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NON-DEFORMABILITY OF EINSTEIN METRICS

Norihito KOISO

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Introduction

Let $M$ be a compact connected $C^\infty$-manifold and $g$ be an Einstein metric on $M$. By an Einstein deformation of $g$ we mean a 1-parameter family $g(t)$ of Einstein metrics on $M$ such that $g(0) = g$ and the volume of $g(t)$ is constant for $t$. If for each Einstein deformation $g(t)$ of $g$ there exists a 1-parameter family $\gamma(t)$ of diffeomorphisms such that $g(t) = \gamma(t)^*g$ (resp. $g'(0) = \frac{d}{dt}|_{0}\gamma(t)^*g$) then $g$ is said to be non-deformable (resp. infinitesimally non-deformable). M. Berger and D. Ebin [1, Lemma 7.4] show that the Einstein structure of the standard sphere is infinitesimally non-deformable, by using the fact that the operator $L$ associated to the curvature tensor of the standard sphere is positive definite. In this paper, the main theorem (Theorem 3.3) gives a criterion for an Einstein structure to be non-deformable, improving their method of estimating eigenvalues of the operator $L$. As an application we see, for example, that the Einstein structure of a compact irreducible locally symmetric space $M$ of non-compact type with $\dim M > 2$ is non-deformable. (Corollary 3.5).

To prove the main theorem we have to relate infinitesimal non-deformability to non-deformability. For this purpose we need a smooth slice theorem. The slice theorem (Theorem 2.1) in the $H^s$-situation (D. Ebin [5, Theorem 7.1], [6, Theorem 8.20]) being in continuous category, we shall improve this continuous slice theorem to a smooth slice theorem (Theorem 2.2) in the ILH-situation. Owing to this we get a theorem (Theorem 2.11) which relates infinitesimal non-deformability to non-deformability.

The author would like to express his sincere gratitude to the referees for their kind advices.

1. Preliminaries

First, we introduce notation which will be used throughout this paper. Let $M$ be an $n$-dimensional, connected and compact $C^\infty$-manifold without boundary, and we always assume $n > 2$. For a riemannian manifold $(M,g)$, we
Consider the Riemannian connection and use the following notation:

- $S^2$: the symmetric covariant 2-tensor bundle over $M$,
- $C^\omega(T)$: the vector space of all $C^\omega$-sections of a tensor bundle $T$ over $M$,
- $S^2_0$: the space of all symmetric covariant 2-tensors whose trace is zero,
- $(\cdot, \cdot)$: the inner product in fibers of a tensor bundle defined by the Riemannian structure,
- $<\cdot, \cdot>$: the global inner product for sections of a tensor bundle over $M$, i.e., $<\cdot, \cdot> = f_M(\cdot, \cdot)v_x v_y$ being the volume element defined by $g$,
- $R$: the curvature tensor,
- $\rho$: the Ricci tensor,
- $\tau$: the scalar curvature,
- $\nabla$: the covariant derivation on $C^\omega(T)$,
- $\delta$: the formal adjoint of $\nabla$ with respect to $(\cdot, \cdot)$,
- $\delta^*$: the formal adjoint of $\delta|_{C^\omega(S^2)}$,
- $\Delta = \delta d + d\delta$: the Laplacian operating on the space $C^\omega(M)$ of $C^\omega$-functions on $M$,
- $\Delta = \delta \nabla$: the rough Laplacian operating on $C^\omega(T)$,
- $\text{Hess} = \nabla^2$: the Hessian on $C^\omega(M)$.

We shall use the Einstein's convention, although we use $\sum$ if necessary. We shall apply the following formulae throughout the paper.

\[
R^l_{ijl} = \nabla_i \nabla_j \xi^l - \nabla_j \nabla_i \xi^l, \quad R_{ijkl} = R^m_{ijkl} g_{mi},
\]
\[
\rho_{ij} = -R^l_{ijl}, \quad \tau = \rho^l_l,
\]
\[
(\delta S)^{i_1 \ldots i_r}_{j_1 \ldots j_s} = -\nabla^i S^{i_1 \ldots i_r}_{j_1 \ldots j_s}, \quad (\delta^* S)^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{1}{2} (\nabla_i \xi_j + \nabla_j \xi_i),
\]
\[
\Delta f = -\nabla^i d_i f, \quad (\Delta S)^{i_1 \ldots i_r}_{j_1 \ldots j_s} = -\nabla^i \nabla_i S^{i_1 \ldots i_r}_{j_1 \ldots j_s}.
\]

(For the standard sphere, $R_{2222} < 0$, $\rho_{11} > 0$ and $\tau > 0$, with respect to orthonormal frame.)

Let $(M, g)$ be an Einstein manifold. If $\text{tr} \ h = 0$ then

\[g^{ij} R_{ij}^k h_{kl} = -\rho^{kl} h_{kl} = 0.\]

Hence we can define the operator $L: S^2_0 \rightarrow S^2_0$ by

\[(Lh)_{ij} = R_{ij}^k h_{kl}.
\]

Next, we recall the following concepts defined by H. Omori [12, pp. 168–169]. A topological vector space $E$ is called an $ILH$-space, if $E$ is an inverse limit of Hilbert spaces $\{E_i\}_{i=1,2,\ldots}$ such that if $j \geq i$, $E_i \subset E_j$ and the inclusion is a bounded linear operator. We denote $E = \lim \nrightarrow E_i$.

A topological space $X$ is called a $C^k$-ILH-manifold modeled on $E$, if $X$ has the following properties C1 and C2.

C1) $X$ is an inverse limit of $C^k$-Hilbert manifolds $\{X_i\}_{i=1,2,\ldots}$ such that
each $X_i$ is modeled on $E_i$ and $X_i \supset X_j$ if $j \geq i$.

2) Let $x$ be any point of $X$. For each $i$ there are an open neighbourhood $U_i(x)$ of $x$ in $X_i$ and a homeomorphism $\psi_i$ from $U_i(x)$ onto an open subset $V_i$ in $E_i$ which gives a $C^k$-coordinate around $x$ in $X_i$ and satisfies $U_i(x) \supset U_j(x)$ if $j \geq i$ and $\psi_{i+1}(y) = \psi_i(y)$ for every $y \in U_{i+1}(x)$.

Let $X$ be a $C^k$-ILH-manifold ($k \geq 1$). Let $TX_i$ be the tangent bundle of $X_i$. Then the inverse limit $TX = \lim_{\leftarrow} TX_i$ is called the ILH-tangent bundle of $X$.

Let $X, Y$ be $C^k$-ILH-manifolds. A mapping $\phi : X \rightarrow Y$ is said to be $C^l$-ILH-differentiable ($l \leq k$), if $\phi$ is an inverse limit of $C^l$-differentiable mappings, that is, for every $i$, there are a positive integer $j(i)$ and a $C^l$-mapping $\phi_i : X_j(i) \rightarrow Y_i$ such that $\phi_i(x) = \phi_{i+1}(x)$ for every $x \in X_{j(i+1)}$ and $\phi = \lim \phi_i$.

If $X$ is a $C^k$-ILH-manifold for all $k \geq 0$, we call $X$ an ILH-manifold. For ILH-manifolds $X, Y$, if $\phi$ is $C^k$-ILH-differentiable for all $k \geq 0$, we say that $\phi$ is ILH-differentiable. We denote by $T_x X_i$ the tangent space of $X_i$ at $x$ and put $T_x X = \lim_{\leftarrow} T_x X_i$. Also we denote by

$$T^r \phi_i(x) : \prod_{i=1}^r T_x X_{j(i)} \rightarrow T_x Y_i$$

the $r$-th derivative of $\phi_i$ at $x \in X$. Then, it is easy to check that $\{T^r \phi_i(x)\}_{i=1,2,\ldots}$ has an inverse limit

$$\lim_{\leftarrow} T^r \phi_i(x) : \prod_{i=1}^r T_x X \rightarrow T^r \phi(x).$$

We call this inverse limit the $r$-th derivative of $\phi$ and denote it by $T^r \phi(x)$.

A topological group is called an ILH-Lie group, if it is an ILH-manifold and the group operations are ILH-mappings.

We can easily see that the space $\mathcal{M}$ of all smooth Riemannian metrics on $M$ is an ILH-manifold. (See D. Ebin [5, p. 15], [6, Proposition 5.8] and H. Omori [12, p. 170].) We know that the group $\mathcal{D}$ of all diffeomorphisms of $M$ is an ILH-Lie group, and the natural action $A : \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$ is ILH-differentiable. (See [12, Lemma 2.5].)

Let $g \in \mathcal{M}$. By a deformation of $g$ we mean a $C^\infty$-curve $g(t) : I \rightarrow \mathcal{M}$ such that $g(0) = g$, where $I$ is an open interval containing $0$ in $\mathbb{R}$. Since $\mathcal{M}$ is a positive cone in the vector space of all symmetric covariant 2-tensors on $M$, we may identify the differential $g'(0)$ of a deformation $g(t)$ with a symmetric covariant 2-tensor field on $M$. We call such a tensor field an infinitesimal deformation, or simply an i-deformation.

When we consider a deformation $g(t)$ of $g$, the covariant derivation, the curvature tensor or the Ricci tensor with respect to each $g(t)$ will be denoted by $\nabla_t$, $R(t)$ or $\rho_{g(t)}$. Also, we always raise or lower indices of tensors with respect to $g(t)$, and we denote by $'$ the differentiation with respect to $t$. It is clear that the differential at $t = 0$ of the tensors $R, \rho, \tau$ etc. depend only on the i-deformation that $g(t)$ defines.
2. Deformations and infinitesimal deformations

Let $M$ be a compact connected $C^\infty$-manifold. We denote by $\mathcal{M}$ the space of all $H^s$-metrics on $M$ and by $\mathcal{D}$ the space of all $H^s$-diffeomorphisms of $M$, where $H^s$ means an object which has partial derivatives defined almost everywhere up to order $s$ and such that each partial derivative is square integrable. We know that the space $\mathcal{M}$ and the space $\mathcal{D}$ are Hilbert manifolds if $s$ is sufficiently large. Moreover, the usual action $A: \mathcal{D} \times \mathcal{M} \to \mathcal{M}$ extends to a continuous mapping $A^*: \mathcal{D}^{s+1} \times \mathcal{M} \to \mathcal{M}$. (See D. Ebin [5,p.18], [6, Proposition 4.24].)

D. Ebin gave the following

**Theorem 2.1** (D. Ebin [5, Theorem 7.1], [6, Theorem 8.20]). For each $g \in \mathcal{M}$, there is a submanifold $S^*_g$ of $\mathcal{M}$ with the following properties.

1. If $\gamma \in I^s_g$, then $\gamma^*(S^*_g) = S^*_g$.
2. Let $\gamma \in \mathcal{D}^{s+1}$. If $\gamma^*(S^*_g) \cap S^*_g \neq \emptyset$, then $\gamma \in I^s_g$.
3. There is a neighbourhood $U^{s+1}$ of the coset $I^s_g$ in the right coset space $\mathcal{D}^{s+1}/I^s_g$ and a local cross section $\chi^{s+1}: \mathcal{D}^{s+1}/I^s_g \to \mathcal{D}^{s+1}$ defined on $U^{s+1}$ such that if the mapping $F^s: U^{s+1} \times S^*_g \to \mathcal{M}$ is defined by $F^s(u,s) = \chi^{s+1}(u)^s$, then $F^s$ is a homeomorphism onto a neighbourhood of $g$.

Outline of the proof. Canonically we can construct a riemannian metric on $\mathcal{M}$ which is invariant under the action $A$ of $\mathcal{D}$. At any point $g \in \mathcal{M}$, $\psi^*_g: \mathcal{D}^{s+1} \to \mathcal{M}$ is defined by $\psi^*_g(\eta) = A^*(\eta,g)$ for $\eta \in \mathcal{D}^{s+1}$. If $g \in \mathcal{M}$, $\psi^*_g$ is smooth. Also for $\eta \in \mathcal{D}^{s+1}$, we identify the tangent spaces $T_\eta(\mathcal{D}^{s+1})$ or $T_{\phi}(\mathcal{M})$ with the space of $H^s$-sections of some vector bundle over $M$. Then, $T_\eta \psi^*_g$ becomes a first order linear differential operator. It turns out that this operator has an injective symbol, and so its range is closed in $T^s(\mathcal{D}^{s+1})$.

The right coset space $\mathcal{D}^{s+1}/I^s_g$ has an induced manifold structure and admits a smooth local cross section $\chi^{s+1}: U^{s+1} \to \mathcal{D}^{s+1}$. $\psi^*_g$ induces a mapping $\phi^*_g: \mathcal{D}^{s+1}/I^s_g \to \mathcal{M}$. $\phi^*_g$ is an injective immersion and we see directly that it is a diffeomorphism onto the closed orbit $O^s_g$.

Using the riemannian metric on $\mathcal{M}$, we obtain a smooth normal bundle $\pi^*: \nu^s \to O^s_g$. Moreover, the exponential mapping $\exp^s$ on $\mathcal{M}$ is defined on a neighbourhood $W^s$ of the zero-section of $\nu^s$ and it is a diffeomorphism. We put $S^*_g = \exp^s W^s$, where $W^s$ is the fibre on $g$.

Also, we know that for any $\eta \in \mathcal{D}^{s+1}$ a smooth mapping $\eta^*: \mathcal{M} \to \mathcal{M}$ is defined by $\eta^*(g) = A(\eta,g)$ and $\eta^*$ is an isometry. Therefore, if $\exp^s$ is defined for a vector $V$ in $T(\mathcal{M})$, $\exp^s$ is defined for $T\eta^*(V)$ and we have $\eta^* \exp^s V = \exp^s T\eta^* V$.

Combining these informations, we can prove the slice theorem in the $H^s$-situation. Moreover, if we define the mapping $F^s: U^{s+1} \times S^*_g \to \mathcal{M}$ by $F^s(u,s) = A^*(\chi^{s+1}(u), s)$, for $z \in \exp^s W^s$ we have
We shall need the following slice theorem which improve Theorem 2.1 to the \( C^\infty \)-situation.

**Theorem 2.2.** We denote by \( \mathcal{M} \) the ILH-manifold formed by all riemannian metrics on \( M \), and by \( \mathcal{D} \) the ILH-Lie group of all diffeomorphisms on \( M \). The group \( \mathcal{D} \) acts on \( \mathcal{M} \) in a canonical way. For each \( g \in \mathcal{M} \), there is an ILH-submanifold \( S_g \) of \( \mathcal{M} \) with the following properties. Let \( I_g \) be the group of all isometries of the riemannian manifold \( (M,g) \).

1. If \( \gamma \) belongs to \( I_g \), then \( \gamma^*(S_g) = S_g \).
2. Let \( \gamma \in \mathcal{D} \). If \( \gamma^*(S_g) \cap S_g \neq \phi \), then \( \gamma \in I_g \).
3. There are a neighbourhood \( U \) of the point \( I_g \) in the right coset space \( \mathcal{D}/I_g \) and a local cross section \( \chi : \mathcal{D}/I_g \to \mathcal{D} \) defined on \( U \) such that if the mapping \( F : U \times S_g \to \mathcal{M} \) is defined by \( F(u,s) = \chi(u)^*s \), then \( F \) is an ILH-diffeomorphism onto a neighbourhood of \( g \).

We need the following lemmas.

**Lemma 2.3.** \( \mathcal{D}/I_g \) is an ILH-manifold.

**Lemma 2.4.** Put \( U = U^\gamma \cap \mathcal{D}/I_g \). Then \( \chi'(U) \) is contained in \( \mathcal{D} \) and the mapping \( \chi = \chi' \mid U \) is ILH-differentiable.

**Lemma 2.5.** Put \( W = W^\gamma \cap T\mathcal{M} \). Then \( \exp'(W) \) is contained in \( \mathcal{M} \) and the mapping \( \exp = \exp' \mid W \) is an ILH-diffeomorphism. Hence \( S_g = S^\gamma_g \cap \mathcal{M} \) is an ILH-submanifold of \( \mathcal{M} \).

These lemmas will be proved in below.

**Lemma 2.6** [12, Lemma 2.5]. \( A'(\mathcal{D} \times \mathcal{M}) \) is contained in \( \mathcal{M} \) and the mapping \( A = A' \mid \mathcal{D} \times \mathcal{M} \) is ILH-differentiable.

**Lemma 2.7** [12, Lemma 1.14]. If the mapping \( \tilde{\iota} : \mathcal{D} \to \mathcal{D} \) is defined by \( \tilde{\iota}(\eta) = \eta^{-1} \) for \( \eta \in \mathcal{D} \), then \( \tilde{\iota} \) is ILH-differentiable.

**Proof of Theorem 2.2.** Combining these lemmas and the proof of Theorem 2.1, the mappings \( F = F^\gamma \mid U \times S_g \) and \( F^{-1} = (F^\gamma)^{-1} \mid \exp W \) are compositions of ILH-mappings, and so \( F \) is an ILH-diffeomorphism, which proves Theorem 2.2.

**Proof of Lemma 2.3.** We know that \( \mathcal{D}/I_g \) is a Hilbert manifold. We shall prove that the inclusion \( \tilde{\iota} : \mathcal{D}^\gamma/I_g \to \mathcal{D}^\gamma/I_g \) is smooth. By [5, Corollary 5.11] or [6, Corollary 7.16], \( \tilde{\iota} \) is smooth if and only if \( \tilde{\iota} \circ p^\gamma : \mathcal{D}^\gamma \to \mathcal{D}^\gamma/I_g \) is smooth, where \( p^\gamma : \mathcal{D}^\gamma \to \mathcal{D}^\gamma/I_g \) is the natural projection. We can easily see \( \tilde{\iota} \circ p^\gamma = \)
$p' \circ i'$, where $i': \mathcal{D}' \to \mathcal{D}$ is the inclusion. Since $i'$ and $p'$ are smooth, $i'$ is smooth.

Proof of Lemma 2.4. By [5, Proposition 5.10] or [6, Proposition 7.15], $\mathcal{D}'/I_g$ admits a smooth local cross section around any coset. We denote by $\chi'_x$ the local cross section around $x \in \mathcal{D}'/I_g$ and put $\chi'=\chi'_1$. Let $U'$ be the domain of $\chi'$ and set $U=U' \cap \mathcal{D}'/I_g$ and $\chi'=\chi'_{|U'}$ for $r \geq s$. If $u \in U'$, there is an element $a \in \mathcal{D}'$ such that $u=I_g a$ and $\chi'(u) \in I_g a \subset \mathcal{D}'$. Hence we have $\chi'(U') \subset \mathcal{D}'$. To prove that $\chi'$ is smooth, we shall show that if we define a mapping $\nu: (p')^{-1}(U') \to I_g$ by $\nu(\eta)=\eta(\chi' \circ p' \eta)^{-1}$, then $\nu$ is smooth. By [5, Lemma 5.5] or [6, Corollary 7.7], the composition: $\mathcal{D}' \times \mathcal{D}' \to \mathcal{D}'$ is smooth. Hence, if we define a mapping $\nu: \mathcal{D}' \times U' \to \mathcal{D}'$ by $\nu(\xi, x)=\chi'\xi'(x)$, then $\nu$ is smooth. On the other hand, we have $\nu^{-1}(\eta)=\nu(\eta)$, $p'(\eta)$ and $p'$ is smooth. For $\nu$, we fix a positive integer $i$ such that the composition: $\mathcal{D}' \times \mathcal{D}' \to \mathcal{D}'$ and the inverse: $\mathcal{D}' \to \mathcal{D}'$ are $C^1$-mappings. ([12, Lemma 1.13 and Lemma 1.14]. Suppose that $s$ is sufficiently large.) Then, we see directly that $\nu$ is a $C^1$-mapping into $\mathcal{D}'$. But $I_g$ contains the image of $\nu$ and $I_g$ is a submanifold of $\mathcal{D}'$ (see [5, Corollary 5.4] or [6, Theorem 7.1]). Hence, $\nu$ is a $C^1$-mapping into $I_g$. Therefore, we know that $\nu$ is smooth and $\nu^{-1}$ is a $C^1$-mapping. By the inverse function theorem, $\nu^{-1}$ is smooth and so $\nu$ is smooth.

Now, we shall prove the smoothness of $\chi'$ around any $x \in U'$. There is a smooth local cross section $\chi'_x$ on a neighbourhood $V$ of $x$. Therefore the mapping $\nu \circ "inclusion" \circ \chi'_x: V \to I_g$ is smooth and we have $\nu \circ "inclusion" \circ \chi'_x(y)=\chi'_x(y)(\chi'(y))^{-1}=\chi'_x(y)^{-1}$ since we know that $\chi'(y)=((\chi'_x(y))(\chi'(y))^{-1})^{-1} \chi'(y)$ and the inverse: $I_g \to I_g$ and the composition: $I_g \times \mathcal{D}' \to \mathcal{D}'$ are smooth, the mapping $\chi': V \to \mathcal{D}'$ is smooth.

Proof of Lemma 2.5. Let $\tilde{W}$ be an open subset of $T\mathcal{M}'$ such that $W'=\nu' \cap \tilde{W}$. Set $\tilde{W}'=\tilde{W} \cap T\mathcal{M}'$, $W'=\tilde{W} \cap \nu'$, $\exp'=\exp' \tilde{W}$ and $(\exp')^{-1}=\exp'(W')^{-1} \exp'(W') \cap \mathcal{M}'$. The mappings $\exp': \tilde{W}' \to \mathcal{M}'$ and $(\exp')^{-1}: \exp'(W') \cap \mathcal{M}' \to \tilde{W}'$ are smooth and commute with the action of $\mathcal{D}$. Hence, by the following Lemma 2.8, $\exp'(\tilde{W}')$ and $(\exp')^{-1}(\exp'(W') \cap \mathcal{M}')$ are contained in $\mathcal{M}'$ and $T\mathcal{M}'$ respectively, and the mappings $\exp': \tilde{W}' \to \mathcal{M}'$ and $(\exp')^{-1}: \exp'(W') \cap \mathcal{M}' \to T\mathcal{M}'$ are smooth for $r \geq s$. But $W'$ is a submanifold of $W'$ and $(\exp')^{-1}(\exp'(W') \cap \mathcal{M}')$ is contained in $W'$, which implies that $\exp': W' \to \exp'(W') \cap \mathcal{M}'$ is a diffeomorphism. Thus we see that $\exp$ is an ILH-diffeomorphism onto $\exp'(W') \cap \mathcal{M}$.

Lemma 2.8. Let $E$ and $F$ be vector bundles over $M$ associated with the frame bundle (e.g., $T$, $T^*$, $S^q$, $T \times T^*$, the $k$-th jet bundle $J^k(T)$ etc.). Any $\eta \in \mathcal{D}$ defines a natural linear mapping $\eta^*: H^k(E) \to H^k(E)$. Let $A$ be an open subset of $H^k(E)$ and let $f: A \to H^k(F)$ be a smooth mapping which commutes with the action of
\[ \text{\textcircled{D.}} \text{ Put } A^r = A \cap H^r(E) \text{ for } r \geq s. \text{ Then } f(A^r) \text{ is contained in } H^r(F) \text{ and } f \mid A^r \rightarrow H^r(F) \text{ is smooth.} \]

Proof of Lemma 2.8. We shall prove that if this lemma holds for \( r = i \), then the same is true for \( r = i + 1 \). The induction will then complete the proof. First, by induction, we shall prove that \( \eta^* \circ T^k f(a) = T^k f(\eta^* a) \cdot \eta^* \) for all positive integer \( k \). If \( \eta^* \circ T^i f(a) = T^i f(\eta^* a) \cdot \eta^* \), then we have

\[
\eta^* \circ T^{i+1} f(a) (v_1, \ldots, v_i) = \eta^* \frac{d}{dt} |_{t=0} T^i f(a + tv_1, \ldots, v_i)
\]

\[
= \frac{d}{dt} |_{t=0} T^i f(\eta^* a + \eta^* v) (\eta^* v_1, \ldots, \eta^* v_i)
\]

\[
= T^{i+1} f(\eta^* a) (\eta^* v, \eta^* v_1, \ldots, \eta^* v_i).
\]

Let \( V \) be a vector field on \( M \) and let \( \eta_t \) be the 1-parameter subgroup of diffeomorphisms generated by \( V \). For sufficiently small \( t \), \( \eta_t^* a \in A^i \) if \( a \in A^i \). Hence we get

\[
\mathcal{L}_v T^k f(a) (v_1, \ldots, v_k) = \frac{d}{dt} |_{t=0} \eta_t^* T^k f(a) (v_1, \ldots, v_k)
\]

\[
= \frac{d}{dt} |_{t=0} \eta_t^* f(\eta_t^* v_1, \ldots, \eta_t^* v_k)
\]

\[
= T^{i+1} f(a) (\mathcal{L}_v v_1, \ldots, v_k) + T^k f(a) (\mathcal{L}_v v_1, v_2, \ldots, v_k)
\]

\[
+ \cdots + T^k f(a) (v_1, \ldots, v_{k-1}, \mathcal{L}_v v_k).
\]

Next, we shall prove that \( f(A^{i+1}) \subset H^{i+1}(F) \), and \( f \mid A^{i+1} : A^{i+1} \rightarrow H^{i+1}(F) \) is continuous and that if \( f \mid A^{i+1} \) is a \( C^k \)-mapping and \( T^k(f \mid A^{i+1}) = T^k f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E), \) then \( f \mid A^{i+1} \) is a \( C^{k+1} \)-mapping and \( T^{k+1}(f \mid A^{i+1}) = T^{k+1} f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E) \). Then, by the hypothesis of the induction, \( f \mid A^{i+1} \) is smooth.

If \( a \in A^{i+1} \), then \( \mathcal{L}_a a \in H^i(E) \) for all \( V \in C^\infty(T) \). Hence \( \mathcal{L}_a f(a) = T f(a) \) \( (\mathcal{L}_a a) \in H^i(F) \), which implies that \( f(A^{i+1}) \subset H^{i+1}(F) \). If a sequence \( \{a_n\} \) converges to \( a \) in \( A^{i+1} \), then \( \{\mathcal{L}_a a_n\} \) converges to \( \mathcal{L}_a a \) in \( H^i(E) \) for all \( V \in C^\infty(T) \). Hence \( \{\mathcal{L}_a f(a_n) = T f(a_n) \} \) converges to \( T f(a) \) \( (\mathcal{L}_a a) \in H^i(F) \), which implies that \( f \mid A^{i+1} \) is continuous. By the same calculation, we check easily that \( T^j f(A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)) \subset H^{i+1}(F) \) and \( T^j f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E) \) is continuous. We assume that \( f \mid A^{i+1} \) is a \( C^k \)-mapping and \( T^k(f \mid A^{i+1}) = T^k f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E) \). Define a mapping

\[
\nu : A^{i+1} \times H^{i+1}(E) \times \{H^{i+1}(E) \times \cdots \times H^{i+1}(E)\} \rightarrow H^{i+1}(F)
\]

by

\[
\nu(a, v, v) = T^k f(a^+ v) (v) - T^k f(a^+ v) (v)
\]

\[
- T^k f(a) (v, v).
\]
Then, by the assumption,

\[ v(a,v,v) = T^k f(a+v)(v) - T^k f(a)(v) \]

and

\[ v(a,tv,v) = T^k f(a+tv,v) - T^k f(a)(v) \]

By differentiation with respect to the \( H^i \)-topology, we get

\[ v(a,v,v) = \int_0^t T^{k+2} f(a+uv)(v,v,v) dudt . \]

Since \( |T^{k+2}f| \) is continuous with respect to the \( H^{i+1} \)-topology, we have

\[ |v(a,v,v)|/|v| \leq \max |T^{k+2}f(b)|/|v|/|v|, \]

where \( \varepsilon \) is sufficiently small and \( |\cdot| \) is the \( H^{i+1} \)-norm. Therefore, \( T^k f | A^{i+1} \) is differentiable and \( T^{k+1} f | A^{i+1} \) coincides with the continuous mapping \( T^{k+1} f | A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E) \).

Q.E.D.

A deformation \( g(t) \) contained in a \( \mathcal{D} \)-orbit \( O_g \) of \( g \) is called trivial, since each \((M,g(t))\) is isometric to \((M,g)\). On the other hand, a deformation contained in \( S_g \) is said to be essential with respect to \( g \). According to M. Berger and D. Ebin [1, §3, (3.1)], we can identify the tangent spaces \( T_g(O_g) \) and \( T_g(S_g) \) at \( g \) with \( \text{Im } \delta^* \) and \( \text{Ker } \delta \). We call therefore an element of \( \text{Im } \delta^* \) a trivial \( i \)-deformation and an element of \( \text{Ker } \delta \) an essential \( i \)-deformation.

Let \( g(t) \) and \( \bar{g}(t) \) be deformations of \( g \). If there is a 1-parameter family of diffeomorphisms \( \gamma(t) \) satisfying \( g(t) = \gamma(t)^* g(t) \), then \( g(t) \) is said to be equivalent to \( \bar{g}(t) \). Theorem 2.2 implies that every deformation is equivalent to an essential deformation (by restricting the range of \( t \) to some open interval containing 0).

**Lemma 2.9.** If \( g'(t) \) is trivial with respect to \( g(t) \) (i.e., \( g'(t) \in \text{Im } \delta^* \)) for each \( t \), then \( g(t) \) is a trivial deformation.

Proof. D. Ebin [5, Theorem 8.1 or 6, Proposit 8.30] shows that for given \( g \in \mathcal{M} \) and any neighbourhood \( V \) of the identity in \( \mathcal{D} \), there is a neighbourhood \( H \) of \( g \) in \( \mathcal{M} \) such that if \( \gamma \in V \) satisfying \( \gamma^{-1} \mathcal{H} \gamma \subset H \). So, we find \( \dim I_{g(t)} \) is upper semi-continuous. Let \( W \) be a connected component of the set of all \( t \) such that \( \dim I_{g(t)} \) is minimum. Then \( W \) is open in \( I \). Fixing \( t_0 \in W \), we shall apply Theorem 2.2 for \( g(t_0) \).

Let \( \bar{g}(t) \) be a deformation equivalent to \( g(t) \) contained in \( S_{g(t_0)} \). First we prove \( g'(t_0) = 0 \) for all \( t_1 \) for which \( g(t_1) \) is defined. If \( \gamma \in I_{g(t_1)} \), then \( \gamma^* g(t_1) = \bar{g}(t_1) \in S_{g(t_0)} \) and so \( \gamma \in I_{g(t_0)} \), because of the property (S2) in Theorem 2.2. This implies \( I_{g(t_2)} \subset I_{g(t_0)} \). Since \( t_0 \in W \), it follows that any Killing vector field with respect to \( g(t_0) \) is a Killing vector field with respect to \( \bar{g}(t_1) \). Now, because \( g'(t_1) \) is trivial with respect to \( \bar{g}(t_1) \), there is \( \xi \in T_{\mathcal{D}}(\mathcal{D}) \) such that \( g'(t_1) = TA_{\mathcal{D}, \mathcal{M}}(\xi, 0) \), where \( A \) is the map \( \mathcal{D} \times \mathcal{M} \to \mathcal{M} \) defined by the action of
\[D\] on \(\mathcal{M}\) and \(TA\) is the differential of \(A\). Denote by \(\pi\) the natural projection from \(D\) to \(D/\mathcal{I}_{g(t_0)}\) and let \(\chi\) be as in Theorem 2.2. Put \(\xi := TX \circ T\pi(\xi)\). Then \(\xi - \tilde{\xi}\) is a Killing vector field with respect to \(g(t_0)\), and so with respect to \(\bar{g}(t)\) also. Therefore \(TA_{(\text{id}, \tilde{r}(t))}(\xi - \tilde{\xi}, 0) = 0\), \(\bar{g}(t)\) being fixed under the action of \(I_{\tilde{r}(t)}\).

Now, set \(F^{-1} := p \times q\) where \(p: \mathcal{M} \to D/\mathcal{I}_{g(t_0)}\) and \(q: \mathcal{M} \to S_{g(t_0)}\). Since \(\bar{g}'(t_1)\) is tangent to \(S_{g(t_0)}\), \(TP(\bar{g}'(t_1)) = 0\). On the other hand,

\[
\begin{align*}
g'(t_1) &= TA_{(\text{id}, \tilde{r}(t_1))}(\xi, 0) \\
&= TA_{(\text{id}, \tilde{r}(t_1))}(\xi - \tilde{\xi}, 0) + TA_{(\text{id}, \tilde{r}(t_1))}(\tilde{\xi}, 0) \\
&= TA_{(\text{id}, \tilde{r}(t_1))}(TX \circ T\pi(\xi), 0) \\
&= TF_{(\text{id}, \tilde{r}(t_1))}(T\pi(\xi), 0),
\end{align*}
\]

hence \(Tq(g'(t_1)) = 0\). But \(Tq \times Tq\) is an isomorphism, and therefore \(g'(t_1) = 0\). We have thus proved that \(g(t)\) is constant on \(W\), and so \(g(t)\) is trivial on \(W\). By [5, Proposition 6.13 or 6, Theorem 8.10], a \(D\)-orbit is closed in \(M\). Let \(a\) be an end point of \(W\). Since \(W\) is open, \(a \in \mathcal{W}\). If \(a \in \mathcal{I}\), then \(g(a) \in \mathcal{O}_{g(t_0)}\), and so \(g(a)\) is isometric to \(g(t_0)\), which contradicts \(a \in \mathcal{W}\). Hence \(W = \mathcal{I}\). Q.E.D.

Let \(\mathcal{P}\) be a subset of \(\mathcal{M}\) invariant under the action of \(D\). For \(g \in \mathcal{P}\), we denote by \(\mathcal{P}_{\bar{g}}\) the vector space which is spanned by all \(t\)-deformations \(g'(0)\) defined by deformations \(g(t)\) contained in \(\mathcal{P}\).

**Definition 2.10.** If all deformations of \(g\) contained in \(\mathcal{P}\) are trivial then \(g\) is said to be non-deformable (in the sense of \(\mathcal{P}\)). If \(\mathcal{P}_{g} \subseteq \text{Im } \delta_{g}^{*}\) then \(g\) is said to be infinitesimally non-deformable (in the sense of \(\mathcal{P}\)).

**Theorem 2.11.** Let \(\mathcal{P}\) be a \(D\)-invariant subset of \(\mathcal{M}\). If there is a \(D\)-invariant open set \(W\) of \(\mathcal{P}\) such that all metrics in \(W\) are infinitesimally non-deformable, then every \(g \in W\) is non-deformable.

**Proof.** Let \(g(t): \mathcal{I} \to \mathcal{P}\) be any deformation of \(g \in W\) contained in \(\mathcal{P}\). Let \(J\) be the subset of \(\mathcal{I}\) of all \(t\) such that \(g(t) \in W\), and \(J_1\) be the connected component of \(J\) containing 0. Then \(g(t)\) is infinitesimally non-deformable for each \(t \in J_1\), and so, by Lemma 2.9, \(g(t)|J_1\) is trivial. If \(J_1\) does not coincide with \(\mathcal{I}\), then there is an end point \(t_0\) of \(J_1\) in \(\mathcal{I}\). Since \(D\)-orbits in \(\mathcal{M}\) are closed, \(g(t_0)\) is isometric to \(g\), which contradicts \(g(t_0) \notin W\). Thus \(J_1 = \mathcal{I}\). Q.E.D.

3. **Einstein deformations**

**Definition 3.1** We denote by \(\mathcal{E}\) the space of all Einstein metrics on \(M\) whose volume is some constant \(c\). A deformation contained in \(\mathcal{E}\) is called an Einstein deformation. If all Einstein deformations of \(g \in \mathcal{E}\) are trivial, then \(g\) is said to be non-deformable. (cf. Definition 2.10)

**Lemma 3.2.** Let \(g(t)\) be an Einstein deformation of \(g\). Then the essential
component $h$ of the i-deformation $g'(0)$ (i.e., $g'(0)=h+\delta^*\xi$ and $\delta h=0$) satisfies the following equalities:

$$\Delta h + 2Lh = 0, \quad \text{tr } h = 0,$$

where the operator $L$: $S_0^2 \to S_0^2$ is defined in 1; $(Lh)_{ij} = R_{ij}^k h_k$.

Proof. See M. Berger and D. Ebin [1, Lemma 7.1, (7.1)].

**Theorem 3.3.** Let $(M,g)$ be a compact Einstein manifold with $\rho=\varepsilon g$, $\rho$ being the Ricci tensor. Denote by $\alpha_0$ the minimum eigenvalue on $M$ of the operator $L$. If $\alpha_0 > \min \left\{ \varepsilon, -\frac{1}{2} \varepsilon \right\}$, then $(M,g)$ is non-deformable.

Proof. Owing to Theorem 2.11 and Lemma 3.2, it is sufficient to prove that if $h$ is an i-deformation of $g$ such that $\delta h=0$, $\Delta h + 2Lh=0$ and $\text{tr } h=0$ then $h=0$. First we define the operators $\Theta\nabla: C^\infty(S^2) \to C^\infty(T^2_0)$ and $\nabla: C^\infty(S^2) \to C^\infty(T^2_0)$ by

$$(\Theta\nabla) (X,Y,Z) = \alpha(\nabla_X h)(Y,Z) + \beta(\nabla_Y h)(Z,X) + \gamma(\nabla_Z h)(X,Y)$$

where, $\alpha, \beta, \gamma \in \mathbb{R}$, $\alpha^2 + \beta^2 + \gamma^2 = 1$. Set $u=\alpha\beta + \beta\gamma + \gamma\alpha$. Then the minimum and the maximum of $u$ are $-\frac{1}{2}$ and 1 respectively. By simple computations, we have

$$\langle \Theta\nabla h, \Theta\nabla h \rangle = \langle \nabla h, \nabla h \rangle + 2u \langle S\nabla h, \nabla h \rangle$$

$$= \langle \Delta h, h \rangle + 2u \langle \delta S\nabla h, h \rangle .$$

Now,

$$(\delta S\nabla h)_{ij} = -\nabla^l (S\nabla h)_{kij} = -\nabla^l \nabla^j h_{ik}$$

$$= g^{km} R_{mi}^l h_{jk} + g^{km} R_{mj}^i h_{ik} - \nabla_i \nabla^l h_{jk}$$

$$= -(Lh)_{ij} - \rho_j^j h_{ij} + \langle \nabla \delta h \rangle_{ij} .$$

Therefore, we get

$$\langle \Delta h - 2uLh - 2u\varepsilon h + 2u\nabla \delta h, h \rangle \geq 0 .$$

Here, we set $\delta h=0$ and $\Delta h=-2Lh$. Then

$$u\varepsilon \langle h, h \rangle \leq -(1+u) \langle Lh, h \rangle .$$

Thus, if $h \neq 0$ then we have $\alpha_0 \leq \varepsilon$ and $\alpha_0 \leq -\frac{1}{2} \varepsilon$, by setting $u=-\frac{1}{2}$, 1, respectively.

Q.E.D.

Let $N$ be a riemannian manifold and $O_p=X_i$ be an orthonormal frame at $p \in N$. Then $\sigma_{ij}=-R_{ij}^i$ is the sectional curvature if $i \neq j$, and is zero if $i=j$. We count the number of $j$ such that $\sigma_{ij}=0$ for an index $i_0$, and call the maximum of such numbers the **flat dimension** $fd(N)$ of $N$ when $p, O_p, i_0$ run over respective sets. For example, if $N$ has negative curvature, then $fd(N)=1$. 


Proposition 3.4. If an Einstein manifold \((M,g)\) has non-positive sectional curvature, and if its universal Riemannian covering \((\tilde{M},\tilde{g})\) is the product of the Riemannian manifolds \(\tilde{M}_a (1 \leq a \leq k)\) satisfying \(2\text{fd}(\tilde{M}_a) < \text{dim} \tilde{M}_a\), then \(g\) is non-deformable. Especially, an Einstein manifold \((M,g)\) is non-deformable, if all irreducible component of \((\tilde{M},\tilde{g})\) have negative sectional curvature and are of dimension \(>2\).

Proof. (I) First, we consider the case that \(\tilde{M}\) itself is such that \(2\text{fd}(\tilde{M}) < \text{dim} \tilde{M}\). Put \(r = \text{fd}(\tilde{M})\). Fix a point \(m\) in \(\tilde{M}\) and let \(Lh = \alpha h\) for a non-zero symmetric bilinear form \(h\) whose trace is zero. Using an orthonormal frame \(\{X_i\}\) at \(m\), we diagonalize \(h\) with respect to \(\tilde{g}\), and set \(\tilde{h}^i = x^i\). Then, \(\sum x^i = 0\) and
\[ R_{ijkl}h_{ij}h_{kl} = \sum_{ij} R_{ijkl}x^ix^j = -\sum_{ij} \sigma_{ij}x^ix^j . \]

Now, let \((y_i)\) be an eigenvector of the matrix \((\sigma_{ij})\) belonging to an eigenvalue \(\lambda\). By changing order of coordinates if necessary, we can assume that \(y_i = \max_j |y_j|\) and \(\sigma_{ij} < 0\) for all \(i > r\). Then,
\[ -\lambda y_i = -\sum \sigma_{ir} y_r \geq \sum \sigma_{ir} y_r = \varepsilon y_r . \]

So \(-\lambda \geq \varepsilon\) and, if the equality holds, then we have \(y_i = -y_r\) for all \(i > r\), which implies
\[ \sum y_i = \sum y_i + \sum y_i \leq -(n-r)y_r + ry_r = -(n-2r)y_r < 0 . \]
Therefore, for \((x_i)\) such that \(\sum x^i = 0\), we have
\[ -\sum \sigma_{ij}x^ix^j > \varepsilon \sum (x^i)^2 . \]
Hence,
\[ \alpha(h,h) = -\sum \sigma_{ij}x^ix^j > \varepsilon \sum (x_i)^2 = \varepsilon(h,h) . \]
Thus we get \(\alpha > \varepsilon\). Our assertion follows then from Theorem 3.3.

(II) Now we consider the general case. Corresponding to the decomposition \((\tilde{M},\tilde{g}) = \bigsqcup (\tilde{M}_a,\tilde{g}_a)\), the curvature tensor decomposes. Hence, the Ricci tensor \(\rho\) of \(\tilde{M}\) has the decomposition \(\rho = \sum \tilde{\rho}_a\) where \(\tilde{\rho}_a\) is the Ricci tensor of \(\tilde{M}_a\). Therefore \(\tilde{\rho}_a = \varepsilon \tilde{g}_a\). Moreover, \(S^2(\tilde{M})\) and the operator \(\tilde{L}\) on \(S^2(\tilde{M})\) decompose as follows;
\[ S^2(\tilde{M}) = (\bigoplus_a S^2(\tilde{M}_a)) \oplus ((\bigoplus_a \tilde{R}\tilde{g}_a) \cap S^2(\tilde{M})) \oplus \sum_{a \neq b} S^2(\tilde{M}_a,\tilde{M}_b) , \]
\[ \tilde{L}|S^2(\tilde{M}_a) = \tilde{L}_a , \]
\[ \tilde{L}|((\bigoplus_a \tilde{R}\tilde{g}_a)) \cap S^2(\tilde{M}) = -\varepsilon , \]
\[ \tilde{L}|S^2(\tilde{M}_a,\tilde{M}_b) = 0 \text{ for } a \neq b , \]
where \(\tilde{L}_a\) is the operator of \(\tilde{M}_a\) and
\[ S^2(\tilde{M}_a,\tilde{M}_b) = \{ h \in S^2(\tilde{M}_a \times \tilde{M}_b); h(TM_a, TM_b) = 0 \text{ for } c = a,b \} . \]
Since the curvature of $(M,g) \leq 0$, $\varepsilon$ is negative. Then, combined with what we have proved in (I), we get $\alpha_0 > \varepsilon$ and our assertion follows from Theorem 3.3 Q.E.D.

**Corollary 3.5.** Let $(M,g)$ be a compact Einstein manifold. If $M$ is a locally symmetric space of non-compact type, and the dimension of every irreducible component of the universal covering $(\tilde{M},\tilde{g})$ of $(M,g)$ is greater than 2, then $(M,g)$ is non-deformable.

Proof. Let $G/K$ be a symmetric space which is the universal covering of $(M,g)$. Since the dimension of every irreducible component of $G/K$ is greater than 2, we may assume that $G$ has no simple factor of dimension 3. On the other hand A. Weil [13, §10] shows that if $G$ has no simple factor of dimension 3, then $\alpha_0 > \varepsilon$. Thus the proof reduces to Theorem 3.3.

**Remark 3.6.** Theorem 24.1' in G.D. Mostow [10] implies that if $(M,g_1)$ and $(M,g_2)$ are locally symmetric spaces of non-compact type without 2-dimensional factors locally, then $g_1$ and $g_2$ are isometric up to normalizing constants. (cf. E. Calabi [3, Theorem 1], A. Weil [13, Theorem 1])

**Corollary 3.7.** If the sectional curvature of a compact Einstein manifold $(M,g)$ ranges in the interval $\left(\frac{n-2}{2n-1}, 1\right]$, then $(M,g)$ is non-deformable.

Proof. We easily see that $\varepsilon = \frac{1}{n} \sum_{i \neq j} \sigma_{ij}$, hence the condition implies $\varepsilon > (n-2)(n-1)/(2n-1)$. By virtue of Theorem 3.3, it is sufficient to prove $\alpha_0 + \frac{1}{2}\varepsilon > 0$. In the same way as for the proof I of Proposition 3.4, we may set $h_i^i = x^i$ with $\sum x^i = 0$. We can assume that there is an integer $c$ such that $y^i = x^i \geq 0$ for any $i \leq c$, and $z^i = -x^i > 0$ for any $i > c$. Set $\sum_{i \leq c} y^i = \sum_{i > c} z^i = A$. Then, since $\sum x^i = 0$,

$$(Lh,h) + \frac{1}{2}(\varepsilon h,h) = -\sum_{i \neq j} \sigma_{ij} x^i x^j + \frac{1}{2} \varepsilon \sum_{i} (x^i)^2 + \sum_{i} x^i \sum_{j} x^j$$

$$= \left(1 + \frac{1}{2}\varepsilon \right) \sum_{i \leq c} (y^i)^2 + \sum_{i \leq c} (z^i)^2 + \sum_{i \leq j, i,j \geq c} (1-\sigma_{ij}) y^i y^j$$

$$+ \sum_{i \leq j, i,j > c} (1-\sigma_{ij}) z^i z^j - 2 \sum_{i \leq c, j > c} (1-\sigma_{ij}) y^i z^j$$

$$> \frac{n(n+1)}{2(2n-1)} \left(\sum_{i \leq c} (y^i)^2 + \sum_{i \leq c} (z^i)^2\right) - 2 \frac{n+1}{2n-1} A^2$$

$$\geq \frac{n(n+1)}{2(2n-1)} \left(\frac{1}{c} A^2 + \frac{1}{n-c} A^2\right) - 2 \frac{n+1}{2n-1} A^2$$

$$\geq \frac{n(n+1)}{2(2n-1)} \frac{4}{n} A^2 - 2 \frac{n+1}{2n-1} A^2 = 0.$$
REMARK 3.8. Y. Muto [11, Theorem] shows that every Einstein metric near a metric with positive constant sectional curvature is of positive constant sectional curvature.

REMARK 3.9. Even if \((M, g)\) is a non-deformable Einstein metric, \(M\) may have an Einstein metric \(\tilde{g}\) which is not isometric to \(g\). In fact, G.R. Jensen [8, pp. 612–613] constructs a non-standard Einstein metric \(\tilde{g}\) on \(S^{4p+3}\). The author does not know whether \(\tilde{g}\) is non-deformable or not.

Finally, by a direct computation, we may apply Theorem 3.3 to the manifold \(M\) whose universal covering \(\tilde{M}\) is an irreducible symmetric space \(G/K\) of compact type.

I. the case where \(\tilde{M}\) is hermitian symmetric

In this case, the eigenvalue of the generalized operator \(\tilde{L}: S^2 S^2\) are calculated by E. Calabi and E. Vesentini [4, p. 502, Table 2] and A. Borel [2, Corollary 4.6, 4.7]. See Table 1. Here we omit 0 and \(-\varepsilon\), which are always eigenvalues of \(\tilde{L}\). The eigenspace corresponding to this eigenvalue \(-\varepsilon\) is generated by \(g\). Hence, this is not an eigenvalue of our operator \(L\) on \(S^6\). We conclude that the following three classes are non-deformable.

- AIII \((p=1), (q=1)\)
- DIII \((p\geq 6)\)
- EVII

<table>
<thead>
<tr>
<th>type</th>
<th>(\dim_{\mathbb{C}}M)</th>
<th>(G/M)</th>
<th>(\alpha \varepsilon^{-1}/\text{multiplicity})</th>
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<tr>
<td>AIII</td>
<td>(pq)</td>
<td>(SU(p+q)) (\times) (U) (\times) (U) (\times) (U)</td>
<td>(2(p+q)^{-1}) ((p+1)^{-1}) ((q+1)^{-1}) (2\left(p\right)\left(q\right))</td>
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<td>DIII</td>
<td>(p)</td>
<td>(SO(2p)) (\times) (U) (\times) (U)</td>
<td>((p-1)^{-1}) (1) (\frac{1}{6} p^2(p^2-1))</td>
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<tr>
<td>CI</td>
<td>(p+1)</td>
<td>(Sp(p)) (\times) (U) (\times) (U)</td>
<td>(2(p+1)^{-1}) (2\left(p+3\right))</td>
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<tr>
<td>BDI</td>
<td>(p)</td>
<td>(SO(p+2)) (\times) (U) (\times) (U)</td>
<td>(2p^{-1}) (2) ((p-1)(p+2))</td>
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<tr>
<td>EIII</td>
<td>(16)</td>
<td>(E_6) (\times) (\mathbb{R}^2) (\times) (U) (\times) (U) (\times) (U)</td>
<td>(\times) (252) (\times) (2)</td>
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<tr>
<td>EVII</td>
<td>(27)</td>
<td>(E_7) (\times) (\mathbb{R}^2) (\times) (U) (\times) (U) (\times) (U)</td>
<td>(\times) (702) (\times) (54)</td>
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</table>
II) Other cases

By easy but complicated computations we can compute \( \alpha_0 \). Let \( g = \mathfrak{k} + \mathfrak{m} \) be the orthogonal decomposition with respect to the Killing form on \( g \), where \( \mathfrak{k} \) is the Lie algebra of \( K \). Then the tangent space \( T_eK(\mathbb{M}) \) at the identity coset is canonically identified with \( \mathfrak{m} \), and we know that \( R(X,Y)Z = -[[X,Y],Z] \) for \( X,Y,Z \subseteq \mathfrak{m} \). (See S. Kobayashi and K. Nomizu [9, p. 231 Theorem 3.2].) We can compute the eigenvalue of the curvature operator \( L \) which is identified with the linear endomorphism on \( S^2_0(\mathfrak{m}) \), and we get Table 2 for the type BDI and CII. Hence the following symmetric spaces are non-deformable, where we assume \( p \geq q \):

- **BDI** \( (p 
\geq 3, q = 1) \), \( (q \geq p - 1, p + q \geq 7) \)
- **CII** \( (p = q = 1) \), \( (p \geq 3, q = 1) \).

<table>
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<tr>
<th>type</th>
<th>n</th>
<th>G/K</th>
<th>( \alpha e^{-1} )</th>
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<td>q</td>
<td>( SO(p+q) ) ( SO(p) \times SO(q) )</td>
<td>( p &gt; q = 1 ) ( (p-1)^{-1} ) ( p \geq q \geq 2 ) ( \pm 2(p+q-2)^{-1}, (2-p)(p+q-2)^{-1}, (2-q)(p+q-2)^{-1} )</td>
</tr>
<tr>
<td>CII</td>
<td>4pq</td>
<td>( Sp(p+q) ) ( Sp(p) \times Sp(q) )</td>
<td>( p = q = 1 ) ( \frac{1}{3} )</td>
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</table>

(\( ^* \) condition)

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References


