, j_1 \wedge \cdots \wedge j_{d-k})$ iff $\varphi_k(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) = (\mathbf{y}, j_1 \wedge \cdots \wedge j_{d-k})$.

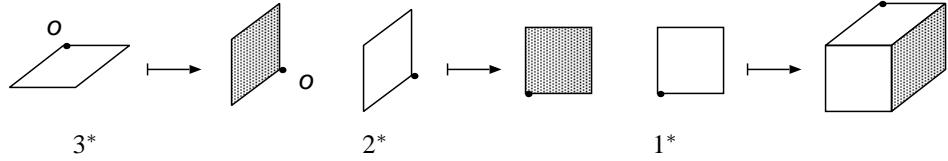
EXAMPLE 2. Let σ_R be the Rauzy substitution given in Example 1. Then

$$\begin{aligned} E_2^*(\sigma_R): \quad & -(-\mathbf{e}_2 - \mathbf{e}_3, 2^* \wedge 3^*) \mapsto -(-\mathbf{e}_1 - \mathbf{e}_2, 1^* \wedge 2^*) \\ & \simeq (\mathbf{o}, 1) \quad \simeq (\mathbf{o}, 3) \\ & (-\mathbf{e}_1 - \mathbf{e}_3, 1^* \wedge 3^*) \mapsto -(-\mathbf{e}_2 - \mathbf{e}_3, 2^* \wedge 3^*) + (-\mathbf{e}_2 - \mathbf{e}_3, 1^* \wedge 2^*) \\ & \simeq (\mathbf{o}, 2) \quad \simeq (\mathbf{o}, 1) - (\mathbf{e}_1 - \mathbf{e}_3, 3) \\ & -(-\mathbf{e}_1 - \mathbf{e}_2, 1^* \wedge 2^*) \mapsto (-\mathbf{e}_1 - \mathbf{e}_3, 1^* \wedge 3^*) + (-\mathbf{e}_1 - \mathbf{e}_3, 1^* \wedge 2^*) \\ & \simeq (\mathbf{o}, 3) \quad \simeq (\mathbf{o}, 2) - (\mathbf{e}_2 - \mathbf{e}_3, 3) \end{aligned}$$

See Fig. 3.

$$\begin{aligned} E_1^*(\sigma_R): \quad & -(-\mathbf{e}_3, 3^*) \mapsto -(-\mathbf{e}_2, 2^*) \\ & \simeq (\mathbf{o}, 1 \wedge 2) \quad \simeq -(\mathbf{o}, 1 \wedge 3) \\ & (-\mathbf{e}_2, 2^*) \mapsto (-\mathbf{e}_1, 1^*) \\ & \simeq (\mathbf{o}, 1 \wedge 3) \quad \simeq -(\mathbf{o}, 2 \wedge 3) \\ & -(-\mathbf{e}_1, 1^*) \mapsto -(-\mathbf{e}_3, 1^*) - (-\mathbf{e}_3, 2^*) - (-\mathbf{e}_3, 3^*) \\ & \simeq (\mathbf{o}, 2 \wedge 3) \quad \simeq (\mathbf{e}_1 - \mathbf{e}_3, 2 \wedge 3) - (\mathbf{e}_2 - \mathbf{e}_3, 1 \wedge 3) + (\mathbf{o}, 1 \wedge 2) \end{aligned}$$

See Fig. 4.

Fig. 4. the map $E_1^*(\sigma_R)$

For $W = s_1 s_2 \cdots s_n \in \widehat{\mathcal{A}}_d^*$, \overline{W} denotes the mirror image of W , i.e.,

$$\overline{W} := s_n s_{n-1} \cdots s_1.$$

For $\sigma \in \text{End}(F_d)$, the endomorphism $\overline{\sigma}$ is given by $\overline{\sigma}(i) = \overline{\sigma(i)}$ ($i \in \mathcal{A}_d$). Now, we can state a result.

Theorem 2. *Let σ be an automorphism on the free group F_d . Then there exists $\mathbf{x} \in \mathbf{Z}^d$ such that*

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} = \det(A_\sigma) \circ T_k(\mathbf{x}) \circ E_k(\overline{\sigma^{-1}}) \quad (0 \leq k \leq d),$$

where the map $T_k(\mathbf{x}) : \mathcal{G}_k \rightarrow \mathcal{G}_k$ with $\mathbf{x} \in \mathbf{Z}^d$ is given by

$$T_k(\mathbf{x}) \left(\sum_{t=1}^m n_t(\mathbf{y}_t, i_1^{(t)} \wedge \cdots \wedge i_k^{(t)}) \right) = \sum_{t=1}^m n_t(\mathbf{x} + \mathbf{y}_t, i_1^{(t)} \wedge \cdots \wedge i_k^{(t)}).$$

We remark that since \mathcal{G}_k is a free \mathbf{Z} -module, an integer a is an operator on \mathcal{G}_k , i.e., $a(\sum_{\lambda \in \Lambda_k} n_\lambda \lambda) = \sum_{\lambda \in \Lambda_k} (a \cdot n_\lambda) \lambda$. For the proof of Theorem 2, we need some lemmas.

Lemma 2. $\overline{\sigma' \circ \sigma} = \overline{\sigma'} \circ \overline{\sigma}$ ($\sigma, \sigma' \in \text{End}(F_d)$).

Proof. Setting $\sigma(i) = w_1^{(i)} \cdots w_{l_i}^{(i)}$, we have

$$\begin{aligned} \overline{\sigma' \circ \sigma}(i) &= \overline{\sigma'(w_1^{(i)} \cdots w_{l_i}^{(i)})} = \overline{\sigma'(w_1^{(i)}) \cdots \sigma'(w_{l_i}^{(i)})} = \overline{\sigma'(w_{l_i}^{(i)})} \cdots \overline{\sigma'(w_1^{(i)})} \\ &= \overline{\sigma'}(w_{l_i}^{(i)} \cdots w_1^{(i)}) = \overline{\sigma'} \circ \overline{\sigma}(i), \end{aligned}$$

so that $\overline{\sigma' \circ \sigma} = \overline{\sigma'} \circ \overline{\sigma}$. □

By $E_k(\sigma_1 \circ \sigma_2) = E_k(\sigma_1) \circ E_k(\sigma_2)$ and the duality $(\Phi_1 \circ \Phi_2)^* = \Phi_2^* \circ \Phi_1^*$, we have

Lemma 3. $E_k^*(\sigma_1 \circ \sigma_2) = E_k^*(\sigma_2) \circ E_k^*(\sigma_1)$ ($\sigma_1, \sigma_2 \in \text{End}(F_d)$).

Lemma 4. *Let σ be one of*

$$\mathcal{N} := \{\alpha_{ij}, \beta_{ij}, \gamma_j \mid 1 \leq i, j \leq d, i \neq j\}.$$

Then

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} = \det(A_\sigma) \circ T_k(v(\sigma)) \circ E_k(\overline{\sigma^{-1}}),$$

where $v: \mathcal{N} \rightarrow \{\mathbf{o}, \mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the map given by $v(\alpha_{ij}) := \mathbf{o}$, $v(\beta_{ij}) := \mathbf{e}_i$, $v(\gamma_j) := \mathbf{e}_j$.

Proof. It suffices to show that

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1}(\mathbf{o}, i_1 \wedge \dots \wedge i_k) = \det(A_\sigma) \circ T_k(v(\sigma)) \circ E_k(\overline{\sigma^{-1}})(\mathbf{o}, i_1 \wedge \dots \wedge i_k).$$

Notice $A_{\overline{\sigma^{-1}}} = A_\sigma^{-1}$. We consider the case of $\sigma = \beta_{ij}$ and $k = 1$. Note that

$$A_{\beta_{ij}}^{-1} = (\mathbf{e}_1, \dots, -\mathbf{e}_i + \mathbf{e}_j, \dots, \mathbf{e}_d), \quad \overline{\beta_{ij}^{-1}}: \begin{cases} j \rightarrow ji^{-1} \\ l \rightarrow l \quad (i \neq j). \\ \text{for all } l \neq j \end{cases}$$

We easily have

$$\varphi_{d-1} \circ E_{d-1}^*(\beta_{ij}) \circ \varphi_{d-1}^{-1}(\mathbf{o}, l) = (\mathbf{o}, l) = E_1(\overline{\beta_{ij}^{-1}})(\mathbf{o}, l) \quad (l \neq j).$$

On the other hand, we get the following equality. On the fourth line in the calculation given below, we must be careful with the location of j^* . If $j > i$, then j^* locates at i th place, otherwise, at $i-1$ th place. Using a permutation, we move j^* to the ordinal place, and then we have the sixth line.

$$\begin{aligned} & \varphi_{d-1} \circ E_{d-1}^*(\beta_{ij}) \circ \varphi_{d-1}^{-1}(\mathbf{o}, j) \\ &= \varphi_{d-1} \circ E_{d-1}^*(\beta_{ij})(-1)^{1+\dots+d-j} \left(-\sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{j^*} \wedge \dots \wedge d^* \right) \\ &= \varphi_{d-1} \left\{ (-1)^{1+\dots+d-j} \left(-\sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{j^*} \wedge \dots \wedge d^* \right) \right. \\ & \quad \left. + (-1)^{1+\dots+d-j} \left(-\sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{i^*} \wedge \dots \wedge j^* \wedge \dots \wedge d^* \right) \right\} \\ &= \varphi_{d-1} \left\{ (-1)^{1+\dots+d-j} \left(-\sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{j^*} \wedge \dots \wedge d^* \right) \right. \\ & \quad \left. + (-1)^{1+\dots+d-i+1} \left(-\sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{i^*} \wedge \dots \wedge d^* \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= (\mathbf{o}, j) - (-\mathbf{e}_i + \mathbf{e}_j, i) \\
&= E_1(\overline{\beta_{ij}^{-1}})(\mathbf{o}, j).
\end{aligned}$$

Hence we get

$$\varphi_{d-1} \circ E_{d-1}^*(\sigma) \circ \varphi_{d-1}^{-1} = \det(A_\sigma) \circ T_1(v(\sigma)) \circ E_1(\overline{\sigma^{-1}})$$

for $\sigma = \beta_{ij}$ and $k = 1$. For other cases, we can do the same, and the technical term can be found in Proof of Proposition 1.1 in [6]. \square

Lemma 5. For $a, a_m \in \mathbf{Z}$, $\mathbf{x}, \mathbf{y}, \mathbf{y}_m \in \mathbf{Z}^d$, $\sigma_m \in \text{End}(F_d)$ ($1 \leq m \leq n$) and for $0 \leq k \leq d$, we have the following formulas:

- (i) $T_k(\mathbf{x}) \circ T_k(\mathbf{y}) = T_k(\mathbf{x} + \mathbf{y})$
- (ii) $a \circ T_k(\mathbf{x}) = T_k(\mathbf{x}) \circ a$
- (iii) $(a_n \circ T_k(\mathbf{y}_n) \circ E_k(\sigma_n)) \circ \cdots \circ (a_1 \circ T_k(\mathbf{y}_1) \circ E_k(\sigma_1)) = a_1 \cdots a_n \circ T_k(\mathbf{y}_n + A_{\sigma_n}(\mathbf{y}_{n-1}) + A_{\sigma_n \sigma_{n-1}}(\mathbf{y}_{n-2}) + \cdots + A_{\sigma_n \cdots \sigma_2}(\mathbf{y}_1)) \circ E_k(\sigma_n \cdots \sigma_1)$.

Proof. The statements (i), (ii) are trivial. For the proof of the third statement, it is enough to show

$$(a_2 \circ T_k(\mathbf{y}_2) \circ E_k(\sigma_2)) \circ (a_1 \circ T_k(\mathbf{y}_1) \circ E_k(\sigma_1)) = a_1 a_2 \circ T_k(\mathbf{y}_2 + A_{\sigma_2}(\mathbf{y}_1)) \circ E_k(\sigma_2 \sigma_1).$$

We can put

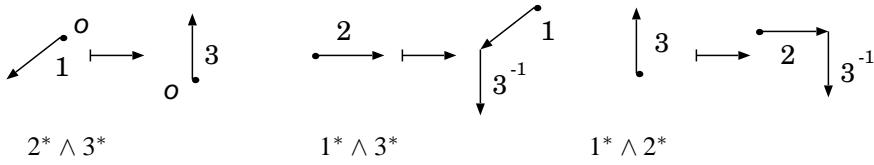
$$E_k(\sigma_1)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) = \sum_{\lambda \in \Lambda_k} n_\lambda(\mathbf{x}_\lambda, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}).$$

Using (i), (ii) in the lemma, we have

$$\begin{aligned}
&(a_2 \circ T_k(\mathbf{y}_2) \circ E_k(\sigma_2)) \circ (a_1 \circ T_k(\mathbf{y}_1) \circ E_k(\sigma_1))(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) \\
&= a_2 \circ T_k(\mathbf{y}_2) \circ E_k(\sigma_2) \left\{ \sum_{\lambda \in \Lambda_k} a_1 n_\lambda(\mathbf{x}_\lambda + \mathbf{y}_1, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}) \right\} \\
&= a_2 \circ T_k(\mathbf{y}_2) \left\{ \sum_{\lambda \in \Lambda_k} a_1 n_\lambda E_k(\sigma_2)(\mathbf{x}_\lambda + \mathbf{y}_1, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}) \right\} \\
&= a_2 \circ T_k(\mathbf{y}_2) \circ a_1 \circ T_k(A_{\sigma_2}(\mathbf{y}_1)) \circ E_k(\sigma_2) \circ E_k(\sigma_1)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) \\
&= a_1 a_2 \circ T_k(\mathbf{y}_2 + A_{\sigma_2}(\mathbf{y}_1)) \circ E_k(\sigma_2 \sigma_1)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k). \quad \square
\end{aligned}$$

Proof of Theorem 2. Let σ be an automorphism. Then σ can be written as $\sigma = \sigma_1 \cdots \sigma_n$ with $\sigma_m \in \mathcal{N}$ ($1 \leq m \leq n$). Using Lemma 2-5, we have

$$\begin{aligned}
&\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} \\
&= \varphi_{d-k} \circ E_{d-k}^*(\sigma_n) \circ \cdots \circ E_{d-k}^*(\sigma_1) \circ \varphi_{d-k}^{-1}
\end{aligned}$$

Fig. 5. the map $E_2^*(\sigma_R)$

$$\begin{aligned}
&= (\det(A_{\sigma_n}) \circ T_k(v(\sigma_n)) \circ E_k(\overline{\sigma_n^{-1}})) \circ \cdots \circ (\det(A_{\sigma_1}) \circ T_k(v(\sigma_1)) \circ E_k(\overline{\sigma_1^{-1}})) \\
&= \det(A_{\sigma_1}) \cdots \det(A_{\sigma_n}) \circ T_k(v(\sigma_n) + A_{\overline{\sigma_n^{-1}}}^{-1}(v(\sigma_{n-1})) + \cdots + A_{\overline{\sigma_n^{-1}} \cdots \overline{\sigma_2^{-1}}}^{-1}(v(\sigma_1))) \\
&\quad \circ E_k(\overline{\sigma_n^{-1}} \cdots \overline{\sigma_1^{-1}}) \\
&= \det(A_{\sigma_1 \cdots \sigma_n}) \circ T_k(v(\sigma_n) + A_{\overline{\sigma_n^{-1}}}^{-1}(v(\sigma_{n-1})) + \cdots + A_{\overline{(\sigma_2 \cdots \sigma_n)^{-1}}}^{-1}(v(\sigma_1))) \circ E_k(\overline{(\sigma_1 \cdots \sigma_n)^{-1}}) \\
&= \det(A_\sigma) \circ T_k(\mathbf{x}) \circ E_k(\overline{\sigma^{-1}}),
\end{aligned}$$

where $\mathbf{x} = v(\sigma_n) + A_{\sigma_n}^{-1}(v(\sigma_{n-1})) + \cdots + A_{\sigma_2 \cdots \sigma_n}^{-1}(v(\sigma_1))$. □

REMARK 5. In the case of $d = 3$, in particular, for a Pisot substitution $\sigma \in \text{Sub}(F_3)$ (i.e., a substitution such that the characteristic polynomial of A_σ is equal to the minimal polynomial of a Pisot number), we are interested in the region $E_1^*(\sigma)^n(\sum_{i=1}^3(\mathbf{o}, i^*))$ ($n \in \mathbb{N}$) in connection with stepped surfaces, cf. [1]. It follows from the assertion (ii) in Proposition 5 that the boundary of $E_1^*(\sigma)^n(\sum_{i=1}^3(\mathbf{o}, i^*))$ coincides with $E_2^*(\sigma)^n(\sum_{i=1}^3 \delta_2^*(\mathbf{o}, i^*))$. On the other hand, Theorem 2 says that $E_2^*(\sigma)^n(\sum_{i=1}^3 \delta_2^*(\mathbf{o}, i^*))$ can be calculated by using the map $\overline{\sigma^{-1}}^n$.

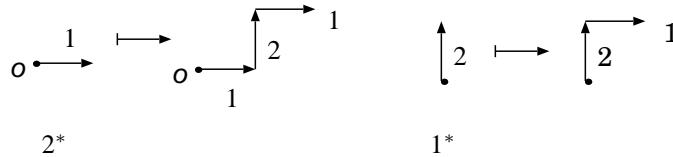
REMARK 6. In this setting we can rephrase Theorem 1 as follows:
If $\sigma \in \text{Aut}(F_d)$ is written as $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ with $\sigma_i \in \mathcal{N}$, then \mathbf{x}_σ in Theorem 1 is given by

$$\mathbf{x}_\sigma = v'(\sigma_1) + A_{\sigma_1}(v'(\sigma_2)) + \cdots + A_{\sigma_1 \cdots \sigma_{n-1}}(v'(\sigma_n)),$$

where $v': \mathcal{N} \rightarrow \{\mathbf{o}, \mathbf{e}_1, \dots, \mathbf{e}_d\}$ is the map given by $v'(\alpha_{ij}) := \mathbf{o}$, $v'(\beta_{ij}) := \mathbf{e}_i$, $v'(\gamma_j) := -\mathbf{e}_j$.

EXAMPLE 2'. For the Rauzy substitution σ_R given in Example 1, we can show $\varphi_2 \circ E_2^*(\sigma_R) \circ \varphi_2^{-1} = E_1(\overline{\sigma_R^{-1}})$. In view of Fig. 5, we easily see that σ_R^{-1} is given by

$$\sigma_R^{-1}: \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 3^{-1}1 \\ 3 \rightarrow 3^{-1}2 \end{cases}.$$

Fig. 6. the map $E_1^*(\sigma)$

We give another example of $E_1^*(\sigma)$ for an endomorphism $\sigma \in \text{End}(F_2)$ which is not a substitution.

EXAMPLE 3. Let $\sigma \in \text{Aut}(F_2)$ be given by

$$\sigma: \begin{cases} 1 \rightarrow 2^{-1}1 \\ 2 \rightarrow 1^{-1}22 \end{cases}.$$

Then

$$\begin{aligned} E_1^*(\sigma): (-e_2, 2^*) &\mapsto -(o, 1^*) + (e_1, 2^*) + (-e_2, 2^*) \\ &\simeq (o, 1) \quad \simeq (e_1, 2) + (e_1 + e_2, 1) + (o, 1) \\ -(-e_1, 1^*) &\mapsto -(-e_1, 1^*) + (o, 2^*) \\ &\simeq (o, 2) \quad \simeq (o, 2) + (e_2, 1) \end{aligned}.$$

We can show $\varphi_1 \circ E_1^*(\sigma) \circ \varphi_1^{-1} = E_1(\overline{\sigma^{-1}})$. In view of Fig. 6, we easily see σ^{-1} is given by

$$\sigma^{-1}: \begin{cases} 1 \rightarrow 121 \\ 2 \rightarrow 12 \end{cases}.$$

As we have already seen in the two examples above, we can construct σ^{-1} , in some cases, by the figure of $E_{d-1}^*(\sigma)$ for $\sigma \in \text{Aut}(F_d)$. In general, we have certain difficulty, cf. Section 3.3.

3. Examples and some comments

In the case of $d = 3$, some difficulties which never occur in the case of $d = 2$, will take place as we shall see through some examples.

3.1. Substitutions given by a matrix. It is easy to see, as is well known, that any unimodular matrix $A \in GL(2, N \cup \{0\})$ (i.e., $\det(A) = \pm 1$) can be decomposed into two matrices

$$A_{\alpha_{12}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{\beta_{12}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Therefore, for any matrix $A \in GL(2, N \cup \{0\})$, there exists at least one invertible substitution σ such that $A_\sigma = A$. On the other hand, in Example 1 in Section 1,

we have seen that any substitution σ satisfying $A_\sigma = A_{\sigma_C}$ is not invertible. We put $A_{ij} = (a_{lm})_{1 \leq l, m \leq d}$ ($1 \leq i, j \leq d, i \neq j$) by

$$a_{lm} := \begin{cases} 1 & \text{if } l = m \\ -1 & \text{if } l = i, m = j \\ 0, & \text{otherwise} \end{cases}.$$

We say that a matrix $M \in GL(d, N \cup \{0\})$ is *non-comparable* if both MA_{ij} , $A_{ij}M$ have negative entries for all A_{ij} ($1 \leq i, j \leq d, i \neq j$). For instance, A_{σ_R} is a comparable matrix, while A_{σ_C} is a non-comparable one. It seems very likely that if A_σ is non-comparable, then σ can not be an invertible substitution, i.e.,

$$\sigma \notin IS(F_d),$$

as far as we know.

3.2. Generators of the invertible substitutions. An invertible substitution σ is called a *prime substitution* if σ cannot be decomposed into 2 invertible substitutions σ_1, σ_2 such that one of σ_1, σ_2 does not belong to the group generated by α_{ij} ($1 \leq i, j \leq d, i \neq j$). Related to generators of the invertible substitutions σ (i.e., $\sigma \in IS(F_d)$), some results are found in [4], [5]. In the case of $d = 2$, generators of the invertible substitutions are given by three prime substitutions:

$$\alpha: \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 1 \end{cases}, \beta: \begin{cases} 1 \rightarrow 12 \\ 2 \rightarrow 1 \end{cases}, \delta: \begin{cases} 1 \rightarrow 21 \\ 2 \rightarrow 1 \end{cases},$$

so that the number of generators is finite, cf. [4]. But, the monoid $IS(F_d)$ for $d \geq 3$ turns out to be quite different from that for $d = 2$. For example, in the case of $d = 3$, we need infinitely many generators. In fact, $\sigma \in IS(F_3)$ defined by

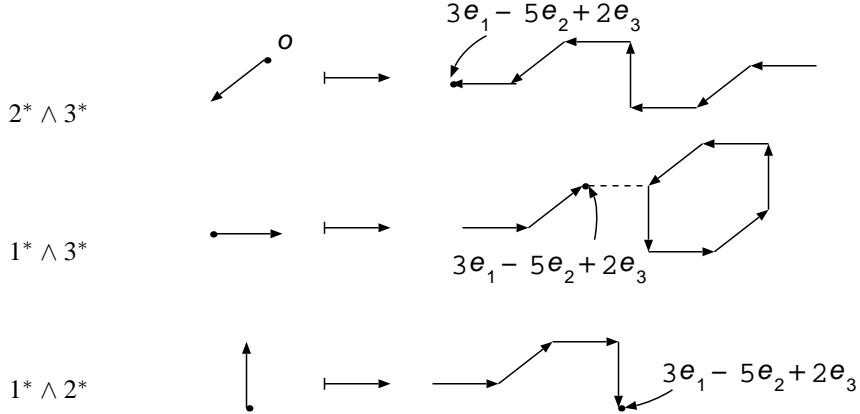
$$\sigma(1) := 12, \sigma(2) := 132, \sigma(3) := 3^n 2 \quad (n \geq 2)$$

are prime substitutions, cf. [5].

3.3. Connectedness of $E_{d-1}^*(\sigma)(o, \mathbf{1}^* \wedge \cdots \wedge \hat{j}^* \cdots \wedge \mathbf{d}^*)$. In the case of $d = 2$, we have shown in [2] the following proposition related to the dual map $E_1^*(\sigma)$.

Proposition 6 ([2]). *A substitution σ over 2 letters is invertible iff all the figures coming from $E_1^*(\sigma)(-\mathbf{e}_1, 1^*), E_1^*(\sigma)(-\mathbf{e}_2, 2^*), E_1^*(\sigma)((-\mathbf{e}_1, 1^*) + (-\mathbf{e}_2, 2^*))$ are connected.*

The figures coming from $E_1^*(\sigma)(-\mathbf{e}_i, i^*)$ ($i = 1, 2$) are parts of the so called stepped surface, cf. [1]. Since a stepped curve (a stepped surface of dimension 1)

Fig. 7. the map $E_2^*(\sigma)$

univalently spreads along a line, any cancellation can not occur in $E_1^*(\sigma)(-\mathbf{e}_i, i^*)$ ($i = 1, 2$). We can easily find the inverse of an invertible substitution $\sigma \in \text{IS}(F_2)$ from the figures coming from $E_1^*(\sigma)(-\mathbf{e}_i, i^*)$ ($i = 1, 2$), provided that $E_1^*(\sigma)(-\mathbf{e}_i, i^*)$ ($i = 1, 2$) contain no cancellations.

On the other hand, in the case of $d = 3$, $E_2^*(\sigma)(-\mathbf{e}_i - \mathbf{e}_j, i^* \wedge j^*)$ ($\sigma \in \text{IS}(F_3)$, $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$) is not always connective:

EXAMPLE 4. Let σ be an invertible substitution given by

$$\sigma: \begin{cases} 1 \rightarrow 1223 \\ 2 \rightarrow 123 \\ 3 \rightarrow 133 \end{cases} \quad \left(\sigma^{-1}: \begin{cases} 1 \rightarrow 21^{-1}23^{-1}21^{-1}2 \\ 2 \rightarrow 2^{-1}12^{-1}32^{-1}13^{-1}21^{-1}2 \\ 3 \rightarrow 2^{-1}12^{-1}3 \end{cases} \right).$$

Since $\varphi_2 \circ E_2^*(\sigma) \circ \varphi_2^{-1} = -T_1(3\mathbf{e}_1 - 5\mathbf{e}_2 + 2\mathbf{e}_3) \circ E_1(\overline{\sigma^{-1}})$, the figure coming from $E_2^*(\sigma)(\mathbf{o}, 1^* \wedge 3^*)$ is not connected, see Fig. 7.

3.4. Open problems. We give two problems for arbitrary $d \geq 3$:

- (i) Does the converse of the statement of Theorem 1 hold?
- (ii) Let $\sigma \in \text{Sub}(F_d)$ be a substitution with a non-comparable matrix A_σ such that A_σ does not belong to the group generated by $A_{\alpha_{ij}}, A_{\beta_{ij}}$ ($1 \leq i, j \leq d, i \neq j$). Then, is σ always not invertible?

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