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NON-DEFORMABILITY OF EINSTEIN METRICS

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Introduction

Let M be a compact connected C^{∞} -manifold and g be an Einstein metric on M. By an Einstein deformation of g we mean a 1-parameter family g(t) of Einstein metrics on M such that g(0)=g and the volume of g(t) is constant for t. If for each Einstein deformation g(t) of g there exists a 1-parameter family $\gamma(t)$ of diffeomorphisms such that $g(t) = \gamma(t)^* g$ (resp. $g'(0) = \frac{d}{dt}|_0 \gamma(t)^* g$) then g is said to be non-deformable (resp. infinitesimally non-deformable). M. Berger and D. Ebin [1, Lemma 7.4] show that the Einstein structure of the standard sphere is infinitesimally non-deformable, by using the fact that the operator L associated to the curvature tensor of the standard sphere is positive definite. In this paper, the main theorem (Theorem 3.3) gives a criterion for an Einstein structure to be non-deformable, improving their method of estimating eigenvalues of the operator L. As an application we see, for example, that the Einstein structure of a compact irreducible locally symmetric space Mof non-compact type with dim M>2 is non-deformable. (Corollary 3.5)

To prove the main theorem we have to relate infinitesimal non-deformability to non-deformability. For this purpose we need a smooth slice theorem. The slice theorem (Theorem 2.1) in the H^s -situation (D. Ebin [5, Theorem 7.1], [6, Theorem 8.20]) being in continuous category, we shall improve this continuous slice theorem to a smooth slice theorem (Theorem 2.2) in the ILHsituation. Owing to this we get a theorem (Theorem 2.11) which relates infinitesimal non-deformability to non-deformability.

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1. Preliminaries

First, we introduce notation which will be used throughout this paper. Let M be an *n*-dimensional, connected and compact C^{∞} -manifold without boundary, and we always assume n>2. For a riemannian manifold (M,g), we consider the riemannian connection and use the following notation;

 S^2 ; the symmetric covariant 2-tensor bundle over M,

 $C^{\infty}(T)$; the vector space of all C^{∞} -sections of a tensor bundle T over M,

 S_0^2 ; the space of all symmetric covariant 2-tensors whose trace is zero, (,); the inner product in fibers of a tensor bundle defined by the riemannian

structure,

 \langle , \rangle ; the global inner product for sections of a tensor bundle over M, i.e., $\langle , \rangle = f_M(,)v_g, v_g$ being the volume element defined by g,

R; the curvature tensor,

 ρ ; the Ricci tensor,

 τ ; the scalar curvature,

 ∇ ; the covariant derivation on $C^{\infty}(T)$,

 δ ; the formal adjoint of ∇ with respect to \langle , \rangle ,

 δ^* ; the formal adjoint of $\delta | C^{\infty}(S^2)$,

 $\Delta = \delta d$; the Laplacian operating on the space $C^{\infty}(M)$ of C^{∞} -functions on M,

 $\overline{\Delta} = \delta \nabla$; the rough Laplacian operating on $C^{\infty}(T)$,

Hess= ∇d ; the Hessian on $C^{\infty}(M)$.

We shall use the Einstein's convention, although we use \sum if necessary. We shall apply the following formulae throughout the paper.

$$\begin{split} R_{ijl}^{k}\xi^{l} &= \nabla_{i}\nabla_{j}\xi^{k} - \nabla_{j}\nabla_{i}\xi^{k}, \ R_{ijkl} = R_{ijk}^{m}g_{ml}, \\ \rho_{ij} &= -R_{ilj}^{l}, \ \tau = \rho_{l}^{l}, \\ (\delta S)_{j_{2}\cdots j_{s}}^{i_{1}\cdots i_{r}} &= -\nabla^{l}S_{lj_{2}\cdots j_{s}}^{i_{1}\cdots i_{r}}, \ (\delta^{*}\xi)_{ij} = \frac{1}{2}(\nabla_{i}\xi_{j} + \nabla_{j}\xi_{i}), \\ \Delta f &= -\nabla^{l}d_{l}f, \ (\overline{\Delta}S)_{j_{1}\cdots j_{s}}^{i_{1}\cdots i_{r}} = -\nabla^{l}\nabla_{l}S_{j_{1}\cdots j_{s}}^{i_{1}\cdots i_{r}}. \end{split}$$

(For the standard sphere, $R_{1212} < 0$, $\rho_{11} > 0$ and $\tau > 0$, with respect to orthonormal frame.)

Let (M,g) be an Einstein manifold. If tr h=0 then

$$g^{ij}R_{i}{}^{k}{}_{j}{}^{l}h_{kl} = -\rho^{kl}h_{kl} = 0.$$

Hence we can define the operator $L: S_0^2 \rightarrow S_0^2$ by

$$(Lh)_{ij} = R_i^{k}{}_j^{l}h_{kl}.$$

Next, we recall the following concepts defined by H. Omori [12, pp. 168– 169]. A topological vector space E is called an *ILH-space*, if E is an inverse limit of Hilbert spaces $\{E_i\}_{i=1,2,\cdots}$ such that if $j \ge i E_i \supset E_j$ and the inclusion is a bounded linear operator. We denote $E=\lim E_i$.

A topological space X is called a C^{*} -ILH-manifold modeled on E, if X has the following properties C1 and C2.

C1) X is an inverse limit of C^k-Hilbert manifolds $\{X_i\}_{i=1,2,\dots}$ such that

each X_i is modeled on E_i and $X_i \supset X_i$ if $j \ge i$.

2) Let x be any point of X. For each *i* there are an open neighbourhood $U_i(x)$ of x in X_i and a homeomorphism ψ_i from $U_i(x)$ onto an open subset V_i in E_i which gives a C^k -coordiante around x in X_i and satisfies $U_i(x) \supset U_j(x)$ if $j \ge i$ and $\psi_{i+1}(y) = \psi_i(y)$ for every $y \in U_{i+1}(x)$.

Let X be a C^{k} -ILH-manifold $(k \ge 1)$. Let TX_{i} be the tangent bundle of X_{i} . Then the inverse limit $TX = \lim TX_{i}$ is called the *ILH-tangent bundle* of X.

Let X, Y be C^k-ILH-manifolds. A mapping $\phi: X \to Y$ is said to be C^l-ILH-differentiable $(l \leq k)$, if ϕ is an inverse limit of C^l-defferentiable mappings, that is, for every *i*, there are a positive integer j(i) and a C^l-mapping $\phi_i: X_{j(i)} \to Y_i$ such that $\phi_i(x) = \phi_{i+1}(x)$ for every $x \in X_{j(i+1)}$ and $\phi = \lim \phi_i$.

If X is a C^k-ILH-manifold for all $k \ge 0$, we call X an *ILH-manifold*. For ILH-manifolds X, Y, if ϕ is C^k-ILH-defferentiable for all $k \ge 0$, we say that ϕ is *ILH-differentiable*. We denote by T_xX_i the tangent space of X_i at x and put $T_xX = \lim T_xX_i$. Also we denote by

$$T^{r}\phi_{i}(x): \prod_{l=1}^{r} T_{x}X_{i}(i) \to T_{\phi_{x}}Y_{i}$$

the r-th derivative of ϕ_i at $x \in X$. Then, it is easy to check that $\{T^r \phi_i(x)\}_{i=1,2,\dots}$ has an inverse limit

$$\lim T^r \phi_i(x) \colon \prod_{l=1}^r T_x X \to T_{\phi_x} Y.$$

We call this inverse limit the *r*-th derivative of ϕ and denote it by $T^r\phi(x)$.

A topological group is called an *ILH-Lie group*, if it is an ILH-manifold and the group operations are ILH-mappings.

We can easily see that the space \mathcal{M} of all smooth riemannian metrics on M is an ILH-manifold. (See D. Ebin [5, p.15], [6, Proposition 5.8] and H. Omori [12, p.170].) We know that the group \mathcal{D} of all diffeomorphisms of M is an ILH-Lie group, and the natural action $A: \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$ is ILH-differentiable. (See [12, Lemma 2.5].)

Let $g \in \mathcal{M}$. By a deformation of g we mean a C^{∞} -curve $g(t): I \to \mathcal{M}$ such that g(0)=g, where I is an open interval containing 0 in **R**. Since \mathcal{M} is a positive cone in the vector space of all symmetric covariant 2-tensors on M, we may identify the differential g'(0) of a deformation g(t) with a symmetric covariant 2-tensor field on M. We call such a tensor field an *infinitesimal deformation*, or simply an *i-deformation*.

When we consider a deformation g(t) of g, the covariant derivation, the curvature tensor or the Ricci tensor with respect to each g(t) will be *denoted by* ∇_t , R(t) or $\rho_{g(t)}$. Also, we always raise or lower indices of tensors with respect to g(t), and we *denote by* ' the differentiation with respect to t. It is clear that the differential at t=0 of the tensors R, ρ , τ etc. depend only on the i-deformation that g(t) defines.

2. Deformations and infinitesimal deformations

Let M be a compact connected C^{∞} -manifold. We denote by \mathcal{M}^{s} the space of all H^{s} -metrics on M and by \mathcal{D}^{s} the space of all H^{s} -diffeomorphisms of M, where H^{s} means an object which has partial derivatives defined almost everywhere up to order s and such that each partial derivative is square integrable. We know that the space \mathcal{M}^{s} and the space \mathcal{D}^{s} are Hilbert manifolds if s is sufficiently large. Moreover, the usual action $A: \mathcal{D} \times \mathcal{M} \rightarrow \mathcal{M}$ extends to a continuous mapping $A^{s}: \mathcal{D}^{s+1} \times \mathcal{M}^{s} \rightarrow \mathcal{M}^{s}$. (See D. Ebin [5,p.18], [6, Proposition 4.24].)

D. Ebin gave the following

Theorem 2.1 (D. Ebin [5, Theorem 7.1], [6, Theorem 8.20]). For each $g \in \mathcal{M}$, there is a submanifold S_g^s of \mathcal{M}^s with the following properties.

- (1) If $\gamma \in I_g$, then $\gamma^*(S_g^s) = S_g^s$.
- (2) Let $\gamma \in \mathcal{D}^{s+1}$. If $\gamma^*(S_g^s) \cap S_g^s \neq \phi$, then $\gamma \in I_g$.

(3) There are a neighbourhood U^{s+1} of the coset I_g in the right coset space \mathcal{D}^{s+1}/I_g and a local cross section $\chi^{s+1}: \mathcal{D}^{s+1}/I_g \to \mathcal{D}^{s+1}$ defined on U^{s+1} such that if the mapping $F^s: U^{s+1} \times S^s_g \to \mathcal{M}^s$ is defined by $F^s(u,s) = \chi^{s+1}(u)^*s$, then F^s is a homeomorphism onto a neighbourhood of g.

Outline of the proof. Canonically we can construct a riemannian metric on \mathcal{M}^s which is invariant under the action A of \mathcal{D} . At any point $g \in \mathcal{M}^s$, $\psi_g^s: \mathcal{D}^{s+1} \to \mathcal{M}^s$ is defined by $\psi_g^s(\eta) = A^s(\eta, g)$ for $\eta \in \mathcal{D}^{s+1}$. If $g \in \mathcal{M}$, ψ_g^s is smooth. Also for $\eta \in \mathcal{D}^{s+1}$, we identify the tangent spaces $T_{\eta}(\mathcal{D}^{s+1})$ or $T_{\psi_g^s(\eta)}(\mathcal{M}^s)$ with the space of H^s -sections of some vector bundle over M. Then, $T_{\eta}\psi_g^s$ becomes a first order linear differential operator. It turns out that this operator has an injective symbol, and so its range is closed in $T_{\psi_g^s(\eta)}(\mathcal{M}^s)$.

The right coset space \mathcal{D}^{s+1}/I_g has an induced manifold structure and admits a smooth local cross section χ^{s+1} : $U^{s+1} \rightarrow \mathcal{D}^{s+1}$. ψ_g^s induces a mapping ϕ_g^s : \mathcal{D}^{s+1}/I_g $\rightarrow \mathcal{M}^s$. ϕ_g^s is an injective immersion and we see directly that it is a diffeomorphism onto the closed orbit O_g^s .

Using the riemannian metric on \mathcal{M}^s , we obtain a smooth normal bundle $\pi^s: v^s \to O_g^s$. Moreover, the exponential mapping \exp^s on \mathcal{M}^s is defined on a neighbourhood W^s of the zero-section of v^s and it is a diffeomorphism. We put $S_g^s = \exp^s W_g^s$, where W_g^s is the fibre on g.

Also, we know that for any $\eta \in \mathcal{D}^{s+1}$ a smooth mapping $\eta^* \colon \mathcal{M}^s \to \mathcal{M}^s$ is defined by $\eta^*(g) = A(\eta, g)$ and η^* is an isometry. Therefore, if \exp^s is defined for a vector V in $T(\mathcal{M}^s)$, \exp^s is defined for $T\eta^*(V)$ and we have $\eta^* \exp^s V = \exp^s T\eta^* V$.

Combining these informations, we can prove the slice theorem in the H^s situation. Moreover, if we define the mapping $F^s: U^{s+1} \times S^s_s \to \mathcal{M}^s$ by $F^s(u,s) = A^s(\chi^{s+1}(u), s)$, for $z \in \exp^s W^s$ we have $(F^{s})^{-1}(z) = ((\phi_{g}^{s})^{-1} \circ \pi^{s} \circ (\exp^{s})^{-1}(z), A((\mathcal{X}^{s+1} \circ (\phi_{g}^{s})^{-1} \circ \pi^{s} \circ (\exp^{s})^{-1}(z))^{-1}, z)).$

We shall need the following slice theorem which improve Theorem 2.1 to the C^{∞} -situation.

Theorem 2.2. We denote by \mathcal{M} the ILH-manifold formed by all riemannian metrics on M, and by \mathcal{D} the ILH-Lie group of all diffeomrophisms on M. The group \mathcal{D} acts on \mathcal{M} in a canonical way. For each $g \in \mathcal{M}$, there is an ILH-submanifold S_g of \mathcal{M} with the following properties. Let I_g be the group of all isometries of the riemannian manifold (M,g).

(S1) If γ belongs to I_g , then $\gamma^*(S_g) = S_g$.

(S2) Let $\gamma \in \mathcal{D}$. If $\gamma^*(S_g) \cap S_g \neq \phi$, then $\gamma \in I_g$.

(S3) There are a neighbourhood U of the point I_g in the right coset space $\mathcal{D}|I_g$ and a local cross section $\chi: \mathcal{D}|I_g \to \mathcal{D}$ defined on U such that if the mapping $F: U \times S_g \to \mathcal{M}$ is defined by $F(u,s) = \chi(u)^*s$, then F is an ILH-diffeomorphism onto a neighbourhood of g.

We need the following lemmas.

Lemma 2.3. \mathcal{D}/I_g is an ILH-manifold.

Lemma 2.4. Put $U=U^s \cap \mathcal{D}/I_g$. Then $\chi^s(U)$ is contained in \mathcal{D} and the mapping $\chi = \chi^s | U$ is ILH-differentiable.

Lemma 2.5. Put $W = W^s \cap T\mathcal{M}$. Then $\exp^s(W)$ is contained in \mathcal{M} and the mapping $\exp = \exp^s | W$ is an ILH-diffeomorphism. Hence $S_g = S_g^s \cap \mathcal{M}$ is an ILH-submanifold of \mathcal{M} .

These lemmas will be proved in below.

Lemma 2.6 [12, Lemma 2.5]. $A^{s}(\mathcal{D} \times \mathcal{M})$ is contained in \mathcal{M} and the mapping $A = A^{s} | \mathcal{D} \times \mathcal{M}$ is ILH-differentiable.

Lemma 2.7 [12, Lemma 1.14]. If the mapping $\tilde{i}: \mathcal{D} \to \mathcal{D}$ is defined by $\tilde{i}(\eta) = \eta^{-1}$ for $\eta \in \mathcal{D}$, then \tilde{i} is ILH-differentiable.

Proof of Theorem 2.2. Combining these lemmas and the proof of Theorem 2.1, the mappings $F=F^s | U \times S_g$ and $F^{-1}=(F^s)^{-1} | \exp W$ are compositions of ILH-mappings, and so F is an ILH-diffeomorphism, which proves Theorem 2.2.

Proof of Lemma 2.3. We know that \mathcal{D}^s/I_g is a Hilbert manifold. We shall prove that the inclusion $i^s: \mathcal{D}^{s+1}/I_g \to \mathcal{D}^sI_g$ is smooth. By [5, Corollary 5.11] or [6, Corollary 7.16], i^s is smooth if and only if $i^s \circ p^{s+1}: \mathcal{D}^{s+1} \to \mathcal{D}^s/I_g$ is smooth, where $p^{s+1}: \mathcal{D}^{s+1} \to \mathcal{D}^{s+1}/I_g$ is the natural projection. We can easily see $i^s \circ p^{s+1} =$ $p^s \circ i^s$, where $i^s: \mathcal{D}^{s+1} \to \mathcal{D}^s$ is the inclusion. Since i^s and p^s are smooth, $\overline{i^s}$ is smooth.

Proof of Lemma 2.4. By [5, Proposition 5.10] or [6, Proposition 7.15], \mathcal{D}^{s}/I_{x} admits a smooth local cross section around any coset. We denote by χ_x^s the local cross section around $x \in \mathcal{D}^s/I_g$ and put $\chi^s = \chi_{I_g}^s$. Let U^s be the domain of χ^s and set $U' = U^s \cap \mathcal{D}'/I_s$ and $\chi^r = \chi^s | U'$ for $r \ge s$. If $u \in U'$, there is an element $a \in D^r$ such that $u = I_g a$ and $\chi^r(u) \in I_g a \subset \mathcal{D}^r$. Hence we have $\chi'(U') \subset \mathcal{D}'$. To prove that χ' is smooth, we shall show that if we define a mapping $\nu: (p^s)^{-1}(U^s) \to I_{\sigma}$ by $\nu(\eta) = \eta(\chi^s \circ p^s \eta)^{-1}$, then ν is smooth. By [5, Lemma 5.5] or [6, Corollary 7.7], the composition: $I_{\sigma} \times \mathcal{D}^{s} \to \mathcal{D}^{s}$ is smooth. Hence, if we define a mapping $\psi: I_{\sigma} \times U^s \to \mathcal{D}^s$ by $\psi(\xi, x) = \xi \chi^s(x)$, then ψ is smooth. On the other hand, we have $\psi^{-1}(\eta) = (\nu(\eta), p^s(\eta))$ and p^s is smooth. For ν , we fix a positive integer *i* such that the composition: $\mathcal{D}^{i} \times \mathcal{D}^{i} \rightarrow \mathcal{D}^{i}$ and the inverse: $\mathfrak{D}^s \rightarrow \mathfrak{D}^i$ are C¹-mappings. ([12, Lemma 1.13 and Lemma 1.14]. Suppose that s is sufficientry large.) Then, we see directry that v is a C^{1} mapping into \mathcal{D}^i . But I_g contains the image of ν and I_g is a submanifold of \mathcal{D}^i (see [5, Corollary 5.4] or [6, Theorem 7.1]). Hence, ν is a C¹-mapping into I_{σ} . Therefore, we know that ψ is smooth and ψ^{-1} is a C¹-mapping. By the inverse function theorem, ψ^{-1} is smooth and so ν is smooth.

Now, we shall prove the smoothness of χ' around any $x \in U'$. There is a smooth local cross section χ'_x on a neighbourhood V of x. Therefore the mapping $\nu \circ$ "inclusion" $\circ \chi'_x$: $V \to I_g$ is smooth and we have $\nu \circ$ "inclusion" $\circ \chi'_x(y) = \chi'_x(y) (\chi^s(y))^{-1} = \chi'_x(y) (\chi^r(y))^{-1}$. Since we know that $\chi^r(y) = ((\chi'_x(y)) (\chi^r(y))^{-1})^{-1} \chi'_x(y)$ and the inverse: $I_g \to I_g$ and the composition: $I_g \times \mathcal{D}^r \to \mathcal{D}^r$ are smooth, the mapping $\chi^r: V \to \mathcal{D}^r$ is smooth.

Proof of Lemma 2.5. Let \overline{W}^s be an open subset of $T\mathcal{M}^s$ such that $W^s = v^s \cap \overline{W}^s$. Set $\overline{W}^r = \overline{W}^s \cap T\mathcal{M}^r$, $W^r = \overline{W}^s \cap v^r$, $\exp^r = \exp^s | \overline{W}^r$ and $(\exp^{-1})^r = (\exp^s | W^s)^{-1} | \exp^s(W^s) \cap \mathcal{M}^r$. The mappings $\exp^s \colon \overline{W}^s \to \mathcal{M}^s$ and $(\exp^{-1})^s \colon \exp^s(W^s) \to T\mathcal{M}^s$ are smooth and commute with the action of \mathcal{D} . Hence, by the following Lemma 2.8, $\exp^r(\overline{W}^r)$ and $(\exp^{-1})^r(\exp(W^s) \cap \mathcal{M}^r)$ are contained in \mathcal{M}^r and $T\mathcal{M}^r$ respectively, and the mappings $\exp^r \colon \overline{W}^r \to \mathcal{M}^r$ and $(\exp^{-1})^r \colon \exp^s(W^s) \cap \mathcal{M}^r \to T\mathcal{M}^r$ are smooth for $r \ge s$. But W^r is a submanifold of \overline{W}^r and $(\exp^{-1})^r(\exp^s(W^s) \cap \mathcal{M}^r) \cap \mathcal{M}^r)$ is contained in W^r , which implies that $\exp^r \colon W^r \to \exp^s(W^s) \cap \mathcal{M}^r$ is a diffeomorphism. Thus we see that exp is an ILH-diffeomorphism onto $\exp^s(W^s) \cap \mathcal{M}$.

Lemma 2.8. Let E and F be vector bundles over M associated with the frame bundle (e.g., T, T*, S², $T \times T^*$, the k-th jet bundle $J^*(T)$ etc.). Any $\eta \in \mathcal{D}$ defines a natural linear mapping η^* : $H^0(E) \rightarrow H^0(E)$. Let A be an open subset of $H^s(E)$ and let $f: A \rightarrow H^s(F)$ be a smooth mapping which commutes with the action of

 \mathcal{D} . Put $A^r = A \cap H^r(E)$ for $r \ge s$. Then $f(A^r)$ is contained in $H^r(F)$ and $f \mid A^r \rightarrow H^r(F)$ is smooth.

Proof of Lemma 2.8. We shall prove that if this lemma holds for r=i, then the same is true for r=i+1. The induction will then complete the proof. First, by induction, we shall prove that $\eta^* \circ T^k f(a) = T^k f(\eta^*a) \circ \eta^*$ for all positive integer k. If $\eta^* \circ T^l f(a) = T^l f(\eta^*a) \circ \eta^*$, then we have

$$\begin{split} \eta^* \circ T^{l+1} f(a) \, (v, v_1, \cdots, v_l) &= \eta^* \frac{d}{dt} |_{\,_0} T^l f(a + tv) \, (v_1, \cdots, v_l) \\ &= \frac{d}{dt} |_{\,_0} T^l f(\eta^* a + t\eta^* v) \, (\eta^* v_1, \cdots, \eta^* v_l) \\ &= T^{l+1} f(\eta^* a) \, (\eta^* v, \eta^* v_1, \cdots, \eta^* v_l) \, . \end{split}$$

Let V be a vector field on M and let η_t be the 1-parameter subgroup of diffeomorphisms generated by V. For sufficiently small $t, \eta_t^* a \in A^i$ if $a \in A^i$. Hence we get

$$\begin{split} \mathcal{L}_{v}T^{k}f(a)\left(v_{1},\cdots,v_{k}\right) &= \frac{d}{dt}|_{0}\eta_{t}^{*}T^{k}f(a)\left(v_{1},\cdots,v_{k}\right) \\ &= \frac{d}{dt}|_{0}T^{k}f(\eta_{t}^{*}a)\left(\eta_{t}^{*}v_{1},\cdots,\eta_{t}^{*}v_{k}\right) \\ &= T^{k+1}f(a)\left(\mathcal{L}_{v}a,v_{1},\cdots,v_{k}\right) + T^{k}f(a)\left(\mathcal{L}_{v}v_{1},v_{2},\cdots,v_{k}\right) \\ &+ \cdots + T^{k}f(a)\left(v_{1},\cdots,v_{k-1},\mathcal{L}_{v}v_{k}\right). \end{split}$$

Next, we shall prove that $f(A^{i+1}) \subset H^{i+1}(F)$, and $f|A^{i+1}: A^{i+1} \rightarrow H^{i+1}(F)$ is continuous and that if $f|A^{i+1}$ is a C^{k} -mapping and $T^{k}(f|A^{i+1}) = T^{k}f|A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$, then $f|A^{i+1}$ is a C^{k+1} -mapping and $T^{k+1}(f|A^{i+1}) = T^{k+1}f|A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$. Then, by the hypothesis of the induction, $f|A^{i+1}$ is smooth.

If $a \in A^{i+1}$, then $\mathcal{L}_v a \in H^i(E)$ for all $V \in C^{\infty}(T)$. Hence $\mathcal{L}_v f(a) = Tf(a)$ $(\mathcal{L}_v a) \in H^i(F)$, which implies that $f(A^{i+1}) \subset H^{i+1}(F)$. If a sequence $\{a_n\}$ converges to a in A^{i+1} , then $\{\mathcal{L}_v a_n\}$ converges to $\mathcal{L}_v a$ in $H^i(E)$ for all $V \in C^{\infty}$ (T). Hence $\{\mathcal{L}_v f(a_n) = Tf(a_n) (\mathcal{L}_v a_n)\}$ converges to $T_f(a) (\mathcal{L}_v a) = \mathcal{L}_v f(a)$ in $H^i(F)$, which implies that $f \mid A^{i+1}$ is continuous. By the same calculation, we check easily that $T^j f(A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)) \subset H^{i+1}(F)$ and $T^j f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$ is continuous. We assume that $f \mid A^{i+1}$ is a C^k -mapping and $T^k(f \mid A^{i+1}) = T^k f \mid A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$. Define a mapping

$$\nu: A^{i+1} \times H^{i+1}(E) \times \{H^{i+1}(E) \times \dots \times H^{i+1}(E)\} \to H^{i+1}(F)$$

k-terms
$$\nu(a, v, v) = T^{k}(f | A^{i+1}) (a+v) (v) - T^{k}(f | A^{i+1}) (a) (v)$$

$$-T^{k+1}f(a) (v, v) .$$

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Then, by the assumption,

$$\begin{split} \nu(a,v,v) &= T^k f(a+v) \, (v) - T^k f(a) \, (v) - T^{k+1} f(a) \, (v,v) \\ \text{and} \qquad \nu(a,tv,v) &= T^k f(a+tv,v) - T^k f(a) \, (v) - t T^{k+1} f(a) \, (v,v) \, . \end{split}$$

By differentiation with respect to the H^i -topology, we get

$$\nu(a,v,v) = \int_0^1 \int_0^t T^{k+2} f(a+uv)(v,v,v) du dt$$

Since $|T^{k+2}f|$ is continuous with respect to the H^{i+1} -topology, we have $|\nu(a, v, v)|/|v| \leq \max_{|b^{-q}| < \varepsilon} |T^{k+2}f(b)||v||v|$, where ε is sufficiently small and || is the H^{i+1} -norm. Therefore, $T^k(f|A^{i+1})$ is differentiable and $T^{k+1}(f|A^{i+1})$ coincides with the continuous mapping $T^{k+1}f|A^{i+1} \times H^{i+1}(E) \times \cdots \times H^{i+1}(E)$. Q.E.D.

A deformation g(t) contained in a \mathcal{D} -orbit O_g of g is called *trivial*, since each (M,g(t)) is isometric to (M,g). On the other hand, a deformation contained in S_g is said to be *essential with respect to g*. According to M. Berger and D. Ebin $[1,\S3,(3.1)]$, we can identify the tangent spaces $T_g(O_g)$ and $T_g(S_g)$ at g with Im δ^* and Ker δ . We call therefore an element of Im δ^* a *trivial i-deformation* and an element of Ker δ an *essential i-deformation*.

Let g(t) and $\tilde{g}(t)$ be deformations of g. If there is a 1-parameter family of diffeomorphisms $\gamma(t)$ satisfying $g(t) = \gamma(t)^* \tilde{g}(t)$, then g(t) is said to be *equivalent* to $\tilde{g}(t)$. Theorem 2.2 implies that every deformation is equivalent to an essential deformation (by restricting the range of t to some open interval containing 0).

Lemma 2.9. If g'(t) is trivial with respect to g(t) (i.e., $g'(t) \in \text{Im } \delta_{g(t)}^*$) for each t, then g(t) is a trivial deformation.

Proof. D. Ebin [5, Theorem 8.1 or 6, Propsition 8.30] shows that for given $g \in \mathcal{M}$ and any neighbourhood V of the identity in \mathcal{D} , there is a neighbourhood H of g in \mathcal{M} such that if $\psi \in H$ there is $\gamma \in V$ satisfying $\gamma^{-1}I_{\psi}\gamma \subset I_{g}$. So, we find dim $I_{g(t)}$ is upper semi-continuous. Let W be a connected component of the set of all t such that dim $I_{g(t)}$ is minimum. Then W is open in I. Fixing $t_{0} \in W$, we shall apply Theorem 2.2 for $g(t_{0})$.

Let $\tilde{g}(t)$ be a deformation equivalent to g(t) contained in $S_{g(t_0)}$. First we prove $\tilde{g}'(t_1)=0$ for all t_1 for which $\tilde{g}(t_1)$ is defined. If $\gamma \in I_{\tilde{g}(t_1)}$, then $\gamma^* \tilde{g}(t_1)=$ $\tilde{g}(t_1) \in S_{g(t_0)}$ and so $\gamma \in I_{g(t_0)}$, because of the property (S2) in Theorem 2.2. This implies $I_{\tilde{g}(t_1)} \subset I_{g(t_0)}$. Since $t_0 \in W$, it follows that any Killing vector field with respect to $g(t_0)$ is a Killing vector field with respect to $\tilde{g}(t_1)$. Now, because $\tilde{g}'(t_1)$ is trivial with respect to $\tilde{g}(t_1)$, there is $\xi \in T_{\mathrm{Id}}(\mathcal{D})$ such that $\tilde{g}'(t_1)=TA_{(\mathrm{Id},\tilde{g}(t_1))}(\xi,0)$, where A is the map $\mathcal{D} \times \mathcal{M} \to \mathcal{M}$ defined by the action of

 \mathcal{D} on \mathcal{M} and TA is the differential of A. Denote by π the natural projection from \mathcal{D} to $\mathcal{D}/I_{g(t_0)}$ and let χ be as in Theorem 2.2. Put $\tilde{\xi} = T\chi \circ T\pi(\xi)$. Then $\xi - \tilde{\xi}$ is a Killing vector field with respect to $g(t_0)$, and so with respect to $\tilde{g}(t_1)$ also. Therefore $TA_{(\mathrm{Id}, \tilde{g}(t_1))}(\xi - \tilde{\xi}, 0) = 0$, $\tilde{g}(t_1)$ being fixed under the action of $I_{\tilde{g}(t_1)}$.

Now, set $F^{-1} = p \times q$ where $p: \mathcal{M} \to \mathcal{D}/I_{g(t_0)}$ and $q: \mathcal{M} \to S_{g(t_0)}$. Since $\tilde{g}'(t_1)$ is tangent to $S_{g(t_0)}$, $Tp(\tilde{g}'(t_1))=0$. On the other hand,

$$\begin{split} \tilde{g}'(t_1) &= TA_{(\mathrm{Id},\tilde{\mathfrak{s}}(t_1))}(\xi,0) \\ &= TA_{(\mathrm{Id},\tilde{\mathfrak{s}}(t_1))}(\xi-\tilde{\xi},0) + TA_{(\mathrm{Id},\tilde{\mathfrak{s}}(t_1))}(\tilde{\xi},0) \\ &= TA_{(\mathrm{Id},\tilde{\mathfrak{s}}(t_1))}(T\chi \circ T\pi(\xi),0) \\ &= TF_{(Ig(t_0),\tilde{\mathfrak{s}}(t_1))}(T\pi(\xi),0) \,, \end{split}$$

hence $Tq(\tilde{g}'(t_1))=0$. But $Tp \times Tq$ is an isomorphism, and therefore $\tilde{g}'(t_1)=0$. We have thus proved that $\tilde{g}(t)$ is constant on W, and so g(t) is trivial on W. By [5, Proposition 6.13 or 6, Theorem 8.10], a \mathcal{D} -orbit is closed in M. Let a be an end point of W. Since W is open, $a \notin W$. If $a \in I$, then $g(a) \in O_{g(t_0)}$, and so g(a) is isometric to $g(t_0)$, which contradicts $a \notin W$. Hence W=I. Q.E.D.

Let \mathcal{P} be a subset of \mathcal{M} invariant under the action of \mathcal{D} . For $g \in \mathcal{P}$, we denote by \mathcal{P}_g the vector space which is spanned by all i-deformations g'(0) defined by deformations g(t) contained in \mathcal{P} .

DEFINITION 2.10. If all deformations of g contained in \mathcal{P} are trivial then g is said to be *non-deformable* (in the sence of \mathcal{P}). If $\mathcal{P}_g \subset \text{Im } \delta_g^*$ then g is said to be *infinitesimally non-deformable* (in the sence of \mathcal{P}).

Theorem 2.11. Let \mathcal{P} be a \mathcal{D} -invariant subset of \mathcal{M} . If there is a \mathcal{D} -invariant open set W of \mathcal{P} such that all metrics in W are infinitesimally non-deformable, then every $g \in W$ is non-deformable.

Proof. Let $g(t): I \to \mathcal{P}$ be any deformation of $g \in W$ contained in \mathcal{P} . Let J be the subset of I of all t such that $g(t) \in W$, and J_1 be the connected component of J containing 0. Then g(t) is infinitesimally non-deformable for each $t \in J_1$, and so, by Lemma 2.9, $g(t) | J_1$ is trivial. If J_1 does not coincide with I, then there is an end point t_0 of J_1 in I. Since \mathcal{D} -orbits in \mathcal{M} are closed, $g(t_0)$ is isometric to g, which contradicts $g(t_0) \notin W$. Thus $J_1 = I$. Q.E.D.

3. Einstein deformations

DEFINITION 3.1 We denote by \mathcal{E} the space of all Einstein metrics on M whose volume is some constant c. A deformation contained in \mathcal{E} is called an *Einstein deformation*. If all Enistein deformations of $g \in \mathcal{E}$ are trivial, then g is said to be *non-deformable*. (cf. Definition 2.10)

Lemma 3.2. Let g(t) be an Einstein deformation of g. Then the essential

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component h of the i-deformation g'(0) (i.e., $g'(0)=h+\delta^{\xi}$ and $\delta h=0$) satisfies the following equalities:

 $\overline{\Delta}h + 2Lh = 0$, tr h = 0,

where the operator L: $S_0^2 \rightarrow S_0^2$ is defined in 1; $(Lh)_{ij} = R_i^{k} h_{kl}$.

Proof. See M. Berger and D. Ebin [1, Lemma 7.1, (7.1)].

Theorem 3.3. Let (M,g) be a compact Einstein manifold with $\rho = \varepsilon g$, ρ being the Ricci tensor. Denote by α_0 the minimum eigenvalue on M of the operator L. If $\alpha_0 > \min\left\{\varepsilon, -\frac{1}{2}\varepsilon\right\}$, then (M,g) is non-deformable.

Proof. Owing to Theorem 2.11 and Lemma 3.2, it is sufficient to prove that if h is an i-deformation of g such that $\delta h=0$, $\Delta h+2Lh=0$ and tr h=0then h=0. First we define the operators $\mathfrak{S}\nabla: C^{\infty}(S^2) \to C^{\infty}(T_3^0)$ and $S\nabla: C^{\infty}(S^2)$ $\to C^{\infty}(T_3^0)$ by

$$(\mathfrak{S}\nabla h) (X, Y, Z) = \alpha(\nabla_X h) (Y, Z) + \beta(\nabla_Y h) (Z, X) + \gamma(\nabla_Z h) (X, Y)$$
$$(S\nabla h) (X, Y, Z) = (\nabla_Y h) (Z, X)$$

where, α , β , $\gamma \in \mathbf{R}$, $\alpha^2 + \beta^2 + \gamma^2 = 1$. Set $u = \alpha\beta + \beta\gamma + \gamma\alpha$. Then the minimum and the maximum of u are $-\frac{1}{2}$ and 1 respectively. By simple computations, we have

$$\begin{split} \langle \mathfrak{S} \nabla h, \mathfrak{S} \nabla h \rangle &= \langle \nabla h, \nabla h \rangle + 2u \langle S \nabla h, \nabla h \rangle \\ &= \langle \overline{\Delta} h, h \rangle + 2u \langle \delta S \nabla h, h \rangle \, . \\ \mathrm{Now}, \qquad (\delta S \nabla h)_{ij} &= -\nabla^k (S \nabla h)_{kij} = -\nabla^k \nabla_i h_{jk} \\ &= g^{km} R^l_{mij} h_{lk} + g^{km} R^l_{mik} h_{jl} - \nabla_i \nabla^k h_{jk} \\ &= -(Lh)_{ij} - \rho_i^l h_{jl} + (\nabla \delta h)_{ij} \, . \end{split}$$

Therefore, we get

$$\langle \overline{\Delta}h - 2uLh - 2u\varepsilon h + 2u\nabla \delta h, h \rangle \geq 0$$
.

Here, we set $\delta h = 0$ and $\Delta h = -2Lh$. Then

$$u \in \langle h, h \rangle \leq -(1+u) \langle Lh, h \rangle$$
.

Thus, if $h \neq 0$ then we have $\alpha_0 \leq \varepsilon$ and $\alpha_0 \leq -\frac{1}{2}\varepsilon$, by setting $u = -\frac{1}{2}$, 1, respectively. Q.E.D.

Let N be a riemannian manifold and $O_p = X_i$ be an orthonormal frame at $p \in N$. Then $\sigma_{ij} = -R_{ijij}$ is the sectional curvature if $i \neq j$, and is zero if i=j. We count the number of j such that $\sigma_{i0j}=0$ for an index i_0 , and call the maximum of such numbers the *flat dimension* fd(N) of N when p, O_p , i_0 run over respective sets. For example, if N has negative curvature, then fd(N)=1.

Proposition 3.4. If an Einstein manifold (M,g) has non-positive sectional curvature, and if its universal riemannian covering (\tilde{M}, \tilde{g}) is the product of the riemannian manifolds $\tilde{M}_a(1 \le a \le k)$ satisfying $2 \operatorname{fd}(\tilde{M}_a) < \dim \tilde{M}_a$, then g is non-deformable. Especially, an Einstein manifold (M,g) is non-deformable, if all irreducible component of (\tilde{M}, \tilde{g}) have negative sectional curvature and are of dimension > 2.

Proof. (I) First, we consider the case that \tilde{M} itself is such that $2fd(\tilde{M}) < \dim \tilde{M}$. Put $r = fd(\tilde{M})$. Fix a point m in \tilde{M} and let $Lh = \alpha h$ for a non-zero symmetric bilinear form h whose trace is zero. Using an orthormal frame $\{X_i\}$ at m, we diagonalize h with respect to \tilde{g} , and set $h^{ii} = x^i$. Then, $\sum x^i = 0$ and

$$R_{i_{jkl}}h^{i_k}h^{j_l} = \sum_{i,j} R_{i_ji_j}x^i x^j = -\sum_{i,j} \sigma_{i_j}x^i x^j$$
.

Now, let (y_i) be an eigenvector of the matrix (σ_{ij}) belonging to an eigenvalue λ . By changing order of coordinates if necessary, we can assume that $y_r = \max_i |y_j|$ and $\sigma_{ri} < 0$ for all i > r. Then,

$$-\lambda y_r = -\sum_i \sigma_{ir} y_i \ge \sum_i \sigma_{ir} y_r = \mathcal{E} y_r \,.$$

So $-\lambda \ge \varepsilon$ and, if the equality holds, then we have $y_i = -y_r$ for all i > r, which implies

$$\sum_{i} y_{i} = \sum_{i>r} y_{i} + \sum_{i \leq r} y_{i} \leq -(n-r)y_{r} + ry_{r} = -(n-2r)y_{r} < 0.$$

Therefore, for (x_i) such that $\sum x^i = 0$, we have

$$-\sum_{i,j}\sigma_{ij}x^ix^j > \varepsilon \sum_i (x^i)^2$$
.

Hence, $\alpha(h,h) = -\sum_{i,j} \sigma_{ij} x^i x^j > \varepsilon \sum_i (x^i)^2 = \varepsilon(h,h).$

Thus we get $\alpha > \varepsilon$. Our assertion follows then from Theorem 3.3.

(II) Now we consider the general case. Corresponding to the decomposition $(\tilde{M}, \tilde{g}) = \prod_{a} (\tilde{M}_{a}, \tilde{g}_{a})$, the curvature tensor decomposes. Hence, the Ricci tensor $\tilde{\rho}$ of \tilde{M} has the decompostion $\tilde{\rho} = \sum_{a} \tilde{\rho}_{a}$ where $\tilde{\rho}_{a}$ is the Ricci tensor of \tilde{M}_{a} . Therefore $\tilde{\rho}_{a} = \varepsilon \tilde{g}_{a}$. Moreover, $S_{0}^{2}(\tilde{M})$ and the operator \tilde{L} on $S_{0}^{2}(\tilde{M})$ decomposes as follows;

$$egin{aligned} S^2_0(ilde{M}) &= (igoplus_a S^2_0(ilde{M}_a)) \oplus ((igoplus_a extbf{R} ilde{g}_a) \cap S^2_0(ilde{M})) \oplus \sum_{a
eq b} S^2(ilde{M}_a, ilde{M}_b) \,, \ & ilde{L} \,|\, S^2_0(ilde{M}_a) &= ilde{L}_a \,, \ & ilde{L} \,|\, (\oplus(extbf{R} ilde{g}_a)) \cap S^2_0(ilde{M}) &= - arepsilon \,, \ & ilde{L} \,|\, S^2(ilde{M}_a, ilde{M}_b) = 0 \quad ext{for} \quad a
eq b \,, \end{aligned}$$

where \tilde{L}_a is the operator of \tilde{M}_a and

$$S^2(ilde{M}_a, ilde{M}_b)=\{h\!\in\!S^2(ilde{M}_a\!\times\! ilde{M}_b);\,h(T ilde{M}_c,T ilde{M}_c)=0\;\; ext{for}\;\;c=a,b\}\;.$$

Since the curvature of $(M,g) \leq 0$, ε is negative. Then, combined with what we have proved in (I), we get $\alpha_0 > \varepsilon$ and our assertion follows from Theorem 3.3 Q.E.D.

Corollary 3.5. Let (M,g) be a compact Einstein manifold. If M is a locally symmetric space of non-compact type, and the dimension of every irreducible component of the universal covering (\tilde{M}, \tilde{g}) of (M,g) is greater than 2, then (M,g) is non-deformable.

Proof. Let G/K be a symmetric space which is the universal covering of (M,g). Since the dimension of every irreducible component of G/K is greater than 2, we may assume that G has no simple factor of dimension 3. On the other hand A. Weil [13, §10] shows that if G has no simple factor of dimension 3, then $\alpha_0 > \varepsilon$. Thus the proof reduces to Theorem 3.3.

REMARK 3.6. Theorem 24.1' in G.D. Mostow [10] implies that if (M,g_1) and (M,g_2) are locally symmetric spaces of non-compact type without 2-dimensional factors locally, then g_1 and g_2 are isometric up to normalizing constants. (cf. E. Calabi [3, Theorem 1], A. Weil [13, Theorem 1])

Corollary 3.7. If the sectional curvature of a compact Einstiein manifold (M,g) ranges in the interval $\left(\frac{n-2}{2n-1},1\right]$, then (M,g) is non-deformable.

Proof. We easily see that $\mathcal{E} = \frac{1}{n} \sum_{i \neq j} \sigma_{ij}$, hence the condition implies $\mathcal{E} > (n-2)(n-1)/(2n-1)$. By virtue of Theorem 3.3, it is sufficient to prove $\alpha_0 + \frac{1}{2}\mathcal{E} > 0$. In the same way as for the proof I of Proposition 3.4, we may set $h^{ii} = x^i$ with $\sum x^i = 0$. We can assume that there is an integer c such that $y^i = x^i \ge 0$ for any $i \le c$, and $z^i = -x^i > 0$ for any i > c. Set $\sum_{i \le c} y^i = \sum_{i > c} z^i = A$. Then, since $\sum x^i = 0$,

$$\begin{split} (Lh,h) + &\frac{1}{2} (\varepsilon h,h) = -\sum_{i,j} \sigma_{ij} x^i x^j + \frac{1}{2} \varepsilon \sum_i (x^i)^2 + \sum_i x^i \sum_j x^j \\ &= \left(1 + \frac{1}{2} \varepsilon \right) \{ \sum_{i \leq \varepsilon} (y^i)^2 + \sum_{i > \varepsilon} (z^i)^2 + \sum_{i \neq j, i, j \leq \varepsilon} (1 - \sigma_{ij}) y^i y^j \\ &+ \sum_{i \neq j, i, j > \varepsilon} (1 - \sigma_{ij}) z^i z^j - 2 \sum_{i \leq \varepsilon, j > \varepsilon} (1 - \sigma_{ij}) y^j y^j \\ &> \frac{n(n+1)}{2(2n-1)} \{ \sum_{i \leq \varepsilon} (y^i)^2 + \sum_{i > \varepsilon} (z^i)^2 \} - 2 \frac{n+1}{2n-1} A^2 \\ &\ge \frac{n(n+1)}{2(2n-1)} \left(\frac{1}{c} A^2 + \frac{1}{n-c} A^2 \right) - 2 \frac{n+1}{2n-1} A^2 \\ &\ge \frac{n(n+1)}{2(2n-1)} \frac{4}{n} A^2 - 2 \frac{n+1}{2n-1} A^2 = 0 \,. \end{split}$$

REMARK 3.8. Y. Muto [11, Theorem] shows that every Einstein metric near a metric with positive constant sectional curvature is of positive constant sectional curvature.

REMARK 3.9. Even if (M,g) is a non-deformable Einstein metric, M may have an Einstein metric \tilde{g} which is not isometric to g. In fact, G.R. Jensen [8, pp. 612-613] constructs a non-standard Einstein metric \tilde{g} on S^{4p+3} . The author does not know whether \tilde{g} is non-deformable or not.

Finally, by a direct computation, we may apply Theorem 3.3 to the manifold M whose universal covering \tilde{M} is an irreducible symmetric space G/K of compact type.

I. the case where \tilde{M} is hermitian symmetric

In this case, the eigenvalue of the generalized operator $\tilde{L}: S^2 \rightarrow S^2$ are calculated by E. Calabi and E. Vesentini [4, p. 502, Table 2] and A. Borel [2, Corollary 4.6, 4.7]. See Table 1. Here we omit 0 and $-\varepsilon$, which are always eigenvalues of \tilde{L} . The eigenspace corresponding to this eigenvalue $-\varepsilon$ is generated by g. Hence, this is not an eigenvalue of our operator L on S_0^2 . We conclude that the following three classes are non-deformable.

```
AIII (p=1), (q=1)
DIII (p \ge 6)
EVII
```

type	dim _C M	G/M	$\alpha \varepsilon^{-1}$ /multiplicity			
AIII	Þq	SU(p+q) S(Up×Uq)	$2(p+q)^{-1} \\ 2\binom{p+1}{2}\binom{q+1}{2}$	$-2(p+q)^{-1}$ $2\binom{p}{2}\binom{q}{2}$	$p(p+q)^{-1}$ q^2-1	$-q(p+q)^{-1}$ p^2-1
DIII	$\binom{p}{2}$	SO(2 <i>p</i>) U(<i>p</i>)	$(p-1)^{-1}$ $\frac{1}{6}p^2(p^2-1)$	$\frac{-2(p-1)^{-1}}{2\binom{p}{4}}$	$\frac{-\frac{1}{2}(p-2)(p-1)^{-1}}{p^2-1}$	
СІ	$\binom{p+1}{2}$	Sp(p) U(p)	$2(p+1)^{-1}$ $2\binom{p+3}{4}$	$\frac{-(p+1)^{-1}}{\frac{1}{6}p^2(p^2-1)}$	$\frac{-\frac{1}{2}(p+2)(p+1)^{-1}}{p^2-1}$	
BDI	Þ	SO(p+2) $SO(p) imes T^1$	$2p^{-1}$ (p-1)(p+2)	$-(p-2)p^{-1}$ 2	$\frac{-2p^{-1}}{\binom{p}{2}}$	
EIII	16	E ₆ Spin (10)• T ¹	$\frac{1}{6}$ 252	$-\frac{1}{2}$ 20	$-\frac{1}{3}$ 45	
EVII	27	E_7 $E_6 imes T^1$	$\frac{1}{9}$ 702	$-\frac{4}{9}$ 54	$\frac{-\frac{1}{3}}{78}$	

Table 1

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II) Other cases

By easy but complicated computations we can compute α_0 . Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$ be the orthogonal decomposition with respect to the Killing form on \mathfrak{g} , where \mathfrak{k} is the Lie algebra of K. Then the tangent space $T_{eK}(\tilde{M})$ at the identity coset is canonically identified with \mathfrak{m} , and we know that R(X,Y)Z=-[[X,Y],Z] for $X,Y,Z\in\mathfrak{m}$. (See S. Kobayashi and K. Nomizu [9, p. 231 Theorem 3.2].) We can compute the eigenvalue of the curvature operator L which is identified with the linear endomorphism on $S_0^2(\mathfrak{m})$, and we get Table 2 for the type BDI and CII. Hence the following symmetric spaces are non-deformable, where we assume $p \ge q$;

BDI
$$(p \ge 3, q=1), (q \ge p-1, p+q \ge 7)$$

CII $(p=q=1), (p \ge 3, q=1).$

type	n	G/K	(*)	αε ⁻¹
BDI q		SO(p+q) SO(p) imes SO(q)	p > q = 1	(<i>p</i> -1) ⁻¹
	q		$p \ge q \ge 2$	$\pm 2(p+q-2)^{-1}, (2-p)(p+q-2)^{-1}, (2-q)(p+q-2)^{-1}$
CII 4pq	Sp(p+q) $Sp(p) \times Sp(q)$	p = q = 1	$\frac{1}{3}$	
		p > q = 1	$-(p+2)^{-1}, (p+2)^{-1}$	
			$p \ge q > 1$	$\pm (p+q+1)^{-1}, -(p+1)(p+q+1)^{-1}, -(q+1)(p+q+1)^{-1},$

Table 2

(*) condition

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