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HARMONIC MAPS FROM S^2 TO HP^2

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In this paper, we describe all harmonic maps from the Riemann 2-sphere to $\boldsymbol{H}P^2$, the quaternion 2-projective space. In example 1.3, all isotropic harmonic maps from S^2 to $\boldsymbol{H}P^n$ are given. A particular class of nonisotropic harmonic maps from S^2 to $\boldsymbol{H}P^n$ are classified by Theorem 1.4. With theorem 1.5, the description of harmonic maps from S^2 to $\boldsymbol{H}P^2$ becomes complete.

Harmonic maps from S^2 to S^n and S^2 to CP^n are classified by Calabi. E [1] and Eells-Wood [2] respectively. Our description of harmonic maps from S^2 to $\boldsymbol{H}P^2$ is not as elegant as those of Calabi and Eells-Wood. Still it gives hope for classifying harmonic maps from S^2 to compact symmetric spaces.

We state our main results in §1. §2 contains some preliminaries. In §3, the proof of theorem 1.4 is given. Theorem 1.5 is proved in §4. Here we use some of the ideas from [6].

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1. Main results

$\boldsymbol{H}^n$ denotes the quaternionic space of dimension n over $\boldsymbol{H}$, the quaternions. We have the quaternion metric $\langle\ ,\ \rangle$ on $\boldsymbol{H}^n$ defined $\langle v, w\rangle=\sum a_i \bar{b}_i$ where $v=(a_1, \cdots, a_n)$, $w=(b_1, \cdots, b_n)\in\boldsymbol{H}^n$. For $a\in\boldsymbol{H}$, $\bar{a}$ denotes the conjugation of a in $\boldsymbol{H}$. Write

$$\langle v, w\rangle = H(v, w)+A(v, w)j \tag{1.1}$$

where $H(v, w)$, $A(v, w)\in C=\boldsymbol{R}+\boldsymbol{R}i$. Define T: $\boldsymbol{H}^n\to C^{2n}$ by $T(x_1+y_1 j, \cdots, x_n+y_n j)=(x_1, y_1, \cdots x_n, y_n)$. T is a C-linear isomorphism of $\boldsymbol{H}^n$ with C^{2n}. Always, we identify C^{2n} and $\boldsymbol{H}^n$ through this isomorphism. Then H defined in (1.1) is the standard Hermitian metric on C^{2n} and A defined in (1.1) is a nondegenerate alternating C-bilinear form on C^{2n}. Let J denote left multiplication by j in $\boldsymbol{H}^n$. Then $H(v,Jw)=A(v,w)$ and $A(v,Jw)=-H(v,w)$ for $v,w\in\boldsymbol{H}^n$.

For a subspace W of C^{2n}, put

$$W^\perp = \{x\in C^{2n}\colon H(x, y) = 0 \quad \text{for all } y\in W\} \text{ and}$$
$$W_A^\perp = \{x\in C^{2n}\colon A(x, y) = 0 \quad \text{for all } y\in W\}\ .$$

$\underline{C}^n$ stands for the trivial bundle $S^2 \times C^n$ with the standard connection ∂ and standard Hermitian metric on each fibre. For a smooth $(=C^\infty)$ subbundle E of $\underline{C}^n$, $E^\perp$ denotes the bundle with $(E^\perp)_x = (E_x)^\perp$, where E_x denotes the fibre of E at $x \in S^2$, and in case n is even, $E_A^\perp$ denotes the bundle with $(E_A^\perp)_x = (E_x)_A^\perp$.

By a chart (U, Z) of S^2, we mean a nonempty open set U of S^2 equipped with a holomorphic coordinate Z: $U \to C$. For a smooth complex vector bundle E over S^2, $\underset{U}{C}(E)$ (resp. $C(E)$) denotes the space of all smooth sections of E over U (resp. S^2). A pair (E, F) of holomorphic subbundles of $\underline{C}^n$ is called a *∂'-pair* of $\underline{C}^n$ if $E \subset F$ and $\frac{\partial}{\partial Z}(\underset{U}{C}(E)) \subset \underset{U}{C}(F)$ for all charts (U, Z) of S^2.

For a subbundle E of $\underline{C}^n$, let D^E denote the induced connection on E. Then, we have operators

$$D_Z^E \text{ (resp. } D_{\bar{Z}}^E\text{)}: \underset{U}{C}(E) \to \underset{U}{C}(E)$$

defined by $D_Z^E(s) = p(\frac{\partial s}{\partial Z})$ (resp. $D_{\bar{Z}}^E s = p(\frac{\partial s}{\partial \bar{Z}})$) for $s \in \underset{U}{C}(E)$. Here, p: $\underline{C}^n = E \oplus E^\perp \to E$ is the orthogonal projection. Also, we have operators

$$A_Z^E \text{ (resp. } A_{\bar{Z}}^E\text{)}: \underset{U}{C}(E) \to \underset{U}{C}(E^\perp)$$

defined by $A_Z^E(s) = q(\frac{\partial s}{\partial Z})$ (resp. $A_{\bar{Z}}^E s = q(\frac{\partial s}{\partial \bar{Z}})$) for $s \in \underset{U}{C}(E)$. Here, q: $C^n = E \oplus E^\perp \to E^\perp$ is the orthogonal projection. A_Z^E and $A_{\bar{Z}}^E$ are tensors (i.e. $A_Z^E(fs) = fA_Z^E(s)$ for all $s \in \underset{U}{C}(E)$ and for any function f: $U \to C$).

For integers k, n with $0 \leqslant k \leqslant n$, $G_k(C^n)$ denotes the Grassmannian of k-dimensional subspaces of C^n. There is a one-to-one correspondence between maps from S^2 to $G_k(C^n)$ and subbundles of $\underline{C}^n$ of rank k. We often denote the map and the corresponding subbundle by the same letter.

A subbundle E of $\underline{C}^n$ is said to be *full* in $\underline{C}^n$ if it is not contained in a proper trivial subbundle of $\underline{C}^n$. First, we give some examples of harmonic maps from S^2 to $\boldsymbol{H}P^{n-1} (n \geqslant 2)$.

EXAMPLE 1.2. Let $S \subset T \subset T_A^\perp$ be a sequence of holomorphic subbundles of $\underline{C}^{2n}$ such that (i) (S, T) and $(T, T_A^\perp)$ are ∂'-pairs of $\underline{C}^{2n}$ and (ii) (rank S)$+1$ $=$rank $T \leq n-1$. Put $\phi(x) = S_x^\perp \cap T_x \oplus J(S_x^\perp \cap T_x)$, for $x \in S^2$. Then ϕ: $S^2 \to \boldsymbol{H}P^{n-1}$ is a harmonic map.

EXAMPLE 1.3. Let F be a holomorphic subbundle of $\underline{C}^{2n}$ of rank $n-1$ such that $(F, F_A^\perp)$ is a ∂'-pair of $\underline{C}^{2n}$. Put $\phi(x) = F_x^\perp \cap (F_A^\perp)_x$. Then ϕ: $S^2 \to \boldsymbol{H}P^{n-1}$ is a harmonic map which is isotropic. All isotropic harmonic maps from S^2 to $\boldsymbol{H}P^{n-1}$ can be described in this way [3].

M.V. Nori conjectured that (1.2) and (1.3) give all harmonic maps from

S^2 to $\boldsymbol{H}P^{n-1}$. This is found to be false when $n=3$. See example 4.15 and remark 4.17.

Now we state our main results.

Theorem 1.4. *Let* ϕ: $S^2 \to \boldsymbol{H}P^{n-1}$ $(n \geqslant 2)$ *be a harmonic map such that* (i) ϕ *is not an isotropic map and* (ii) *rank*$[\partial\phi]=1$. *Then there exists a holomorphic line subbundle* F *of* $\underline{C}^{2n}$ *such that* $(F, F_A^\perp)$ *is a* ∂'*-pair of* $\underline{C}^{2n}$ *and* ϕ *is given by*

$$\phi(x) = F_x + JF_x$$

for $x \in S_2$.

Theorem 1.5. *Let* ϕ: $S^2 \to \boldsymbol{H}P^2$ *be a harmonic map such that* (i) ϕ *is not an isotropic map* (ii) *rank*$[\partial\phi]=2$. *Then there exist*

(a) *a unique holomorphic map* h: $S^2 \to CP^5$ *with the property that* $Jh_2 = h_3$

(b) *a line bundle* $R \subset (h_2 \oplus h_3 \oplus h_4)^\perp = H$ *satisfying, for all charts* (U, Z) *of* S^2,

(1) $\frac{\partial}{\partial \bar{Z}} (\underset{U}{C}(R)) \subset \underset{U}{C}(R_A^\perp)$ (2) $D_Z^H(\underset{U}{C}(R)) \subset \underset{U}{C}(R)$.

such that ϕ *is given by*

$$\phi(x) = (R \oplus JR \oplus h_2 \oplus h_3)_x^\perp$$

for $x \in S^2$.

REMARKS. (1) For the definitions of $[\partial\phi]$ and rank$[\partial\phi]$, see §2C.

(2) Isotropic maps are defined in §2D.

(3) For a holomorphic map h: $S^2 \to CP^n$, there are harmonic maps h_k's associated to it. See §2E.

2. Preliminaries

A. Harmonic maps. Let M, N be two compact smooth $(=C^\infty)$ Riemannian manifolds. A smooth map ϕ: $M \to N$ is *harmonic* if its tension field

$$\tau(\phi) = \text{Trace } Dd\phi$$

vanishes identically. Here D is the connection on the bundle $T^*M \otimes \phi^{-1}(TN)$ induced from the Riemannian connections on TM and TN.

Now let $\dim M=2$, M orientable and N a complex manifold with a Hermitian metric. Since vanishing of $\tau(\phi)$ depends only on the conformal class of the metric on M, we can talk of harmonic maps whose domain is a Riemann surface. Let $(d\phi)^C$: $TM \otimes C \to TN \otimes C$ be the C-linear extension of the differential $d\phi$: $TM \to TN$. $(d\phi)^C$ gives, in particular (see [2] for the notation below)

$$\partial\phi: T'M \to T'N \text{ and } \bar{\partial}\phi: T''M \to T'N \tag{2.1}$$

where $T'M$ is the holomorphic tangent bundle of M and $T''M$ is its conjugate in $TM\otimes C$. Let (U, Z) be a chart of M. Then

$$\partial'\phi = (\partial\phi)\left(\frac{\partial}{\partial Z}\right) \text{ and } \partial''\phi = (\bar{\partial}\phi)\left(\frac{\partial}{\partial \bar{Z}}\right) \in \underset{\cap}{\mathcal{C}}(\phi^{-1}(T'N)) \tag{2.2}$$

Taking N to be Kähler, ϕ is harmonic if and only if

$$D_{\frac{\partial}{\partial \bar{Z}}} \partial'\phi = 0 \tag{2.3}$$

where D is the connection on $\phi^{-1}(T'N)$ induced from the Hermitian connection on $T'N$. See [2].

B. Consider $G_k(C^n)$ with the standard Kähler metric [4]. Then $T'G_k(C^n)$ gets a unique Hermitian connection. Let ν be the tautological k-plane bundle over $G_k(C^n)$. Equip ν and $\nu^\perp$ with metrices and connections induced from $G_k(C^n)\times C^n$. There is a canonical linear transformation

$$\eta\colon T'G_k(C^n) \to \mathrm{Hom}(\nu, \nu^\perp)$$

defined by, for $X \in T'_W G_k(C^n)$ and for a section s of ν defined in a neighbourhood of W,

$$\eta(X)\,(s) = p(D_X s)\,.$$

Here D stands for the standard connection on $G_k(C^n)\times C^n$ and $p\colon G_k(C^n)\times C^n = \nu\oplus\nu^\perp \to \nu^\perp$ is the orthogonal projection. η is a connection-preserving isometric isomorphism ([2]).

Let $\phi\colon M \to G_k(C^n)$ be a smooth map from a Riemann surface M. Give $\phi^{-1}(T'G_k(C^n))$ and $\phi^{-1}(\mathrm{Hom}(\nu, \nu^\perp))$ pull-back metrics and pull-back connections. $\phi^{-1}(\eta)\colon \phi^{-1}(T'G_k(C^n)) \to \phi^{-1}(\mathrm{Hom}(\nu, \nu^\perp))$ is a connection-preserving isometric isomorphism. Let ∇ denote the connection in either of the two bundles. Let (U, Z) be a chart of M and $s \in \underset{U}{\mathcal{C}}(\phi^{-1}(\nu))$. Through the isomorphism $\phi^{-1}(\eta)$,

$$(\partial'\phi)\,(s) = A_Z(s) \tag{2.4}$$

and

$$\begin{aligned}(\nabla_{\frac{\partial}{\partial \bar{Z}}}\partial'\phi)\,(s) &= D_{\bar{Z}}\circ A_Z(s) - A_Z\circ D_{\bar{Z}}(s)\\ &= [D_{\bar{Z}}, A_Z]\,(s)\,.\end{aligned}$$

Here A_Z, $D_{\bar{Z}}$ are defined with respect to the decomposition $M\times C^n = \phi^{-1}(\nu) \oplus\phi^{-1}(\nu^\perp)$. See §1. From (2.3), ϕ is harmonic if and only if

$$[D_{\bar{Z}}, A_Z]\,(s) = 0 \tag{2.5}$$

for all charts (U, Z) of M and for all $s \in \underset{U}{\mathcal{C}}(\phi^{-1}(\nu))$.

Let E be a subbundle of $M\times C^n$. Over a chart (U, Z) of M let $s\in \underset{U}{\mathcal{C}}(E)$ or $\underset{U}{\mathcal{C}}(E^\perp)$. Then

$$\frac{\partial}{\partial \bar{Z}}(s)=(D_{\bar{Z}}+A_{\bar{Z}})(s),\ \frac{\partial}{\partial Z}(s)=(D_Z+A_Z)(s)\,.$$

Here $D_{\bar{Z}}, D_Z, A_{\bar{Z}}, A_Z$ are defined with respect to the decomposition $E\oplus E^\perp$ of $M\times C^n$. The identity $\left[\frac{\partial}{\partial \bar{Z}}, \frac{\partial}{\partial Z}\right]\equiv 0$ implies

$$([D_{\bar{Z}}, D_Z]+[A_{\bar{Z}}, A_Z])(s)=0 \tag{2.6}$$

and

$$([D_{\bar{Z}}, A_Z]+[A_{\bar{Z}}, D_Z])(s)=0\,. \tag{2.7}$$

Taking $E=\phi^{-1}(\nu)$, where $\phi\colon M\to G_k(C^n)$ is a smooth map, (2.5) and (2.7) imply that ϕ is harmonic if and only if

$$[D_Z, A_{\bar{Z}}]\,(s)=0 \tag{2.8}$$

for all charts (U, Z) of M and for all $s\in\underset{U}{\mathcal{C}}(\partial^{-1}(\nu))$.

C. $\phi\colon M\to G_k(C^n)$ is a harmonic map from a Riemann surface. It is well known that, if E is a C^∞ complex vector bundle over a Riemann surface, a complex connection D on E induces a unique holomorphic structure on E whose $\bar{\partial}$ operator is the (0, 1) part of D (See [5]). With respect to the connection described in §2B, $\phi^{-1}(\nu)$, $\phi^{-1}(\nu^\perp)$ get unique holomorphic structures. Then a section $s\in\underset{U}{\mathcal{C}}(\phi^{-1}(\nu))$ or $\underset{U}{\mathcal{C}}(\phi^{-1}(\nu^\perp))$ is holomorphic if and only if $D_{\bar{Z}}s=0$, (U, Z) being a chart of M. Since ϕ is harmonic, (2.5) implies that $A_Z^{\phi^{-1}}(\nu)$: $\phi^{-1}(\nu)|_U\to\phi^{-1}(\nu^\perp)|_U$ is holomorphic. Define $[\partial\phi]\colon T'M\otimes\phi^{-1}(\nu)\to\phi^{-1}(\nu^\perp)$ by $[\partial\phi]\left(\frac{\partial}{\partial Z}\otimes s\right)=(\partial'\phi)(s)$ for $s\in\underset{U}{\mathcal{C}}(\phi^{-1}(\nu))$, (U, Z) being any chart of M. By (2.4), $[\partial\phi]$ is holomorphic. Hence, we have

(2.9) If $\dim\,([\partial\phi]\,(T'M\otimes\phi^{-1}(\nu))_x)=r$ for all points of a nonempty openset of M, then it is so at all but a discrete set of points of M.

DEFINITION 2.10. Define rank $[\partial\phi]=r$ if (2.9) holds.

D. Isotropic maps. Let $\phi\colon M\to G_k(C^n)$ be a smooth map from a Riemann surface. Define

$$(\phi'_{(r)})(x)=\operatorname{span}\{D_{\bar{Z}}^m A_Z s(x)\colon 0\leqslant m\leqslant r,\ s\in\underset{U}{\mathcal{C}}(\phi^{-1}(\nu))\}\ \text{and}$$
$$(\phi''_{(r)})(x)=\operatorname{span}\{D_Z^m A_{\bar{Z}} s(x)\colon 0\leqslant m\leqslant r,\ s\in\underset{U}{\mathcal{C}}(\phi^{-1}(\nu))\}\ \text{for } x\in M\,,$$

(U, Z) being a chart of M around x. Let $(\phi'_{(\infty)})(x)=\bigcup_{r\geqslant 0}(\phi'_{(r)})(x)$ and $(\phi''_{(\infty)})(x)=\bigcup_{r\geqslant 0}(\phi''_{(r)})(x)$.

DEFINITION 2.11. We say that a smooth map $\phi\colon M\to G_k(C^n)$ is (strongly) *isotropic* if $(\phi'_{(\infty)})(x)$ is orthogonal to $(\phi''_{(\infty)})(x)$ for each $x\in M$.

Suppose $\phi\colon M\to \boldsymbol{H}P^{n-1}$ is a smooth map. Let $i\colon \boldsymbol{H}P^{n-1}\to G_2(C^{2n})$ be the inclusion and denote $i\circ\phi$ by ϕ itself. For any $s\in \underset{U}{\mathcal{C}}(\phi^{-1}(\nu))$, $\dfrac{\partial}{\partial Z}(Js)=J\dfrac{\partial s}{\partial \bar{Z}}$, (U, Z) being a chart of M. Then $(\phi'_{(\infty)})(x)=J(\phi''_{(\infty)})(x)$. Hence, we have

REMARK 2.12. $\phi\colon M\to \boldsymbol{H}P^{n-1}$ is an isotropic map if and only if $(\phi'_{(\infty)})(x)$ is an isotropic space, i.e., $A(v, w)=0$ for all $v, w\in(\phi'_{(\infty)})(x)$.

E. Harmonic maps from CP^1 to CP^n. Let $h\colon CP^1\to CP^n$ be a full holomorphic map (i.e. image h is not contained in a proper projective subspace of CP^n). Let (U, Z) be a chart of CP^1 and $\bar{h}$ be a lift of h to $C^{n+1}-0$. Let

$E_k(Z)=\text{span}\{\dfrac{\partial^r \bar{h}}{Z^r}: 0\leqslant r\leqslant k\}$, $k=0, 1, \cdots, n$. E_k is independent of the co-ordinate Z and the lift chosen and $\dim. E_k=k+1$ except possibly at a discrete set of points. E_k gives rise to a unique complex vector bundle denoted again by E_k. Write

$$h_k(Z)=E_k(Z)\cap(E_{k-1}(Z))^{\perp},\ k=0, 1, \cdots, n$$

(Put $E_{-1}=0$).

Then for each $k=0, 1, \cdots, n$, $h_k\colon CP^1\to CP^n$ is harmonic. This construction works for any Riemann surface in place of CP^1.

Conversely, if $\phi\colon CP^1\to CP^n$ is a full harmonic map, then there exists a unique full holomorphic map $h\colon CP^1\to CP^n$ and an integer k, $0\leqslant k\leqslant n$, such that $\phi=h_k$. See [2] for details.

3. Let $\phi\colon S^2\to \boldsymbol{H}P^{n-1}$ be a map and $i\colon \boldsymbol{H}P^{n-1}\to G_2(C^{2n})$ be the inclusion. Throughout §3 and §4, $E(\phi)$ stands for the bundle $(i\circ\phi)^{-1}(\nu)$ (where ν is the tautological 2-plane bundle over $G_2(C^{2n})$) and $D_Z, D_{\bar{Z}}, A_Z, A_{\bar{Z}}$ are defined with respect to the decomposition $E(\phi)+E(\phi)^{\perp}$ of C^{2n} (See §1). Further (U, Z) stands for an arbitrary chart of S^2.

We start with a lemma.

Lemma 3.1. *Let $\phi\colon S^2\to \boldsymbol{H}P^{n-1}$ be a harmonic map. Then for any chart (U, Z) of S^2 and $x\in U$, $A_Z(E(\phi)_x)$ is an isotropic space.*

Proof. With respect to the holomorphic structures on $E(\phi)$ and $E(\phi)^{\perp}$ given in §2 C, $A_Z\colon E(\phi)|_U\to E(\phi)^{\perp}|_U$ is holomorphic. Let $s, t\in \underset{U}{\mathcal{C}}(E(\phi))$ be two linearly independent holomorphic sections. For $x\in U$, putting

$$\beta(x)=\frac{A(A_Z s(x), A_Z t(x))}{A(s(x), t(x))}\,dZ^2,$$

$\beta\in\underset{U}{C}(K^2)$. K always stands for the canonical line bundle of S^2. Since A_Z is holomorphic and s, t are holomorphic sections, $A(A_Zs(x), A_Zt(x))$ and $A(s(x), t(x))$ are holomorphic functions on U. Hence β is a holomorphic section of K^2 over U. Also β is independent of the linearly independent holomorphic sections s, $t\in\underset{U}{C}(E(\phi))$. Further, it is easily seen that β is a global holomorphic section of K^2. Since K^2 has no nonzero holomorphic section, $\beta\equiv 0$. The lemma now follows.

Corollary 3.2. *$A_Z^{E(\phi)^\perp}\circ A_Z^{E(\phi)}\equiv 0$ over any chart (U, Z) of S^2.*

Proof. For s, $t\in\underset{U}{C}(E(\phi))$ and $x\in U$,

$$A(A_Z^2s(x), t(x)) = -A(A_Zs(x), A_Zt(x)) = 0\,.$$

Since A is a nondegenerate alternating form on $E(\phi)_x$, $A_Z^2s(x)=0$.

Proposition 3.3. *Let ϕ: $S^2\to\boldsymbol{H}P^{n-1}$ be a harmonic map with rank $[\partial\phi]=1$. Then $A_Z^{E(\phi)}\circ A_Z^{E(\phi)^\perp}\equiv 0$ over any chart (U, Z) of S^2.*

Proof. Let $t\in\underset{U}{C}(E(\phi))$ be a section which is nowhere zero on U and $A_Zt\equiv 0$. Consider $s\in\underset{U}{C}(E(\phi)^\perp)$.

$$A(A_Zs(x), t(x)) = -A(s(x), A_Zt(x)) = 0\,.$$

Since A is nondegenerate on $E(\phi)_x$ and dimension $E(\phi)_x=2$, the proposition now follows.

There is an isometry f: $G_k(C^n)\to G_{n-k}(C^n)$ given by $f(W)=W^\perp$. If ϕ: $S^2\to G_k(C^n)$ is a harmonic map, then $\phi^\perp=f\circ\phi$ is also harmonic.

Let ϕ: $S^2\to\boldsymbol{H}P^{n-1}$ be a harmonic map with rank $[\partial\phi]=1$. Then $[\partial\phi]$: $T'S^2\otimes E(\phi)\to E(\phi)^\perp$ and $[\partial\phi^\perp]$: $T'S^2\otimes E(\phi)^\perp\to E(\phi)$ are holomorphic maps (See §2C). Since Rank $[\partial\phi]=1$, the kernel of $[\partial\phi]$ gives a unique line subbundle $\ker[\partial\phi]$ of $T'S^2\otimes E(\phi)$ which will coriespond to a line subbundle L of $E(\phi)$. Similarly, let W be the subbundle of $E(\phi)^\perp$ corresponding to ker $[\partial\phi^\perp]$. Let $\mathrm{Im}[\partial\phi]$ (resp. $\mathrm{Im}[\partial\phi^\perp]$) denote the unique bundle obtained from the image of $[\partial\phi]$ (resp. $[\partial\phi^\perp]$). L, $\mathrm{Im}[\partial\phi^\perp]$ (resp. W, $\mathrm{Im}[\partial\phi]$) are holomorphic subbundles of $E(\phi)$ (resp. $E(\phi)^\perp$). Also by Corollary (3.2), $\mathrm{Im}[\partial\phi]\subset W$ and by proposition (3.3) $\mathrm{Im}[\partial\phi^\perp]=L$. Then, rank $W=2n-3$.

Let η: $E(\phi)\to\dfrac{E(\phi)}{L}$ and μ: $E(\phi)^\perp\to\dfrac{E(\phi)^\perp}{W}$,

be the canonical maps. Define $\hat{D}_Z$: $\underset{U}{C}(E(\phi))\to\underset{U}{C}\Big(\dfrac{E(\phi)}{L}\Big)$ by $\hat{D}_Z=\eta\circ D_Z$ and define $\tilde{D}_Z^k=\mu\circ D_Z^k$: $\underset{U}{C}(E(\phi)^\perp)\to\underset{U}{C}\Big(\dfrac{E(\phi)^\perp}{W}\Big)$ for $k=1,2,\cdots$. Let i: $L\to E(\phi)$ and j: $\mathrm{Im}[\partial\phi]$

$\to E(\phi)^\perp$ be the inclusion maps. $\hat{D}_Z \circ i$ gives a linear map $\hat{D}\colon L \to \frac{E(\phi)}{L} \otimes K$ and $\tilde{D}^1 \circ j$ gives a linear map $\tilde{D}^1\colon \mathrm{Im}[\partial\phi] \to \frac{E(\phi)^\perp}{W} \otimes K$ since $\mathrm{Im}[\partial\phi] \subset W$.

Proposition 3.4. *$\hat{D}$ is a holomorphic map.*

Proof. For $s \in \underset{\cup}{C}(L)$, $[D_{\bar{Z}}, D_Z]s = [A_Z, A_{\bar{Z}}]s$ by (2.6). So, $D_{\bar{Z}} D_Z s = D_Z D_{\bar{Z}} s + [A_Z, A_{\bar{Z}}]s$. $[A_Z, A_{\bar{Z}}]s = A_Z A_{\bar{Z}} s$, and by proposition (3.3), $A_Z A_{\bar{Z}} s \in \underset{\cup}{C}(L)$. Hence, if s is a holomorphic section of L (i.e. $D_{\bar{Z}} s = 0$), then $\hat{D}s$ is a holomorphic section of $\frac{E(\phi)}{L} \otimes K$. This proves that $\hat{D}$ is a holomorphic map.

We want to show that $\hat{D} \equiv 0$. For each integer $k \geqslant 0$, define $\phi'_{(k)}(x) = \mathrm{span}\,\{D_Z^r A_Z s(x)\colon 0 \leqslant r \leqslant k,\ s \in \underset{\cup}{C}(E(\phi))\}$. Put $\phi'_{(\infty)}(x) = \bigcup_{k \geqslant 0} \phi'_{(k)}(x)$.

Proposition 3.5. *If $\hat{D} \neq 0$, then $\phi'_{(\infty)}(x) \subset W_x$.*

Proof. By induction on k.
By Corollary (3.2), $\phi'_{(0)}(x) \subset W_x$. Assume by induction that $\phi'_{(k)}(x) \subset W_x$. Then $\tilde{D}_Z^{k+1} \circ j$ gives a linear map

$$\tilde{D}^{k+1}\colon \mathrm{Im}[\partial\phi] \to \frac{E(\phi)^\perp}{W} \otimes K^{k+1}.$$

Now, for $s \in \underset{\cup}{C}(E(\phi)) D_{\bar{Z}}(D_Z^{k+1} A_Z s) = D_Z D_{\bar{Z}}(D_Z^k A_Z s) + [A_Z, A_{\bar{Z}}] D_Z^k A_Z s$ (by 2.6). By induction assumption, $[A_Z, A_{\bar{Z}}] D_Z^k A_Z s = A_Z(A_{\bar{Z}} D_Z^k A_Z s) \in \underset{\cup}{C}(W)$. Using the induction repeatedly, we get

$$D_{\bar{Z}}(D_Z^{k+1} A_Z s) = D_Z^{k+1} D_{\bar{Z}} A_Z s + t(s) \tag{3.6}$$

where $t(s) \in \underset{\cup}{C}(W)$. From (3.6) we see that $\tilde{D}^{k+1}$ is a holomorphic map.

We have isomorphisms $T'S^2 \otimes \frac{E(\phi)^\perp}{W} \xrightarrow[\text{canonical}]{} \frac{T'S^2 \otimes E(\phi)^\perp}{\ker[\partial\phi^\perp]}$ and $\frac{T'S^2 \otimes E(\phi)^\perp}{\ker[\partial\phi^\perp]} \xrightarrow{[\partial\phi^\perp]} L$. Denote the composite of these two maps by $[\partial\phi^\perp]$ itself.

$$[\partial\phi^\perp]\colon T'S^2 \otimes \frac{E(\phi)^\perp}{W} \to L\,.$$

Then we have

$$\hat{D} \circ [\partial\phi^\perp]\colon T'S^2 \otimes \frac{E(\phi)^\perp}{W} \to \frac{E(\phi)}{L} \otimes K\,. \tag{3.7}$$

Also, denote the composite of $T'S^2 \otimes \frac{E(\phi)}{L} \xrightarrow[\text{canonical}]{} \frac{T'S^2 \otimes E(\phi)}{\ker[\partial\phi]}$ and $\frac{T'S^2 \otimes E(\phi)}{\ker[\partial\phi]} \xrightarrow{[\partial\phi]} \mathrm{Im}[\partial\phi]$ by $[\partial\phi]$ itself and form the composite

$$\tilde{D}^{k+1} \circ [\partial\phi]\colon T'S^2 \otimes \frac{E(\phi)}{L} \to \frac{E(\phi)^\perp}{W} \otimes K^{k+1}\,. \tag{3.8}$$

All bundles involved in (3–7), (3.8) are holomorphic line bundles and the linear maps between them are holomorphic. Let a and b be the first chern classes of $\frac{E(\phi)^{\perp}}{W}$ and $\frac{E(\phi)}{L}$ respectively. It is well known that, over a compact Riemann surface, a holomorphic line bundle with negative first chern class does not admit any nonzero holomorphic section. Using this, since $\hat{D}\circ[\partial\phi^{\perp}]\neq 0$, we have $2+a\leqslant b-2$. This implies that $2+b>a+(-2)(k+1)$ for $k\geqslant 0$. Hence $\tilde{D}^{k+1}\circ[\partial\phi]=0$. So $\tilde{D}^{k+1}=0$ giving $\phi'_{(k+1)}(x)\subset W_x$. We conclude that $\phi'_{(\infty)}(x)\subset W_x$ if $\hat{D}\neq 0$.

Proposition 3.9. *If $\hat{D}\neq 0$, ϕ is an isotropic map.*

Proof. By (2.12), we have to show that $\phi'_{(\infty)}(x)$ is a isotropic space. Let $s\in\underset{\cup}{\mathcal{C}}(E(\phi))$ be arbitrary. For any nonnegative integer p, we show that $P_{k,m}(x)=A(D_Z^k A_Z s(x), D_Z^m A_Z s(x))=0$ for all k, m s.t. $0\leqslant k$, $m\leqslant k+m\leqslant p$, by induction on p.

For $p=0$, we are through by lemma 3.1. Assume that $P_{k,m}(x)=0$ for k, m s.t. $k+m\leqslant p$. Then for k, m s.t. $k+m=p+1$, using the induction assumption repeatedly,

$$P_{k,m}(x)=-P_{k-1,m+1}(x)=\cdots=(-1)^k P_{0,m+k}(x)$$

But $P_{0,m+k}(x)=-A(s(x), A_Z(D_Z^{k+m}A_Z s(x)))$ which is zero by proposition (3.5). Hence $P_{k,m}(x)=0$ for $k+m=p+1$. The proof is now complete.

If ϕ is assumed to be nonisotropic, we conclude that $\hat{D}=0$.

Proof of theorem 1.4. For $s\in\underset{\cup}{\mathcal{C}}(L)$, $A_Z s=0$ by definition of L, and $\hat{D}=0$ implies that $D_Z s\in\underset{\cup}{\mathcal{C}}(L)$. Then $\underset{\cup}{\mathcal{C}}(L)$ is $\frac{\partial}{\partial Z}$-closed i.e. L is an antiholomorphic subbundle of $\underline{C}^{2n}$. Since L is a holomorphic subbundle of $E(\phi)$, $\underset{\cup}{\mathcal{C}}(L)$ is $D_{\bar{Z}}$-closed which implies that $\frac{\partial}{\partial\bar{Z}}(\underset{\cup}{\mathcal{C}}(L))\subset\underset{\cup}{\mathcal{C}}(L_A^{\perp})$. Put $JL=F$. Then F is a holomorphic line subbundle of $\underline{C}^{2n}$ and $(F, F_A^{\perp})$ is a ∂'-pair of $\underline{C}^{2n}$. Also, $E(\phi)=F\oplus JF$. This completes the proof of theorem 1.4.

In the following, we prove that the map ϕ given by example 1.2 is harmonic.

$$\text{Put } H_1=S,\ H_2=S^{\perp}\cap T,\ H_3=T^{\perp}\cap T_A^{\perp}\,.$$

Then $E(\phi)=H_2\oplus JH_2$ and $E(\phi)^{\perp}=H_1\oplus H_3\oplus JH_1$. It is clear that

$$D_{\bar{Z}}\underset{\cup}{\mathcal{C}}(H_3)\subset\underset{\cup}{\mathcal{C}}(H_3),\ D_{\bar{Z}}\underset{\cup}{\mathcal{C}}(H_2)\subset\underset{\cup}{\mathcal{C}}(H_2),\ A_Z\underset{\cup}{\mathcal{C}}(H_2)\subset\underset{\cup}{\mathcal{C}}(H_3)\,.$$

Hence $[D_{\bar{Z}}, A_Z]\underset{\cup}{\mathcal{C}}(H_2)\subset\underset{\cup}{\mathcal{C}}(H_3)$ (3.10)

Also,

$$D_Z\underset{U}{C}(H_2)\subset\underset{U}{C}(H_2),\ D_Z\underset{U}{C}(H_1)\subset\underset{U}{C}(H_1),\ A_{\bar{Z}}\underset{U}{C}(H_2)\subset\underset{U}{C}(H_2).$$

Hence,

$$[D_Z, A_{\bar{Z}}]\underset{U}{C}(H_2)\subset\underset{U}{C}(H_1). \tag{3.11}$$

From (3.10), (3.11), (2.7), we get $[D_{\bar{Z}}, A_Z]=0=[D_Z, A_{\bar{Z}}]$ on $\underset{U}{C}(H_2)$. We conclude that $[D_{\bar{Z}}, A_Z]=0$ on $\underset{U}{C}(E(\phi))$. Thus ϕ is harmonic.

We end this section with the following two remarks.

REMARK 3.12. Let W be a maximal isotropic subspace of C^{2n} ($n\geqslant 2$) and T be a holomorphic line subbundle of $S^2\times W$ which is nontrivial (i.e. T is not a constant line bundle). Then $(T, T_A^\perp)$ is a ∂'-pair of $\underline{C}^{2n}$. Then, ϕ given by $\phi(x)=T_x\oplus J\,T_x$ is a harmonic map with rank$[\partial\phi]=1$. But ϕ is an isotropic map.

Proof. $\phi'_{(\infty)}(x)\subset W$. Hence ϕ is isotropic.

REMARK 3.13. Let T be a holomorphic line subbundle of $\underline{C}^4$ such that T is full in $\underline{C}^4$ and $(T, T_A^\perp)$ is a ∂'-pair of $\underline{C}^4$. As in example 1.2, let ϕ be the harmonic map given by $\phi(x)=T_x\oplus J\,T_x$. Then, rank$[\partial\phi]=1$ and ϕ is not isotropic.

Proof. We prove that ϕ is not an isotropic map. Let $h\colon S^2\to CP^3$ be the holomorphic map defined by $h(x)=T_x$. By proposition 4.18, $J\,h_1=h_2$. By (4.13), $Jh_0=h_3$. Hence $\phi'_{(\infty)}(x)=E(\phi)_x^\perp$. So, $\phi'_{(\infty)}(x)$ is not an isotropic space.

4. Let ϕ: $S^2\to \boldsymbol{H}P^2$ be a harmonic map such that ϕ is not isotropic and rank$[\partial\phi]=2$. Then Im$[\partial\phi]$ (See §3) is a holomorphic subbundle of $E(\phi)^\perp$ and is of rank two. There is a holomorphic map

$$\tilde{D}^1\colon \mathrm{Im}[\partial\phi]\to\frac{E(S)^\perp}{\mathrm{Im}[\partial\phi]}\otimes K \ \text{(See §3)}.$$

Proposition 4.1. *Dimension* $\tilde{D}^1(\mathrm{Im}[\partial\phi])=1$ *everywhere except possibly at a discrete set of points of* S^2.

Proof. If $\tilde{D}^1=0$, then $D_Z(\underset{U}{C}(\mathrm{Im}[\partial\phi]))\subset\underset{U}{C}(\mathrm{Im}[\partial\phi])$. Then $(\phi'_{(\infty)})(x)\subset \mathrm{Im}[\partial\phi]_x$ for all $x\in S^2$. By lemma 3.1 and remark (2.12), ϕ is an isotropic map which is a contradiction. Hence $\tilde{D}^1\neq 0$. Now it is enouth to prove that

$$\phi'_{(1)}(x)=\mathrm{span}\,\{D_Z^k A_Z s(x)\colon\ s\in\underset{U}{C}(E(\phi)),\ k=0, 1\}$$

is a proper subspace of $E(\phi)_x^\perp$.

Let $s_1, s_2\in\underset{U}{C}(E(\phi))$ be two linearly independent sections. Put $v_1=A_Z s_1$,

$v_2=A_Z s_2$, $v_3=D_Z A_Z s_1$ and $v_4=D_Z A_Z s_2$. Put

$$P(x)=\begin{bmatrix} A(v_1(x),\ v_3(x)), & A(v_1(x),\ v_4(x)) \\ A(v_2(x),\ v_3(x)), & A(v_2(x),\ v_4(x)) \end{bmatrix}.$$

Then,

$$[A(v_i(x),\ v_j(x))]=\begin{bmatrix} 0 & P(x) \\ {}^tP(x) & * \end{bmatrix}$$

by lemma 3.1. Define β by

$$\beta(x)=\frac{\text{determinant } P(x)}{A(s_1(x),\ s_2(x))^2}(d\,Z)^6$$

for $x\in U$. Using lemma 3.1, one can verify that β is independent of the linearly independent pair of sections of $E(\phi)$ over U. Again, using lemma 3.1, β is independent of the chart (U, Z) chosen. Hence β is a global section of K^6. We prove that β is a holomorphic section.

Choose $s_1, s_2\in \underset{U}{C}(E(\phi))$ such that $D_{\bar{Z}}s_1=0=D_{\bar{Z}}s_2$. By (2.6),

$$D_{\bar{Z}}D_Z A_Z s_i=D_Z D_{\bar{Z}}A_Z s_i+[A_Z,\ A_{\bar{Z}}]A_Z s_i\,.$$

Then by (2.5) and lemma 3.1, $D_{\bar{Z}}D_Z A_Z s_i=A_Z A_{\bar{Z}}A_Z s_i$. Now,

$$\begin{aligned}\frac{\partial}{\partial\bar{Z}}A(A_Z s_i,\ D_Z A_Z s_j)&=A(D_{\bar{Z}}A_Z s_i,\ D_Z A_Z s_j)+A(A_Z s_i,\ D_{\bar{Z}}D_Z A_Z s_j)\\ &=A(0,\ D_Z A_Z s_j)+A(A_Z s_i;\ A_Z A_{\bar{Z}}A_Z s_j)\\ &=0\,.\end{aligned}$$

Each entry being holomorphic, det. $P(x)$ is a holomorphic function on U. Since s_1, s_2 are holomorphic sections, $A(s_1(x),\ s_2(x))$ is a holomorphic function on U. Thus β is a holomorphic section of K^6. Then $\beta=0$ and hence determinant $P(x)=0$. It follows that $[A(v_i(x),\ v_j(x))]$ is a singular matrix. This implies that $\phi'_{(1)}(x)$ is a proper subspace of $E(\phi)^{\perp}_x$. This completes the proof.

The kernel of $\tilde{D}^1$ gives a unique line bundle R of $\mathrm{Im}[\partial\phi]$. We have, then

(4.2) R is a holomorphic line subbundle of $\mathrm{Im}[\partial\phi]$. So, $D_{\bar{Z}}\underset{U}{C}(R)\subset\underset{U}{C}(R)$.

(4.3) By Corollary 3.2 and definition of R, $\frac{\partial}{\partial Z}\underset{U}{C}(R)\subset\underset{U}{C}(\mathrm{Im}[\partial\phi])$. Put $M=R^{\perp}\cap\mathrm{Im}[\partial\phi]$.

Define $\alpha\colon S^2\to G_2(C^6)$ by $\alpha(x)=(\mathrm{Im}[\partial\phi])_x$.

Proposition 4.4. *$\alpha\colon S^2\to G_2(C^6)$ is a harmonic map.*

Proof. Over a chart (U, Z) of S^2, A'_Z always denotes either $A_Z^{\mathrm{Im}[\partial\phi]}$ or $A_Z^{\mathrm{Im}[\partial\phi]^{\perp}}$ (See §1). Similarly we have the operators $A'_{\bar{Z}}$, D'_Z and $D'_{\bar{Z}}$. α is

harmonic if and only if $[D'_Z, A'_{\bar{Z}}]=0$ on $\underset{U}{\mathcal{C}}(\mathrm{Im}[\partial\phi])$
Since $\mathrm{Im}[\partial\phi]$ is a holomorphic subbundle of $E(\phi)^{\perp}$,

$$\text{for all } s\in\underset{U}{\mathcal{C}}(\mathrm{Im}[\partial\phi]),\ A'_{\bar{Z}}s = A_{\bar{Z}}s\,. \tag{4.5}$$

By Corollary 3.2,

$$\text{for all } s\in\underset{U}{\mathcal{C}}(E(\phi)),\ D'_Z s = D_Z s\,. \tag{4.6}$$

$A_{\bar{Z}}Js=JA_Z s$, and $A_Z s=0$ for any $s\in\underset{U}{\mathcal{C}}(\mathrm{Im}[\partial\phi)$ (Corollary 3.2). So

$$\text{for } s\in\underset{U}{\mathcal{C}}(\mathrm{Im}[\partial\phi]),\ A_{\bar{Z}}Js = 0\,. \tag{4.7}$$

By lemma 3.1, $\mathrm{Im}[\partial\phi]$ and $J\ \mathrm{Im}[\partial\phi]$ are mutually orthogonal. Hence

$$(\mathrm{Im}[\partial\phi])^{\perp} = E(\phi)\oplus J\ \mathrm{Im}[\partial\phi]\,. \tag{4.8}$$

For $s\in\underset{U}{\mathcal{C}}(\mathrm{Im}[\partial\phi])$, $D'_Z A'_{\bar{Z}}s=D_Z A_{\bar{Z}}s$ by 4.5, 4.6. Since $\phi^{\perp}$ is harmonic, by 2.8, $D_Z A_{\bar{Z}}s=A_{\bar{Z}}D_Z s$. By 4.8, and 4.7 and 4.5, $A_{\bar{Z}}D_Z s=A'_{\bar{Z}}D'_Z s$. Thus $[D'_Z, A'_{\bar{Z}}]s=0$. This completes the proof.

Recall, $M=R^{\perp}\cap\mathrm{Im}[\partial\phi]$.

Proposition 4.9. *M: $S^2\to CP^5$ is a harmonic map.*

Proof. Over a chart (U, Z) of S^2, let A''_Z denote either A^M_Z or $A^{M^{\perp}}_Z$ (See §1). Similarly define $A''_{\bar{Z}}$, D''_Z, $D''_{\bar{Z}}$.
By (2.5), M is a harmonic map if and only if $[D''_{\bar{Z}}, A''_Z]s=0$ for any $s\in\underset{U}{\mathcal{C}}(M)$. Let the operators A'_Z, $A'_{\bar{Z}}$, D'_Z, $D'_{\bar{Z}}$ be as in proposition (4.4). By (4.2), $D'_Z\underset{U}{\mathcal{C}}(M)\subset\underset{U}{\mathcal{C}}(M)$. Hence for $s\in\underset{U}{\mathcal{C}}(M)$,

$$A''_Z s = A'_Z s\,. \tag{4.10}$$

By (4.3),

$$A'_Z s=0 \text{ for } s\in\underset{U}{\mathcal{C}}(R)\,. \tag{4.11}$$

Now, for $s\in\underset{U}{\mathcal{C}}(M)$, by (4.10), $D''_{\bar{Z}}A''_Z s=D''_{\bar{Z}}(A'_Z s)$. By (4.3), $D''_{\bar{Z}}(A'_Z s)=D'_{\bar{Z}}(A'_Z s)$. By (2.8) and proposition (4.4), $D'_{\bar{Z}}A'_Z s=A'_Z D'_{\bar{Z}}s$. Using (4.10), $A'_Z(D'_{\bar{Z}}s)=A''_Z D''_{\bar{Z}}s$. So, $[D''_{\bar{Z}}, A''_Z]s=0$ as needed.

REMARK 4.10. M is not a holomorphic map. For, let $x\in S^2$ and (U, Z) be a chart with $x\in U$. Since rank $[\partial\phi]=2$, $A_{\bar{Z}}$: $E(\phi)^{\perp}_x\to E(\phi)_x$ is surjective except possibly at a finite number of points of S^2. Since $A_{\bar{Z}}s=0$ for $s\in\underset{U}{\mathcal{C}}(J\ \mathrm{Im}[\partial\phi])$, the remark follows. Recall the map α: $S^2\to G_2(C^6)$ defined by $\alpha(x)=\mathrm{Im}[\partial\phi]_x$. Define $\partial\alpha$: $T'S^2\to T'G_2(C^6)$ by $\partial\alpha=p\circ(d\alpha)^c$ where p: $TG_2(C^6)\otimes C\to T'G_2(C^6)$ is the projection. $\partial\alpha$ gives a map $[\partial\alpha]$: $T'S^2\otimes\mathrm{Im}[\partial\phi]\to(\mathrm{Im}[\partial\phi])^{\perp}$ (See §2C). By Proposition (4.1), rank $[\partial\alpha]=1$. The image of $[\partial\alpha]$

gives a unique line subbundle N of $(\mathrm{Im}[\partial\phi])^{\perp}$. By (4.8) and Corollary (3.2), $N\subset J\,\mathrm{Im}[\partial\phi]$.

Proposition 4.11. $JM=N$.

Proof. R, M are mutually orthogonal and $\mathrm{Im}[\partial\phi]=R\oplus M$. It is enough to show that JR and N are mutually orthogonal. Over a chart (U, Z) of S^2, $[\partial\alpha]\left(\frac{\partial}{\partial Z}\otimes s\right)=A'_Z s$ for $s\in \underset{U}{C}(\mathrm{Im}[\partial\phi])$. Then for $t\in \underset{U}{C}(R)$, $H(A'_Z s(x), J\,t(x))=-H(s(x), A'_{\bar{Z}}Jt(x))$. $A'_{\bar{Z}}(Jt(x))=JA'_Z(t(x))=0$ by (4.11). Hence $H(A'_Z s(x), Jt(x))=0$. This implies that JR and N are mutually orthogonal.

For an odd positive integer n, consider C^{n+1} with H, A, J as in §1.

Proposition 4.12. *Let h: $S^2\to CP^n$ be a holomorphic map such that for some integer k, $0\leqslant k\leqslant n$, $Jh_k=h_{k+1}$. Then h is a full holomorphic map if and only if $2k+1=n$.*

Proof. If $k=0$ or $k+1=n$, then $Jh_k=h_{k+1}$ implies that h_k is a holomorphic map and h_{k+1} is an antiholomorphic map. Then, $h_k\oplus h_{k+1}$ is an antiholomorphic bundle and being J-stable (i.e. $J(h_k\oplus h_{k+1})=h_k\oplus h_{k+1}$), it is a holomorphic subbundle of $\underline{C}^{n+1}$. Thus $h_k\oplus h_{k+1}$ is a trivial bundle of rank two. Then h is full if and only if $n=1=2k+1$.

Now assume that $1<k+1<n$. For $x\in S^2$ and a chart (U, Z) around x, define

$$h'_i(x) = \mathrm{span}\left\{\frac{\partial s}{\partial Z}(x) : s\in \underset{U}{C}(h_i)\right\} \text{ and}$$

$$h''_i(x) = \mathrm{span}\left\{\frac{\partial s}{\partial \bar{Z}}(x) : s\in \underset{U}{C}(h_i)\right\}.$$

For $i\neq n$, dimension of $h'_i(x)$ is equal to two except possibly at a finite number of points of S^2, hence gives a unique bundle h'_i. We have $h'_i=h_i\oplus h_{i+1}$. Similarly, for $i\neq 0$, $h''_i=h_i\oplus h_{i-1}$. Now for $i=k+1$, $h'_{k+1}=h_{k+1}\oplus h_{k+2}$ and since $h_{k+1}=Jh_k$, $h'_{k+1}=Jh''_k$. So, we get $h_{k+1}\oplus h_{k+2}=Jh_k\oplus Jh_{k-1}$. Since the bundles h_i are mutually orthogonal, we get $Jh_{k-1}=h_{k+2}$. Continuing this procedure, for i such that $k+1\leqslant k+1+i\leqslant n$,

$$Jh_{k-i} = h_{k+1+i}\,. \tag{4.13}$$

If $k-i=0$, then by 4.13, $Jh_0=h_{2k+1}$. Since $h_0=h$ is a holomorphic map, h_{2k+1} is an antiholomorphic map. Then $\underset{U}{C}(h_0\oplus\cdots\oplus h_{2k+1})$ is stable under $\frac{\partial}{\partial Z}$ and $\frac{\partial}{\partial \bar{Z}}$. Thus $h_0\oplus\cdots\oplus h_{2k+1}$ is a trivial subbundle of $\underline{C}^{n+1}$. Hence we conclude that h is a full map if and only if $n=2k+1$.

By proposition 4.9, M: $S^2\to CP^5$ is a harmonic map. By a result of Eells-

Wood (§2E), there exists a holomorphic map $h\colon S^2 \to CP^5$ such that $M=h_k$ for some integer k, $0 \leqslant k \leqslant 5$. Since M is not an antiholomorphic map $0 \leqslant k < 5$. $D'_{\bar{Z}}\underset{U}{C}(M) \subset \underset{U}{C}(M)$ by (4.2) and then by Corollary 3.2 $N=h_{k+1}$. Then by proposition 4.11, $Jh_k=h_{k+1}$. By proposition 4.12, $0 \leqslant k \leqslant 2$ and by remark 4.10, $k \neq 0$. Thus, $k=1$ or $k=2$ i.e. $M=h_1$ or $M=h_2$. In the following proposition we prove that $M=h_2$.

Proposition 4.14. $M=h_2$.

Proof. Suppose on the contrary that $M=h_1$. Using proposition 4.12 and 4.13, we see that $h_0 \oplus h_1 \oplus h_2 \oplus h_3$ is a J-stable trivial subbundle of $\underline{C}^6$. Put $W=(h_0 \oplus h_1 \oplus h_2 \oplus h_3)^{\perp}$. Rank $W=2$. We have, $R \subset (h_1 \oplus h_2 \oplus h_3)^{\perp}=H$. Then, $H=h_0 \oplus W$.
By (4.3), $D^H_{\bar{Z}}\underset{U}{C}(R) \subset \underset{U}{C}(R)$. Hence $R \subset h_0 \oplus T$ where T is a trivial line subbundle of W. Let $x \in S^2$ and let (U, Z) be a chart arround x. Put

$$S_x = \text{span}\,\{A_{\bar{Z}}\, s(x)\colon s \in \underset{U}{C}(R)\}$$

Dimension $S_x=1$ except possibly at a finite number of points of S. Hence we get a line subbundle S of $E(\phi)$. Since $h_0 \oplus T$ is holomorphic and $R \subset h_0 \oplus T$, we get $h_0 \oplus T=R \oplus S$. Now,

$$\frac{\partial}{\partial \bar{Z}}\underset{U}{C}(h_1) \subset \underset{U}{C}(h_0 \oplus h_1) \text{ and } h_0 \oplus h_1 \oplus T = h_1 \oplus R \oplus S.$$

It follows that $A_{\bar{Z}}\underset{U}{C}(h_1) \subset \underset{U}{C}(S)$. Thus $A_{\bar{Z}}\underset{U}{C}(h_1 \oplus R) \subset \underset{U}{C}(S)$. This contradicts the assumption that rank$[\partial\phi]=2$. We conclude that $M=h_2$.

Proof of theorem 1.5. Let h be a holomorphic map from S^2 to CP^5 such that $M=h_2$. Then by proposition 4.12, h is a full holomorphic map and hence M is a full harmonic map from S^2 to CP^5. Thus the map h is unique (§2E). Put $H=(h_2 \oplus h_3 \oplus h_4)^{\perp}$. Using (4.2) $R \subset H$ and $\frac{\partial}{\partial \bar{Z}}\underset{U}{C}(R) \subset \underset{U}{C}(R^{\perp}_{A})$. By (4.3), $D^H_{\bar{Z}}\underset{U}{C}(R) \subset \underset{U}{C}(R)$. Finally, $\phi(x)=(R \oplus JR \oplus h_2 \oplus h_3)^{\perp}_x$. The proof of theorem 1.5 is now complete.

EXAMPLE 4.15. Let $h\colon S^2 \to CP^5$ be a holomorphic map such that $Jh_k = h_{k+1}$ for some $0 \leqslant k \leqslant 2$. Define

$$H = \begin{cases} (h_k \oplus h_{k+1})^{\perp} & \text{if } k=0 \\ (h_k \oplus h_{k+1} \oplus h_{k+2})^{\perp} & \text{otherwise.} \end{cases}$$

Let $R \subset H$ be a line bundle such that $D^H_{\bar{Z}}\underset{U}{C}(R) \subset \underset{U}{C}(R)$ and $\frac{\partial}{\partial \bar{Z}}\underset{U}{C}(R) \subset \underset{U}{C}(R^{\perp}_{A})$ for all charts (U, Z) of S^2. Define $\phi\colon S^2 \to \boldsymbol{H}P^2$ by $\phi(x)=(R \oplus JR \oplus h_k \oplus h_{k+1})^{\perp}_x$

for $x\in S^2$. Then ϕ is a harmonic map.

We briefly sketch a proof of this. One can check that $A_Z(\underset{\cup}{C}(R\oplus h_k))=0$ and $D_{\bar{Z}}(\underset{\cup}{C}(R\oplus h_k)\subset\underset{\cup}{C}(R\oplus h_k)$. Then $[D_{\bar{Z}}, A_Z]=0$ on $\underset{\cup}{C}(R\oplus h_k)$. Using (2.7), $[D_{\bar{Z}}, A_Z]=0$ on $\underset{\cup}{C}(JR\oplus h_{k+1})$. By (2.5), $\phi^{\perp}$ is harmonic. So, ϕ is a harmonic map.

REMARKS 4.16. (1) In example 4.15, if $Jh_k=h_{k+1}$ for $k=0, 1$, then rank $[\partial\phi]\leqslant 1$. If $Jh_2=h_3$ and $R=h_1$, then ϕ is not an isotropic map, but rank $[\partial\phi]=1$.

(2) In example 4.15, consider the case when $Jh_2=h_3$. Then, $R\subset h_0\oplus h_1\subset H$ if and only if rank $[\partial\phi]=1$.

REMARK 4.17. Consider example 1.2 with $n=3$. There are nonisotropic harmonic maps ϕ with rank $[\partial\phi]=2$ (e.g. Take S to be a full holomorphic line subbundle of $\underline{C}^6$.) For any such map ϕ, R (Recall that R is given by the kernel of $\hat{D}^1$, proposition (4.11)) is an antiholomorphic subbundle of $\underline{C}^6$. In fact, $R=JS$.

In example 4.15, consider the case when $Jh_2=h_3$ and $R\neq h_5$, $R\not\subset h_0\oplus h_1$. Such a harmonic map ϕ is nonisotropic and rank $[\partial\phi]=2$. Im $[\partial\phi]=R\oplus h_2$ and $R=\ker\hat{D}^1$ is not an antiholomorphic line bundle. So examples 1.2, 1.3 do not cover all the harmonic maps from S^2 to $\boldsymbol{H}P^{n-1}$.

Following proposition describes holomorphic maps h: $S^2\rightarrow CP^5$ having the property that $Jh_k=h_{k+1}$ for some $0\leqslant k\leqslant 2$. Let $F\subset\underline{C}^6$ be the holomorphic linebundle corresponding to h (i.e. $F_x=h(x)$ for $x\in S^2$). Let $F_{(r)}$ be the r-th associated bundle of F (See [3] for the notation $F_{(r)}$).

Proposition 4.18. (a) *$Jh_0=h_1$ if and only if F is full in $S^2\times W_0$ where W_0 is a J-invariant subspace of C^6 of dimension 2.*
(b) *$Jh_1=h_2$ if and only if $(F, F_A^{\perp})$ is a ∂'-pair of $\underline{C}^6$ and F is full in $S^2\times W$, where W is a J-invariant subspace of C^6 of dimension 4.*
(c) *$Jh_2=h_3$ if and only if F is full in $\underline{C}^6$ and $(F_{(1)}, (F_{(1)})_A^{\perp})$ is a ∂'-pair of $\underline{C}^6$.*

Proof. We prove (c).
$\Rightarrow$. Suppose $Jh_2=h_3$. By proposition 4.12, F is full in $\underline{C}^6$. By (4.13), $(F_{(2)})_x$ is an isotropic space for any $x\in S^2$. But, $F_{(2)}\subseteq(F_{(2)})_A^{\perp}\Leftrightarrow(F_{(1)}, (F_{(1)})_A^{\perp})$ is a ∂'-pair of $\underline{C}^6$.
$\Leftarrow$. Let $s_i\in\underset{\cup}{C}(h_i)$ for $0\leqslant i\leqslant 3$. Since $(F_{(2)})_x$ is an isotropic space, $A(\frac{\partial s_2}{\partial Z}(x), s_i(x))=-A(s_2(x), \frac{\partial s_i(x)}{\partial Z})=0$ for $0\leqslant i\leqslant 1$. This implies that $h_3\subset(F_{(1)})_A^{\perp}$.
Hence $(F_{(1)})_A^{\perp}=F_{(3)}$. For $0\leqslant i\leqslant 1$,

$$A(Js_2(x), s_i(x)) = H(\overline{s_2(x)}, s_i(x)) = 0\,.$$

This implies that $Jh_2 \subset (F_{(1)})_A^{\perp} = F_{(3)}$. Also for $0 \leqslant i \leqslant 2$,

$$H(s_i(x), Js_2(x)) = A(s_i(x), s_2(x)) = 0$$

since $(F_{(2)})_x$ is an isotropic space. We conclude that

$$Jh_2 = F_{(2)}^{\perp} \cap F_{(3)} = h_3\,.$$

The proof for (b) is similar to that of (c). (a) is obvious.

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