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SOME PROPERTIES OF INVERTIBLE SUBSTITUTIONS OF rank d, AND HIGHER DIMENSIONAL SUBSTITUTIONS

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0. Introduction

We denote by $\mathcal{A}_d{}^*$ (resp., F_d) the free monoid (resp., the free group), with the empty word as unit, generated by an alphabet $\mathcal{A}_d := \{1, 2, \dots, d\}$ consisting of d letters. We consider an endomorphism σ on F_d, i.e., a group homomorphism from F_d to itself. An endomorphism σ will be referred to as a *substitution* if we can take a nonempty word $\sigma(i) \in \mathcal{A}_d{}^*$ for all $i \in \mathcal{A}_d$, cf. the first paragraph of Section 1. When is a substitution σ invertible as an endomorphism on F_d? An answer to this question is known when $d = 2$, cf. Proposition 1. Our objective is to generalize Proposition 1 for arbitrary $d \geq 2$. We introduce a geometrical method in [2]; and we use a general method given in [6], where the so called *higher dimensional substitutions* $E_k(\sigma)$ $(0 \leq k \leq d)$ are established for a given substitution σ on F_d.

Throughout the paper, we denote by $\boldsymbol{Z}$ (resp., $\boldsymbol{N}$, $\boldsymbol{R}$) the set of integers (resp., positive integers, real numbers), and by $\mathrm{End}(F_d)$ (resp., $\mathrm{Sub}(F_d)$, $\mathrm{Aut}(F_d)$, $\mathrm{IS}(F_d)$) the set of endomorphisms (resp., substitutions, automorphisms, invertible substitutions) on F_d.

Let $d \geq 2$ be an integer. We mean by $(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k)$ the positively oriented unit cube of dimension k translated by $\boldsymbol{x}$ in the Euclidean space $\boldsymbol{R}^d$:

$$(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) := \{\boldsymbol{x} + t_1\boldsymbol{e}_{i_1} + \cdots + t_k\boldsymbol{e}_{i_k} \mid 0 \leq t_n \leq 1,\ 1 \leq n \leq k\},$$
$$\boldsymbol{x} \in \boldsymbol{Z}^d,\ 0 \leq k \leq d,\ 1 \leq i_1 < \cdots < i_k \leq d,$$

where $\{\boldsymbol{e}_i\}_{i=1,\dots,d}$ is the canonical basis of $\boldsymbol{R}^d$. In particular, for $k = 0$, the k dimensional unit cube $(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k)$, which will be denoted by $(\boldsymbol{x}, \bullet)$, is considered to turn out a point $\boldsymbol{x}$. In general, for $\{i_1, i_2, \dots, i_k\}$ with $1 \leq i_m \leq d$, $1 \leq m \leq k$, we define

$$(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) := 0, \quad \text{if } i_n = i_m \text{ for some } n \neq m,$$
$$(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) := \epsilon(\tau)(\boldsymbol{x}, i_{\tau(1)} \wedge \cdots \wedge i_{\tau(k)}) \quad (1 \leq i_{\tau(1)} < \cdots < i_{\tau(k)} \leq d), \quad \text{otherwise},$$

where τ is a permutation on $\{1, \dots, k\}$, and $\epsilon(\tau)$ is the signature of τ, which designates the orientation. We put

$$\Lambda_0 := \boldsymbol{Z}^d \times \{\bullet\},$$

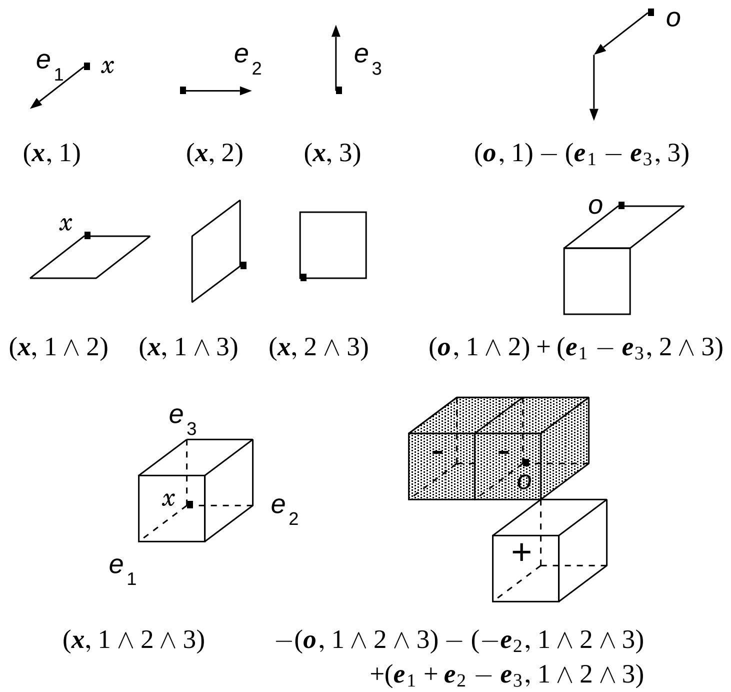


Fig. 1. elements $\sum_{\lambda \in \Lambda_k} n_\lambda \lambda \in \mathcal{G}_k$, $k = 1, 2, 3$

$$\Lambda_k := \boldsymbol{Z}^d \times \{i_1 \wedge i_2 \wedge \cdots \wedge i_k \mid 1 \le i_1 < \cdots < i_k \le d\} \ (1 \le k \le d).$$

We denote by $\mathcal{G}_k$ the free $\boldsymbol{Z}$-module generated by the elements of Λ_k:

$$\mathcal{G}_k := \left\{ \sum_{\lambda \in \Lambda_k} n_\lambda \lambda \;\middle|\; n_\lambda \in \boldsymbol{Z}, \sharp\{\lambda \in \Lambda_k \mid n_\lambda \ne 0\} < \infty \right\} \ (0 \le k \le d).$$

We can identify the element $\sum_{\lambda \in \Lambda_k} n_\lambda \lambda \in \mathcal{G}_k$ with the union of oriented d dimensional unit cubes with their multiplicity, cf. Fig. 1.

For a word $P \in \mathcal{A}_d{}^*$, $|P|$ denotes the length of the word P. For $\sigma \in \mathrm{Sub}(F_d)$, $i \in \mathcal{A}_d$ and $0 \le k \le l_i := |\sigma(i)|$, we define a word $P_k^{(i)} \in \mathcal{A}_d{}^*$ to be a prefix

$$P_k^{(i)} := w_1^{(i)} \cdots w_{k-1}^{(i)}$$

of $\sigma(i) = w_1^{(i)} \cdots w_k^{(i)} \cdots w_{l_i}^{(i)}$ ($w_j^{(i)} \in \mathcal{A}_d$ $(1 \le j \le l_i)$). A higher dimensional substitution $E_d(\sigma)\colon \mathcal{G}_d \to \mathcal{G}_d$ is a $\boldsymbol{Z}$-linear map (an endomorphism on a free $\boldsymbol{Z}$-module) defined by

$$E_d(\sigma)(\boldsymbol{x}, 1 \wedge \cdots \wedge d) :=$$
$$\sum_{n_1=1}^{|\sigma(1)|} \cdots \sum_{n_d=1}^{|\sigma(d)|} \left(A_\sigma(\boldsymbol{x}) + \mathbf{f}(P_{n_1}^{(1)}) + \cdots + \mathbf{f}(P_{n_d}^{(d)}), w_{n_1}^{(1)} \wedge \cdots \wedge w_{n_d}^{(d)}\right),$$

where A_σ is the linear representation (or the so called characteristic matrix; see the beginning of Section 1.) of σ, so that it is of size $d \times d$ with integer entries; and where $\mathbf{f}(W) := {}^t(x_1, \ldots, x_d) \in \boldsymbol{Z}^d$, $x_i = x_i(W)$ is the number of the occurrence of a letter i appearing in a word $W \in \mathcal{A}_d{}^*$. Now, we can state a result:

Proposition 1 ([2]). *Let $\sigma \in \mathrm{Sub}(F_2)$ be a substitution with 2 letters. Then σ is invertible iff there exists $\boldsymbol{x} = \boldsymbol{x}_\sigma \in \boldsymbol{Z}^2$ such that*

$$E_2(\sigma)(\boldsymbol{o}, 1 \wedge 2) = \det(A_\sigma)(\boldsymbol{x}, 1 \wedge 2).$$

Related to generators of the group $\mathrm{Aut}(F_d)$, the following result is well known.

Proposition 2 ([3]). *$\sigma \in \mathrm{Aut}(F_d)$ iff σ is decomposed into the following three kinds of automorphisms*:

$$\alpha_{ij}: \begin{cases} i \to j \\ j \to i \\ k \to k \\ \textit{for all } k \neq i, \ j \end{cases} (i \neq j), \quad \beta_{ij}: \begin{cases} j \to ij \\ k \to k \\ \textit{for all } k \neq j \end{cases} (i \neq j), \quad \gamma_j: \begin{cases} j \to j^{-1} \\ k \to k \\ \textit{for all } k \neq j \end{cases}.$$

α_{ij}, β_{ij}, γ_j are called *Nielsen's generators*. We shall use Proposition 2 for the proof of our main results (the following theorems). Noting that γ_i is not a substitution, we define $E_k(\sigma)$ $(0 \leq k \leq d)$ not only for substitutions σ but also for endomorphisms σ. The map $E_k(\sigma)$ $(\sigma \in \mathrm{End}(F_d))$ plays an important role in this paper.

In Section 1, we define $E_k(\sigma)$ $(0 \leq k \leq d)$ for $\sigma \in \mathrm{End}(F_d)$; and we prove

Theorem 1. *Let $\sigma \in \mathrm{End}(F_d)$. If σ is invertible, then there exists $\boldsymbol{x} = \boldsymbol{x}_\sigma \in \boldsymbol{Z}^d$ such that*

$$E_d(\sigma)(\boldsymbol{o}, 1 \wedge 2 \wedge \cdots \wedge d) = \det(A_\sigma)(\boldsymbol{x}, 1 \wedge 2 \wedge \cdots \wedge d).$$

Roughly speaking, Theorem 1 says that the unit cube $(\boldsymbol{o}, 1 \wedge \cdots \wedge d)$ of dimension d is mapped to a unit cube of dimension d by $E_d(\sigma)$ if $\sigma \in \mathrm{Aut}(F_d)$.

In Section 2, we consider the dual map $E_k^*(\sigma)$ of $E_k(\sigma)$. $E_k^*(\sigma)$ acts on a union of oriented $(d-k)$ dimensional unit cubes with their multiplicity; and by the map φ_{d-k} $(0 \leq k \leq d)$, we can consider $\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1}$ as a map on $\mathcal{G}_k$. We apply

$E_k^*(\sigma)$ to getting the following theorem, which describes the relation between $E_k(\overline{\sigma^{-1}})$ and the dual map $E_{d-k}^*(\sigma)$, where $\overline{W}$ is the mirror image of a word W and $\overline{\sigma}(i) := \overline{\sigma(i)}$, $i \in \mathcal{A}_d$.

Theorem 2. *Let σ be an automorphism on the free group F_d. Then there exists $\boldsymbol{x} \in \boldsymbol{Z}^d$ such that*

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} = \det(A_\sigma) \circ T(\boldsymbol{x}) \circ E_k(\overline{\sigma^{-1}}) \quad (0 \le k \le d),$$

where the map $T(\boldsymbol{x})$ is a translation by $\boldsymbol{x}$.

In the case of $k = 1$, Theorem 2 says that we can construct σ^{-1} by the figure of $E_{d-1}^*(\sigma)$.

When we study invertible substitutions $\sigma \in \mathrm{Sub}(F_d)$ with $d \ge 3$, we encounter phenomena which do not occur in the case of $d = 2$. Accordingly, some results for $d = 2$ can not be extended for the case of $d \ge 3$. In Section 3, we see the gap between the cases of $d = 2$ and of $d \ge 3$ through some examples.

1. Map $E_k(\sigma)$ for an endomorphism σ on F_d

We put $\widehat{\mathcal{A}}_d := \{1^{\pm 1}, 2^{\pm 1}, \dots, d^{\pm 1}\}$, which is an alphabet consisting of $2d$ letters. We say that a word $W \in \widehat{\mathcal{A}}_d^*$ is a *reduced word* if W is the empty word, or $W = w_1 \cdots w_n$ $(w_i \in \widehat{\mathcal{A}}_d)$ such that we can not find a number $1 \le i \le n-1$ satisfying $w_i = s^\rho$, $w_{i+1} = s^{-\rho}$, $s \in \mathcal{A}_d$ and $\rho \in \{-1, 1\}$. We write $W \doteq W'$ for two words W, $W' \in \widehat{\mathcal{A}}_d^*$ satisfying

$$W = W'; \quad W = UV \text{ and } W' = Us^\rho s^{-\rho} V; \quad \text{or } W = Us^\rho s^{-\rho} V \text{ and } W' = UV,$$

with $s \in \mathcal{A}_d$, $\rho \in \{-1, 1\}$. Two words W, $V \in \widehat{\mathcal{A}}_d^*$ are referred to be equivalent, and written as $W \approx V$, if there exist words $U_1, \dots, U_n \in \widehat{\mathcal{A}}_d^*$ such that $W \doteq U_1$, $U_i \doteq U_{i+1}$ $(1 \le i \le n-1)$, $U_n \doteq V$. The relation $\approx$ is an equivalence one, and $F_d = \widehat{\mathcal{A}}_d^* / \approx$ holds by the definition of free groups. For a given word $W \in \widehat{\mathcal{A}}_d^*$, $[W]$ denotes the element of F_d determined by $[W] \ni W$. Note that $\sigma \in \mathrm{End}(F_d)$ is a substitution iff there exists a nonempty word $W(i) \in \mathcal{A}_d{}^*$ such that $W(i) \in [\sigma(i)]$ for each $1 \le i \le d$. In what follows, a word $W \in \widehat{\mathcal{A}}_d^*$ will be identified with the element $[W]$, cf. the definition of substitutions given in Section 0.

For $\sigma \in \mathrm{End}(F_d)$, we can set

$$\sigma(i) = w_1^{(i)} \cdots w_k^{(i)} \cdots w_{l_i}^{(i)} \in \widehat{\mathcal{A}}_d^* \quad (w_k^{(i)} \in \widehat{\mathcal{A}}_d)$$

such that the word on the right-hand side is reduced one in $\widehat{\mathcal{A}}_d^*$ for each $i \in \mathcal{A}_d$. We define $P_k^{(i)}$, $S_k^{(i)} \in \widehat{\mathcal{A}}_d^*$ by

$$P_k^{(i)} = w_1^{(i)} \cdots w_{k-1}^{(i)}, \quad S_k^{(i)} = w_{k+1}^{(i)} \cdots w_{l_i}^{(i)}.$$

$P_k^{(i)}$ (resp., $S_k^{(i)}$) will be referred to as the *k-prefix* (resp., the *k-suffix*) of $\sigma(i)$. Note that $P_1^{(i)}$ is the empty word for any $i \in \mathcal{A}_d$. A canonical homomorphism $\mathbf{f}\colon F_d \to \boldsymbol{Z}^d$ is defined by $\mathbf{f}(i^{\pm 1}) = \pm \boldsymbol{e}_i$ $(i \in \mathcal{A}_d)$. Then there exists a unique linear representation A_σ on $\boldsymbol{Z}^d$ associated with σ such that the following diagram becomes commutative:

$$\begin{array}{ccc} F_d & \xrightarrow{\sigma} & F_d \\ \mathbf{f}\downarrow & & \downarrow\mathbf{f} \\ \boldsymbol{Z}^d & \xrightarrow[A_\sigma]{} & \boldsymbol{Z}^d \end{array}$$

We introduce $\widehat{\Lambda}_k$ $(0 \le k \le d)$ formally defined by:

$$\begin{aligned} \widehat{\Lambda}_0 &:= \Lambda_0 = \boldsymbol{Z}^d \times \{\bullet\}, \\ \widehat{\Lambda}_k &:= \boldsymbol{Z}^d \times \{i_1 \wedge i_2 \wedge \cdots \wedge i_k \mid i_n \in \widehat{\mathcal{A}}_d\} \qquad (1 \le k \le d). \end{aligned}$$

We denote by $\widehat{\mathcal{G}}_k$ the free $\boldsymbol{Z}$-module generated by the elements of $\widehat{\Lambda}_k$:

$$\widehat{\mathcal{G}}_k := \left\{ \sum_{\lambda' \in \widehat{\Lambda}_k} n_{\lambda'} \lambda' \;\middle|\; n_{\lambda'} \in \boldsymbol{Z}, \sharp\{\lambda' \in \widehat{\Lambda}_k \mid n_{\lambda'} \ne 0\} < \infty \right\} \; (0 \le k \le d).$$

DEFINITION 1. We denote by $\iota\colon \widehat{\mathcal{G}}_k \to \mathcal{G}_k$ the $\boldsymbol{Z}$-homomorphism (the $\boldsymbol{Z}$-linear map) defined by

$$\iota(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) := \begin{cases} 0 & \text{if } \|i_n\| = \|i_m\| \text{ for some } n \ne m, \\ \operatorname{sgn}(i_1) \cdots \operatorname{sgn}(i_k) \epsilon(\tau)(\boldsymbol{x} + \sum_{j=1}^k \chi(i_j), \|i_{\tau(1)}\| \wedge \cdots \wedge \|i_{\tau(k)}\|) & \\ \qquad (1 \le \|i_{\tau(1)}\| < \cdots < \|i_{\tau(k)}\| \le d), & \text{otherwise,} \end{cases}$$

where τ is a permutation on $\{1, \ldots, k\}$, $\epsilon(\tau)$ is the signature of τ, $\operatorname{sgn}(i^a)$ and $\|i^a\|$ means $\operatorname{sgn}(i^a) := a$, $\|i^a\| := i$, and

$$\chi(i^a) := \begin{cases} \boldsymbol{o} & \text{if } a = 1 \\ \mathbf{f}(i^a) & \text{if } a = -1 \end{cases} \; (a \in \{-1, 1\},\ i \in \mathcal{A}_d).$$

For two elements g_1, $g_2 \in \widehat{\mathcal{G}}_k$, we write $g_1 \sim g_2$ if $\iota(g_1) = \iota(g_2)$. It is easy to see that $\sim$ is an equivalence relation. For example,

$$(\boldsymbol{o}, 2^{-1} \wedge 1) \sim \operatorname{sgn}(2^{-1}) \operatorname{sgn}(1)(\chi(2^{-1}) + \chi(1), 2 \wedge 1) \sim -(-\boldsymbol{e}_2, 2 \wedge 1) \sim (-\boldsymbol{e}_2, 1 \wedge 2).$$

Then, $\mathcal{G}_k$ can be identified with a complete set of representatives of $\widehat{\mathcal{G}}_k/\sim$.

The geometrical meaning of the elements of $\mathcal{G}_k$ $(0 \le k \le d)$, we have already mentioned, leads us the following definition of a map $\delta_k\colon \mathcal{G}_k \to \mathcal{G}_{k-1}$, which is considered to be a boundary map.

Definition 2. Boundary maps $\delta_k\colon \widehat{\mathcal{G}}_k \to \widehat{\mathcal{G}}_{k-1}$ $(1 \le k \le d)$ are $\mathbf{Z}$-homomorphisms defined by

$$\begin{aligned}&\delta_k(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k)\\ &:= \sum_{n=1}^{k} (-1)^n \{(\boldsymbol{x}, i_1 \wedge \cdots \wedge \widehat{i_n} \wedge \cdots \wedge i_k) - (\boldsymbol{x} + \mathbf{f}(i_n), i_1 \wedge \cdots \wedge \widehat{i_n} \wedge \cdots \wedge i_k)\}.\end{aligned}$$

We note that $\delta_{k-1} \circ \delta_k = 0$ $(1 \le k \le d)$ holds.

Remark 1. The value of the map δ_k is independent of the choice of a representative, i.e., $g_1 \sim g_2$ $(g_1,\ g_2 \in \widehat{\mathcal{G}}_k)$ implies $\delta_k(g_1) \sim \delta_k(g_2)$.

Let V and V' be $\mathbf{Z}$-modules. We mean by $\mathrm{Hom}_{\mathbf{Z}}(V, V')$ (resp., $\mathrm{End}_{\mathbf{Z}}(V)$) the set of $\mathbf{Z}$-linear maps from V to V' (resp., from V to itself). Now, we can define a map $E_k(\sigma) \in \mathrm{End}_{\mathbf{Z}}(\mathcal{G}_k)$ for an endomorphism σ on F_d.

Definition 3. Let $\sigma \in \mathrm{End}(F_d)$. $E_k(\sigma)\colon \mathcal{G}_k \to \mathcal{G}_k$ $(0 \le k \le d)$ are $\mathbf{Z}$-linear maps defined by

$$E_0(\sigma)(\boldsymbol{x}, \bullet) := (A_\sigma(\boldsymbol{x}), \bullet),$$

$$\begin{aligned}&E_k(\sigma)(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) :=\\ &\sum_{n_1=1}^{|\sigma(i_1)|} \cdots \sum_{n_k=1}^{|\sigma(i_k)|} (A_\sigma(\boldsymbol{x}) + \mathbf{f}(P_{n_1}^{(i_1)}) + \cdots + \mathbf{f}(P_{n_k}^{(i_k)}), w_{n_1}^{(i_1)} \wedge \cdots \wedge w_{n_k}^{(i_k)}) \quad (1 \le k \le d).\end{aligned}$$

We call $E_k(\sigma)$ a substitution of dimension k with respect to $\sigma \in \mathrm{End}(F_d)$. We remark that in a certain sense, our definition is compatible with the boundary map δ_k:

Proposition 3. *For $\sigma \in \mathrm{End}(F_d)$ and for each $1 \le k \le d$, the following diagram is commutative*:

$$\begin{tikzcd} \mathcal{G}_k \arrow[r, "E_k(\sigma)"] \arrow[d, "\delta_k"'] & \mathcal{G}_k \arrow[d, "\delta_k"] \\ \mathcal{G}_{k-1} \arrow[r, "E_{k-1}(\sigma)"] & \mathcal{G}_{k-1} \end{tikzcd}$$

See Theorem 2.1 in [6], where the commutative diagram with $\sigma \in \mathrm{Sub}(F_d)$ is given. The following theorem is one of our main results.

Theorem 1. *Let $\sigma \in \mathrm{End}(F_d)$. If σ is invertible, then there exists $\boldsymbol{x} = \boldsymbol{x}_\sigma \in \boldsymbol{Z}^d$ such that*

$$E_d(\sigma)(\boldsymbol{o}, 1 \wedge 2 \wedge \cdots \wedge d) = \det(A_\sigma)(\boldsymbol{x}, 1 \wedge 2 \wedge \cdots \wedge d).$$

We need a lemma for the proof of Theorem 1.

Lemma 1. *Let an automorphism σ be decomposed as $\sigma = \sigma' \circ \sigma''$. Put*

$$\sigma(i) = P_n^{(i)} w_n^{(i)} S_n^{(i)},$$
$$\sigma'(i) = {P'}_n^{(i)} {w'}_n^{(i)} {S'}_n^{(i)}, \sigma''(i) = {P''}_n^{(i)} {w''}_n^{(i)} {S''}_n^{(i)}.$$

Then the following statements are valid:
(i) $A_\sigma = A_{\sigma'} A_{\sigma''}$.
(ii) *For any pair (i, k) $(i \in \mathcal{A}_d, 1 \le k \le |\sigma(i)|)$, there exists a unique pair (m, n) such that*

$$P_k^{(i)} = \sigma'({P''}_m^{(i)}) \quad {P'}_n^{({w''}_m^{(i)})}, 0 \le m \le |\sigma''(i)|, 0 \le n \le |\sigma'({w''}_m^{(i)})|.$$

(iii) $E_k(\sigma) = E_k(\sigma') \circ E_k(\sigma'') \quad (0 \le k \le d)$.

Proof. The assertions (i), (ii) can be easily seen. We show the equality

$$E_k(\sigma)(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) = E_k(\sigma') \circ E_k(\sigma'')(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k).$$

For simplicity, we put

$$\sum := \sum_{m_1=1}^{|\sigma''(i_1)|} \cdots \sum_{m_k=1}^{|\sigma''(i_k)|} \sum_{n_1=1}^{|\sigma'({w''}_{m_1}^{(i_1)})|} \cdots \sum_{n_k=1}^{|\sigma'({w''}_{m_k}^{(i_k)})|}.$$

Using (i), (ii) in this lemma, we get

$$\begin{aligned}
&E_k(\sigma)(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) \\
&= \sum_{p_1=1}^{|\sigma(i_1)|} \cdots \sum_{p_k=1}^{|\sigma(i_k)|} \Big(A_\sigma(\boldsymbol{x}) + \mathbf{f}(P_{p_1}^{(i_1)}) + \cdots + \mathbf{f}(P_{p_k}^{(i_k)}), w_{p_1}^{(i_1)} \wedge \cdots \wedge w_{p_k}^{(i_k)} \Big) \\
&= \sum \Big(A_{\sigma'} A_{\sigma''}(\boldsymbol{x}) + \mathbf{f}\big(\sigma'({P''}_{m_1}^{(i_1)}) \quad {P'}_{n_1}^{({w''}_{m_1}^{(i_1)})}\big) + \cdots + \mathbf{f}\big(\sigma'({P''}_{m_k}^{(i_k)}) \quad {P'}_{n_k}^{({w''}_{m_k}^{(i_k)})}\big), \\
&\qquad {w'}_{n_1}^{({w''}_{m_1}^{(i_1)})} \wedge \cdots \wedge {w'}_{n_k}^{({w''}_{m_k}^{(i_k)})} \Big) \\
&= \sum \Big(A_{\sigma'} \left(A_{\sigma''}(\boldsymbol{x}) + \mathbf{f}\big({P''}_{m_1}^{(i_1)}\big) + \cdots + \mathbf{f}\big({P''}_{m_k}^{(i_k)}\big) \right) + \mathbf{f}\big({P'}_{n_1}^{({w''}_{m_1}^{(i_1)})}\big) + \cdots + \mathbf{f}\big({P'}_{n_k}^{({w''}_{m_k}^{(i_k)})}\big), \\
&\qquad {w'}_{n_1}^{({w''}_{m_1}^{(i_1)})} \wedge \cdots \wedge {w'}_{n_k}^{({w''}_{m_k}^{(i_k)})} \Big)
\end{aligned}$$

$$= E_k(\sigma')\left\{\sum_{m_1=1}^{|\sigma''(i_1)|}\cdots\sum_{m_k=1}^{|\sigma''(i_k)|}\left(A_{\sigma''}(\boldsymbol{x})+\mathbf{f}\big(P''^{(i_1)}_{m_1}\big)+\cdots+\mathbf{f}\big(P''^{(i_k)}_{m_k}\big),\, w''^{(i_1)}_{m_1}\wedge\cdots\wedge w''^{(i_k)}_{m_k}\right)\right\}$$
$$= E_k(\sigma')\circ E_k(\sigma'')(\boldsymbol{x}, i_1\wedge\cdots\wedge i_k). \qquad \square$$

Proof of Theorem 1. It suffices to show Theorem 1 only when $\sigma = \alpha_{ij},\ \beta_{ij},\ \gamma_j$. By easy calculation, we have

$$\begin{aligned} E_d(\alpha_{ij})(\boldsymbol{o}, 1\wedge\cdots\wedge i\cdots\wedge j\cdots\wedge d) &= (\boldsymbol{o}, 1\wedge\cdots\wedge j\cdots\wedge i\cdots\wedge d) \\ &= -(\boldsymbol{o}, 1\wedge 2\wedge\cdots\wedge d), \\ E_d(\beta_{ij})(\boldsymbol{o}, 1\wedge\cdots\wedge j\wedge\cdots\wedge d) &= (\boldsymbol{e}_i, 1\wedge\cdots\wedge j\wedge\cdots\wedge d), \\ E_d(\gamma_j)(\boldsymbol{o}, 1\wedge\cdots\wedge j\wedge\cdots\wedge d) &= (\boldsymbol{o}, 1\wedge\cdots\wedge j^{-1}\wedge\cdots\wedge d) \\ &= -(-\boldsymbol{e}_j, 1\wedge\cdots\wedge j\wedge\cdots\wedge d). \end{aligned}$$

In view of the assertion (iii) in Lemma 1, we get Theorem 1, cf. Remark 6 in Section 2. $\square$

REMARK 2. It is not clear whether what the unit cube $(\boldsymbol{o}, 1\wedge\cdots\wedge d)$ of dimension d is mapped to one unit cube of dimension d by $E_d(\sigma)$ implies that σ is an automorphism.

EXAMPLE 1. Let σ_R be the so called *Rauzy substitution*:

$$\sigma_R\colon \left\{\begin{array}{lcl} 1 & \to & 12 \\ 2 & \to & 13 \\ 3 & \to & 1 \end{array}\right. \quad \left(A_{\sigma_R} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right).$$

Then

$$E_3(\sigma_R)(\boldsymbol{o}, 1\wedge 2\wedge 3) = (2\boldsymbol{e}_1, 1\wedge 2\wedge 3).$$

By Remark 2, this doesn't imply that σ_R is invertible. But for the substitution σ_R, we have the inverse

$$\sigma_R^{-1}\colon \left\{\begin{array}{lcl} 1 & \to & 3 \\ 2 & \to & 3^{-1}1 \\ 3 & \to & 3^{-1}2 \end{array}\right. .$$

We define another substitution σ_C given by

$$\sigma_C\colon \left\{\begin{array}{lcl} 1 & \to & 123 \\ 2 & \to & 112 \\ 3 & \to & 2333 \end{array}\right. \quad \left(A_{\sigma_C} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 3 \end{pmatrix}\right).$$

Then

$$E_3(\sigma_C)(\boldsymbol{o}, 1 \wedge 2 \wedge 3) = -(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3, 1 \wedge 2 \wedge 3) - (\boldsymbol{e}_1 + \boldsymbol{e}_2 + 2\boldsymbol{e}_3, 1 \wedge 2 \wedge 3) + (2\boldsymbol{e}_1 + \boldsymbol{e}_2, 1 \wedge 2 \wedge 3).$$

This together with Theorem 1 implies that σ_C is not invertible.

We consider all of substitutions satisfying $A_\sigma = A_{\sigma_C}$, whose cardinal number is 72. Then, we see that there does not exist a substitution σ satisfying $A_\sigma = A_{\sigma_C}$ such that

$$E_3(\sigma)(\boldsymbol{o}, 1 \wedge 2 \wedge 3) = -(\boldsymbol{x}, 1 \wedge 2 \wedge 3)$$

for some $\boldsymbol{x} \in \boldsymbol{Z}^3$. By Theorem 1, it means there does not exist an invertible substitution satisfying $A_\sigma = A_{\sigma_C}$, cf. Section 3.1. The matrix $A_\sigma = A_{\sigma_C}$ can be found in [5] as an example which does not give an invertible substitution.

2. Dual map $E_k^*(\sigma)$ for an endomorphism σ on F_d

We can define in the obvious manner a dual space. Since we are in an infinite dimensional $\boldsymbol{Z}$-module, this defines a complicated space; and restrict ourself to the set of dual maps with finite support. We denote this set by $\mathcal{G}_k^*$. It has a natural basis, and we write $(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*)$ for the dual vector of $(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k)$.

In fact, we introduce Λ_k^* $(0 \le k \le d)$ formally defined by

$$\Lambda_0^* := \boldsymbol{Z}^d \times \{\bullet^*\},$$
$$\Lambda_k^* := \boldsymbol{Z}^d \times \{i_1^* \wedge i_2^* \wedge \cdots \wedge i_k^* \mid 1 \le i_1 < \cdots < i_k \le d\} \quad (1 \le k \le d).$$

We denote by $\mathcal{G}_k^*$ the free $\boldsymbol{Z}$-module generated by the elements of Λ_k^* as follows:

$$\mathcal{G}_k^* := \left\{ \sum_{\lambda^* \in \Lambda_k^*} n_{\lambda^*} \lambda^* \;\middle|\; n_{\lambda^*} \in \boldsymbol{Z}, \sharp\{\lambda^* \in \Lambda_k^* \mid n_{\lambda^*} \ne 0\} < \infty \right\}.$$

REMARK 3. Following convention, we define a pairing by

$$\langle (\boldsymbol{y}, j_1 \wedge \cdots \wedge j_k), (\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle := \begin{cases} 1 & \text{if } \boldsymbol{x} = \boldsymbol{y} \text{ and } i_t = j_t \text{ for all } 1 \le t \le k \\ 0, & \text{otherwise} \end{cases},$$

$(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_k^*$, $(\boldsymbol{y}, j_1 \wedge \cdots \wedge j_k) \in \mathcal{G}_k$.

Thus $\mathcal{G}_k^*$ can be considered as the set of dual maps with finite support.

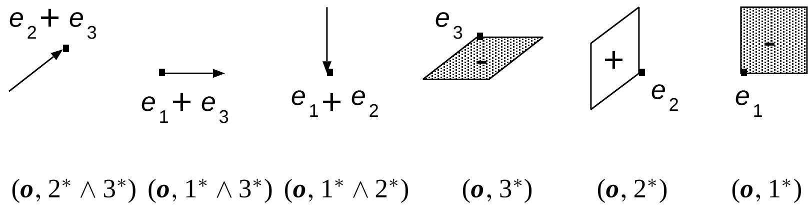


Fig. 2. elements of $\mathcal{G}_1^*$, $\mathcal{G}_2^*$ in the case of $d = 3$

We can define the $\boldsymbol{Z}$-linear isomorphism $\varphi_k \colon \mathcal{G}_k^* \to \mathcal{G}_{d-k}$ $(0 \le k \le d)$ by

$$\varphi_0(\boldsymbol{x}, \bullet^*) := (\boldsymbol{x}, 1 \wedge \cdots \wedge d),$$
$$\varphi_k(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*) := (-1)^{i_1+\cdots+i_k}(\boldsymbol{x} + \boldsymbol{e}_{i_1} + \cdots + \boldsymbol{e}_{i_k}, j_1 \wedge \cdots \wedge j_{d-k}) \quad (1 \le k \le d),$$

where $\{j_1, j_2, \ldots, j_{d-k}\} = \mathcal{A}_d \setminus \{i_1, i_2, \ldots, i_k\}$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{d-k}$. By virtue of the isomorphism φ_k, we can imagine an element $(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_k^*$ as the element $\varphi_k(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*) \in \mathcal{G}_{d-k}$ with geometrical meaning for each $0 \le k \le d$, see Fig. 2.

REMARK 4. In general, for $\Phi \in \mathrm{Hom}_{\boldsymbol{Z}}(\mathcal{G}_s, \mathcal{G}_t)$ $(0 \le s, t \le d)$, the dual map Φ^* of Φ is an element of $\mathrm{Hom}_{\boldsymbol{Z}}(\mathcal{G}_t^*, \mathcal{G}_s^*)$ determined by $\langle F, \Phi^*(G^*)\rangle = \langle \Phi(F), G^*\rangle$ $(F \in \mathcal{G}_s, G^* \in \mathcal{G}_t^*)$.

Proposition 4 ([6]). (i) *The dual boundary map* $\delta_k^* \colon \mathcal{G}_{k-1}^* \to \mathcal{G}_k^*$ $(1 \le k \le d)$ *is given by*

$$\delta_1^*(\boldsymbol{x}, \bullet^*) = \sum_{i=1}^{d} \{(\boldsymbol{x} - \boldsymbol{e}_i, i^*) - (\boldsymbol{x}, i^*)\},$$
$$\begin{aligned}\delta_k^*&(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_{k-1}^*)\\ &= \sum_{n=1}^{d-k+1} (-1)^{j_n-n+1}\{(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_{j_n-n}^* \wedge j_n^* \wedge i_{j_n-n+1}^* \wedge \cdots \wedge i_{k-1}^*)\\ &\qquad - (\boldsymbol{x} - \boldsymbol{e}_{j_n}, i_1^* \wedge \cdots \wedge i_{j_n-n}^* \wedge j_n^* \wedge i_{j_n-n+1}^* \wedge \cdots \wedge i_{k-1}^*)\},\end{aligned}$$

where $\{j_1, j_2, \ldots, j_{d-k+1}\} = \mathcal{A}_d \setminus \{i_1, i_2, \ldots, i_{k-1}\}$ *with* $i_1 < \cdots < i_{k-1}$, $j_1 < \cdots < j_{d-k+1}$ *and* $i_{j_n-n} < j_n < i_{j_n-n+1}$ $(2 \le k \le d)$.
(ii) *The following diagram commutes for each* $1 \le k \le d$:

$$\begin{tikzcd} \mathcal{G}_{k-1}^* \arrow[r, "\varphi_{k-1}"] \arrow[d, "\delta_k^*"'] & \mathcal{G}_{d-k+1} \arrow[d, "\delta_{d-k+1}"] \\ \mathcal{G}_k^* \arrow[r, "\varphi_k"'] & \mathcal{G}_{d-k} \end{tikzcd}$$

From the commutativity (ii) given above, we can see that δ_k^* is a boundary map with a geometrical sense.

By Remark 4, we can determine the dual map $E_k^*(\sigma)$ (on $\mathcal{G}_k^*$) of $E_k(\sigma)$ (on $\mathcal{G}_k$) for $\sigma \in \mathrm{End}(F_d)$ under a minor condition on $\det(A_\sigma)$:

Proposition 5. (i) *Let σ be an endomorphism on the free group F_d satisfying* $\det(A_\sigma) = \pm 1$. *Then dual maps $E_k^*(\sigma)\colon \mathcal{G}_k^* \to \mathcal{G}_k^*$ $(0 \le k \le d)$ satisfies*

$$
\begin{aligned}
&E_0^*(\sigma)(\boldsymbol{x}, \bullet^*) = (A_\sigma^{-1}\boldsymbol{x}, \bullet^*),\\
&E_k^*(\sigma)(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*)\\
&\quad = \sum_{\tau \in S_k} \sum_{\|w_{n_1}^{(j_1)}\| = i_{\tau(1)}} \cdots \sum_{\|w_{n_k}^{(j_k)}\| = i_{\tau(k)}} \mathrm{sgn}(w_{n_1}^{(j_1)}) \cdots \mathrm{sgn}(w_{n_k}^{(j_k)}) \epsilon(\tau)\\
&\qquad \left(A_\sigma^{-1} \left(\boldsymbol{x} - \sum_{m=1}^{k} \{ \mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)}) \} \right), j_1^* \wedge \cdots \wedge j_k^* \right) \quad (1 \le j_1 < \cdots < j_k \le d),
\end{aligned}
$$

where S_k is the symmetric group of rank k.

(ii) *The following diagram is commutative for each $1 \le k \le d$:*

$$
\begin{array}{ccc}
\mathcal{G}_{k-1}^* & \xrightarrow{E_{k-1}^*(\sigma)} & \mathcal{G}_{k-1}^* \\
\delta_k^* \downarrow & & \downarrow \delta_k^* \\
\mathcal{G}_k^* & \xrightarrow[E_k^*(\sigma)]{} & \mathcal{G}_k^*
\end{array}
$$

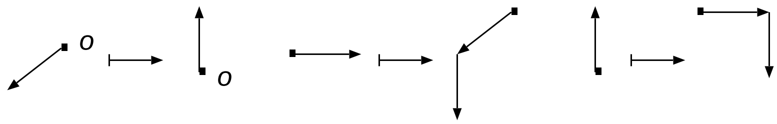
We remark that $\sigma \in \mathrm{Aut}(F_d)$ implies $\det(A_\sigma) = \pm 1$.

Proof. We can prove the proposition in a similar fashion as that given in [6]. The dual map $E_k^*(\sigma) \in \mathrm{End}_{\mathbf{Z}}(\mathcal{G}_k^*)$ of $E_k(\sigma)$ is given by the identity

$$\langle F, E_k^*(\sigma)(G^*) \rangle = \langle E_k(\sigma)(F), G^* \rangle$$

for $F \in \mathcal{G}_k$, $G^* \in \mathcal{G}_k^*$ by Remark 4. Therefore, we get

$$
\begin{aligned}
&\langle (\boldsymbol{y}, j_1 \wedge \cdots \wedge j_k), E_k^*(\sigma)(\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle\\
&\quad = \langle E_k(\sigma)(\boldsymbol{y}, j_1 \wedge \cdots \wedge j_k), (\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*) \rangle\\
&\quad = \sum_{n_1=1}^{|\sigma(j_1)|} \cdots \sum_{n_k=1}^{|\sigma(j_k)|} \mathrm{sgn}\big(w_{n_1}^{(j_1)}\big) \cdots \mathrm{sgn}\big(w_{n_k}^{(j_k)}\big)\\
&\qquad \left\langle \left(A_\sigma(\boldsymbol{y}) + \sum_{m=1}^{k} \{ \mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)}) \}, \|w_{n_1}^{(j_1)}\| \wedge \cdots \wedge \|w_{n_k}^{(j_k)}\| \right), (\boldsymbol{x}, i_1^* \wedge \cdots \wedge i_k^*) \right\rangle.
\end{aligned}
$$



$2^* \wedge 3^*$ $\qquad$ $1^* \wedge 3^*$ $\qquad$ $1^* \wedge 2^*$

Fig. 3. the map $E_2^*(\sigma_R)$

The pairing appearing in the summation is equal to

$$\mathrm{sgn}(w_{n_1}^{(j_1)}) \cdots \mathrm{sgn}(w_{n_k}^{(j_k)})\epsilon(\tau)$$

if $A_\sigma(\boldsymbol{y}) + \sum_{m=1}^{k}\{\mathbf{f}(P_{n_m}^{(j_m)}) + \chi(w_{n_m}^{(j_m)})\} = \boldsymbol{x}$, and $\|w_{n_l}^{(j_l)}\| = i_{\tau(l)}$ $(1 \le l \le k)$, $\tau \in S_k$; it is equal to 0, otherwise. Hence we get (i). The commutative diagram (ii) comes from Proposition 3. $\square$

We write $(\boldsymbol{x}, \boldsymbol{i}_1^* \wedge \cdots \wedge \boldsymbol{i}_k^*) \simeq (\boldsymbol{y}, j_1 \wedge \cdots \wedge j_{d-k})$ iff $\varphi_k(\boldsymbol{x}, \boldsymbol{i}_1^* \wedge \cdots \wedge \boldsymbol{i}_k^*) = (\boldsymbol{y}, j_1 \wedge \cdots \wedge j_{d-k})$.

EXAMPLE 2. Let σ_R be the Rauzy substitution given in Example 1. Then

$$\begin{aligned} E_2^*(\sigma_R):\ -(-\boldsymbol{e}_2 - \boldsymbol{e}_3, 2^* \wedge 3^*) &\mapsto -(-\boldsymbol{e}_1 - \boldsymbol{e}_2, 1^* \wedge 2^*) \\ \simeq (\boldsymbol{o}, 1) \quad &\quad \simeq (\boldsymbol{o}, 3) \\ (-\boldsymbol{e}_1 - \boldsymbol{e}_3, 1^* \wedge 3^*) &\mapsto -(-\boldsymbol{e}_2 - \boldsymbol{e}_3, 2^* \wedge 3^*) + (-\boldsymbol{e}_2 - \boldsymbol{e}_3, 1^* \wedge 2^*) \\ \simeq (\boldsymbol{o}, 2) \quad &\quad \simeq (\boldsymbol{o}, 1) - (\boldsymbol{e}_1 - \boldsymbol{e}_3, 3) \\ -(-\boldsymbol{e}_1 - \boldsymbol{e}_2, 1^* \wedge 2^*) &\mapsto (-\boldsymbol{e}_1 - \boldsymbol{e}_3, 1^* \wedge 3^*) + (-\boldsymbol{e}_1 - \boldsymbol{e}_3, 1^* \wedge 2^*) \\ \simeq (\boldsymbol{o}, 3) \quad &\quad \simeq (\boldsymbol{o}, 2) - (\boldsymbol{e}_2 - \boldsymbol{e}_3, 3) \end{aligned}$$

See Fig. 3.

$$\begin{aligned} E_1^*(\sigma_R):\ -(-\boldsymbol{e}_3, 3^*) &\mapsto -(-\boldsymbol{e}_2, 2^*) \\ \simeq (\boldsymbol{o}, 1 \wedge 2) \quad &\quad \simeq -(\boldsymbol{o}, 1 \wedge 3) \\ (-\boldsymbol{e}_2, 2^*) &\mapsto (-\boldsymbol{e}_1, 1^*) \\ \simeq (\boldsymbol{o}, 1 \wedge 3) \quad &\quad \simeq -(\boldsymbol{o}, 2 \wedge 3) \\ -(-\boldsymbol{e}_1, 1^*) &\mapsto -(-\boldsymbol{e}_3, 1^*) - (-\boldsymbol{e}_3, 2^*) - (-\boldsymbol{e}_3, 3^*) \\ \simeq (\boldsymbol{o}, 2 \wedge 3) \quad &\quad \simeq (\boldsymbol{e}_1 - \boldsymbol{e}_3, 2 \wedge 3) - (\boldsymbol{e}_2 - \boldsymbol{e}_3, 1 \wedge 3) + (\boldsymbol{o}, 1 \wedge 2) \end{aligned}$$

See Fig. 4.

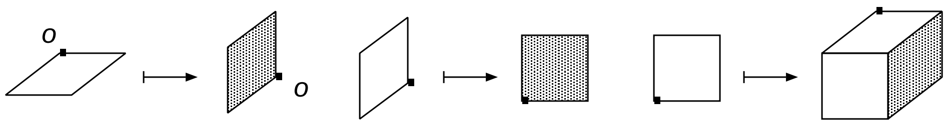


Fig. 4. the map $E_1^*(\sigma_R)$

For $W = s_1 s_2 \cdots s_n \in \widehat{\mathcal{A}}_d^*$, $\overline{W}$ denotes the mirror image of W, i.e.,

$$\overline{W} := s_n s_{n-1} \cdots s_1.$$

For $\sigma \in \mathrm{End}(F_d)$, the endomorphism $\overline{\sigma}$ is given by $\overline{\sigma}(i) = \overline{\sigma(i)}$ $(i \in \mathcal{A}_d)$. Now, we can state a result.

Theorem 2. *Let σ be an automorphism on the free group F_d. Then there exists $\boldsymbol{x} \in \boldsymbol{Z}^d$ such that*

$$\varphi_{d-k} \circ E_{d-k}^*(\sigma) \circ \varphi_{d-k}^{-1} = \det(A_\sigma) \circ T_k(\boldsymbol{x}) \circ E_k(\overline{\sigma^{-1}}) \quad (0 \le k \le d),$$

where the map $T_k(\boldsymbol{x})\colon \mathcal{G}_k \to \mathcal{G}_k$ with $\boldsymbol{x} \in \boldsymbol{Z}^d$ is given by

$$T_k(\boldsymbol{x}) \left(\sum_{t=1}^{m} n_t \big(\boldsymbol{y}_t, i_1^{(t)} \wedge \cdots \wedge i_k^{(t)}\big) \right) = \sum_{t=1}^{m} n_t \big(\boldsymbol{x} + \boldsymbol{y}_t, i_1^{(t)} \wedge \cdots \wedge i_k^{(t)}\big).$$

We remark that since $\mathcal{G}_k$ is a free $\boldsymbol{Z}$-module, an integer a is an operator on $\mathcal{G}_k$, i.e., $a(\sum_{\lambda \in \Lambda_k} n_\lambda \lambda) = \sum_{\lambda \in \Lambda_k} (a \cdot n_\lambda)\lambda$. For the proof of Theorem 2, we need some lemmas.

Lemma 2. $\overline{\sigma' \circ \sigma} = \overline{\sigma'} \circ \overline{\sigma}$ $(\sigma,\ \sigma' \in \mathrm{End}(F_d))$.

Proof. Setting $\sigma(i) = w_1^{(i)} \cdots w_{l_i}^{(i)}$, we have

$$\begin{aligned} \overline{\sigma' \circ \sigma}(i) &= \overline{\sigma'(w_1^{(i)} \cdots w_{l_i}^{(i)})} = \overline{\sigma'(w_1^{(i)}) \cdots \sigma'(w_{l_i}^{(i)})} = \overline{\sigma'(w_{l_i}^{(i)})} \cdots \overline{\sigma'(w_1^{(i)})} \\ &= \overline{\sigma'}(w_{l_i}^{(i)} \cdots w_1^{(i)}) = \overline{\sigma'} \circ \overline{\sigma}(i), \end{aligned}$$

so that $\overline{\sigma' \circ \sigma} = \overline{\sigma'} \circ \overline{\sigma}$. □

By $E_k(\sigma_1 \circ \sigma_2) = E_k(\sigma_1) \circ E_k(\sigma_2)$ and the duality $(\Phi_1 \circ \Phi_2)^* = \Phi_2^* \circ \Phi_1^*$, we have

Lemma 3. $E_k^*(\sigma_1 \circ \sigma_2) = E_k^*(\sigma_2) \circ E_k^*(\sigma_1)$ $(\sigma_1, \sigma_2 \in \mathrm{End}(F_d))$.

Lemma 4. *Let σ be one of*

$$\mathcal{N} := \{\alpha_{ij},\ \beta_{ij},\ \gamma_j \mid 1 \le i,\ j \le d,\ i \ne j\}.$$

Then

$$\varphi_{d-k} \circ E^*_{d-k}(\sigma) \circ \varphi^{-1}_{d-k} = \det(A_\sigma) \circ T_k(v(\sigma)) \circ E_k(\overline{\sigma^{-1}}),$$

where $v\colon \mathcal{N} \to \{\boldsymbol{o}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_d\}$ is the map given by $v(\alpha_{ij}) := \boldsymbol{o}$, $v(\beta_{ij}) := \boldsymbol{o}$, $v(\gamma_j) := \boldsymbol{e}_j$.

Proof. It suffices to show that

$$\varphi_{d-k} \circ E^*_{d-k}(\sigma) \circ \varphi^{-1}_{d-k}(\boldsymbol{o}, i_1 \wedge \cdots \wedge i_k) = \det(A_\sigma) \circ T_k(v(\sigma)) \circ E_k(\overline{\sigma^{-1}})(\boldsymbol{o}, i_1 \wedge \cdots \wedge i_k).$$

Notice $A_{\overline{\sigma^{-1}}} = A^{-1}_\sigma$. We consider the case of $\sigma = \beta_{ij}$ and $k = 1$. Note that

$$A^{-1}_{\beta_{ij}} = \begin{array}{c} \qquad\qquad\qquad j \text{ th} \\ \big(\boldsymbol{e}_1, \dots, \ -\boldsymbol{e}_i + \boldsymbol{e}_j\ , \dots, \boldsymbol{e}_d\big) \end{array}, \qquad \overline{\beta^{-1}_{ij}}\colon \begin{cases} j \to j i^{-1} \\ l \to l \\ \text{for all } l \ne j \end{cases} \quad (i \ne j).$$

We easily have

$$\varphi_{d-1} \circ E^*_{d-1}(\beta_{ij}) \circ \varphi^{-1}_{d-1}(\boldsymbol{o}, l) = (\boldsymbol{o}, l) = E_1(\overline{\beta^{-1}_{ij}})(\boldsymbol{o}, l) \quad (l \ne j).$$

On the other hand, we get the following equality. On the fourth line in the calculation given below, we must be careful with the location of j^*. If $j > i$, then j^* locates at i th place, otherwise, at $i-1$ th place. Using a permutation, we move j^* to the ordinal place, and then we have the sixth line.

$$\begin{aligned}
&\varphi_{d-1} \circ E^*_{d-1}(\beta_{ij}) \circ \varphi^{-1}_{d-1}(\boldsymbol{o}, j) \\
&\quad = \varphi_{d-1} \circ E^*_{d-1}(\beta_{ij})(-1)^{1+\cdots+d-j}\left(-\sum_{m=1}^d \boldsymbol{e}_m + \boldsymbol{e}_j, 1^* \wedge \cdots \wedge \widehat{j^*} \wedge \cdots \wedge d^*\right) \\
&\quad = \varphi_{d-1}\Bigg\{(-1)^{1+\cdots+d-j}\left(-\sum_{m=1}^d \boldsymbol{e}_m + \boldsymbol{e}_j, 1^* \wedge \cdots \wedge \widehat{j^*} \wedge \cdots \wedge d^*\right) \\
&\qquad\quad + (-1)^{1+\cdots+d-j}\left(-\sum_{m=1}^d \boldsymbol{e}_m + \boldsymbol{e}_j, 1^* \wedge \cdots \wedge \widehat{i^*} \cdots \wedge j^* \cdots \wedge d^*\right)\Bigg\} \\
&\quad = \varphi_{d-1}\Bigg\{(-1)^{1+\cdots+d-j}\left(-\sum_{m=1}^d \boldsymbol{e}_m + \boldsymbol{e}_j, 1^* \wedge \cdots \wedge \widehat{j^*} \wedge \cdots \wedge d^*\right) \\
&\qquad\quad + (-1)^{1+\cdots+d-i+1}\left(-\sum_{m=1}^d \boldsymbol{e}_m + \boldsymbol{e}_j, 1^* \wedge \cdots \wedge \widehat{i^*} \wedge \cdots \wedge d^*\right)\Bigg\}
\end{aligned}$$

$$= (\boldsymbol{o}, j) - (-\boldsymbol{e}_i + \boldsymbol{e}_j, i)$$
$$= E_1(\overline{\beta_{ij}^{-1}})(\boldsymbol{o}, j).$$

Hence we get
$$\varphi_{d-1} \circ E^*_{d-1}(\sigma) \circ \varphi^{-1}_{d-1} = \det(A_\sigma) \circ T_1(v(\sigma)) \circ E_1(\overline{\sigma^{-1}})$$
for $\sigma = \beta_{ij}$ and $k = 1$. For other cases, we can do the same, and the technical term can be found in Proof of Proposition 1.1 in [6]. □

Lemma 5. *For* a, $a_m \in \boldsymbol{Z}$, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{y}_m \in \boldsymbol{Z}^d$, $\sigma_m \in \mathrm{End}(F_d)$ $(1 \leq m \leq n)$ *and for* $0 \leq k \leq d$, *we have the following formulas*:
(i) $T_k(\boldsymbol{x}) \circ T_k(\boldsymbol{y}) = T_k(\boldsymbol{x}+\boldsymbol{y})$
(ii) $a \circ T_k(\boldsymbol{x}) = T_k(\boldsymbol{x}) \circ a$
(iii) $(a_n \circ T_k(\boldsymbol{y}_n) \circ E_k(\sigma_n)) \circ \cdots \circ (a_1 \circ T_k(\boldsymbol{y}_1) \circ E_k(\sigma_1)) = a_1 \cdots a_n \circ T_k(\boldsymbol{y}_n + A_{\sigma_n}(\boldsymbol{y}_{n-1}) + A_{\sigma_n\sigma_{n-1}}(\boldsymbol{y}_{n-2}) + \cdots + A_{\sigma_n\cdots\sigma_2}(\boldsymbol{y}_1)) \circ E_k(\sigma_n \cdots \sigma_1)$.

Proof. The statements (i), (ii) are trivial. For the proof of the third statement, it is enough to show

$$(a_2 \circ T_k(\boldsymbol{y}_2) \circ E_k(\sigma_2)) \circ (a_1 \circ T_k(\boldsymbol{y}_1) \circ E_k(\sigma_1)) = a_1a_2 \circ T_k(\boldsymbol{y}_2 + A_{\sigma_2}(\boldsymbol{y}_1)) \circ E_k(\sigma_2\sigma_1).$$

We can put

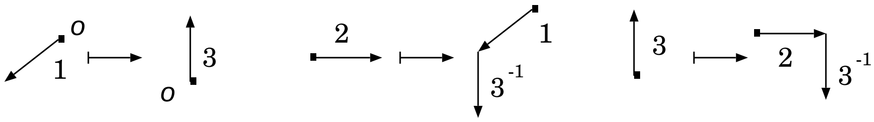
$$E_k(\sigma_1)(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k) = \sum_{\lambda \in \Lambda_k} n_\lambda(\boldsymbol{x}_\lambda, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}).$$

Using (i), (ii) in the lemma, we have

$$\begin{aligned}
&(a_2 \circ T_k(\boldsymbol{y}_2) \circ E_k(\sigma_2)) \circ (a_1 \circ T_k(\boldsymbol{y}_1) \circ E_k(\sigma_1))(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k)\\
&= a_2 \circ T_k(\boldsymbol{y}_2) \circ E_k(\sigma_2)\left\{\sum_{\lambda \in \Lambda_k} a_1 n_\lambda\big(\boldsymbol{x}_\lambda + \boldsymbol{y}_1, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}\big)\right\}\\
&= a_2 \circ T_k(\boldsymbol{y}_2)\left\{\sum_{\lambda \in \Lambda_k} a_1 n_\lambda E_k(\sigma_2)\big(\boldsymbol{x}_\lambda + \boldsymbol{y}_1, i_1^{(\lambda)} \wedge \cdots \wedge i_k^{(\lambda)}\big)\right\}\\
&= a_2 \circ T_k(\boldsymbol{y}_2) \circ a_1 \circ T_k(A_{\sigma_2}(\boldsymbol{y}_1)) \circ E_k(\sigma_2) \circ E_k(\sigma_1)(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k)\\
&= a_1a_2 \circ T_k(\boldsymbol{y}_2 + A_{\sigma_2}(\boldsymbol{y}_1)) \circ E_k(\sigma_2\sigma_1)(\boldsymbol{x}, i_1 \wedge \cdots \wedge i_k).
\end{aligned}$$ □

Proof of Theorem 2. Let σ be an automorphism. Then σ can be written as $\sigma = \sigma_1 \cdots \sigma_n$ with $\sigma_m \in \mathcal{N}$ $(1 \leq m \leq n)$. Using Lemma 2–5, we have

$$\begin{aligned}
&\varphi_{d-k} \circ E^*_{d-k}(\sigma) \circ \varphi^{-1}_{d-k}\\
&= \varphi_{d-k} \circ E^*_{d-k}(\sigma_n) \circ \cdots \circ E^*_{d-k}(\sigma_1) \circ \varphi^{-1}_{d-k}
\end{aligned}$$



$2^* \wedge 3^*$ $\quad$ $1^* \wedge 3^*$ $\quad$ $1^* \wedge 2^*$

Fig. 5. the map $E_2^*(\sigma_R)$

$$
\begin{aligned}
&= \left(\det(A_{\sigma_n}) \circ T_k(v(\sigma_n)) \circ E_k(\overline{\sigma_n^{-1}})\right) \circ \cdots \circ \left(\det(A_{\sigma_1}) \circ T_k(v(\sigma_1)) \circ E_k(\overline{\sigma_1^{-1}})\right) \\
&= \det(A_{\sigma_1}) \cdots \det(A_{\sigma_n}) \circ T_k\left(v(\sigma_n) + A_{\overline{\sigma_n^{-1}}}(v(\sigma_{n-1})) + \cdots + A_{\overline{\sigma_n^{-1}} \cdots \overline{\sigma_2^{-1}}}(v(\sigma_1))\right) \\
&\qquad\qquad \circ E_k\left(\overline{\sigma_n^{-1}} \cdots \overline{\sigma_1^{-1}}\right) \\
&= \det(A_{\sigma_1 \cdots \sigma_n}) \circ T_k\left(v(\sigma_n) + A_{\overline{\sigma_n^{-1}}}(v(\sigma_{n-1})) + \cdots + A_{\overline{(\sigma_2 \cdots \sigma_n)^{-1}}}(v(\sigma_1))\right) \circ E_k\left(\overline{(\sigma_1 \cdots \sigma_n)^{-1}}\right) \\
&= \det(A_\sigma) \circ T_k(\boldsymbol{x}) \circ E_k\left(\overline{\sigma^{-1}}\right),
\end{aligned}
$$

where $\boldsymbol{x} = v(\sigma_n) + A_{\sigma_n}^{-1}(v(\sigma_{n-1})) + \cdots + A_{\sigma_2 \cdots \sigma_n}^{-1}(v(\sigma_1))$. □

REMARK 5. In the case of $d = 3$, in particular, for a Pisot substitution $\sigma \in \mathrm{Sub}(F_3)$ (i.e., a substitution such that the characteristic polynomial of A_σ is equal to the minimal polynomial of a Pisot number), we are interested in the region $E_1^*(\sigma)^n(\sum_{i=1}^3 (\boldsymbol{o}, i^*))$ ($n \in \boldsymbol{N}$) in connection with stepped surfaces, cf. [1]. It follows from the assertion (ii) in Proposition 5 that the boundary of $E_1^*(\sigma)^n(\sum_{i=1}^3 (\boldsymbol{o}, i^*))$ coincides with $E_2^*(\sigma)^n(\sum_{i=1}^3 \delta_2^*(\boldsymbol{o}, i^*))$. On the other hand, Theorem 2 says that $E_2^*(\sigma)^n(\sum_{i=1}^3 \delta_2^*(\boldsymbol{o}, i^*))$ can be calculated by using the map $\overline{\sigma^{-1}}^{\,n}$.

REMARK 6. In this setting we can rephrase Theorem 1 as follows:
If $\sigma \in \mathrm{Aut}(F_d)$ is written as $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ with $\sigma_i \in \mathcal{N}$, then $\boldsymbol{x}_\sigma$ in Theorem 1 is given by

$$\boldsymbol{x}_\sigma = v'(\sigma_1) + A_{\sigma_1}(v'(\sigma_2)) + \cdots + A_{\sigma_1 \cdots \sigma_{n-1}}(v'(\sigma_n)),$$

where $v' \colon \mathcal{N} \to \{\boldsymbol{o}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d\}$ is the map given by $v'(\alpha_{ij}) := \boldsymbol{o}$, $v'(\beta_{ij}) := \boldsymbol{e}_i$, $v'(\gamma_j) := -\boldsymbol{e}_j$.

EXAMPLE 2′. For the Rauzy substitution σ_R given in Example 1, we can show $\varphi_2 \circ E_2^*(\sigma_R) \circ \varphi_2^{-1} = E_1(\overline{\sigma_R^{-1}})$. In view of Fig. 5, we easily see that σ_R^{-1} is given by

$$\sigma_R^{-1} \colon \begin{cases} 1 \to 3 \\ 2 \to 3^{-1}1 \\ 3 \to 3^{-1}2 \end{cases} .$$

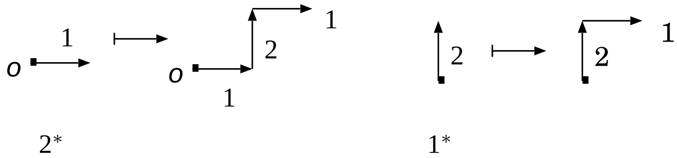


Fig. 6. the map $E_1^*(\sigma)$

We give another example of $E_1^*(\sigma)$ for an endomorphism $\sigma \in \mathrm{End}(F_2)$ which is not a substitution.

EXAMPLE 3. Let $\sigma \in \mathrm{Aut}(F_2)$ be given by

$$\sigma\colon \begin{cases} 1 & \to & 2^{-1}1 \\ 2 & \to & 1^{-1}22 \end{cases}.$$

Then

$$\begin{array}{rcl} E_1^*(\sigma)\colon (-\boldsymbol{e}_2, 2^*) & \mapsto & -(\boldsymbol{o}, 1^*) + (\boldsymbol{e}_1, 2^*) + (-\boldsymbol{e}_2, 2^*) \\ \simeq (\boldsymbol{o}, 1) & & \simeq (\boldsymbol{e}_1, 2) + (\boldsymbol{e}_1 + \boldsymbol{e}_2, 1) + (\boldsymbol{o}, 1) \\ -(-\boldsymbol{e}_1, 1^*) & \mapsto & -(-\boldsymbol{e}_1, 1^*) + (\boldsymbol{o}, 2^*) \\ \simeq (\boldsymbol{o}, 2) & & \simeq (\boldsymbol{o}, 2) + (\boldsymbol{e}_2, 1) \end{array} .$$

We can show $\varphi_1 \circ E_1^*(\sigma) \circ \varphi_1^{-1} = E_1(\overline{\sigma^{-1}})$. In view of Fig. 6, we easily see σ^{-1} is given by

$$\sigma^{-1}\colon \begin{cases} 1 & \to & 121 \\ 2 & \to & 12 \end{cases}.$$

As we have already seen in the two examples above, we can construct σ^{-1}, in some cases, by the figure of $E_{d-1}^*(\sigma)$ for $\sigma \in \mathrm{Aut}(F_d)$. In general, we have certain difficulty, cf. Section 3.3.

3. Examples and some comments

In the case of $d = 3$, some difficulties which never occur in the case of $d = 2$, will take place as we shall see through some examples.

3.1. Substitutions given by a matrix. It is easy to see, as is well known, that any unimodular matrix $A \in GL(2, \boldsymbol{N} \cup \{0\})$ (i.e., $\det(A) = \pm 1$) can be decomposed into two matrices

$$A_{\alpha_{12}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_{\beta_{12}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Therefore, for any matrix $A \in GL(2, \boldsymbol{N} \cup \{0\})$, there exists at least one invertible substitution σ such that $A_\sigma = A$. On the other hand, in Example 1 in Section 1, we have seen that any substitution σ satisfying $A_\sigma = A_{\sigma_C}$ is not invertible. We put $A_{ij} = (a_{lm})_{1\le l,m\le d}$ $(1 \le i,\ j \le d,\ i \ne j)$ by

$$a_{lm} := \begin{cases} 1 & \text{if } l = m \\ -1 & \text{if } l = i,\ m = j \\ 0, & \text{otherwise} \end{cases}.$$

We say that a matrix $M \in GL(d, \mathbf{N} \cup \{0\})$ is *non-comparable* if both $MA_{ij},\ A_{ij}M$ have negative entries for all A_{ij} $(1 \le i,\ j \le d,\ i \ne j)$. For instance, A_{σ_R} is a comparable matrix, while A_{σ_C} is a non-comparable one. It seems very likely that if A_σ is non-comparable, then σ can not be an invertible substitution, i.e.,

$$\sigma \notin \mathrm{IS}(F_d),$$

as far as we know.

3.2. Generators of the invertible substitutions. An invertible substitution σ is called a *prime substitution* if σ cannot be decomposed into 2 invertible substitutions $\sigma_1,\ \sigma_2$ such that one of σ_1, and σ_2 does not belong to the group generated by α_{ij} $(1 \le i,\ j \le d,\ i \ne j)$. Related to generators of the invertible substitutions σ (i.e., $\sigma \in \mathrm{IS}(F_d)$), some results are found in [4], [5]. In the case of $d = 2$, generators of the invertible substitutions are given by three prime substitutions:

$$\alpha\colon \begin{cases} 1 \to 2 \\ 2 \to 1 \end{cases},\ \beta\colon \begin{cases} 1 \to 12 \\ 2 \to 1 \end{cases},\ \delta\colon \begin{cases} 1 \to 21 \\ 2 \to 1 \end{cases},$$

so that the number of generators is finite, cf. [4]. But, the monoid $\mathrm{IS}(F_d)$ for $d \ge 3$ turns out to be quite different from that for $d = 2$. For example, in the case of $d = 3$, we need infinitely many generators. In fact, $\sigma \in \mathrm{IS}(F_3)$ defined by

$$\sigma(1) := 12,\ \sigma(2) := 132,\ \sigma(3) := 3^n2 \quad (n \ge 2)$$

are prime substitutions, cf. [5].

3.3. Connectedness of $E^*_{d-1}(\sigma)(o, 1^* \wedge \cdots \wedge \widehat{j^*} \cdots \wedge d^*)$. In the case of $d = 2$, we have shown in [2] the following proposition related to the dual map $E_1^*(\sigma)$.

Proposition 6 ([2]). *A substitution σ over 2 letters is invertible iff all the figures coming from $E_1^*(\sigma)(-\boldsymbol{e}_1, 1^*)$, $E_1^*(\sigma)(-\boldsymbol{e}_2, 2^*)$, $E_1^*(\sigma)((-\boldsymbol{e}_1, 1^*) + (-\boldsymbol{e}_2, 2^*))$ are connected.*

The figures coming from $E_1^*(\sigma)(-\boldsymbol{e}_i, i^*)$ $(i = 1,\ 2)$ are parts of the so called stepped surface, cf. [1]. Since a stepped curve (a stepped surface of dimension 1)

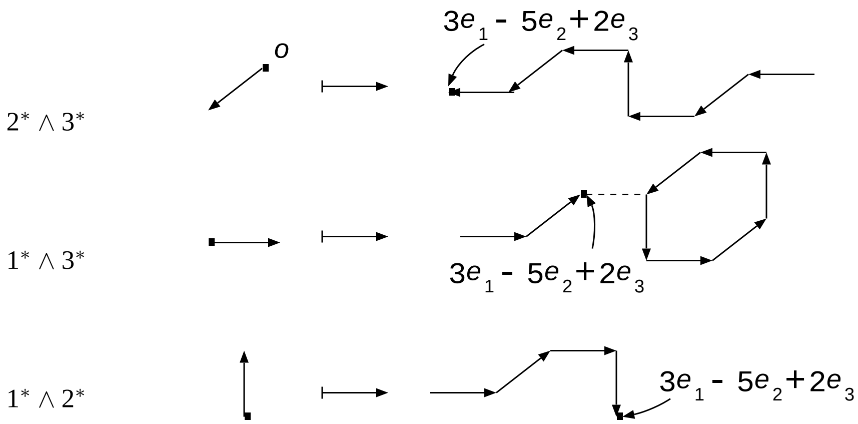


Fig. 7. the map $E_2^*(\sigma)$

univalently spreads along a line, any cancellation can not occur in $E_1^*(\sigma)(-\boldsymbol{e}_i, i^*)$ $(i = 1,\ 2)$. We can easily find the inverse of an invertible substitution $\sigma \in \mathrm{IS}(F_2)$ from the figures coming from $E_1^*(\sigma)(-\boldsymbol{e}_i, i^*)$ $(i = 1,\ 2)$, provided that $E_1^*(\sigma)(-\boldsymbol{e}_i, i^*)$ $(i = 1,\ 2)$ contain no cancellations.

On the other hand, in the case of $d = 3$, $E_2^*(\sigma)(-\boldsymbol{e}_i - \boldsymbol{e}_j, i^* \wedge j^*)$ $(\sigma \in \mathrm{IS}(F_3),\ (i, j) \in \{\ (1,2), (1,3), (2,3)\ \})$ is not always connective:

EXAMPLE 4. Let σ be an invertible substitution given by

$$\sigma\colon \begin{cases} 1 \to 1223 \\ 2 \to 123 \\ 3 \to 133 \end{cases} \qquad \left(\sigma^{-1}\colon \begin{cases} 1 \to 21^{-1}23^{-1}21^{-1}2 \\ 2 \to 2^{-1}12^{-1}32^{-1}13^{-1}21^{-1}2 \\ 3 \to 2^{-1}12^{-1}3 \end{cases}\right).$$

Since $\varphi_2 \circ E_2^*(\sigma) \circ \varphi_2^{-1} = -T_1(3\boldsymbol{e}_1 - 5\boldsymbol{e}_2 + 2\boldsymbol{e}_3) \circ E_1(\overline{\sigma^{-1}})$, the figure coming from $E_2^*(\sigma)(\boldsymbol{o}, 1^* \wedge 3^*)$ is not connected, see Fig. 7.

3.4. Open problems. We give two problems for arbitrary $d \geq 3$:

(i) Does the converse of the statement of Theorem 1 hold?

(ii) Let $\sigma \in \mathrm{Sub}(F_d)$ be a substitution with a non-comparable matrix A_σ such that A_σ does not belong to the group generated by $A_{\alpha_{ij}}, A_{\beta_{ij}}$ $(1 \leq i,\ j \leq d,\ i \neq j)$. Then, is σ always not invertible?

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