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## SOME PROPERTIES OF INVERTIBLE SUBSTITUTIONS OF rank $d$ , AND HIGHER DIMENSIONAL SUBSTITUTIONS

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### 0. Introduction

We denote by  $\mathcal{A}_d^*$  (resp.,  $F_d$ ) the free monoid (resp., the free group), with the empty word as unit, generated by an alphabet  $\mathcal{A}_d := \{1, 2, \dots, d\}$  consisting of  $d$  letters. We consider an endomorphism  $\sigma$  on  $F_d$ , i.e., a group homomorphism from  $F_d$  to itself. An endomorphism  $\sigma$  will be referred to as a *substitution* if we can take a nonempty word  $\sigma(i) \in \mathcal{A}_d^*$  for all  $i \in \mathcal{A}_d$ , cf. the first paragraph of Section 1. When is a substitution  $\sigma$  invertible as an endomorphism on  $F_d$ ? An answer to this question is known when  $d = 2$ , cf. Proposition 1. Our objective is to generalize Proposition 1 for arbitrary  $d \geq 2$ . We introduce a geometrical method in [2]; and we use a general method given in [6], where the so called *higher dimensional substitutions*  $E_k(\sigma)$  ( $0 \leq k \leq d$ ) are established for a given substitution  $\sigma$  on  $F_d$ .

Throughout the paper, we denote by  $\mathbf{Z}$  (resp.,  $\mathbf{N}$ ,  $\mathbf{R}$ ) the set of integers (resp., positive integers, real numbers), and by  $\text{End}(F_d)$  (resp.,  $\text{Sub}(F_d)$ ,  $\text{Aut}(F_d)$ ,  $\text{IS}(F_d)$ ) the set of endomorphisms (resp., substitutions, automorphisms, invertible substitutions) on  $F_d$ .

Let  $d \geq 2$  be an integer. We mean by  $(\mathbf{x}, i_1 \wedge \cdots \wedge i_k)$  the positively oriented unit cube of dimension  $k$  translated by  $\mathbf{x}$  in the Euclidean space  $\mathbf{R}^d$ :

$$(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) := \{\mathbf{x} + t_1 \mathbf{e}_{i_1} + \cdots + t_k \mathbf{e}_{i_k} \mid 0 \leq t_n \leq 1, \ 1 \leq n \leq k\},$$

$$\mathbf{x} \in \mathbf{Z}^d, \ 0 \leq k \leq d, \ 1 \leq i_1 < \cdots < i_k \leq d,$$

where  $\{\mathbf{e}_i\}_{i=1, \dots, d}$  is the canonical basis of  $\mathbf{R}^d$ . In particular, for  $k = 0$ , the  $k$  dimensional unit cube  $(\mathbf{x}, i_1 \wedge \cdots \wedge i_k)$ , which will be denoted by  $(\mathbf{x}, \bullet)$ , is considered to turn out a point  $\mathbf{x}$ . In general, for  $\{i_1, i_2, \dots, i_k\}$  with  $1 \leq i_m \leq d$ ,  $1 \leq m \leq k$ , we define

$$(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) := 0, \text{ if } i_n = i_m \text{ for some } n \neq m,$$

$$(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) := \epsilon(\tau)(\mathbf{x}, i_{\tau(1)} \wedge \cdots \wedge i_{\tau(k)}) \quad (1 \leq i_{\tau(1)} < \cdots < i_{\tau(k)} \leq d), \quad \text{otherwise,}$$

where  $\tau$  is a permutation on  $\{1, \dots, k\}$ , and  $\epsilon(\tau)$  is the signature of  $\tau$ , which designates the orientation. We put

$$\Lambda_0 := \mathbf{Z}^d \times \{\bullet\},$$

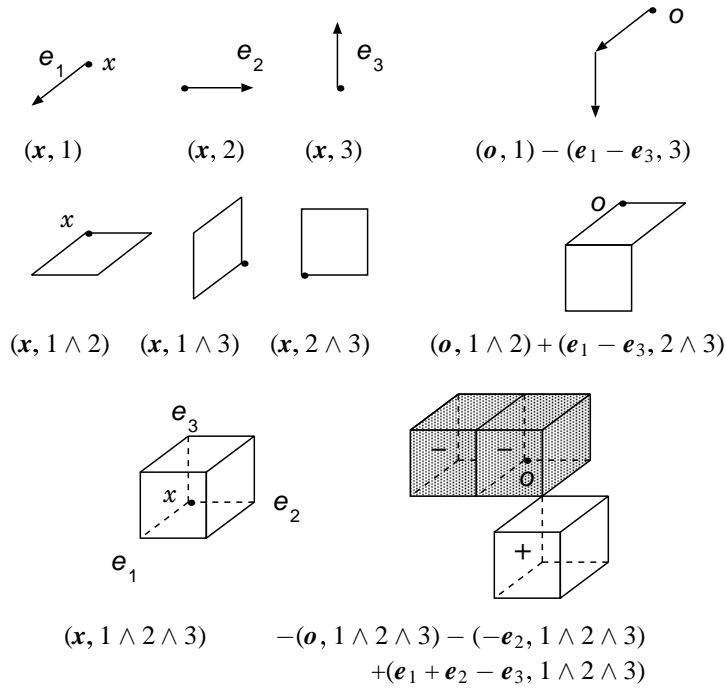


Fig. 1. elements  $\sum_{\lambda \in \Lambda_k} n_\lambda \lambda \in \mathcal{G}_k$ ,  $k = 1, 2, 3$

$$\Lambda_k := \mathbf{Z}^d \times \{i_1 \wedge i_2 \wedge \cdots \wedge i_k \mid 1 \leq i_1 < \cdots < i_k \leq d\} \quad (1 \leq k \leq d).$$

We denote by  $\mathcal{G}_k$  the free  $\mathbf{Z}$ -module generated by the elements of  $\Lambda_k$ :

$$\mathcal{G}_k := \left\{ \sum_{\lambda \in \Lambda_k} n_\lambda \lambda \mid n_\lambda \in \mathbf{Z}, \#\{\lambda \in \Lambda_k \mid n_\lambda \neq 0\} < \infty \right\} \quad (0 \leq k \leq d).$$

We can identify the element  $\sum_{\lambda \in \Lambda_k} n_\lambda \lambda \in \mathcal{G}_k$  with the union of oriented  $d$  dimensional unit cubes with their multiplicity, cf. Fig. 1.

For a word  $P \in \mathcal{A}_d^*$ ,  $|P|$  denotes the length of the word  $P$ . For  $\sigma \in \text{Sub}(F_d)$ ,  $i \in \mathcal{A}_d$  and  $0 \leq k \leq l_i := |\sigma(i)|$ , we define a word  $P_k^{(i)} \in \mathcal{A}_d^*$  to be a prefix

$$P_k^{(i)} := w_1^{(i)} \cdots w_{k-1}^{(i)}$$

of  $\sigma(i) = w_1^{(i)} \cdots w_k^{(i)} \cdots w_{l_i}^{(i)}$  ( $w_j^{(i)} \in \mathcal{A}_d$  ( $1 \leq j \leq l_i$ )). A higher dimensional substitution  $E_d(\sigma): \mathcal{G}_d \rightarrow \mathcal{G}_d$  is a  $\mathbf{Z}$ -linear map (an endomorphism on a free  $\mathbf{Z}$ -module)

defined by

$$E_d(\sigma)(\mathbf{x}, 1 \wedge \cdots \wedge d) := \sum_{n_1=1}^{|\sigma(1)|} \cdots \sum_{n_d=1}^{|\sigma(d)|} (A_\sigma(\mathbf{x}) + \mathbf{f}(P_{n_1}^{(1)}) + \cdots + \mathbf{f}(P_{n_d}^{(d)}), w_{n_1}^{(1)} \wedge \cdots \wedge w_{n_d}^{(d)}),$$

where  $A_\sigma$  is the linear representation (or the so called characteristic matrix; see the beginning of Section 1.) of  $\sigma$ , so that it is of size  $d \times d$  with integer entries; and where  $\mathbf{f}(W) := {}^t(x_1, \dots, x_d) \in \mathbf{Z}^d$ ,  $x_i = x_i(W)$  is the number of the occurrence of a letter  $i$  appearing in a word  $W \in \mathcal{A}_d^*$ . Now, we can state a result:

**Proposition 1** ([2]). *Let  $\sigma \in \text{Sub}(F_2)$  be a substitution with 2 letters. Then  $\sigma$  is invertible iff there exists  $\mathbf{x} = \mathbf{x}_\sigma \in \mathbf{Z}^2$  such that*

$$E_2(\sigma)(\mathbf{o}, 1 \wedge 2) = \det(A_\sigma)(\mathbf{x}, 1 \wedge 2).$$

Related to generators of the group  $\text{Aut}(F_d)$ , the following result is well known.

**Proposition 2** ([3]).  *$\sigma \in \text{Aut}(F_d)$  iff  $\sigma$  is decomposed into the following three kinds of automorphisms:*

$$\alpha_{ij}: \begin{cases} i \rightarrow j \\ j \rightarrow i \\ k \rightarrow k \\ \text{for all } k \neq i, j \end{cases} \quad (i \neq j), \quad \beta_{ij}: \begin{cases} j \rightarrow ij \\ k \rightarrow k \\ \text{for all } k \neq j \end{cases} \quad (i \neq j), \quad \gamma_j: \begin{cases} j \rightarrow j^{-1} \\ k \rightarrow k \\ \text{for all } k \neq j \end{cases}.$$

$\alpha_{ij}$ ,  $\beta_{ij}$ ,  $\gamma_j$  are called *Nielsen's generators*. We shall use Proposition 2 for the proof of our main results (the following theorems). Noting that  $\gamma_i$  is not a substitution, we define  $E_k(\sigma)$  ( $0 \leq k \leq d$ ) not only for substitutions  $\sigma$  but also for endomorphisms  $\sigma$ . The map  $E_k(\sigma)$  ( $\sigma \in \text{End}(F_d)$ ) plays an important role in this paper.

In Section 1, we define  $E_k(\sigma)$  ( $0 \leq k \leq d$ ) for  $\sigma \in \text{End}(F_d)$ ; and we prove

**Theorem 1.** *Let  $\sigma \in \text{End}(F_d)$ . If  $\sigma$  is invertible, then there exists  $\mathbf{x} = \mathbf{x}_\sigma \in \mathbf{Z}^d$  such that*

$$E_d(\sigma)(\mathbf{o}, 1 \wedge 2 \wedge \cdots \wedge d) = \det(A_\sigma)(\mathbf{x}, 1 \wedge 2 \wedge \cdots \wedge d).$$

Roughly speaking, Theorem 1 says that the unit cube  $(\mathbf{o}, 1 \wedge \cdots \wedge d)$  of dimension  $d$  is mapped to a unit cube of dimension  $d$  by  $E_d(\sigma)$  if  $\sigma \in \text{Aut}(F_d)$ .

In Section 2, we consider the dual map  $E_k^*(\sigma)$  of  $E_k(\sigma)$ .  $E_k^*(\sigma)$  acts on a union of oriented  $(d-k)$  dimensional unit cubes with their multiplicity; and by the map  $\varphi_{d-k}$  ( $0 \leq k \leq d$ ), we can consider  $\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1}$  as a map on  $\mathcal{G}_k$ . We apply

$E_k^*(\sigma)$  to getting the following theorem, which describes the relation between  $E_k(\overline{\sigma^{-1}})$  and the dual map  $E_{d-k}^*(\sigma)$ , where  $\overline{W}$  is the mirror image of a word  $W$  and  $\overline{\sigma(i)} := \overline{\sigma(i)}$ ,  $i \in \mathcal{A}_d$ .

**Theorem 2.** *Let  $\sigma$  be an automorphism on the free group  $F_d$ . Then there exists  $\mathbf{x} \in \mathbf{Z}^d$  such that*

$$\varphi_{d-k} \circ \widehat{E}_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} = \det(A_\sigma) \circ T(\mathbf{x}) \circ E_k(\overline{\sigma^{-1}}) \quad (0 \leq k \leq d),$$

where the map  $T(\mathbf{x})$  is a translation by  $\mathbf{x}$ .

In the case of  $k = 1$ , Theorem 2 says that we can construct  $\sigma^{-1}$  by the figure of  $E_{d-1}^*(\sigma)$ .

When we study invertible substitutions  $\sigma \in \text{Sub}(F_d)$  with  $d \geq 3$ , we encounter phenomena which do not occur in the case of  $d = 2$ . Accordingly, some results for  $d = 2$  can not be extended for the case of  $d \geq 3$ . In Section 3, we see the gap between the cases of  $d = 2$  and of  $d \geq 3$  through some examples.

### 1. Map $E_k(\sigma)$ for an endomorphism $\sigma$ on $F_d$

We put  $\widehat{\mathcal{A}}_d := \{1^{\pm 1}, 2^{\pm 1}, \dots, d^{\pm 1}\}$ , which is an alphabet consisting of  $2d$  letters. We say that a word  $W \in \widehat{\mathcal{A}}_d^*$  is a *reduced word* if  $W$  is the empty word, or  $W = w_1 \cdots w_n$  ( $w_i \in \widehat{\mathcal{A}}_d$ ) such that we can not find a number  $1 \leq i \leq n-1$  satisfying  $w_i = s^\rho$ ,  $w_{i+1} = s^{-\rho}$ ,  $s \in \mathcal{A}_d$  and  $\rho \in \{-1, 1\}$ . We write  $W \doteq W'$  for two words  $W, W' \in \widehat{\mathcal{A}}_d^*$  satisfying

$$W = W'; \quad W = UV \text{ and } W' = Us^\rho s^{-\rho} V; \quad \text{or } W = Us^\rho s^{-\rho} V \text{ and } W' = UV,$$

with  $s \in \mathcal{A}_d$ ,  $\rho \in \{-1, 1\}$ . Two words  $W, V \in \widehat{\mathcal{A}}_d^*$  are referred to be equivalent, and written as  $W \approx V$ , if there exist words  $U_1, \dots, U_n \in \widehat{\mathcal{A}}_d^*$  such that  $W \doteq U_1$ ,  $U_i \doteq U_{i+1}$  ( $1 \leq i \leq n-1$ ),  $U_n \doteq V$ . The relation  $\approx$  is an equivalence one, and  $F_d = \widehat{\mathcal{A}}_d^* / \approx$  holds by the definition of free groups. For a given word  $W \in \widehat{\mathcal{A}}_d^*$ ,  $[W]$  denotes the element of  $F_d$  determined by  $[W] \ni W$ . Note that  $\sigma \in \text{End}(F_d)$  is a substitution iff there exists a nonempty word  $W(i) \in \mathcal{A}_d^*$  such that  $W(i) \in [\sigma(i)]$  for each  $1 \leq i \leq d$ . In what follows, a word  $W \in \widehat{\mathcal{A}}_d^*$  will be identified with the element  $[W]$ , cf. the definition of substitutions given in Section 0.

For  $\sigma \in \text{End}(F_d)$ , we can set

$$\sigma(i) = w_1^{(i)} \cdots w_k^{(i)} \cdots w_{l_i}^{(i)} \in \widehat{\mathcal{A}}_d^* \quad (w_k^{(i)} \in \widehat{\mathcal{A}}_d)$$

such that the word on the right-hand side is reduced one in  $\widehat{\mathcal{A}}_d^*$  for each  $i \in \mathcal{A}_d$ . We define  $P_k^{(i)}, S_k^{(i)} \in \widehat{\mathcal{A}}_d^*$  by

$$P_k^{(i)} = w_1^{(i)} \cdots w_{k-1}^{(i)}, \quad S_k^{(i)} = w_{k+1}^{(i)} \cdots w_{l_i}^{(i)}.$$

$P_k^{(i)}$  (resp.,  $S_k^{(i)}$ ) will be referred to as the  $k$ -prefix (resp., the  $k$ -suffix) of  $\sigma(i)$ . Note that  $P_1^{(i)}$  is the empty word for any  $i \in \mathcal{A}_d$ . A canonical homomorphism  $\mathbf{f}: F_d \rightarrow \mathbf{Z}^d$  is defined by  $\mathbf{f}(i^{\pm 1}) = \pm \mathbf{e}_i$  ( $i \in \mathcal{A}_d$ ). Then there exists a unique linear representation  $A_\sigma$  on  $\mathbf{Z}^d$  associated with  $\sigma$  such that the following diagram becomes commutative:

$$\begin{array}{ccc} F_d & \xrightarrow{\sigma} & F_d \\ \mathbf{f} \downarrow & & \downarrow \mathbf{f} \\ \mathbf{Z}^d & \xrightarrow{A_\sigma} & \mathbf{Z}^d \end{array}$$

We introduce  $\widehat{\Lambda}_k$  ( $0 \leq k \leq d$ ) formally defined by:

$$\begin{aligned} \widehat{\Lambda}_0 &:= \Lambda_0 = \mathbf{Z}^d \times \{\bullet\}, \\ \widehat{\Lambda}_k &:= \mathbf{Z}^d \times \{i_1 \wedge i_2 \wedge \cdots \wedge i_k \mid i_n \in \widehat{\mathcal{A}}_d\} \quad (1 \leq k \leq d). \end{aligned}$$

We denote by  $\widehat{\mathcal{G}}_k$  the free  $\mathbf{Z}$ -module generated by the elements of  $\widehat{\Lambda}_k$ :

$$\widehat{\mathcal{G}}_k := \left\{ \sum_{\lambda' \in \widehat{\Lambda}_k} n_{\lambda'} \lambda' \mid n_{\lambda'} \in \mathbf{Z}, \#\{\lambda' \in \widehat{\Lambda}_k \mid n_{\lambda'} \neq 0\} < \infty \right\} \quad (0 \leq k \leq d).$$

DEFINITION 1. We denote by  $\iota: \widehat{\mathcal{G}}_k \rightarrow \mathcal{G}_k$  the  $\mathbf{Z}$ -homomorphism (the  $\mathbf{Z}$ -linear map) defined by

$$\iota(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) := \begin{cases} 0 & \text{if } \|i_n\| = \|i_m\| \text{ for some } n \neq m, \\ \text{sgn}(i_1) \cdots \text{sgn}(i_k) \epsilon(\tau)(\mathbf{x} + \sum_{j=1}^k \chi(i_j), \|i_{\tau(1)}\| \wedge \cdots \wedge \|i_{\tau(k)}\|) & (1 \leq \|i_{\tau(1)}\| < \cdots < \|i_{\tau(k)}\| \leq d), \text{ otherwise,} \end{cases}$$

where  $\tau$  is a permutation on  $\{1, \dots, k\}$ ,  $\epsilon(\tau)$  is the signature of  $\tau$ ,  $\text{sgn}(i^a)$  and  $\|i^a\|$  means  $\text{sgn}(i^a) := a$ ,  $\|i^a\| := i$ , and

$$\chi(i^a) := \begin{cases} \mathbf{o} & \text{if } a = 1 \\ \mathbf{f}(i^a) & \text{if } a = -1 \end{cases} \quad (a \in \{-1, 1\}, i \in \mathcal{A}_d).$$

For two elements  $g_1, g_2 \in \widehat{\mathcal{G}}_k$ , we write  $g_1 \sim g_2$  if  $\iota(g_1) = \iota(g_2)$ . It is easy to see that  $\sim$  is an equivalence relation. For example,

$$(\mathbf{o}, 2^{-1} \wedge 1) \sim \text{sgn}(2^{-1}) \text{sgn}(1)(\chi(2^{-1}) + \chi(1), 2 \wedge 1) \sim -(-\mathbf{e}_2, 2 \wedge 1) \sim (-\mathbf{e}_2, 1 \wedge 2).$$

Then,  $\mathcal{G}_k$  can be identified with a complete set of representatives of  $\widehat{\mathcal{G}}_k / \sim$ .

The geometrical meaning of the elements of  $\mathcal{G}_k$  ( $0 \leq k \leq d$ ), we have already mentioned, leads us the following definition of a map  $\delta_k: \mathcal{G}_k \rightarrow \mathcal{G}_{k-1}$ , which is considered to be a boundary map.

DEFINITION 2. Boundary maps  $\delta_k: \widehat{\mathcal{G}}_k \rightarrow \widehat{\mathcal{G}}_{k-1}$  ( $1 \leq k \leq d$ ) are  $\mathbf{Z}$ -homomorphisms defined by

$$\begin{aligned} & \delta_k(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) \\ &:= \sum_{n=1}^k (-1)^n \{(\mathbf{x}, i_1 \wedge \cdots \wedge \widehat{i}_n \wedge \cdots \wedge i_k) - (\mathbf{x} + \mathbf{f}(i_n), i_1 \wedge \cdots \wedge \widehat{i}_n \wedge \cdots \wedge i_k)\}. \end{aligned}$$

We note that  $\delta_{k-1} \circ \delta_k = 0$  ( $1 \leq k \leq d$ ) holds.

REMARK 1. The value of the map  $\delta_k$  is independent of the choice of a representative, i.e.,  $g_1 \sim g_2$  ( $g_1, g_2 \in \widehat{\mathcal{G}}_k$ ) implies  $\delta_k(g_1) \sim \delta_k(g_2)$ .

Let  $V$  and  $V'$  be  $\mathbf{Z}$ -modules. We mean by  $\text{Hom}_{\mathbf{Z}}(V, V')$  (resp.,  $\text{End}_{\mathbf{Z}}(V)$ ) the set of  $\mathbf{Z}$ -linear maps from  $V$  to  $V'$  (resp., from  $V$  to itself). Now, we can define a map  $E_k(\sigma) \in \text{End}_{\mathbf{Z}}(\mathcal{G}_k)$  for an endomorphism  $\sigma$  on  $F_d$ .

DEFINITION 3. Let  $\sigma \in \text{End}(F_d)$ .  $E_k(\sigma): \mathcal{G}_k \rightarrow \mathcal{G}_k$  ( $0 \leq k \leq d$ ) are  $\mathbf{Z}$ -linear maps defined by

$$\begin{aligned} & E_0(\sigma)(\mathbf{x}, \bullet) := (A_{\sigma}(\mathbf{x}), \bullet), \\ & E_k(\sigma)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) := \\ & \sum_{n_1=1}^{|\sigma(i_1)|} \cdots \sum_{n_k=1}^{|\sigma(i_k)|} (A_{\sigma}(\mathbf{x}) + \mathbf{f}(P_{n_1}^{(i_1)}) + \cdots + \mathbf{f}(P_{n_k}^{(i_k)}), w_{n_1}^{(i_1)} \wedge \cdots \wedge w_{n_k}^{(i_k)}) \quad (1 \leq k \leq d). \end{aligned}$$

We call  $E_k(\sigma)$  a substitution of dimension  $k$  with respect to  $\sigma \in \text{End}(F_d)$ . We remark that in a certain sense, our definition is compatible with the boundary map  $\delta_k$ :

**Proposition 3.** For  $\sigma \in \text{End}(F_d)$  and for each  $1 \leq k \leq d$ , the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{G}_k & \xrightarrow{E_k(\sigma)} & \mathcal{G}_k \\ \delta_k \downarrow & & \downarrow \delta_k \\ \mathcal{G}_{k-1} & \xrightarrow{E_{k-1}(\sigma)} & \mathcal{G}_{k-1} \end{array}$$

See Theorem 2.1 in [6], where the commutative diagram with  $\sigma \in \text{Sub}(F_d)$  is given. The following theorem is one of our main results.

**Theorem 1.** *Let  $\sigma \in \text{End}(F_d)$ . If  $\sigma$  is invertible, then there exists  $\mathbf{x} = \mathbf{x}_\sigma \in \mathbf{Z}^d$  such that*

$$E_d(\sigma)(\mathbf{o}, 1 \wedge 2 \wedge \cdots \wedge d) = \det(A_\sigma)(\mathbf{x}, 1 \wedge 2 \wedge \cdots \wedge d).$$

We need a lemma for the proof of Theorem 1.

**Lemma 1.** *Let an automorphism  $\sigma$  be decomposed as  $\sigma = \sigma' \circ \sigma''$ . Put*

$$\begin{aligned}\sigma(i) &= P_n^{(i)} w_n^{(i)} S_n^{(i)}, \\ \sigma'(i) &= P'_n{}^{(i)} w'_n{}^{(i)} S'_n{}^{(i)}, \sigma''(i) = P''_n{}^{(i)} w''_n{}^{(i)} S''_n{}^{(i)}.\end{aligned}$$

Then the following statements are valid:

- (i)  $A_\sigma = A_{\sigma'} A_{\sigma''}$ .
- (ii) For any pair  $(i, k)$  ( $i \in \mathcal{A}_d, 1 \leq k \leq |\sigma(i)|$ ), there exists a unique pair  $(m, n)$  such that

$$P_k^{(i)} = \sigma'(P''_m{}^{(i)}) P'_n{}^{(w''_m{}^{(i)})}, 0 \leq m \leq |\sigma''(i)|, 0 \leq n \leq |\sigma'(w''_m{}^{(i)})|.$$

- (iii)  $E_k(\sigma) = E_k(\sigma') \circ E_k(\sigma'') \quad (0 \leq k \leq d)$ .

Proof. The assertions (i), (ii) can be easily seen. We show the equality

$$E_k(\sigma)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) = E_k(\sigma') \circ E_k(\sigma'')(\mathbf{x}, i_1 \wedge \cdots \wedge i_k).$$

For simplicity, we put

$$\sum := \sum_{m_1=1}^{|\sigma''(i_1)|} \cdots \sum_{m_k=1}^{|\sigma''(i_k)|} \sum_{n_1=1}^{|\sigma'(w''_{m_1}{}^{(i_1)})|} \cdots \sum_{n_k=1}^{|\sigma'(w''_{m_k}{}^{(i_k)})|}.$$

Using (i), (ii) in this lemma, we get

$$\begin{aligned}E_k(\sigma)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) &= \sum_{p_1=1}^{|\sigma(i_1)|} \cdots \sum_{p_k=1}^{|\sigma(i_k)|} \left( A_\sigma(\mathbf{x}) + \mathbf{f}(P_{p_1}^{(i_1)}) + \cdots + \mathbf{f}(P_{p_k}^{(i_k)}), w_{p_1}^{(i_1)} \wedge \cdots \wedge w_{p_k}^{(i_k)} \right) \\ &= \sum \left( A_{\sigma'} A_{\sigma''}(\mathbf{x}) + \mathbf{f}(\sigma'(P''_{m_1}{}^{(i_1)})) P'_{n_1}{}^{(w''_{m_1}{}^{(i_1)})} + \cdots + \mathbf{f}(\sigma'(P''_{m_k}{}^{(i_k)})) P'_{n_k}{}^{(w''_{m_k}{}^{(i_k)})}, \right. \\ &\quad \left. w'_{n_1}{}^{(w''_{m_1}{}^{(i_1)})} \wedge \cdots \wedge w'_{n_k}{}^{(w''_{m_k}{}^{(i_k)})} \right) \\ &= \sum \left( A_{\sigma'} (A_{\sigma''}(\mathbf{x}) + \mathbf{f}(P''_{m_1}{}^{(i_1)}) + \cdots + \mathbf{f}(P''_{m_k}{}^{(i_k)})) + \mathbf{f}(P'_{n_1}{}^{(w''_{m_1}{}^{(i_1)})}) + \cdots + \mathbf{f}(P'_{n_k}{}^{(w''_{m_k}{}^{(i_k)})}), \right. \\ &\quad \left. w'_{n_1}{}^{(w''_{m_1}{}^{(i_1)})} \wedge \cdots \wedge w'_{n_k}{}^{(w''_{m_k}{}^{(i_k)})} \right)\end{aligned}$$



$$\begin{aligned}
&= E_k(\sigma') \left\{ \sum_{m_1=1}^{|\sigma''(i_1)|} \cdots \sum_{m_k=1}^{|\sigma''(i_k)|} \left( A_{\sigma''}(\mathbf{x}) + \mathbf{f}(P_{m_1}''(i_1)) + \cdots + \mathbf{f}(P_{m_k}''(i_k)), w_{m_1}''(i_1) \wedge \cdots \wedge w_{m_k}''(i_k) \right) \right\} \\
&= E_k(\sigma') \circ E_k(\sigma'')(\mathbf{x}, i_1 \wedge \cdots \wedge i_k). \quad \square
\end{aligned}$$

Proof of Theorem 1. It suffices to show Theorem 1 only when  $\sigma = \alpha_{ij}, \beta_{ij}, \gamma_j$ . By easy calculation, we have

$$\begin{aligned}
E_d(\alpha_{ij})(\mathbf{o}, 1 \wedge \cdots \wedge i \cdots \wedge j \cdots \wedge d) &= (\mathbf{o}, 1 \wedge \cdots \wedge j \cdots \wedge i \cdots \wedge d) \\
&= -(\mathbf{o}, 1 \wedge 2 \wedge \cdots \wedge d), \\
E_d(\beta_{ij})(\mathbf{o}, 1 \wedge \cdots \wedge j \wedge \cdots \wedge d) &= (\mathbf{e}_i, 1 \wedge \cdots \wedge j \wedge \cdots \wedge d), \\
E_d(\gamma_j)(\mathbf{o}, 1 \wedge \cdots \wedge j \wedge \cdots \wedge d) &= (\mathbf{o}, 1 \wedge \cdots \wedge j^{-1} \wedge \cdots \wedge d) \\
&= -(-\mathbf{e}_j, 1 \wedge \cdots \wedge j \wedge \cdots \wedge d).
\end{aligned}$$

In view of the assertion (iii) in Lemma 1, we get Theorem 1, cf. Remark 6 in Section 2.  $\square$

REMARK 2. It is not clear whether what the unit cube  $(\mathbf{o}, 1 \wedge \cdots \wedge d)$  of dimension  $d$  is mapped to one unit cube of dimension  $d$  by  $E_d(\sigma)$  implies that  $\sigma$  is an automorphism.

EXAMPLE 1. Let  $\sigma_R$  be the so called *Rauzy substitution*:

$$\sigma_R: \begin{cases} 1 \rightarrow 12 \\ 2 \rightarrow 13 \\ 3 \rightarrow 1 \end{cases} \quad \left( A_{\sigma_R} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right).$$

Then

$$E_3(\sigma_R)(\mathbf{o}, 1 \wedge 2 \wedge 3) = (2\mathbf{e}_1, 1 \wedge 2 \wedge 3).$$

By Remark 2, this doesn't imply that  $\sigma_R$  is invertible. But for the substitution  $\sigma_R$ , we have the inverse

$$\sigma_R^{-1}: \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 3^{-1}1 \\ 3 \rightarrow 3^{-1}2 \end{cases}.$$

We define another substitution  $\sigma_C$  given by

$$\sigma_C: \begin{cases} 1 \rightarrow 123 \\ 2 \rightarrow 112 \\ 3 \rightarrow 2333 \end{cases} \quad \left( A_{\sigma_C} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix} \right).$$

Then

$$E_3(\sigma_C)(\mathbf{o}, 1 \wedge 2 \wedge 3) = -(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, 1 \wedge 2 \wedge 3) - (\mathbf{e}_1 + \mathbf{e}_2 + 2\mathbf{e}_3, 1 \wedge 2 \wedge 3) \\ + (2\mathbf{e}_1 + \mathbf{e}_2, 1 \wedge 2 \wedge 3).$$

This together with Theorem 1 implies that  $\sigma_C$  is not invertible.

We consider all of substitutions satisfying  $A_\sigma = A_{\sigma_C}$ , whose cardinal number is 72. Then, we see that there does not exist a substitution  $\sigma$  satisfying  $A_\sigma = A_{\sigma_C}$  such that

$$E_3(\sigma)(\mathbf{o}, 1 \wedge 2 \wedge 3) = -(\mathbf{x}, 1 \wedge 2 \wedge 3)$$

for some  $\mathbf{x} \in \mathbf{Z}^3$ . By Theorem 1, it means there does not exist an invertible substitution satisfying  $A_\sigma = A_{\sigma_C}$ , cf. Section 3.1. The matrix  $A_\sigma = A_{\sigma_C}$  can be found in [5] as an example which does not give an invertible substitution.

## 2. Dual map $E_k^*(\sigma)$ for an endomorphism $\sigma$ on $F_d$

We can define in the obvious manner a dual space. Since we are in an infinite dimensional  $\mathbf{Z}$ -module, this defines a complicated space; and restrict ourself to the set of dual maps with finite support. We denote this set by  $\mathcal{G}_k^*$ . It has a natural basis, and we write  $(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*)$  for the dual vector of  $(\mathbf{x}, i_1 \wedge \cdots \wedge i_k)$ .

In fact, we introduce  $\Lambda_k^*$  ( $0 \leq k \leq d$ ) formally defined by

$$\Lambda_0^* := \mathbf{Z}^d \times \{\bullet^*\}, \\ \Lambda_k^* := \mathbf{Z}^d \times \{i_1^* \wedge i_2^* \wedge \cdots \wedge i_k^* \mid 1 \leq i_1 < \cdots < i_k \leq d\} \quad (1 \leq k \leq d).$$

We denote by  $\mathcal{G}_k^*$  the free  $\mathbf{Z}$ -module generated by the elements of  $\Lambda_k^*$  as follows:

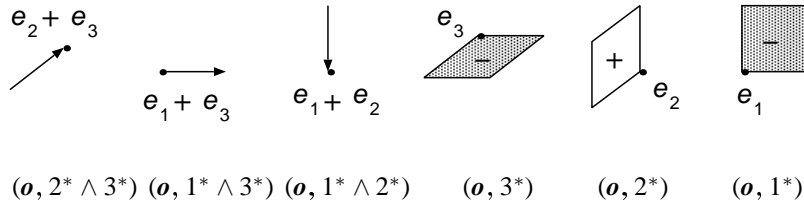
$$\mathcal{G}_k^* := \left\{ \sum_{\lambda^* \in \Lambda_k^*} n_{\lambda^*} \lambda^* \mid n_{\lambda^*} \in \mathbf{Z}, \#\{\lambda^* \in \Lambda_k^* \mid n_{\lambda^*} \neq 0\} < \infty \right\}.$$

REMARK 3. Following convention, we define a pairing by

$$\langle (\mathbf{y}, j_1 \wedge \cdots \wedge j_k), (\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle \\ := \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{y} \text{ and } i_t = j_t \text{ for all } 1 \leq t \leq k, \\ 0, & \text{otherwise} \end{cases},$$

$$(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_k^*, (\mathbf{y}, j_1 \wedge \cdots \wedge j_k) \in \mathcal{G}_k.$$

Thus  $\mathcal{G}_k^*$  can be considered as the set of dual maps with finite support.

Fig. 2. elements of  $\mathcal{G}_1^*$ ,  $\mathcal{G}_2^*$  in the case of  $d = 3$ 

We can define the  $\mathbf{Z}$ -linear isomorphism  $\varphi_k: \mathcal{G}_k^* \rightarrow \mathcal{G}_{d-k}$  ( $0 \leq k \leq d$ ) by

$$\varphi_0(\mathbf{x}, \bullet^*) := (\mathbf{x}, 1 \wedge \cdots \wedge d),$$

$$\varphi_k(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) := (-1)^{i_1 + \cdots + i_k} (\mathbf{x} + \mathbf{e}_{i_1} + \cdots + \mathbf{e}_{i_k}, j_1 \wedge \cdots \wedge j_{d-k}) \quad (1 \leq k \leq d),$$

where  $\{j_1, j_2, \dots, j_{d-k}\} = \mathcal{A}_d \setminus \{i_1, i_2, \dots, i_k\}$  with  $i_1 < \cdots < i_k$  and  $j_1 < \cdots < j_{d-k}$ . By virtue of the isomorphism  $\varphi_k$ , we can imagine an element  $(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_k^*$  as the element  $\varphi_k(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_{d-k}$  with geometrical meaning for each  $0 \leq k \leq d$ , see Fig. 2.

REMARK 4. In general, for  $\Phi \in \text{Hom}_{\mathbf{Z}}(\mathcal{G}_s, \mathcal{G}_t)$  ( $0 \leq s, t \leq d$ ), the dual map  $\Phi^*$  of  $\Phi$  is an element of  $\text{Hom}_{\mathbf{Z}}(\mathcal{G}_t^*, \mathcal{G}_s^*)$  determined by  $\langle F, \Phi^*(G^*) \rangle = \langle \Phi(F), G^* \rangle$  ( $F \in \mathcal{G}_s, G^* \in \mathcal{G}_t^*$ ).

**Proposition 4** ([6]). (i) *The dual boundary map  $\delta_k^*: \mathcal{G}_{k-1}^* \rightarrow \mathcal{G}_k^*$  ( $1 \leq k \leq d$ ) is given by*

$$\begin{aligned} \delta_1^*(\mathbf{x}, \bullet^*) &= \sum_{i=1}^d \{(\mathbf{x} - \mathbf{e}_i, i^*) - (\mathbf{x}, i^*)\}, \\ \delta_k^*(\mathbf{x}, i_1^* \wedge \cdots \wedge i_{k-1}^*) &= \sum_{n=1}^{d-k+1} (-1)^{j_n - n + 1} \{(\mathbf{x}, i_1^* \wedge \cdots \wedge i_{j_n - n}^* \wedge j_n^* \wedge i_{j_n - n + 1}^* \wedge \cdots \wedge i_{k-1}^*) \\ &\quad - (\mathbf{x} - \mathbf{e}_{j_n}, i_1^* \wedge \cdots \wedge i_{j_n - n}^* \wedge j_n^* \wedge i_{j_n - n + 1}^* \wedge \cdots \wedge i_{k-1}^*)\}, \end{aligned}$$

where  $\{j_1, j_2, \dots, j_{d-k+1}\} = \mathcal{A}_d \setminus \{i_1, i_2, \dots, i_{k-1}\}$  with  $i_1 < \cdots < i_{k-1}$ ,  $j_1 < \cdots < j_{d-k+1}$  and  $i_{j_n - n} < j_n < i_{j_n - n + 1}$  ( $2 \leq k \leq d$ ).

(ii) *The following diagram commutes for each  $1 \leq k \leq d$ :*

$$\begin{array}{ccc} \mathcal{G}_{k-1}^* & \xrightarrow{\varphi_{k-1}} & \mathcal{G}_{d-k+1} \\ \delta_k^* \downarrow & & \downarrow \delta_{d-k+1} \\ \mathcal{G}_k^* & \xrightarrow{\varphi_k} & \mathcal{G}_{d-k} \end{array}$$

From the commutativity (ii) given above, we can see that  $\delta_k^*$  is a boundary map with a geometrical sense.

By Remark 4, we can determine the dual map  $E_k^*(\sigma)$  (on  $\mathcal{G}_k^*$ ) of  $E_k(\sigma)$  (on  $\mathcal{G}_k$ ) for  $\sigma \in \text{End}(F_d)$  under a minor condition on  $\det(A_\sigma)$ :

**Proposition 5.** (i) *Let  $\sigma$  be an endomorphism on the free group  $F_d$  satisfying  $\det(A_\sigma) = \pm 1$ . Then dual maps  $E_k^*(\sigma): \mathcal{G}_k^* \rightarrow \mathcal{G}_k^*$  ( $0 \leq k \leq d$ ) satisfies*

$$\begin{aligned} E_0^*(\sigma)(\mathbf{x}, \bullet^*) &= (A_\sigma^{-1} \mathbf{x}, \bullet^*), \\ E_k^*(\sigma)(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) &= \sum_{\tau \in S_k} \sum_{\|w_{n_1}^{(j_1)}\| = i_{\tau(1)}^*} \cdots \sum_{\|w_{n_k}^{(j_k)}\| = i_{\tau(k)}^*} \text{sgn}(w_{n_1}^{(j_1)}) \cdots \text{sgn}(w_{n_k}^{(j_k)}) \epsilon(\tau) \\ &\quad \left( A_\sigma^{-1} \left( \mathbf{x} - \sum_{m=1}^k \{ \mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)}) \} \right), j_1^* \wedge \cdots \wedge j_k^* \right) \quad (1 \leq j_1 < \cdots < j_k \leq d), \end{aligned}$$

where  $S_k$  is the symmetric group of rank  $k$ .

(ii) *The following diagram is commutative for each  $1 \leq k \leq d$ :*

$$\begin{array}{ccc} \mathcal{G}_{k-1}^* & \xrightarrow{E_{k-1}^*(\sigma)} & \mathcal{G}_{k-1}^* \\ \delta_k^* \downarrow & & \downarrow \delta_k^* \\ \mathcal{G}_k^* & \xrightarrow{E_k^*(\sigma)} & \mathcal{G}_k^* \end{array}$$

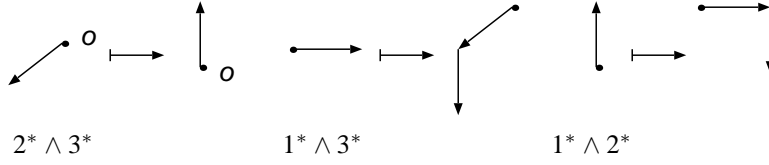
We remark that  $\sigma \in \text{Aut}(F_d)$  implies  $\det(A_\sigma) = \pm 1$ .

**Proof.** We can prove the proposition in a similar fashion as that given in [6]. The dual map  $E_k^*(\sigma) \in \text{End}_{\mathbb{Z}}(\mathcal{G}_k^*)$  of  $E_k(\sigma)$  is given by the identity

$$\langle F, E_k^*(\sigma)(G^*) \rangle = \langle E_k(\sigma)(F), G^* \rangle$$

for  $F \in \mathcal{G}_k$ ,  $G^* \in \mathcal{G}_k^*$  by Remark 4. Therefore, we get

$$\begin{aligned} &\langle (\mathbf{y}, j_1 \wedge \cdots \wedge j_k), E_k^*(\sigma)(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle \\ &= \langle E_k(\sigma)(\mathbf{y}, j_1 \wedge \cdots \wedge j_k), (\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle \\ &= \sum_{n_1=1}^{|\sigma(j_1)|} \cdots \sum_{n_k=1}^{|\sigma(j_k)|} \text{sgn}(w_{n_1}^{(j_1)}) \cdots \text{sgn}(w_{n_k}^{(j_k)}) \\ &\quad \left\langle \left( A_\sigma(\mathbf{y}) + \sum_{m=1}^k \{ \mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)}) \}, \|w_{n_1}^{(j_1)}\| \wedge \cdots \wedge \|w_{n_k}^{(j_k)}\| \right), (\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \right\rangle. \end{aligned}$$

Fig. 3. the map  $E_2^*(\sigma_R)$ 

The pairing appearing in the summation is equal to

$$\text{sgn}(w_{n_1}^{(j_1)}) \cdots \text{sgn}(w_{n_k}^{(j_k)}) \epsilon(\tau)$$

if  $A_\sigma(\mathbf{y}) + \sum_{m=1}^k \{\mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)})\} = \mathbf{x}$ , and  $\|w_{n_l}^{(j_l)}\| = i_{\tau(l)}$  ( $1 \leq l \leq k$ ),  $\tau \in S_k$ ; it is equal to 0, otherwise. Hence we get (i). The commutative diagram (ii) comes from Proposition 3.  $\square$

We write  $(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) \simeq (\mathbf{y}, j_1 \wedge \cdots \wedge j_{d-k})$  iff  $\varphi_k(\mathbf{x}, i_1^* \wedge \cdots \wedge i_k^*) = (\mathbf{y}, j_1 \wedge \cdots \wedge j_{d-k})$ .

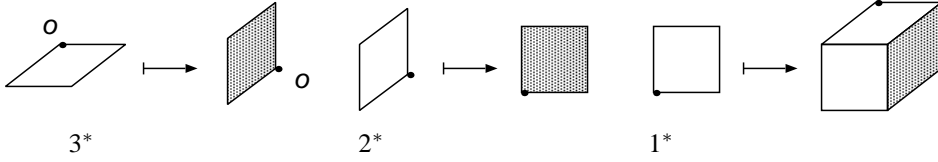
EXAMPLE 2. Let  $\sigma_R$  be the Rauzy substitution given in Example 1. Then

$$\begin{aligned} E_2^*(\sigma_R): & -(-\mathbf{e}_2 - \mathbf{e}_3, 2^* \wedge 3^*) \mapsto -(-\mathbf{e}_1 - \mathbf{e}_2, 1^* \wedge 2^*) \\ & \simeq (\mathbf{o}, 1) \quad \simeq (\mathbf{o}, 3) \\ & (-\mathbf{e}_1 - \mathbf{e}_3, 1^* \wedge 3^*) \mapsto -(-\mathbf{e}_2 - \mathbf{e}_3, 2^* \wedge 3^*) + (-\mathbf{e}_2 - \mathbf{e}_3, 1^* \wedge 2^*) \\ & \simeq (\mathbf{o}, 2) \quad \simeq (\mathbf{o}, 1) - (\mathbf{e}_1 - \mathbf{e}_3, 3) \\ & -(-\mathbf{e}_1 - \mathbf{e}_2, 1^* \wedge 2^*) \mapsto (-\mathbf{e}_1 - \mathbf{e}_3, 1^* \wedge 3^*) + (-\mathbf{e}_1 - \mathbf{e}_3, 1^* \wedge 2^*) \\ & \simeq (\mathbf{o}, 3) \quad \simeq (\mathbf{o}, 2) - (\mathbf{e}_2 - \mathbf{e}_3, 3) \end{aligned}$$

See Fig. 3.

$$\begin{aligned} E_1^*(\sigma_R): & -(-\mathbf{e}_3, 3^*) \mapsto -(-\mathbf{e}_2, 2^*) \\ & \simeq (\mathbf{o}, 1 \wedge 2) \quad \simeq -(\mathbf{o}, 1 \wedge 3) \\ & (-\mathbf{e}_2, 2^*) \mapsto (-\mathbf{e}_1, 1^*) \\ & \simeq (\mathbf{o}, 1 \wedge 3) \quad \simeq -(\mathbf{o}, 2 \wedge 3) \\ & -(-\mathbf{e}_1, 1^*) \mapsto -(-\mathbf{e}_3, 1^*) - (-\mathbf{e}_3, 2^*) - (-\mathbf{e}_3, 3^*) \\ & \simeq (\mathbf{o}, 2 \wedge 3) \quad \simeq (\mathbf{e}_1 - \mathbf{e}_3, 2 \wedge 3) - (\mathbf{e}_2 - \mathbf{e}_3, 1 \wedge 3) + (\mathbf{o}, 1 \wedge 2) \end{aligned}$$

See Fig. 4.

Fig. 4. the map  $E_1^*(\sigma_R)$ 

For  $W = s_1 s_2 \cdots s_n \in \hat{\mathcal{A}}_d^*$ ,  $\overline{W}$  denotes the mirror image of  $W$ , i.e.,

$$\overline{W} := s_n s_{n-1} \cdots s_1.$$

For  $\sigma \in \text{End}(F_d)$ , the endomorphism  $\overline{\sigma}$  is given by  $\overline{\sigma}(i) = \overline{\sigma(i)}$  ( $i \in \mathcal{A}_d$ ). Now, we can state a result.

**Theorem 2.** *Let  $\sigma$  be an automorphism on the free group  $F_d$ . Then there exists  $\mathbf{x} \in \mathbf{Z}^d$  such that*

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} = \det(A_\sigma) \circ T_k(\mathbf{x}) \circ E_k(\overline{\sigma^{-1}}) \quad (0 \leq k \leq d),$$

where the map  $T_k(\mathbf{x}): \mathcal{G}_k \rightarrow \mathcal{G}_k$  with  $\mathbf{x} \in \mathbf{Z}^d$  is given by

$$T_k(\mathbf{x}) \left( \sum_{t=1}^m n_t (\mathbf{y}_t, i_1^{(t)} \wedge \cdots \wedge i_k^{(t)}) \right) = \sum_{t=1}^m n_t (\mathbf{x} + \mathbf{y}_t, i_1^{(t)} \wedge \cdots \wedge i_k^{(t)}).$$

We remark that since  $\mathcal{G}_k$  is a free  $\mathbf{Z}$ -module, an integer  $a$  is an operator on  $\mathcal{G}_k$ , i.e.,  $a(\sum_{\lambda \in \Lambda_k} n_\lambda \lambda) = \sum_{\lambda \in \Lambda_k} (a \cdot n_\lambda) \lambda$ . For the proof of Theorem 2, we need some lemmas.

**Lemma 2.**  $\overline{\sigma' \circ \sigma} = \overline{\sigma'} \circ \overline{\sigma}$  ( $\sigma, \sigma' \in \text{End}(F_d)$ ).

Proof. Setting  $\sigma(i) = w_1^{(i)} \cdots w_l^{(i)}$ , we have

$$\begin{aligned} \overline{\sigma' \circ \sigma}(i) &= \overline{\sigma'(w_1^{(i)} \cdots w_l^{(i)})} = \overline{\sigma'(w_1^{(i)}) \cdots \sigma'(w_l^{(i)})} = \overline{\sigma'(w_l^{(i)})} \cdots \overline{\sigma'(w_1^{(i)})} \\ &= \overline{\sigma'}(w_l^{(i)} \cdots w_1^{(i)}) = \overline{\sigma'} \circ \overline{\sigma}(i), \end{aligned}$$

so that  $\overline{\sigma' \circ \sigma} = \overline{\sigma'} \circ \overline{\sigma}$ . □

By  $E_k(\sigma_1 \circ \sigma_2) = E_k(\sigma_1) \circ E_k(\sigma_2)$  and the duality  $(\Phi_1 \circ \Phi_2)^* = \Phi_2^* \circ \Phi_1^*$ , we have

**Lemma 3.**  $E_k^*(\sigma_1 \circ \sigma_2) = E_k^*(\sigma_2) \circ E_k^*(\sigma_1)$  ( $\sigma_1, \sigma_2 \in \text{End}(F_d)$ ).

**Lemma 4.** *Let  $\sigma$  be one of*

$$\mathcal{N} := \{\alpha_{ij}, \beta_{ij}, \gamma_j \mid 1 \leq i, j \leq d, i \neq j\}.$$

*Then*

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} = \det(A_\sigma) \circ T_k(v(\sigma)) \circ E_k(\overline{\sigma^{-1}}),$$

where  $v: \mathcal{N} \rightarrow \{\mathbf{o}, \mathbf{e}_1, \dots, \mathbf{e}_d\}$  is the map given by  $v(\alpha_{ij}) := \mathbf{o}$ ,  $v(\beta_{ij}) := \mathbf{o}$ ,  $v(\gamma_j) := \mathbf{e}_j$ .

**Proof.** It suffices to show that

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1}(\mathbf{o}, i_1 \wedge \dots \wedge i_k) = \det(A_\sigma) \circ T_k(v(\sigma)) \circ E_k(\overline{\sigma^{-1}})(\mathbf{o}, i_1 \wedge \dots \wedge i_k).$$

Notice  $A_{\sigma^{-1}} = A_\sigma^{-1}$ . We consider the case of  $\sigma = \beta_{ij}$  and  $k = 1$ . Note that

$$A_{\beta_{ij}}^{-1} = \begin{matrix} j \text{ th} \\ (\mathbf{e}_1, \dots, -\mathbf{e}_i + \mathbf{e}_j, \dots, \mathbf{e}_d), \end{matrix} \quad \overline{\beta_{ij}^{-1}}: \begin{cases} j \rightarrow ji^{-1} \\ l \rightarrow l \quad (i \neq j). \\ \text{for all } l \neq j \end{cases}$$

We easily have

$$\varphi_{d-1} \circ E_{d-1}^*(\beta_{ij}) \circ \varphi_{d-1}^{-1}(\mathbf{o}, l) = (\mathbf{o}, l) = E_1(\overline{\beta_{ij}^{-1}})(\mathbf{o}, l) \quad (l \neq j).$$

On the other hand, we get the following equality. On the fourth line in the calculation given below, we must be careful with the location of  $j^*$ . If  $j > i$ , then  $j^*$  locates at  $i$  th place, otherwise, at  $i-1$  th place. Using a permutation, we move  $j^*$  to the ordinal place, and then we have the sixth line.

$$\begin{aligned} & \varphi_{d-1} \circ E_{d-1}^*(\beta_{ij}) \circ \varphi_{d-1}^{-1}(\mathbf{o}, j) \\ &= \varphi_{d-1} \circ E_{d-1}^*(\beta_{ij})(-1)^{1+\dots+d-j} \left( - \sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{j^*} \wedge \dots \wedge d^* \right) \\ &= \varphi_{d-1} \left\{ (-1)^{1+\dots+d-j} \left( - \sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{j^*} \wedge \dots \wedge d^* \right) \right. \\ & \quad \left. + (-1)^{1+\dots+d-j} \left( - \sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{i^*} \wedge \dots \wedge j^* \wedge \dots \wedge d^* \right) \right\} \\ &= \varphi_{d-1} \left\{ (-1)^{1+\dots+d-j} \left( - \sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{j^*} \wedge \dots \wedge d^* \right) \right. \\ & \quad \left. + (-1)^{1+\dots+d-i+1} \left( - \sum_{m=1}^d \mathbf{e}_m + \mathbf{e}_j, 1^* \wedge \dots \wedge \widehat{i^*} \wedge \dots \wedge d^* \right) \right\} \end{aligned}$$

$$\begin{aligned}
&= (\mathbf{o}, j) - (-\mathbf{e}_i + \mathbf{e}_j, i) \\
&= E_1(\overline{\beta_{ij}^{-1}})(\mathbf{o}, j).
\end{aligned}$$

Hence we get

$$\varphi_{d-1} \circ E_{d-1}^*(\sigma) \circ \varphi_{d-1}^{-1} = \det(A_\sigma) \circ T_1(v(\sigma)) \circ E_1(\overline{\sigma^{-1}})$$

for  $\sigma = \beta_{ij}$  and  $k = 1$ . For other cases, we can do the same, and the technical term can be found in Proof of Proposition 1.1 in [6].  $\square$

**Lemma 5.** For  $a, a_m \in \mathbf{Z}$ ,  $\mathbf{x}, \mathbf{y}, \mathbf{y}_m \in \mathbf{Z}^d$ ,  $\sigma_m \in \text{End}(F_d)$  ( $1 \leq m \leq n$ ) and for  $0 \leq k \leq d$ , we have the following formulas:

- (i)  $T_k(\mathbf{x}) \circ T_k(\mathbf{y}) = T_k(\mathbf{x} + \mathbf{y})$
- (ii)  $a \circ T_k(\mathbf{x}) = T_k(\mathbf{x}) \circ a$
- (iii)  $(a_n \circ T_k(\mathbf{y}_n) \circ E_k(\sigma_n)) \circ \cdots \circ (a_1 \circ T_k(\mathbf{y}_1) \circ E_k(\sigma_1)) = a_1 \cdots a_n \circ T_k(\mathbf{y}_n + A_{\sigma_n}(\mathbf{y}_{n-1}) + A_{\sigma_n \sigma_{n-1}}(\mathbf{y}_{n-2}) + \cdots + A_{\sigma_n \cdots \sigma_2}(\mathbf{y}_1)) \circ E_k(\sigma_n \cdots \sigma_1).$

*Proof.* The statements (i), (ii) are trivial. For the proof of the third statement, it is enough to show

$$(a_2 \circ T_k(\mathbf{y}_2) \circ E_k(\sigma_2)) \circ (a_1 \circ T_k(\mathbf{y}_1) \circ E_k(\sigma_1)) = a_1 a_2 \circ T_k(\mathbf{y}_2 + A_{\sigma_2}(\mathbf{y}_1)) \circ E_k(\sigma_2 \sigma_1).$$

We can put

$$E_k(\sigma_1)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) = \sum_{\lambda \in \Lambda_k} n_\lambda(\mathbf{x}_\lambda, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}).$$

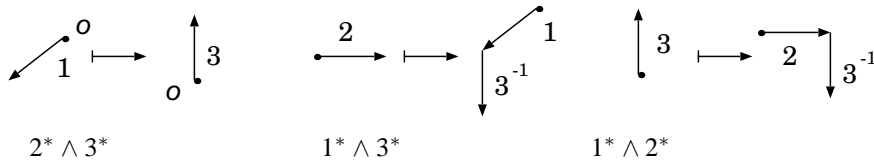
Using (i), (ii) in the lemma, we have

$$\begin{aligned}
&(a_2 \circ T_k(\mathbf{y}_2) \circ E_k(\sigma_2)) \circ (a_1 \circ T_k(\mathbf{y}_1) \circ E_k(\sigma_1))(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) \\
&= a_2 \circ T_k(\mathbf{y}_2) \circ E_k(\sigma_2) \left\{ \sum_{\lambda \in \Lambda_k} a_1 n_\lambda(\mathbf{x}_\lambda + \mathbf{y}_1, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}) \right\} \\
&= a_2 \circ T_k(\mathbf{y}_2) \left\{ \sum_{\lambda \in \Lambda_k} a_1 n_\lambda E_k(\sigma_2)(\mathbf{x}_\lambda + \mathbf{y}_1, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}) \right\} \\
&= a_2 \circ T_k(\mathbf{y}_2) \circ a_1 \circ T_k(A_{\sigma_2}(\mathbf{y}_1)) \circ E_k(\sigma_2) \circ E_k(\sigma_1)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k) \\
&= a_1 a_2 \circ T_k(\mathbf{y}_2 + A_{\sigma_2}(\mathbf{y}_1)) \circ E_k(\sigma_2 \sigma_1)(\mathbf{x}, i_1 \wedge \cdots \wedge i_k). \quad \square
\end{aligned}$$

**Proof of Theorem 2.** Let  $\sigma$  be an automorphism. Then  $\sigma$  can be written as  $\sigma = \sigma_1 \cdots \sigma_n$  with  $\sigma_m \in \mathcal{N}$  ( $1 \leq m \leq n$ ). Using Lemma 2–5, we have

$$\begin{aligned}
&\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} \\
&= \varphi_{d-k} \circ E_{d-k}^*(\sigma_n) \circ \cdots \circ E_{d-k}^*(\sigma_1) \circ \varphi_{d-k}^{-1}
\end{aligned}$$



Fig. 5. the map  $E_2^*(\sigma_R)$ 

$$\begin{aligned}
 &= (\det(A_{\sigma_n}) \circ T_k(v(\sigma_n)) \circ E_k(\overline{\sigma_n^{-1}})) \circ \cdots \circ (\det(A_{\sigma_1}) \circ T_k(v(\sigma_1)) \circ E_k(\overline{\sigma_1^{-1}})) \\
 &= \det(A_{\sigma_1}) \cdots \det(A_{\sigma_n}) \circ T_k(v(\sigma_n) + A_{\sigma_n^{-1}}(v(\sigma_{n-1})) + \cdots + A_{\sigma_n^{-1} \cdots \sigma_2^{-1}}(v(\sigma_1))) \\
 &\quad \circ E_k(\overline{\sigma_n^{-1}} \cdots \overline{\sigma_1^{-1}}) \\
 &= \det(A_{\sigma_1 \cdots \sigma_n}) \circ T_k(v(\sigma_n) + A_{\sigma_n^{-1}}(v(\sigma_{n-1})) + \cdots + A_{(\sigma_2 \cdots \sigma_n)^{-1}}(v(\sigma_1))) \circ E_k(\overline{(\sigma_1 \cdots \sigma_n)^{-1}}) \\
 &= \det(A_\sigma) \circ T_k(\mathbf{x}) \circ E_k(\overline{\sigma^{-1}}),
 \end{aligned}$$

where  $\mathbf{x} = v(\sigma_n) + A_{\sigma_n^{-1}}(v(\sigma_{n-1})) + \cdots + A_{\sigma_2^{-1} \cdots \sigma_n^{-1}}(v(\sigma_1))$ .  $\square$

REMARK 5. In the case of  $d = 3$ , in particular, for a Pisot substitution  $\sigma \in \text{Sub}(F_3)$  (i.e., a substitution such that the characteristic polynomial of  $A_\sigma$  is equal to the minimal polynomial of a Pisot number), we are interested in the region  $E_1^*(\sigma)^n(\sum_{i=1}^3(\mathbf{o}, i^*))$  ( $n \in \mathbb{N}$ ) in connection with stepped surfaces, cf. [1]. It follows from the assertion (ii) in Proposition 5 that the boundary of  $E_1^*(\sigma)^n(\sum_{i=1}^3(\mathbf{o}, i^*))$  coincides with  $E_2^*(\sigma)^n(\sum_{i=1}^3 \delta_2^*(\mathbf{o}, i^*))$ . On the other hand, Theorem 2 says that  $E_2^*(\sigma)^n(\sum_{i=1}^3 \delta_2^*(\mathbf{o}, i^*))$  can be calculated by using the map  $\overline{\sigma^{-1}}^n$ .

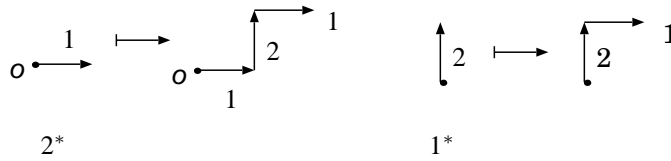
REMARK 6. In this setting we can rephrase Theorem 1 as follows:  
If  $\sigma \in \text{Aut}(F_d)$  is written as  $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$  with  $\sigma_i \in \mathcal{N}$ , then  $\mathbf{x}_\sigma$  in Theorem 1 is given by

$$\mathbf{x}_\sigma = v'(\sigma_1) + A_{\sigma_1}(v'(\sigma_2)) + \cdots + A_{\sigma_1 \cdots \sigma_{n-1}}(v'(\sigma_n)),$$

where  $v': \mathcal{N} \rightarrow \{\mathbf{o}, \mathbf{e}_1, \dots, \mathbf{e}_d\}$  is the map given by  $v'(\alpha_{ij}) := \mathbf{o}$ ,  $v'(\beta_{ij}) := \mathbf{e}_i$ ,  $v'(\gamma_j) := -\mathbf{e}_j$ .

EXAMPLE 2'. For the Rauzy substitution  $\sigma_R$  given in Example 1, we can show  $\varphi_2 \circ E_2^*(\sigma_R) \circ \varphi_2^{-1} = E_1(\overline{\sigma_R^{-1}})$ . In view of Fig. 5, we easily see that  $\sigma_R^{-1}$  is given by

$$\sigma_R^{-1}: \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 3^{-1}1 \\ 3 \rightarrow 3^{-1}2 \end{cases}.$$

Fig. 6. the map  $E_1^*(\sigma)$ 

We give another example of  $E_1^*(\sigma)$  for an endomorphism  $\sigma \in \text{End}(F_2)$  which is not a substitution.

EXAMPLE 3. Let  $\sigma \in \text{Aut}(F_2)$  be given by

$$\sigma: \begin{cases} 1 \rightarrow 2^{-1}1 \\ 2 \rightarrow 1^{-1}22 \end{cases}.$$

Then

$$\begin{aligned} E_1^*(\sigma): (-e_2, 2^*) &\mapsto -(o, 1^*) + (e_1, 2^*) + (-e_2, 2^*) \\ &\simeq (o, 1) \simeq (e_1, 2) + (e_1 + e_2, 1) + (o, 1) \\ -(-e_1, 1^*) &\mapsto -(-e_1, 1^*) + (o, 2^*) \\ &\simeq (o, 2) \simeq (o, 2) + (e_2, 1) \end{aligned}.$$

We can show  $\varphi_1 \circ E_1^*(\sigma) \circ \varphi_1^{-1} = E_1(\overline{\sigma^{-1}})$ . In view of Fig. 6, we easily see  $\sigma^{-1}$  is given by

$$\sigma^{-1}: \begin{cases} 1 \rightarrow 121 \\ 2 \rightarrow 12 \end{cases}.$$

As we have already seen in the two examples above, we can construct  $\sigma^{-1}$ , in some cases, by the figure of  $E_{d-1}^*(\sigma)$  for  $\sigma \in \text{Aut}(F_d)$ . In general, we have certain difficulty, cf. Section 3.3.

### 3. Examples and some comments

In the case of  $d = 3$ , some difficulties which never occur in the case of  $d = 2$ , will take place as we shall see through some examples.

**3.1. Substitutions given by a matrix.** It is easy to see, as is well known, that any unimodular matrix  $A \in GL(2, N \cup \{0\})$  (i.e.,  $\det(A) = \pm 1$ ) can be decomposed into two matrices

$$A_{\alpha_{12}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_{\beta_{12}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Therefore, for any matrix  $A \in GL(2, N \cup \{0\})$ , there exists at least one invertible substitution  $\sigma$  such that  $A_\sigma = A$ . On the other hand, in Example 1 in Section 1,

we have seen that any substitution  $\sigma$  satisfying  $A_\sigma = A_{\sigma_C}$  is not invertible. We put  $A_{ij} = (a_{lm})_{1 \leq l, m \leq d}$  ( $1 \leq i, j \leq d, i \neq j$ ) by

$$a_{lm} := \begin{cases} 1 & \text{if } l = m \\ -1 & \text{if } l = i, m = j \\ 0, & \text{otherwise} \end{cases}.$$

We say that a matrix  $M \in GL(d, N \cup \{0\})$  is *non-comparable* if both  $MA_{ij}$ ,  $A_{ij}M$  have negative entries for all  $A_{ij}$  ( $1 \leq i, j \leq d, i \neq j$ ). For instance,  $A_{\sigma_R}$  is a comparable matrix, while  $A_{\sigma_C}$  is a non-comparable one. It seems very likely that if  $A_\sigma$  is non-comparable, then  $\sigma$  can not be an invertible substitution, i.e.,

$$\sigma \notin \text{IS}(F_d),$$

as far as we know.

**3.2. Generators of the invertible substitutions.** An invertible substitution  $\sigma$  is called a *prime substitution* if  $\sigma$  cannot be decomposed into 2 invertible substitutions  $\sigma_1, \sigma_2$  such that one of  $\sigma_1$ , and  $\sigma_2$  does not belong to the group generated by  $\alpha_{ij}$  ( $1 \leq i, j \leq d, i \neq j$ ). Related to generators of the invertible substitutions  $\sigma$  (i.e.,  $\sigma \in \text{IS}(F_d)$ ), some results are found in [4], [5]. In the case of  $d = 2$ , generators of the invertible substitutions are given by three prime substitutions:

$$\alpha: \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 1 \end{cases}, \beta: \begin{cases} 1 \rightarrow 12 \\ 2 \rightarrow 1 \end{cases}, \delta: \begin{cases} 1 \rightarrow 21 \\ 2 \rightarrow 1 \end{cases},$$

so that the number of generators is finite, cf. [4]. But, the monoid  $\text{IS}(F_d)$  for  $d \geq 3$  turns out to be quite different from that for  $d = 2$ . For example, in the case of  $d = 3$ , we need infinitely many generators. In fact,  $\sigma \in \text{IS}(F_3)$  defined by

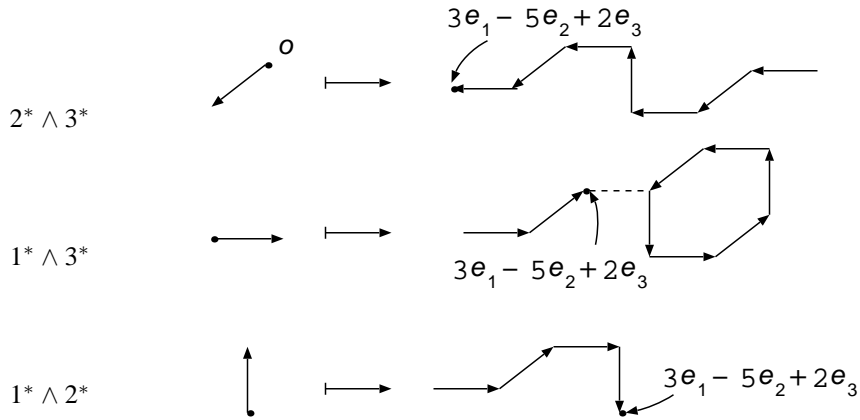
$$\sigma(1) := 12, \sigma(2) := 132, \sigma(3) := 3^n 2 \quad (n \geq 2)$$

are prime substitutions, cf. [5].

**3.3. Connectedness of  $E_{d-1}^*(\sigma)(o, \mathbf{1}^* \wedge \cdots \wedge \widehat{\mathbf{j}^*} \cdots \wedge \mathbf{d}^*)$ .** In the case of  $d = 2$ , we have shown in [2] the following proposition related to the dual map  $E_1^*(\sigma)$ .

**Proposition 6** ([2]). *A substitution  $\sigma$  over 2 letters is invertible iff all the figures coming from  $E_1^*(\sigma)(-\mathbf{e}_1, \mathbf{1}^*)$ ,  $E_1^*(\sigma)(-\mathbf{e}_2, \mathbf{2}^*)$ ,  $E_1^*(\sigma)((-\mathbf{e}_1, \mathbf{1}^*) + (-\mathbf{e}_2, \mathbf{2}^*))$  are connected.*

The figures coming from  $E_1^*(\sigma)(-\mathbf{e}_i, \mathbf{i}^*)$  ( $i = 1, 2$ ) are parts of the so called stepped surface, cf. [1]. Since a stepped curve (a stepped surface of dimension 1)

Fig. 7. the map  $E_2^*(\sigma)$ 

univalently spreads along a line, any cancellation can not occur in  $E_1^*(\sigma)(-e_i, i^*)$  ( $i = 1, 2$ ). We can easily find the inverse of an invertible substitution  $\sigma \in \text{IS}(F_2)$  from the figures coming from  $E_1^*(\sigma)(-e_i, i^*)$  ( $i = 1, 2$ ), provided that  $E_1^*(\sigma)(-e_i, i^*)$  ( $i = 1, 2$ ) contain no cancellations.

On the other hand, in the case of  $d = 3$ ,  $E_2^*(\sigma)(-e_i - e_j, i^* \wedge j^*)$  ( $\sigma \in \text{IS}(F_3)$ ,  $(i, j) \in \{(1, 2), (1, 3), (2, 3)\}$ ) is not always connective:

EXAMPLE 4. Let  $\sigma$  be an invertible substitution given by

$$\sigma: \begin{cases} 1 \rightarrow 1223 \\ 2 \rightarrow 123 \\ 3 \rightarrow 133 \end{cases} \quad \left( \sigma^{-1}: \begin{cases} 1 \rightarrow 21^{-1}23^{-1}21^{-1}2 \\ 2 \rightarrow 2^{-1}12^{-1}32^{-1}13^{-1}21^{-1}2 \\ 3 \rightarrow 2^{-1}12^{-1}3 \end{cases} \right).$$

Since  $\varphi_2 \circ E_2^*(\sigma) \circ \varphi_2^{-1} = -T_1(3e_1 - 5e_2 + 2e_3) \circ E_1(\overline{\sigma^{-1}})$ , the figure coming from  $E_2^*(\sigma)(o, 1^* \wedge 3^*)$  is not connected, see Fig. 7.

**3.4. Open problems.** We give two problems for arbitrary  $d \geq 3$ :

- (i) Does the converse of the statement of Theorem 1 hold?
- (ii) Let  $\sigma \in \text{Sub}(F_d)$  be a substitution with a non-comparable matrix  $A_\sigma$  such that  $A_\sigma$  does not belong to the group generated by  $A_{\alpha_{ij}}, A_{\beta_{ij}}$  ( $1 \leq i, j \leq d, i \neq j$ ). Then, is  $\sigma$  always not invertible?

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