Title	Weakly compact linear operators on function spaces
Author(s)	Wada, Junzo
Citation	Osaka Mathematical Journal. 1961, 13(1), p. 169-183
Version Type	VoR
URL	https://doi.org/10.18910/6576
rights	
Note	

Wada, Junzo
Osaka Math. J.
13 (1961), 169-183.

Weakly Compact Linear Operators on Function Spaces

By Junzo WADA

R. G. Bartle [3], Grothendieck [14] and Bartle, Dunford and J. Schwartz [4] have considered various problems on weakly compact (or compact) linear operators on function spaces (especially, Banach spaces of continuous functions). Our main purpose is to establish some extensions of these results. After some preliminaries in §1 we give in §§2 and 3 representations of weakly compact (or compact) linear operators on the locally convex topological spaces of all continuous functions on general topological spaces, obtaining as results some extensions of theorems of Grothendieck (cf. Theorems 2 and 4). As an application of Theorem 1 in §2 we consider in §4 the simultaneous extension of continuous functions. Michael [17] has proved namely the following theorem: if X is a metric space and if F is a closed subset in X, then there is a simultaneous extension of $C_{\mathfrak{S}}(F)$ into $C_{\mathfrak{X}}(X)$ (cf. §4). We show here that this theorem remains true if X and F are taken to be more general topological spaces and if $C_{\mathfrak{S}}(F)$ is replaced by a relative compact subset (cf. Theorem 5). Finally, we deal in §5 with the spaces of summable functions, giving representations of weakly compact linear operators of the space of summable functions on a Kakutani space (cf. §1) into a Banach space (cf. Theorem 6).

§1. Preliminaries

Let E, F be locally convex topological linear spaces. A continuous linear operator T of E into F is said to be *weakly compact* (or *compact*) if T maps a neighborhood of 0 in E into a relative weakly compact (or relative compact) subset in F. A locally convex topological linear space E is said to be *barrelled* if any closed, symmetric[1] convex and absorbing subset in E is a neighborhood of 0 in E. Let E be a locally convex topological linear space and let $\mathcal{E}$ be an equicontinuous symmetric convex w^*-compact subset in the dual space E' of E. Then $E'_{\mathcal{E}}$ denotes a Banach space whose unit sphere is $\mathcal{E}$. Let X be a topological space and let $\mathfrak{S}$ be

1) Let E be a locally convex topological space. Then a subset A in E is said to be symmetric if $\lambda x \in A$ for any $x \in A$ and for any real number λ with $|\lambda| \leqq 1$.

a set of compact sets in X. "$\bigcup \mathfrak{S}=X$" denotes that the sum of all sets in $\mathfrak{S}$ is X. By $C_{\mathfrak{S}}(X)$ we denote the locally convex topological linear space of all real-valued continuous functions on X with the topology of uniform convergence of sets in $\mathfrak{S}$. Also, by $C_u(X)$ we denote the Banach space of all bounded real-valued continuous functions on X with the norm $\|f\|=\sup_{x\in X}|f(x)|$. A subset A in $C_{\mathfrak{S}}(X)$ is said to be *equicontinuous* if for any $\varepsilon>0$ and for any point x_0 in X there is a neighborhood $U(x_0)$ such that $|f(x)-f(x_0)|<\varepsilon$ for any $x\in U(x_0)$ and for any $f\in A$. Let E be a locally convex topological space and let T be a linear operator (not necessarily continuous) of E into $C_{\mathfrak{S}}(X)$. Then T is said to be *equicontinuous* if there is a neighborhood V of 0 in E such that $T(V)$ is contained in an equicontinuous set in $C_{\mathfrak{S}}(X)$.

Let X be a topological space. Then X is said to be a *k-space* if whenever $U\cap K$ is an open set in K for a subset U in X and for any compact subset K in X, U is an open subset in X (cf. Kelley [16]). Also X is said to be a *k_0-space* if whenever $U\cap K$ is a neighborhood of x_0 in K for a subset $U(\ni x_0)$ and for any compact subset $K(\ni x_0)$, U is a neighborhood of x_0 in X. A neighborhood need not be here an open set (cf. [19]). A k_0-space is a k-space and any completely regular space satisfying the 1st axiom of countability or any locally compact Hausdorff space is always a k_0-space (and therefore a k-space). Let X be a topological space. Then we consider a topological space X satisfying the following condition: if a real-valued function f on X is continuous on any compact subspace K in X, then f is continuous on X. Let $\mathfrak{S}$ be the set of all compact subsets in X. Then Warner [20] has proved that $C_{\mathfrak{S}}(X)$ is complete if and only if X satisfies the above condition. Therefore if X is a k_0-space (or a k-space) and if $\mathfrak{S}$ is the set of all compact subsets in X, then $C_{\mathfrak{S}}(X)$ is complete.

We first prove the following generalized Ascoli's theorem.

Lemma 1. *Let X be a k_0-space and let $\mathfrak{S}$ be the set of all compact subsets in X. Then a set A in $C_{\mathfrak{S}}(X)$ is relative compact if and only if A is an equicontinuous set in $C_{\mathfrak{S}}(X)$ and $A(x)=\{f(x)\mid f\in A\}$ is bounded for any $x\in X$.*

Proof. If A is an equicontinuous set in $C_{\mathfrak{S}}(X)$ and $A(x)$ is bounded for any $x\in X$, then A is relative compact (cf. [6] § 4). Therefore we have only to prove that if A is compact, then it is equicontinuous. By Bourbaki (cf. [6] § 4) we have only to prove that for any $f\in A$ and for any x in X, $(f, x)\rightarrow f(x)$ is continuous. Since X is a k_0-space, $C_{\mathfrak{S}}(X)$ is complete. By Bourbaki ([7] § 4. 1.) the closed convex envelope $\Gamma(A)$ of A is compact, so we can assume that A is convex. $B=A-A=\{f-g\mid f, g\in A\}$

is convex and compact. For any x_0 in X and any $\varepsilon>0$ we put $U=\{x : |f(x)|<2/3\,\varepsilon \text{ if } f\in B\cap W(x_0, 1/3\,\varepsilon)\}$[2]. Then U contains x_0. We here assert that U is a neighborhood of x_0 in X. Suppose that U is not a neighborhood of x_0 in X. Then there is a compact subset $K\ni x_0$ such that $U\cap K$ is not a neighborhood of x_0 in K, since X is a k_0-space. Therefore there is a directed set $\{x_j\}$ in K such that x_j converges to x_0 and $x_j\notin U$. By the definition of U, there is a directed set f_j in B such that $|f_j(x_j)|\geqq 2/3\,\varepsilon$ and $|f_j(x_0)|\leqq 1/3\,\varepsilon$. Since B is compact, the directed set $\{f_j\}$ has a cluster point f_0 in B. For any $\eta>0$, there is a j_0 such that $|f_0(x_j)-f_0(x_0)|<\eta/2$ for any $j\geqq j_0$. Since f_0 is a cluster point of $\{f_j\}$, $|(f_{j_1}-f_0)(K)|<\eta/2$ for a $j_1\geqq j_0$. Therefore $|f_0(x_0)-f_{j_1}(x_{j_1})|<\eta$. Since η is arbitrarily small and $|f_j(x_j)|\geqslant 2/3\,\varepsilon$ for any j, $|f_0(x_0)|\geqslant 2/3\,\varepsilon$. This is a contradiction since $|f_0(x_0)|\leqslant 1/3\,\varepsilon$. Therefore U is a neighborhood of x_0 in X. By the definition of U, we easily see that $B\cap W(x_0, 1/3\,\varepsilon)\subset W(U, 2/3\,\varepsilon)$. We shall here prove that $(f, x)\to f(x)$ is continuous on $A\times X$. If $A\ni f, f_0$ and $f-f_0\in W(x_0, 1/3\,\varepsilon)$, then $f-f_0\in B\cap W(x_0, 1/3\,\varepsilon)\subset W(U, 2/3\,\varepsilon)$, so $|f(x)-f_0(x_0)|\leqq|f(x)-f_0(x)|+|f_0(x)-f_0(x_0)|<\varepsilon$ if $x\in V(x_0)\cap U$ ($V(x_0)=\{x : |f_0(x)-f_0(x_0)|<1/3\,\varepsilon\}$). Thus the lemma is proved.

Let E be an arbitrary set and let ν be a positive measure on a σ-algebra on E such that E is measurable (but E has not necessarily a finite measure). Let $L^1(E, \nu)$ be the space of all ν-summable functions on E. Then there are a compact space $\bar{X}$ such that $L^\infty(E, \nu)$ is equivalent to $C(\bar{X})$ as Banach algebra. Moreover, there is a positive measure μ on a dense open set X in $\bar{X}$ such that $L^1(E, \nu)$ is isometric to $L^1(X, \mu)$ and $L^\infty(X. \mu)$ is identical with $C(\bar{X})$ (cf. [10], [11]). Thus the space X is a Kakutani space according to Dieudonné (cf. [10]). $\bar{X}$ is the Čech compactification βX of X and is a stonian space (cf. [11]). A completely regular Hausdorff space is said to be extremally disconnected (cf. [15]) if the closure $\bar{U}$ of any open set U is also open. An extremally disconnected compact Hausdorff space is stonian.

Let J be a set of indices. Then we denote by $m(J)$ the space of all bounded real-valued functions on J with $\|x\|=\sup_{j\in J}|x(j)|$, and denote by $c_0(J)$ the subspace of those x in $m(J)$ for which the set of j with $|x(j)|>\varepsilon$ is finite for each $\varepsilon>0$; that is, $c_0(J)$ is the set of functions vanishing at infinity on the discrete space J.

§ 2. The spaces of continuous functions[3].

We here deal with weakly compact (or compact) linear operators

2) By $W(K, \varepsilon)$ we denote the set of functions f in $C_{\mathfrak{S}}(X)$ with $|f(K)|\leqq\varepsilon$.

3) The most part of §§ 2 and 3 was announced in [19].

on the spaces of all continuous functions on general topological spaces. Theorem 2 below is an extension of a result of Grothendieck [14]. Bartle [3] gave general forms of weakly compact (or compact) linear operators of a Banach space into a Banach space $C_u(X)$. We first extend this theorem to the case of locally convex topological linear spaces.

Theorem 1. (i) *Let E be a barrelled locally convex linear space. Let Y be a completely regular Hausdorff space and let $\mathfrak{S}$ be a set of compact sets in Y with $\bigcup \mathfrak{S} = Y$. Then a linear operator T of E into $C_{\mathfrak{S}}(Y)$ is continuous if and only if there is a continuous mapping τ of Y into E' with respect to the topology $\sigma(E', E)$ such that $(Te)y = \langle \tau y, e\rangle$ for any $e \in E$ and for any $y \in Y$.*

(ii) *Let E be a locally convex topological linear space. Let Y be a completely regular Hausdorff space and let $\mathfrak{S}$ be a set of compact subsets in Y with $\bigcup \mathfrak{S} = Y$. Then a continuous linear operator T of E into $C_{\mathfrak{S}}(Y)$ is weakly compact if and only if there is a continuous mapping τ of Y into $E'_{\mathcal{E}}$ (Cf. § 1) with respect to the topology $\sigma(E'_{\mathcal{E}}, E''_{\mathcal{E}})$ for a symmetric convex w^*-closed equicontinuous set $\mathcal{E}$ in E' and $(Te)y = \langle \tau y, e\rangle$ for $e \in E$ and $y \in Y$.*

(iii) *Let E be a locally convex topological linear space. Let Y be a k_0-space which is completely regular, Hausdorff and let $\mathfrak{S}$ be the set of all compact subsets in Y. Then a continuous linear operator T of E into $C_{\mathfrak{S}}(Y)$ is compact if and only if there is a continuous mapping τ of Y into $E'_{\mathcal{E}}$ for a symmetric convex w^*-closed equicontinuous set $\mathcal{E}$ in E', and $(Te)y = \langle \tau y, e\rangle$ for $e \in E$ and $y \in Y$.*

Proof. (i) If T is a continuous linear operator of E into $C_{\mathfrak{S}}(Y)$, its transposition T' is a continuous linear operators of $C_{\mathfrak{S}}(Y)'$ (with the topology $\sigma(C_{\mathfrak{S}}(Y)', C_{\mathfrak{S}}(Y))$) into E' (with the topology $\sigma(E', E)$). Since Y may be regarded as a subspace in $C_{\mathfrak{S}}(Y)'$ (with the topology $\sigma(C_{\mathfrak{S}}(Y)', C_{\mathfrak{S}}(Y))$), if we put $\tau y = T'\mu_y$ [4], then τ is a continuous mapping of Y into E' (with the topology $\sigma(E', E)$) and $(Te)y = \langle \tau y, e\rangle$ for $e \in E$ and $y \in Y$. Conversely, let τ be a continuous mapping of Y into E' (with the topology $\sigma(E', E)$) and let $(Te)y = \langle \tau y, e\rangle$ for $e \in E$ and $y \in Y$. Let K be a compact subset in Y. Then $\tau(K)$ is $\sigma(E', E)$-compact. Since E is barrelled, there is a neighborhood U of 0 in E such that $\tau(K) \subset U^0$ [5] (cf. [7]). For any $e \in U$ and for any $y \in K$, $|(Te)y| = |\langle \tau y, e\rangle| \leqq 1$. Therefore $T(U) \subset W(K, 1)$, so T is continuous.

---

4) μ_x denotes the Dirac measure, i. e. $\mu_x(f) = f(x)$ for auy continuous function f.

5) Let A be a subset in a locally convex topological space E. A^0 denotes the set of elements f in E' with $|f(A)| \leqq 1$.

(ii) We first assume that E is a normed linear space. Let T be a weakly compact linear operator of E into $C_{\mathfrak{S}}(Y)$. Then by Grothendieck ([14]. Lemma 1), T' is a continuous linear operator of $C_{\mathfrak{S}}(Y)'$ (with the topology $\sigma(C_{\mathfrak{S}}(Y)', C_{\mathfrak{S}}(Y))$) into E' (with the topology $\sigma(E', E''))$. If we put $\tau y = T'\mu_y$, τ is the required mapping. We prove the converse: if we put $(Te)y = \langle \tau y, e\rangle$, we have only to prove that T' is a continuous linear operator of $C_{\mathfrak{S}}(Y)'$ (with topology $\sigma(C_{\mathfrak{S}}(Y)', C_{\mathfrak{S}}(Y))$) into E' (with the topology $\sigma(E', E'')$) (cf. [14]). For any $\mu \in C_{\mathfrak{S}}(Y)'$, there are a $K \in \mathfrak{S}$ and a real number $\delta > 0$ such that $\mu \in W(K, \delta)^0$. We define $\mu_K(g)$ as follows: $\mu_K(g) = \mu(\tilde{g})$ if $\tilde{g}|K$[6] $= g$ for $\tilde{g} \in C_{\mathfrak{S}}(Y)$ and $g \in C(K)$. We see that $\mu_K(g)$ can be defined as a Borel measure on K. We first prove that $T'\mu = \int \tau_K(y) d\mu_K(y)$, where τ_K denotes the restriction of τ on K. For any $e'' \in E''$ we have $\langle\langle \tau(y), e''\rangle, \mu\rangle = \int \langle \tau_K(y), e''\rangle d\mu_K(y)$. Now, E' (with the topology $\sigma(E', E'')$) satisfies the condition (EC)[7]. For, by Krein's theorem the closed convex envelope of a $\sigma(E', E'')$-compact subset in E' is also $\sigma(E', E'')$-compact. Therefore the vector valued integral $\int \tau_K(y) d\mu_K(y)$ exists and belongs to E'. For any $e \in E$

$$\begin{aligned} \langle e, T'\mu\rangle &= \langle Te, \mu\rangle = \langle\langle \tau(y), e\rangle, \mu\rangle \\ &= \int \langle \tau_K(y), e\rangle d\mu_K(y) = \langle e, \int \tau_K(y) d\mu_K(y)\rangle, \end{aligned}$$

so $T'\mu = \int \tau_K(y) d\mu_K(y)$. For any $e'' \in E''$

$$\begin{aligned} \langle T'\mu, e''\rangle &= \langle \int \tau_K(y) d\mu_K(y), e''\rangle \\ &= \int \langle \tau_K(y), e''\rangle d\mu_K(y) = \langle\langle \tau(y), e''\rangle, \mu\rangle. \end{aligned}$$

We see here that $\alpha(y) = \langle \tau(y), e''\rangle$ is a continuous function on Y. If a directed set μ_j converges to 0 with respect to the topology $\sigma(C(Y)', C(Y))$, then $\langle\langle \tau(y), e''\rangle, \mu_j\rangle, \to 0$, so $\langle T'\mu_j, e''\rangle \to 0$. Therefore, if E is a normed linear space, (ii) is proved.

Let E be any locally convex topological linear space and let T be a weakly compact linear operator of E into $C_{\mathfrak{S}}(Y)$. Then there is a convex, symmetric and closed neighborhood V of 0 in E such that $T(V)$ is contained in a weakly compact subset in $C_{\mathfrak{S}}(Y)$. Therefore $T = Sj$ and j is

6) $f|K$ denotes the restriction of f on K.

7) A locally convex topological space E is said to satisfy the condition (EC) if the convex closed envelop of any compact subset in E is also compact (cf. [8] § 4).

the canonical linear operator of E onto E_V [8] and S is a weakly compact linear operator of E_V into $C_{\mathfrak{S}}(Y)$. Since E_V is a normed linear space, there is a continuous mapping τ_1 of Y into E_V' with respect to the topology $\sigma(E_V', E_V'')$ such that $(S\dot{e})y=\langle\tau_1 y, \dot{e}\rangle$ for any $y\in Y$ and for any $\dot{e}\in E_V$. If we put $\mathcal{E}=V^0$, $E_V'=E'_{\mathcal{E}}$, then there is a continuous mapping τ of Y into $E'_{\mathcal{E}}$ with respect to the topology $\sigma(E'_{\mathcal{E}}, E''_{\mathcal{E}})$ such that $(Te)y=\langle\tau y, e\rangle$ for any $y\in Y$ and for any $e\in E$. The converse may be proved similarly.

(iii) We can easily prove the following lemma.

Lemma 2. *A continuous linear operator T of E into $C_{\mathfrak{S}}(Y)$ is equicontinuous if and only there are a mapping τ of Y into E' and a symmetric convex w^*-closed equicontinuous set $\mathcal{E}$ in E' such that for any $\lambda>0$ and for any y_0 in Y $\tau(y)-\tau(y_0)\in\lambda\mathcal{E}$ for any y in some neighborhood $U(y_0)$ of y_0, and $(Te)y=\langle\tau y, e\rangle$ for $e\in E$ and $y\in Y$.*

Now let T be a compact linear operator of E into $C_{\mathfrak{S}}(Y)$. Then there is a symmetric convex neighborhood V of 0 in E such that $T(V)$ is contained in a compact subset A in $C_{\mathfrak{S}}(Y)$. If we put $\tau(y)=T'\mu_y$, $|\langle\tau(y), x\rangle|=|\langle T'\mu_y, x\rangle|=|\langle\mu_y, Tx\rangle|\leqq\sup_{g\in A}|g(y)|$ for any $x\in V$. By Lemma 1, $|\langle\tau(y), x\rangle|\leqq\sup_{g\in A}|g(y)|<+\infty$ for any $x\in V$. Therefore $\tau(y)\in\sup_{g\in A}|g(y)|\cdot V^0=\sup_{g\in A}|g(y)|\mathcal{E}$ ($\mathcal{E}$ is an equicontinuous set in E'). By Lemma 2 τ is a continuous mapping of Y into $E'_{\mathcal{E}}$. The converse is clear by Lemmas 1 and 2.

From Theorem 1 we have the following

Theorem 2. *Let X be a stonian space (cf. § 1). Let Y be a completely regular Hausdorff space satisfying the 1st axiom of countability and let $\mathfrak{S}$ be a set of compact set in Y with $\bigcup\mathfrak{S}=Y$. Then any continuous linear operator of $C(X)$ into $C_{\mathfrak{S}}(T)$ is weakly compact.*

Proof. Let T be a continuous linear operator of $C(X)$ into $C_{\mathfrak{S}}(Y)$. By Theorem 1, (i) there is a continuous mapping of Y into $C(X)'$ with respect to the topology $\sigma(C(X)', C(X))$ and $(Tf)y=\langle\tau y, f\rangle$. If a sequence $\{y_n\}$ converges to y_0 in Y, then $\tau(y_n)$ converges to $\tau(y_0)$ on the topology $\sigma(C(X)', C(X))$. By Grothendieck [14], $\tau(y_n)$ converges to $\tau(y_0)$ on the topology $\sigma(C(X)', C(X)'')$, so T is weakly compact (Theorem 1, (ii)).

The following corollary is proved by Grothendieck [14].

Corollary 1. *Let X be a stonian space and let E be a separable*[9] *com-*

---

8) We put $N=\bigcap_{n=1}^{\infty}n^{-1}V$. Then E_V denotes the quotient space E/N with the norm induced by the semi-norm $\|x\|_V=\inf_{x\in\lambda V}|\lambda|$ on E.

9) A topological space is said to be separable if it has a countable dense subset.

plete Hausdorff locally convex linear space. Then any continuous linear operator of $C(X)$ into E is weakly compact.

Proof. Any element e in E may be regarded as a continuous functions f_e on E' with respect to the topolgy $\sigma(E', E)$, i.e. $f_e(e')=\langle e, e'\rangle$ for any $e' \in E'$. If $\mathfrak{S}$ is the set of all w^*-compact equicontinuous subset in E', then E can be embedded in $C_{\mathfrak{S}}(E')$. Since E is separable, E' (with the topology $\sigma(E', E)$) is metrizable, so by Theorem 2 the Corollary is proved.

Corollary 2. *Let X be an extremally disconnected space (cf. § 1) and let $\mathfrak{S}$ be a non-empty set of non-void compact subsets in X. Let Y be a compact Hausdorff space satisfying the 1st axiom of countability. Then any continuous linear operator of $C_{\mathfrak{S}}(X)$ into $C(Y)$ is weakly compact.*

Proof. Let T be a continuous linear operator of $C_{\mathfrak{S}}(X)$ into $C(Y)$. Then there are a $K \in \mathfrak{S}$ and a positive number ε such that $T(W(K, \varepsilon)) \subset U$, where U denotes the unit sphere in $C(Y)$. Therefore $T=Sj$ and j is the canonical linear operator of $C_{\mathfrak{S}}(X)$ onto $C(K)$, i.e. for any $f \in C_{\mathfrak{S}}(X)$ jf is the restriction of f on K, and S is a continuous linear operator of $C(K)$ into $C(Y)$. By the similar proof as Hewitt ([15]. p. 66), we can prove that if X is extremally discounnected, then the Čech compactification βX is stonian. Therefore K is a compact subset in βX. If j_1 is the canonical linear operator of $C(\beta X)$ onto $C(K)$, i.e. for any $f \in C(\beta X)$ $j_1 f$ is the restriction of f on K, then Sj_1 is a continuous linear operator of $C(\beta X)$ into $C(Y)$. Therefore Sj_1 is weakly compact (Theorem 2). This shows that S is weakly compact and so does T.

REMARK. (i) If a completely regular space X satisfies the 1st axiom of countability, then $C_{\mathfrak{S}}(X)$ is not, in general separable. There is a compact Hansdorff space which satisfies the 1st axiom of countability but is not metrizable (cf. [1]).

(ii) In Corollary 2 of Theorem 2, the compactness of Y is necessary : let X be the discrete space of all integers and let $\mathfrak{S}$ be the set of all compact subsets in X (that is, the set of all finite subsets). Then the identical mapping of $C_{\mathfrak{S}}(X)$ onto $C_{\mathfrak{S}}(X)$ is not weakly compact. But X is extremally disconnected and satisfies the 1st axiom of countability.

§ 3. Kernel functions.

Bartle, Dunford and J. Schwartz [4] gave the representations of weakly compact (or compact) linear operators of the space of continuous functions on a compact space X into the space of continuous functions on another compact space Y. They are represented by kernels. We here deal with the case that X, Y are general topological spaces.

Theorem 4 below is an extension of a result of Grothendieck.

We first extend a theorem of Bartle, Dounford and J. Schwartz to the case of locally convex topological linear spaces.

Theorem 3. (a) *Let X, Y be completely regular Hausdorff spaces and let Y be a σ-compact k-space. Let $\mathfrak{S}$ be the set of all compact subsets in X and let $\mathfrak{T}$ be the set of all compact subsets in Y. Then a continuous linear operator T of $C_{\mathfrak{S}}(X)$ into $C_{\mathfrak{T}}(Y)$ is weakly compact if and only if there are a kernel function $k(x, y)$ on $K\times Y$ (for some $K\in\mathfrak{S}$) and a non-negative Borel measure ν on K such that*

$$(*)\ (Tf)y = \int (f|K)(x)k(x, y)\nu(dx)$$

and k satisfies the conditions:

(i) *for any $y\in Y$, $k(x, y)\in L^1(K, \nu)$,*

(ii) *for any Borel set E in K, $\int_E k(x, y)\nu(dx)$ is continuous function on Y,*

(iii) *for any $H\in\mathfrak{T}$, $\sup_{y\in H}\int|k(x, y)|\nu(dx)<+\infty$.*

(b) *Let X, Y be completely regular Hausdoff spaces and let Y be a σ-compact k_0-space. Let $\mathfrak{S}$ be the set of all compact subsets in X and let $\mathfrak{T}$ be the set of all compact subset in Y. Then a continuous linear operator T of $C_{\mathfrak{S}}(X)$ into $C_{\mathfrak{T}}(Y)$ is compact if and only if there is a kernel function $k(x, y)$ on $K\times Y$ (for some $K\in\mathfrak{S}$) and a non-negative Borel measure ν on K such that the equation* (*) *is satisfied and k satifies the condition* (i) *and*

(iv) *if $y_\lambda\to y$ in Y, then*

$$\lim_{y_\lambda\to y_0}\int|k(x, y)-k(x, y_0)|\nu(dx) = 0 .$$

Proof. a) Let T be a weakly compact linear operator of $C_{\mathfrak{S}}(X)$ into $C_{\mathfrak{T}}(Y)$. Then there is a $K\in\mathfrak{S}$ such that $T(W(K, 1))\subset$ some weakly compact subset B in $C_{\mathfrak{T}}(Y)$. Since B is weakly compact, it is pointwise bounded, i.e. $\{g(y)|g\in B\}$ is bounded for any y in Y. If we put $A=\bigcap_{n=1}^{\infty} n^{-1}W(K, 1)=\{f: f(K)=0\}$, then $T(A)\subset\bigcap_{n=1}^{\infty} n^{-1}B=\{0\}$. Therefore $T=Sj$, and j is the canonical linear mapping of $C_{\mathfrak{S}}(X)$ onto $C(K)$ (i.e. for $f\in C_{\mathfrak{S}}(X)$ jf is the restriction of f on K) and S is a weakly compact linear operator of $C(K)$ into $C_{\mathfrak{T}}(Y)$. By Theorem 1, there is a continuous mapping τ of Y into $C(K)'$ with the topology $\sigma(C(K)', C(K)'')$ such that $(Sg)y=\langle\tau y, g\rangle$ for any $g\in C(K)$ and for any $y\in Y$. For any $f\in C_{\mathfrak{S}}(X)$,

$(Tf)y=(Sjf)y=\langle\tau y, jf\rangle=\int(f|K)\tau y(dx)$, since τy is a Borel measure on K. Since Y is a σ-compact space, Y can be represented as a sum of a sequence $\{Y_n\}$ of compact subsets. Since τY_n is $\sigma(C(K)', C(K)'')$-compact, by Bartle, Dunford and J. Schwartz ([4] Theorem 1.4), there is a positive Borel measure ν_n on K such that $\nu_n(E)=0$ implies $|\tau y|(E)=0$ for any $y\in Y_n$. If we put $\nu=\sum_{n=1}^{\infty}1/2^n\,\nu_n/||\nu_n||$, then we have that $\nu(E)=0$ implies $|\tau y|(E)=0$ for any $y\in Y$. By Radon-Nikodym theorem, $\tau y=k(x, y)\nu$ and $k(x, y)\in L^1(K, \nu)$ for any $y\in Y$, so $(Tf)y=\int(f|K)\tau y(dx)=\int(f|K)k(x, y)\nu(dx)$. Let E be a Borel set in K. Then we can regard the characteristic function φ_E of E as an element in $C''(K)$. Since for a directed set $\{f_j\}\subset C(K)$ f_j converges to φ_E with the topology $\sigma(C''(K), C'(K))$, we have $(S''\varphi_E)y=\int_E k(x, y)\nu(dx)$. For, $(Sf_j)y=\int f_j(x)k(x, y)\nu(dx)$ converges to $\int_E k(x, y)\nu(dx)$ and $(Sf_j)y=\langle Sf_j, \mu_y\rangle=\langle f_j, S'\mu_y\rangle\rightarrow\langle\varphi_E, S'\mu_y\rangle=\langle S''\varphi_E, \mu_y\rangle=(S''\varphi_E)y$, so $(S''\varphi_E)y=\int_E k(x, y)\nu(dx)$. Since $S''\varphi_E\in C_{\mathfrak{T}}(Y)$, $\int_E k(x, y)\nu(dx)$ is a continuous function on Y for any Borel set E. Next, we put $M=\{f\nu\,|\,f\in L^1(K, \nu)\}$, then $C'(K)\supset M$. Any $f\in L^\infty(K, \nu)$ may be regarded as a continuous linear functional on M, i.e. for any $\mu=g\nu\in M$, we define $\langle f, \mu\rangle=\int f(x)g(x)d\nu(x)$. Therefore f may be regarded as an element in $C''(K)$. For any $y\in Y$, $(S''f)y=\langle S''f, \mu_y\rangle=\langle f, S'\mu_y\rangle=\langle f, \tau y\rangle=\langle f, k(x, y)\nu\rangle=\int f(x)k(x, y)\nu(dx)$. Now we put $f(x)=\operatorname{sgn}k(x, y)$. Then $(S''f)y=\int|k(x, y)|\nu(dx)$. Since $S''f\in C_{\mathfrak{T}}(Y)$, we have that $\sup_{y\in H}\int|k(x, y)|\nu(dx)<+\infty$ for any $H\in\mathfrak{T}$. Conversely, if k satisfies the conditions (i), (ii) and (iii), and $(Tf)y=\int(f|K)k(x, y)\nu(dx)$, then $T=Sj$ and j is the canonical linear mapping of $C_{\mathfrak{S}}(X)$ onto $C(K)$ and S is a continuous linear operator of $C(K)$ into $C_{\mathfrak{T}}(Y)$. Let $B(K)$ be the space of bounded Baire functions of the 1st class, i.e. bounded functions which are limits of sequences of continuous functions. Then for any $g\in B(K)$, $(S''g)y=\int g(x)k(x, y)\nu(dx)$. For, for any $\varepsilon>0$ there is a finite set of Borel sets $\{E_i\}$ in K such that $||g-\sum\alpha_j\varphi_{Ei}||_\infty\leqq\varepsilon$. For any $y\in H$,

$$\left|\int g(x)k(x, y)\nu(dx)-\int(\textstyle\sum\alpha_i\varphi_{Ei})(x)k(x, y)\nu(dx)\right|$$
$$\leqq||g-\textstyle\sum\alpha_i\varphi_{Ei}||_\infty\cdot\displaystyle\int_H|k(x, y)|\nu(dx)\leqq\varepsilon\int_H|k(x, y)|\nu(dx)\,.$$

Therefore $\sum \alpha_i \int_{E_i} k(x, y)\nu(dx)$ converges to $\int g(x)k(x, y)\nu(dx)$ uniformly in H, so $\int g(x)k(x, y)\nu(dx)$ is continuous on H. Since Y is a k-space, $\int g(x)k(x, y)\nu(dx)$ is also continuous on Y (cf. § 1). Therefore for any $\xi \in C(K)''$ $S''\xi \in C_{\mathfrak{T}}(Y)$ since $C_{\mathfrak{T}}(Y)$ is complete (cf. [14] § 3). This shows that S is weakly compact (cf. [14] Lemma 1).

(b) We have

$$\begin{aligned}\|\tau y_\lambda - \tau y_0\| &= \|k(\cdot, y_\lambda) - k(\cdot, y_0)\| \\ &= \int |k(x, y_\lambda) - k(x, y_0)|\nu(dx)\,.\end{aligned}$$

By the proof of Theorem 3 (a) and Theorem 1, (b) is then obvious.

From this theorem we obtain the following

Theorem 4. *Let E be a separable metrizable locally convex linear space and let J be a set of indices. Then any weakly compact linear operator of $c_0(J)$ (cf. § 1) into E is compact.*

Proof. If X is infinite we put $X = J \cup (p_0)$, where p_0 is an abstract point. Let X be the compact space whose all points is isolated except p_0, i.e. a neighborhood of p_0 in X is a subset of the form $X-(a_1, \cdots, a_n)$, where $\{a_1, \cdots, a_n\}$ is a finite set in J. Then we may consider $C(X)$ insted of $c_0(J)$. Next, yince E is a separable metrizable locally convex linear space, its completion $\hat{E}$ can be imbedded into $C_{\mathfrak{T}}(Y)$ for a σ-compact metrizable space Y ($\mathfrak{T}$ is the set of all compact subsets in Y[10]). Now, let T be a weakly compact linear operator of $c_0(J)$ into E, then we can assume that T is a weakly compact linear operator of $C(X)$ into $C_{\mathfrak{T}}(Y)$. By Theorem 3 (a), we can find a kernel $k(x, y)$ and a positive measure ν on X. By a sequence $\{x_n\}$ in X we denote elements with $\nu((x)) \neq 0$. We put $N = \{x_n\}$. For any subset $B = \{x_{jn}\} \subset N$, we set

$$\mu_i(B) = \sum_n [k(x_{jn}, y_i) - k(x_{jn}, y_0)]\nu(x_{jn}) \qquad (\text{for any } i)\,.$$

By Theorem 3 (a), (ii), if a sequence $\{y_j\}$ converges to y_0 in Y, $\mu_i(B)$ converges to 0, and by (a) (iii), the norms of μ_i are bounded as linear functionals on $C^\infty(N)$. Therefore, by Phillips ([19]. Lemma) $\lim_i \sum_j |\mu_i(x_j)| = 0$. This shows that Theorem 3. (b), (iv) is satisfied, so T is compact.

The following corollary is proved by Grothendieck [14].

Corollary. *Any weakly compact linear operator of c_0 into a locally*

10) Since $\hat{E}$ is a Fréchet space, it is barrelled (cf. [7]).

convex Hausdorff topological linear space is compact.

Proof. Let E be a locally convex Hausdorff topological linear space and let V be a closed symmetric convex neighborhood of 0 in E. Let $\hat{E}_v$ be the completion of E_v. Then $\hat{E}_v$ is a Banach space. We put $T_v=j_vT$, where j_v is the canonical linear mapping of E into $\hat{E}_v$. Since c_0 is separable, if F_v is the closure of $T_v(c_0)$ in $\hat{E}_v$, then F_v is a separable Banach space. Therefore, by Theorem 4, T_v is a compact linear operator of c_0 into F_v. If U is the unit sphere of c_0, then $T_v(U)$ is contained in a compact subset K_v in F_v. Since E contained in the topological product space ΠF_v of F_v, we see that $T(U)\subset\prod_v K_v$. Since $T(U)$ is contained in a weakly compact subset in E, T is compact.

REMARK. From the above Corollary we have: let E be a Banach space whose dual E' is separable. Then any continuous linear operator of the space m into E' is compact.

§ 4. Simultaneous extension theorem[11]

Let X be a metric space and let F be a closed subspace in X. Then there is a simultaneous extension T of $C_{\mathfrak{S}}(F)$ into $C_{\mathfrak{T}}(X)$ ($\mathfrak{S}$, $\mathfrak{T}$ denote the set of all compact subsets in X, Y respectively), i.e. T is a non-negative continuous linear operator of $C_{\mathfrak{S}}(F)$ into $C_{\mathfrak{T}}(X)$ and Tf is a continuous extension of f for any $f\in C_{\mathfrak{S}}(F)$ (cf. Michael [17] p. 802). On the other hand, Day [9] gave an example of a compact Hausdorff space X and of a closed subspace F such that there is no linear mapping $C_u(F)$ into $C_u(X)$ which is a simultaneous extension of all elements of $C_u(F)$. His example is the following: Let X be the topological product space of the closed unit interval I_λ ($\lambda\in\Lambda$) and let the set Λ of indices be uncountable. Let S be the unit sphere of $l_q(\Lambda)$ with the topology $\sigma(l_q(\Lambda), l_p(\Lambda))$, where $p>1$, $q>1$ and $p^{-1}+q^{-1}=1$. Then we may regard S as a closed subset in X and $l_p(\Lambda)$ as a linear subspace of $C(S)$. Day showed that there is no continuous linear operator T from $L=l_p(\Lambda)$ into $C_u(X)$ such that, for any f in L, Tf is an extension of f. If we put $x(\xi)=\langle x, \xi\rangle$, $\xi\in S$, then $x(\xi)$ is a continuous function on S. Let U be the unit sphere of $l_p(\Lambda)$ and let $\tilde{U}=\{x(\xi): x\in U\}$. Then $\tilde{U}$ is a weakly compact subset in $C_u(S)$. For, since U is a $\sigma(l_p(\Lambda), l_q(\Lambda))$-compact subset, any directed set in U has a cluster point in the topology $\sigma(l_p(\Lambda), l_q(\Lambda))$, so $\tilde{U}$ is compact in the simple topology on $C_u(S)$ and for

11) The simulteneous extension theorem was considered by Dugundji [12], Arens [2] and Michael [17] on the case of bounded coniinuous functions.

any $x \in U$, $|x(\xi)| = |\langle x, \xi\rangle| \leqq 1$ $(\xi \in S)$. Therefore, by Grothendieck [13], $\tilde{U}$ is a weakly compact subset.

We see that there is no simultaneous extension of $\tilde{U}$ into $C_u(X)$. But we have the following.

Theorem 5. *Let X be a paracompact Hausdorff space and let F be a k_0-space which is closed in X. Let A be a relative compac tsubset in $C_{\mathfrak{S}}(F)$, $\mathfrak{S}$ denoting the set of all compact subset in F. Then there is a simultaneous extension T of A into $C_{\mathfrak{T}}(X)$, $\mathfrak{T}$ denoting the set of all compact subsets in X, i.e. if f, g and $\alpha f+\beta g$ (α, β real) are contained in A, then $T(\alpha f+\beta g) = \alpha Tf+\beta Tg$, and Tf is a continuous extension of f for any $f \in A$. Moreover, T is a continuous operator of A (with the topology induced by $C_{\mathfrak{S}}(F)$) into $C_{\mathfrak{T}}(X)$.*

Proof. Since F is a k_0-space, $C_{\mathfrak{S}}(F)$ is complete. Therefore the symmetric convex closed envelope of A is compact, so we can assume that A is symmetric, convex and compact. $E=[C_{\mathfrak{S}}(F)]_A$ [12] is a Banach space. Let S be a canonical linear mapping of E into $C_{\mathfrak{S}}(F)$, i.e. $Sf=f$ for any $f \in E$. Then S is a compact linear operator. By Theorem 1 there is a continuous mapping τ of F into E' such that $(Sf)y=\langle \tau y, f\rangle$ for any f in E and for any y in F. We put $q(x, y)=\|\tau x-\tau y\|_{E'}$. This pseudo-metric is extended on X (cf. Arens [2] p. 18), and we denote the extension also by q. We put $F_0=\{x : q(x, F)=0\}$. Then τ can be extended to F_0 continuously. This mapping is denoted by τ_1. We divide elements of X into equivalent classes, by making x equivalent to y if $q(x, y)=0$. Let X^* be the set of equivalent classes. If we define $q^*(x^*, y^*)=q(x, y)$ for any $x \in x^*$ and $y \in y^*$, then X^* is a metric space with the metric function q^* (cf. Arens [2] p. 18). The canonical mapping j of X onto X^* is continuous. Set $jF_0=F_0^*$. Then F_0^* is closed in X^*. For any $a \in F_0$ we put $\tau_1^*(a^*)=\tau_1(a)$, then τ_1^* is continuous in F_0^*. τ_1^* can be extended on X^* continuously, and we denote the extension also by τ_1^*. By Michael ([17] p. 803) we can assume that for any compact C^* in X^* there is a compact subset C_1^* in F_0^* such that $\tau_1^*(C^*) \subset \Gamma(\tau_1^*(C^*))$. We here put $\tau_1(x)=\tau_1^*(x^*)$ for any $x \in X$. Then τ_1 is continuous on X and is an extension of τ. We define $(Tf)x=\langle \tau_1 x, f\rangle$ for any $f \in A$ and $x \in X$. Then Tf is a continuous extension of f, and if f, g and $\alpha f+\beta g \in A$, then $T(\alpha f+Bg)=\alpha Tf+\beta Tg$.

We finally prove that T is a continuous operator of A (with the topology induced by $C_{\mathfrak{S}}(F)$) into $C_{\mathfrak{T}}(X)$, i.e. for any compact C in X there is a compact subset K in F such that $T[W(K, 1/2)\cap A] \subset W(C, 1)$.

---

12) By E_A we denote the Banach space whose unit sphere is A.

Supposing the contrary, let $\mathfrak{K}=\{K_j\}$ be the directed set[13] of all compact subset in F. Then for any K_j there is an $f_j \in W(K_j, 1/2)\cap A$ and $Tf_j \notin W(C, 1)$ (for some compact subset C in X), i.e. there is a $c_j \in C$ such that

$$|Tf_j(c_j)| \geqq 1 \tag{1}$$

and

$$|f_j(K_j)| \leqq 1/2 . \tag{2}$$

Since $jC=C^*$ is compact, there is a compact subset C_1^* in F_0^* such that

$$\tau_1^*(C^*) \subset \Gamma(\tau_1^*(C_1^*)) \tag{3}$$

By (3) we have that for any $\varepsilon>0$ and for any c_j there are finite real numbers λ_i^j with $\sum_i |\lambda_i^j| \leqq 1$ and $c_i^j \in F_0$ (with $(c_i^j)^* \in C_1^*$) such that $\|\tau_1^* c_j^* - \sum_i \lambda_i^j \tau_1(c_i^j)\|_{E'} < \varepsilon$. Since $(Tf_j)(c_j) = \langle \tau_1^* c_j^*, f_j \rangle$,

$$\begin{aligned}&|(Tf_j)(c_j) - \sum_i \lambda_i^j (Tf_j)(c_i^j)| \\ &\leqq \|f_j\|_E \cdot \|\tau_1^* c_j^* - \sum_i \lambda_i^j \tau_1(c_i^j)\|_{E'} < \varepsilon \ \|f_j\|_E = \varepsilon .\end{aligned}$$

Therefore there is a $d_j \in \{c_i^j\}$ such that

$$|(Tf_j)(d_j)| \geqq 1-\varepsilon . \tag{4}$$

Since $d_j^* \in C_1^*$ and C_1^* is compact in F_0^*, there is a cluster point d^* of $\{d_j^*\}$. $d \in F_0$. For any j there is a $j'(>j)$ such that $\|\tau_1 d_{j'} - \tau_1 d\|_{E'} < \varepsilon$, so

$$\begin{aligned}|(Tf_{j'})(d_{j'}) - (Tf_{j'})(d)| &= |\langle \tau_1 d_{j'} - \tau_1 d, f_{j'} \rangle| \\ &\leqq \|\tau_1 d_{j'} - \tau_1 d\|_{E'} < \varepsilon .\end{aligned}$$

Now, since $d \in F_0$, there is a $d_0 \in F$ such that $\|\tau_1 d - \tau d_0\|_{E'} < \varepsilon$, i.e. $\sup_{f \in A} |Tf(d) - f(d_0)| < \varepsilon$. For any j' $|Tf_{j'}(d) - f_{j'}(d_0)| < \varepsilon$, so $|f_{j'}(d_0)| \geqq 1-3\varepsilon$ (by (4)). Put $\varepsilon < 1/6$. Then for any j' $|f_{j'}(d_0)| > 1/2$. By (2) $d_0 \notin K_{j'}$. Since $\{K_j\}$ is a directed set, $d_0 \notin F$. This contradiction proves the theorem.

§ 5. The spaces of summable functions

J. Dieudonné [10] has proved the following theorem: let E and E' be dual linear spaces and let X be a Kakutani space (cf. § 1). Then any continuous linear operator of $L^1(X, \mu)$ into $\hat{E}$[14] is of the form $f \to \int f(x)\tau x \, d\mu(x)$, where τ is a weakly summable, weakly bounded mapping of

13) The ordering is defined by inclusion, i.e. $j_1 > j_2$ if $K_{j_1} \supset K_{j_2}$.

14) $\hat{E}$ denotes the completion of E with the topology $\sigma(E, E')$.

X into $\hat{E}$. By Theorem 1 we can prove the following

Theorem 6. *Let E be a Banach space and let X be a Kakutani space. Then any weakly compact linear operator of $L^1(X, \mu)$ (cf. § 1) into E is of the form $f \to \int f(x)\tau x\, d\mu(x)$*[15], *where τ is a weakly continuous mapping*[16] *of X into E and $\tau(X)$ is contained in a weakly compact subset in E.*

Proof. Let T be a weakly compact linear operator of $L^1(X, \mu)$ into E. Then its transposition T' is a continuous linear operator of E' into $C(\bar{X})$ (Cf. § 1). By Theorem 1 there is a continuous mapping of $\bar{X}$ into E'' (with the topology $\sigma(E'', E')$) and $(T'e')x = \langle \tau x, e' \rangle$ for $e' \in E'$ and $x \in X$. Since T is weakly compact and $\tau x = T''\mu_x$, τx is contained in E (cf. [14] Lemma 1). Therefore τ is a continuous mapping of $\bar{X}$ into E (with the topology $\sigma(E, E')$). We have that for $f \in L^1(X, \mu)$ and $e' \in E'$ $\langle Tf, e' \rangle = \langle f, T'e' \rangle = \int f(x) \langle \tau x, e' \rangle d\mu(x)$, so $Tf = \int f(x)\tau x\, d\mu(x)$. Conversely, let τ be a weakly continuous mapping of X into E and $\tau(X)$ is contained in a weakly compact subset in E. Since $\bar{X}$ is the Čech compactification of X (cf. § 1), τ is extended to a weakly continuous mapping τ_1 of $\bar{X}$ into E. By Bourbaki ([8] § 4) for any $\xi \in C(\bar{X})'$ $\int \tau_1 x \xi(dx)$ exists as an element in E and for any $e' \in E'$

$$\langle \int \tau_1 x \xi(dx), e' \rangle = \int \langle \tau_1 x, e' \rangle \xi(dx).$$ For any $f \in L^1(X, \mu)$

and $e' \in E'$

$$\langle T'e', f \rangle = \langle e', Tf \rangle = \int f(x) \langle \tau x, e' \rangle d\mu(x),$$

so $T'e' = \langle \tau_1 x, e' \rangle$ (for, $\langle \tau_1 x, e' \rangle$ is the continuous extension of $\langle \tau x, e' \rangle$). For any $\xi \in C(\bar{X})'$

$$\begin{aligned} \langle T''\xi, e' \rangle &= \langle \xi, T'e' \rangle = \int \langle \tau_1 x, e' \rangle \xi(dx) \\ &= \langle \int \tau_1 x \xi(dx), e' \rangle, \end{aligned}$$

---

15) $\int f(x)\tau x d\mu(x)$ is an element in E which $\langle \int f(x)\tau x d\mu(x), e' \rangle = \int f(x) \langle \tau x, e' \rangle d\mu(x)$ for any $e' \in E'$.

16) τ is said to be a weakly continuous mapping of X into E if it is continuous mapping of X into E with topology $\sigma(E, E')$.

so $T''\xi=\int \tau_1 x\xi(dx)\in E$. Then by [14] Lemma 1, T is weakly compact.

REMARK. By a similar method as Dieudonné ([10], 32-35), we can prove a theorem of Phillips ([18], Theorem 5.4.) from Theorem 6.

(Received March 22, 1961)

Bibliography

[1] P. Alexandorff and P. Urysohn: Mémoire sur les espaces topologiques compacts, Verh. Konink. Akad. Afd. Nat. **14**, No. 1 (1929).
[2] R. Arens: Extension of functions on fully normal spaces, Pacific J. Math. **2** (1952), 11-22.
[3] R. G. Bartle: On compactness in functional analysis, Trans. Amer. Math. Soc. **79** (1955), 35-57.
[4] R. G. Bartle, N. Dunford, and J. Schwartz: Weak compactness and vector measures, Canad. J. Math. **7** (1955), 289-305.
[5] N. Bourbaki: Topologie générale, livre III, Chap. X, 1949.
[6] ———: Espaces vectoriels topologiques, livre V, Chaps. I, II, 1953.
[7] ———: Espaces vectoriels topologiques, livre V, Chaps. III-V, 1955.
[8] ———: Intégration, livre VI, Paris, 1952.
[9] M. M. Day: Strict convexity and smoothness of normed spaces, Trans. Amer. Math. Soc. **78** (1955), 516-528.
[10] J. Dieudonné: Sur le théorème de Lebesgue-Nikodym (III), Ann. Univ. Grenoble, **23** (1948), 25-53.
[11] J. Dixmier: Sur certains espaces considérés par M. H. Stone, Summa Brasil. Math. **2** (1951), 151-182.
[12] J. Dugaindji: An extension of Tietze's theorem, Pacific J. Math. **1** (1951), 353-367.
[13] A. Grothendieck: Critères de compacité dans les espaces fonctionnels généraux, Amer. J. Math. **74** (1952), 168-186.
[14] ———: Sur les applications linéaires faiblement compactes d'espaces du type C(K), Canad. J. Math. **5** (1954), 129-173.
[15] E. Hewitt: Rings of real-valued continuous functions I, Trans. Amer. Math. Soc. **64** (1948), 45-99.
[16] J. L. Kelley: General topology, New York 1955.
[17] E. Michael: Some extension theorems for continuous functions, Pacific J. Math. **3** (1952), 789-805.
[18] R. S. Phillips: On linear transformations, Trans. Amer. Math. Soc. **48** (1940), 516-541.
[19] J. Wada: Weakly compact operators on the spaces of continuous functions, Proc. Japan Acad. **36** (1960), 574-577.
[20] S. Warner: The theory of compact convergence on continuous function spaces, Duke Math. J. **25** (1958), 265-282.