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## A GENERALIZATION OF A THEOREM OF MILNOR

Dedicated to Professor Seiya Sasao on his 60th birthday

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### 1. Introduction

We work in the smooth category with free actions by groups in the present paper. Let us recall Milnor's theorem:

**Theorem 1.1** ([6; Corollary 12.13]). *Any  $h$ -cobordism  $W$  between lens spaces  $L$  and  $L'$  must be diffeomorphic to  $L \times [0,1]$  if the dimension of  $L$  is greater than or equal to 5.*

Let  $\mathbf{Z}_m$  be the cyclic group of order  $m$ . Then we see that Theorem 1.1 is put in another way as follows:

**Theorem 1.2.** *Let  $S(V)$  and  $S(V')$  be free linear  $\mathbf{Z}_m$ -spheres of dimension  $2n-1 \geq 5$ . Then any  $\mathbf{Z}_m$ - $h$ -cobordism  $W$  between  $S(V)$  and  $S(V')$  must be  $\mathbf{Z}_m$ -diffeomorphic to  $S(V) \times I$ , where  $I = [0,1]$ .*

Let  $R$  be a ring with unit,  $G$  a finite group. Put  $GL(R) = \varinjlim GL_n(R)$  and  $E(R) = [GL(R), GL(R)]$  the commutator subgroup of  $GL(R)$ . Then  $K_1(R)$  denotes the quotient group  $GL(R)/E(R)$ . Let  $\mathbf{Z}$  be the ring of integers and  $\mathbf{Q}$  the ring of rational numbers. Let  $\mathbf{Z}[G]$  and  $\mathbf{Q}[G]$  denote the group rings of  $G$  over  $\mathbf{Z}$  and  $\mathbf{Q}$ . The Whitehead group of  $G$  is the quotient group

$$Wh(G) = K_1(\mathbf{Z}[G]) / \langle \pm g : g \in G \rangle.$$

The natural inclusion map  $i: GL(\mathbf{Z}[G]) \rightarrow GL(\mathbf{Q}[G])$  gives rise to a group homomorphism  $i_*: K_1(\mathbf{Z}[G]) \rightarrow K_1(\mathbf{Q}[G])$ . Then  $SK_1(\mathbf{Z}[G])$  is defined by setting

$$SK_1(\mathbf{Z}[G]) = \ker[i_*: K_1(\mathbf{Z}[G]) \rightarrow K_1(\mathbf{Q}[G])].$$

In [15], C.T.C. Wall showed that  $SK_1(\mathbf{Z}[G])$  is isomorphic to the torsion

subgroup of  $Wh(G)$ . We will apply the following algebraic result to extend Theorem 1.2.

**Theorem A.** *Let  $G$  be a finite group which can act linearly and freely on spheres. Then  $SK_1(\mathbb{Z}[G])=0$  if and only if  $G$  is isomorphic to one of the following groups.*

- (1) *A cyclic group.*
- (2) *A group of type I in Appendix(a metacyclic group with certain condition).*
- (3) *A quaternionic group  $\mathbf{Q}(8t)$  with generators  $B, R$  and relations  $B^{4t}=1, B^{2t}=R^2=(BR)^2$ , where  $t \geq 1$ .*
- (4) *A group  $\mathbf{Q}(8t, m_1, m_2)$  generated by  $A, B, R$  with relations  $A^{m_1 m_2}=B^{4t}=1, BAB^{-1}=A^{-1}, R^2=B^{2t}, RAR^{-1}=A^l, RBR^{-1}=B^{-1}$ , where  $m_1, m_2 \geq 1, m_1 m_2 > 1, (m_1, m_2)=1, (2t, m_1 m_2)=1, l \equiv -1(m_1), l \equiv 1(m_2)$ .*
- (5) *The binary tetrahedral group  $\mathbf{T}^*$ .*
- (6) *A generalized binary octahedral group  $\mathbf{O}^*(48t)$  generated by  $B, P, Q, R$  with relations  $B^{3t}=1, P^2=Q^2=(PQ)^2=R^2, BPB^{-1}=Q, BQB^{-1}=PQ, RPR^{-1}=QP, RQR^{-1}=Q^{-1}, RBR^{-1}=B^{-1}$ , where  $t$  is odd.*
- (7) *The binary icosahedral group  $\mathbf{I}^*=SL(2,5)$ .*
- (8) *The group generated by  $SL(2,5)$  and an element  $S$ , where  $S^2=-1 \in SL(2,5), SLS^{-1}=\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} L \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}^{-1}$  for  $L \in SL(2,5)$ .*

We obtain the following applications of Theorem A as generalizations of Theorem 1.2.

**EXAMPLE B.** Let  $G$  be a finite group in Theorem A. Let  $X$  be a free  $G$ -homotopy sphere of dimension  $2n-1 \geq 5$ , and let  $S(V)$  and  $S(V')$  be free linear  $G$ -spheres of dimension  $2n-1 \geq 5$ . Then,

- (1) Any  $G$ - $h$ -cobordism  $W$  between  $X$  and itself must be  $G$ -diffeomorphic to  $X \times I$ .
- (2) Any  $G$ - $h$ -cobordism  $W$  between  $S(V)$  and  $S(V')$  must be  $G$ -diffeomorphic to  $S(V) \times I$ .

**EXAMPLE C.** Let  $G$  be a finite group. Let  $X$  be a free  $G$ -homotopy sphere of dimension  $4n+1 \geq 5$ , and let  $S(V)$  and  $S(V')$  be free linear  $G$ -spheres of dimension  $4n+1 \geq 5$ . Then,

- (1) Any  $G$ - $h$ -cobordism  $W$  between  $X$  and itself must be  $G$ -diffeomorphic to  $X \times I$ .
- (2) Any  $G$ - $h$ -cobordism  $W$  between  $S(V)$  and  $S(V')$  must be  $G$ -diffeomorphic to  $S(V) \times I$ .

When  $G$  is a compact Lie group of positive dimension, a generalization of Theorem 1.2 is:

**Theorem D.** *Let  $G$  be a compact Lie group of positive dimension which can act freely on spheres. Let  $X^m$  and  $X'^m$  be free  $G$ -homotopy spheres of dimension  $m$ , and let  $(W; X, X')$  be a  $G$ - $h$ -cobordism of a free  $G$ -action.*

- (1) *If  $G = S^1$  and  $m = 2n - 1 \geq 7$ , then  $W$  must be  $S^1$ -diffeomorphic to  $X \times I$ .*
- (2) *If  $G = NS^1$  and  $m = 4n - 1 \geq 7$ , then  $W$  must be  $NS^1$ -diffeomorphic to  $X \times I$  where  $NS^1$  is the normalizer of  $S^1$  in  $S^3$ .*
- (3) *If  $G = S^3$  and  $m = 4n - 1 \geq 11$ , then  $W$  must be  $S^3$ -diffeomorphic to  $X \times I$ .*

This paper is organized as follows: Section 2 presents the proof of Theorem A. In section 3 we prove Examples B and C, and state some results on  $G$ - $h$ -cobordisms between  $G$ -homotopy spheres. We prove Theorem D in section 4. Appendix is devoted to quoting the table of the finite solvable groups which can act linearly and freely on odd dimensional spheres from [16].

## 2. Proof of Theorem A

First, let  $G$  be a finite solvable group which can act linearly and freely on spheres. As in [16; Theorem 6.1.11], there are 4 types for such kinds of groups. For the convenience of the readers, the table of these groups are cited in Appendix. We now recall the structure of  $SK_1(\mathbf{Z}[G])$  of these groups  $G$ . We must prepare the following notations.

Let  $G_1, G_2, G_3$  and  $G_4$  denote the groups of type I, II, III and IV respectively mentioned in the table in Appendix. Let  $(a_1, a_2, \dots, a_\lambda)$  denote the greatest common divisor of integers  $\{a_1, a_2, \dots, a_\lambda\}$ , and let  $m, n, r, l, k, u, v$  and  $d$  be the integers appeared in the definition of  $G_1, G_2, G_3$  and  $G_4$ . For positive integers  $\alpha, \beta, \gamma$  and  $\delta$ , put

$$M_\beta = (r^\beta - 1, m),$$

$$D(\alpha) = \{x \in \mathbf{N} \mid x \text{ is a divisor of } \alpha\},$$

$$D(\alpha, \beta) = \{x \in D(\alpha) \mid x \text{ can be divided by } \beta\},$$

$$D(\alpha)_\gamma^\delta = \{x \in D(\alpha) \mid x\gamma \equiv 0 \pmod{\delta}\},$$

$$D(\alpha, \beta)_\gamma^\delta = \{x \in D(\alpha, \beta) \mid x\gamma \equiv 0 \pmod{\delta}\}.$$

If  $d$  is an even integer, we put  $d' = d/2$ , and put

$$t(2) = \#\{(\alpha, \beta) \mid \beta \in D(v)_{k-1}^v, \alpha \in D(M_{2u\beta}),$$

$$(\alpha + aM_{2u\beta})(l-1, r^{n/4} - 1) \equiv 0(m)$$

$$\text{for some integer } a \text{ with } 0 \leq a < m/M_{2u\beta}\}$$

$$- \# \bigcup_{\substack{0 \leq b < d \\ \lambda = 0, 1}} D(m)_{(l-1, r^{n/4} - 1, l^2 r^b + 1)}^m,$$

$$t'(2) = \#\{(\alpha, \beta) \mid \beta \in D(v)_{k-1}^v, \alpha \in D(M_{2u\beta}),$$

$$(\alpha + aM_{2u\beta})(l-1, r^{n/4} - 1) \equiv 0(m) \text{ or}$$

$$(\alpha + aM_{2u\beta})(lr^{d'} - 1, r^{n/4} - 1) \equiv 0(m)$$

$$\text{for some integer } a \text{ with } 0 \leq a < m/M_{2u\beta}\}$$

$$- \# \bigcup_{\substack{0 \leq b < d \\ \lambda = 0, 1}} (D(m)_{(l-1, r^{n/4} - 1, l^2 r^b + 1)}^m \cup D(m)_{(lr^{d'} - 1, r^{n/4} - 1, l^2 r^b + 1)}^m),$$

$$t(3) = \sum_{\beta \in D(n, 3)} \#D(M_\beta) - 1,$$

$$t(4) = \sum_{\beta \in D(n, 3)} \#D(M_\beta) - \sum_{\beta \in D(n, 3)_{k+1}^n} \#D(M_\beta)_{i+1}^m.$$

Then we have:

**Theorem 2.1** ([12; Theorem]). *Let  $G_1, G_2, G_3$  and  $G_4$  denote the groups of type I, II, III and IV respectively.*

- (1)  $SK_1(\mathbf{Z}[G_1]) = 0.$
- (2)  $SK_1(\mathbf{Z}[G_2]) \cong \mathbf{Z}_2^{(2)}$  if  $d$  is an odd integer,  
 $SK_1(\mathbf{Z}[G_2]) \cong \mathbf{Z}_2^{t'(2)}$  if  $d$  is an even integer.
- (3)  $SK_1(\mathbf{Z}[G_3]) \cong \mathbf{Z}_2^{(3)}.$
- (4)  $SK_1(\mathbf{Z}[G_4]) \cong \mathbf{Z}_2^{(4)}.$

By Theorem 2.1, we get (1) and (2) of Theorem A. Let  $G_2^1$  be a group  $G_2$  such that  $d$  is odd. At first, we determine the group  $G_2^1$  satisfying  $SK_1(\mathbf{Z}[G_2^1]) = 0.$  Put

$$\mathcal{F}_+ = \{(\alpha, \beta) \mid \beta \in D(v)_{k-1}^v, \alpha \in D(M_{2u\beta}),$$

$$(\alpha + aM_{2^u\beta})(l-1, r^{n/4} - 1) \equiv 0(m)$$

for some integer  $a$  with  $0 \leq a < m/M_{2^u\beta}$ ,

and

$$\mathcal{F}_- = \{(\alpha, v) \mid \alpha \in \bigcup_{\substack{0 \leq b < d \\ \lambda = 0, 1}} D(m)_{(l-1, r^{n/4} - 1, l^{2^u r^b + 1})}^m\}.$$

By [12; §3],  $t(2)$  the 2-rank of  $SK_1(\mathbf{Z}[G_2^1])$  is calculated by

$$t(2) = \#\mathcal{F}_+ - \#\mathcal{F}_-.$$

It is easy to see that  $\mathcal{F}_-$  is a subset of  $\mathcal{F}_+$ . Suppose that  $t(2) = 0$ . Then it is necessary that  $D(v)_{k-1}^v = \{v\}$ . In fact, if there exists an element  $\beta$  of  $D(v)_{k-1}^v$  which is different from  $v$ , we see that the ordered pair of numbers  $(M_{2^u\beta}, \beta)$  is in  $\mathcal{F}_+$ , but is not in  $\mathcal{F}_-$ . Hence, if  $\beta$  in  $D(v)$  satisfies  $\beta(k-1) \equiv 0 (v)$ , it must be equal to  $v$ . Thus we have  $(k-1, v) = 1$ . Since  $k^2 \equiv 1 (n)$  and  $k \equiv -1 (2^u)$ , it holds that  $k \equiv -1 (n)$ . Since  $d$  is a divisor of  $k-1$  and  $d$  is odd, by [12; Observation 3.1]  $(k-1, v)$  is divisible by  $d$ . Hence we have  $d = 1$ , thereby  $r \equiv 1 (m)$ . By using  $(n(r-1), m) = 1$ , we get  $m = 1$ , that is,  $A$  is equal to the identity element of  $G_2^1$ . Thus if  $SK_1(\mathbf{Z}[G_2^1]) = 0$ ,  $G_2^1$  must be isomorphic to a group of order  $2n$  which is generated by the elements of the form  $B$  and  $R$ , and which has relations:

$$B^n = 1, R^2 = B^{n/2}, RBR^{-1} = B^{-1},$$

where  $n$  is a number of the form  $2^u v$  for some  $u \geq 2, (v, 2) = 1, v \geq 1$ . Conversely, we can easily check that  $SK_1$  for this group vanishes. By putting  $t = n/4$ , we have (3) of Theorem A.

Let  $G_2^0$  be a group  $G_2$  such that  $d$  is even. Next, we determine the group  $G_2^0$  satisfying  $SK_1(\mathbf{Z}[G_2^0]) = 0$ . Since  $d$  is even, we have  $m > 1$ . Put

$$\begin{aligned} \mathcal{F}'_+ = \{ & (\alpha, \beta) \mid \beta \in D(v)_{k-1}^v, \alpha \in D(M_{2^u\beta}), \\ & (\alpha + aM_{2^u\beta})(l-1, r^{n/4} - 1) \equiv 0(m) \text{ or} \\ & (\alpha + aM_{2^u\beta})(lr^{d'} - 1, r^{n/4} - 1) \equiv 0(m) \\ & \text{for some integer } a \text{ with } 0 \leq a < m/M_{2^u\beta} \}, \end{aligned}$$

and

$$\mathcal{F}'_- = \{(\alpha, v) \mid \alpha \in \bigcup_{\substack{0 \leq b < d \\ \lambda = 0, 1}} (D(m)_{(l-1, r^{n/4} - 1, l^{2^u r^b + 1})}^m \cup D(m)_{(lr^{d'} - 1, r^{n/4} - 1, l^{2^u r^b + 1})}^m)\}.$$

By [12; §3],  $t'(2)$  the 2-rank of  $SK_1(\mathbf{Z}[G_2^0])$  is calculated by

$$t'(2) = \#\mathcal{F}'_+ - \#\mathcal{F}'_- .$$

It is easy to see that  $\mathcal{F}'_-$  is a subset of  $\mathcal{F}'_+$ . Then by the same argument as before, we have  $D(v)_{k-1}^v = \{v\}$  and  $(k-1, v) = 1$ . Since  $k^2 \equiv 1 \pmod{n}$  and  $k+1 \equiv 0 \pmod{2^u}$ , it holds that  $k \equiv -1 \pmod{n}$ . Since  $d$  is even, by [12; Observation 3.1],  $d' = d/2$  is a divisor of  $(k-1, v)$ . Hence, we have  $d = 2$ , thereby  $r \not\equiv 1 \pmod{m}$  and  $r^2 \equiv 1 \pmod{m}$ . Now we claim that  $r \equiv -1 \pmod{m}$ . In fact, since  $(r+1)(r-1) \equiv 0 \pmod{m}$  and  $(r-1, m) = 1$ , it holds that  $r+1 \equiv 0 \pmod{m}$  or  $m = 1$ . However, it must hold  $r \equiv -1 \pmod{m}$  because  $m > 1$ . Therefore, we have

$$(lr^{d'} - 1, r^{n/4} - 1) = (l+1, (-1)^{n/4} - 1).$$

Thus, for  $\#\mathcal{F}'_+ = \#\mathcal{F}'_-$ , it is necessary that

$$\begin{aligned} & \#\{\alpha \in D(m) \mid \alpha(l-1, (-1)^{n/4} - 1) \equiv 0 \pmod{m} \\ & \quad \text{or } \alpha(l+1, (-1)^{n/4} - 1) \equiv 0 \pmod{m}\} \\ &= \# \bigcup_{\substack{b=0,1 \\ \lambda=0,1}} (D(m)_{(l-1, (-1)^{n/4} - 1, (-1)^{b l^{\lambda} + 1})}^m \cup D(m)_{(l+1, (-1)^{n/4} - 1, (-1)^{b l^{\lambda} + 1})}^m). \end{aligned}$$

However, we can easily check that this formula always holds. Thus, if  $SK_1(\mathbf{Z}[G_2^0]) = 0$ ,  $G_2^0$  must be isomorphic to a group of order  $2n$  which is generated by the elements of the form  $A, B$  and  $R$ , and which has relations:

$$\begin{aligned} A^m &= B^n = 1, \quad BAB^{-1} = A^{-1}, \\ R^2 &= B^{n/2}, \quad RAR^{-1} = A^l, \quad RBR^{-1} = B^{-1}, \end{aligned}$$

where  $m, n$  and  $l$  satisfy the following conditions:

$$\begin{aligned} m &> 1, \quad (n, m) = 1, \quad l^2 \equiv 1 \pmod{m}, \\ n &= 2^u v \ (u \geq 2, \ (v, 2) = 1, \ v \geq 1). \end{aligned}$$

Conversely, we can easily check that  $SK_1$  for this group vanishes. Now, we put  $t = n/4$ . Since  $l^2 \equiv 1 \pmod{m}$ , there exist two integers  $m_1$  and  $m_2$  such that  $m = m_1 m_2$ ,  $(m_1, m_2) = 1$ ,  $l \equiv -1 \pmod{m_1}$ , and  $l \equiv 1 \pmod{m_2}$ . Conversely if we write  $m = m_1 m_2$  where  $(m_1, m_2) = 1$ , there exists an integer  $l$  uniquely modulo  $m$  such that  $l \equiv -1 \pmod{m_1}$  and  $l \equiv 1 \pmod{m_2}$ . We denote this group by  $\mathbf{Q}(8t, m_1, m_2)$  (This notation is based on [11]). Thus we get (4) of

Theorem A.

Next, we determine the group  $G_3$  satisfying  $SK_1(\mathbf{Z}[G_3])=0$ . Assume that

$$t(3) = \sum_{\beta \in D(n,3)} \#D(M_\beta) - 1 = 0.$$

Since  $\#D(M_\beta) \geq 1$  for every  $\beta \in D(n,3)$ , it is necessary that  $\#D(n,3)=1$ . Hence,  $n$  must be 3, thereby  $d$  is 1 or 3. However, if  $d=3$ ,  $n/d$  is not divisible by 3. Hence  $d$  must be equal to 1, thereby  $r \equiv 1(m)$ . By using  $(n(r-1),m)=1$ , we have  $m=1$ , that is,  $A$  is equal to the identity element of  $G_3$ . Thus, if  $SK_1(\mathbf{Z}[G_3])=0$ ,  $G_3$  must be isomorphic to a group of order 24 which is generated by the elements of the form  $B, P$  and  $Q$ , and which has relations:

$$B^3=1, P^2=Q^2=(PQ)^2, BPB^{-1}=Q, BQB^{-1}=PQ.$$

This group is the binary tetrahedral group  $\mathbf{T}^*$ . Conversely, we can easily see that  $SK_1(\mathbf{Z}[\mathbf{T}^*])=0$ . This proves (5) of Theorem A.

Next, we determine the group  $G_4$  satisfying  $SK_1(\mathbf{Z}[G_4])=0$ . Suppose that

$$t(4) = \sum_{\beta \in D(n,3)} \#D(M_\beta) - \sum_{\beta \in D(n,3)_{k+1}^n} \#D(M_\beta)_{l+1}^m = 0.$$

Then it is necessary that  $D(n,3)=D(n,3)_{k+1}^n$ . In fact, if there exists an element  $\beta_0$  of  $D(n,3)-D(n,3)_{k+1}^n$ , since  $\#D(M_{\beta_0}) \geq 1$ , we have  $t(4) \neq 0$ . Hence, for every element  $\beta$  in  $D(n,3)$ , it must hold that  $\beta(k+1) \equiv 0(n)$ . In particular, we have  $3(k+1) \equiv 0(n)$ . Thus  $k$  must satisfy  $k+1 \equiv 0(n/3)$ . We claim that  $k \equiv -1(n)$ . In fact, if  $k$  is congruent to  $n/3-1$  or  $2n/3-1$  modulo  $n$ , the conditions  $k+1 \equiv 0(3)$  and  $n \equiv 0(3)$  imply  $n \equiv 0(9)$ , but it is a contradiction to the condition  $k^2 \equiv 1(n)$ . Therefore we have

$$r^{k-1} \equiv r^{n-2} \equiv r^n \equiv 1(m),$$

which implies  $d$  is a divisor of  $(n-2, n)$ . Since a group  $G_4$  has odd  $n$ , we have  $d=1$ . By the same argument as above, we have  $m=1$ , that is,  $A$  is equal to the identity element of  $G_4$ . Thus if  $SK_1(\mathbf{Z}[G_4])=0$ ,  $G_4$  must be a group of order  $16n$  which is generated by the elements of the



form  $B, P, Q$  and  $R$ , and which has relations:

$$\begin{aligned} B^n &= 1, P^2 = Q^2 = (PQ)^2 = R^2, \\ BPB^{-1} &= Q, BQB^{-1} = PQ, RPR^{-1} = QP, \\ RQR^{-1} &= Q^{-1}, RBR^{-1} = B^{-1}, \end{aligned}$$

where  $n$  is divisible by 3, but is not divisible by 2. This group is the generalized binary octahedral group  $\mathbf{O}^*(48t)$ . Conversely, we can easily check that  $SK_1(\mathbf{Z}[\mathbf{O}^*(48t)])=0$ . This proves (6) of Theorem A.

Next, we consider the case that  $G$  is non-solvable.

**Lemma 2.2** ([16; 6.3.1 Theorem]). *Let  $G$  be a finite non-solvable group. If  $G$  has a fixed point free representation, then  $G$  is one of the following two types.*

TYPE V.  $G = K \times SL(2,5)$  where  $K$  is a solvable fixed point free group of type I in Appendix and order prime to 30.

TYPE VI.  $G = \langle G_5, S \rangle$  where  $G_5 = K \times SL(2,5)$  is a normal subgroup of index 2 and type V,  $S^2 = -1 \in SL(2,5), SLS^{-1} = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} L \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}^{-1}$  for  $L \in SL(2,5)$ , and  $S$  normalizes  $K$ .

Let  $G$  be a finite group of type V or VI. For an odd prime  $p$ , since  $p$ -Sylow subgroups of  $G$  are cyclic,  $SK_1(\mathbf{Z}[G])_{(p)}=0$ . Hence by [7; Theorem 3],  $SK_1(\mathbf{Z}[G])$  is generated by induction from 2-elementary subgroups of  $G$ , that is,  $SK_1(\mathbf{Z}[G])=0$  if and only if  $G$  has not a subgroup which is isomorphic to  $\Gamma \times S_2$  where  $\Gamma$  is a cyclic group of order prime to 2 and  $S_2$  is a 2-group. In these cases,  $SK_1(\mathbf{Z}[G])=0$  if and only if  $G$  has not a subgroup of the form  $\Gamma \times \mathbf{Q}_8$  (see [5]). Hence  $K$  must be  $\{1\}$  which proves (7) and (8) of Theorem A.

### 3. $G$ - $h$ -cobordisms between $G$ -homotopy spheres

Let  $Wh(G)$  be the Whitehead group of  $G$ ,  $L_m^s(G)$  and  $L_m^h(G)$  the Wall groups (for the Wall groups, see [2], [14]).  $\mathbf{Z}[G]$  is the integral group ring with involution  $-$  defined by  $\overline{\sum a_g g} = \sum a_g g^{-1}$  where  $a_g \in \mathbf{Z}$  and  $g \in G$ . For a matrix  $(x_{ij})$  with coefficients in  $\mathbf{Z}[G]$ ,  $(\overline{x_{ij}})$  is defined by  $(\overline{x_{ji}})$ . Then  $Wh(G)$  has the induced involution also denoted by  $-$ . We define a subgroup  $\tilde{A}_m(G)$  of  $Wh(G)$  by

$$\tilde{A}_m(G) = \{ \tau \in Wh(G) \mid \bar{\tau} = (-1)^m \tau \},$$

and put

$$A_m(G) = \tilde{A}_m(G) / \{ \tau + (-1)^m \bar{\tau} \mid \tau \in Wh(G) \}.$$

Let  $c: A_{2n+1}(G) \rightarrow L_{2n}^s(G)$  be the map in the Rothenberg exact sequence

$$\cdots \rightarrow A_{2n+1}(G) \xrightarrow{c} L_{2n}^s(G) \xrightarrow{d} L_{2n}^h(G) \rightarrow \cdots,$$

and  $\tilde{c}: \tilde{A}_{2n+1}(G) \rightarrow L_{2n}^s(G)$  the map determining  $c$  (for this exact sequence, see [8; Proposition 4.1]).

**Proposition 3.1.** *Let  $G$  be a finite group such that  $SK_1(\mathbf{Z}[G]) = 0$ . Then the following hold:*

- (1) *If  $X$  is a free  $G$ -homotopy sphere of dimension  $2n - 1 \geq 5$ , any  $G$ - $h$ -cobordism  $W$  between  $X$  and itself must be  $G$ -diffeomorphic to  $X \times I$ .*
- (2) *If  $S(V)$  and  $S(V')$  are free linear  $G$ -spheres of dimension  $2n - 1 \geq 5$ , any  $G$ - $h$ -cobordism  $W$  between  $S(V)$  and  $S(V')$  must be  $G$ -diffeomorphic to  $S(V) \times I$ .*

*Proof.* (1) In the case  $|G| \leq 2$ , since it holds that  $Wh(G) = 0$ , the conclusion follows from the  $s$ -cobordism theorem. Our proof will be done under  $|G| \geq 3$ . Let  $W$  be a  $G$ - $h$ -cobordism between  $X$  and itself, with  $\dim W = 2n \geq 6$ . To distinguish the inclusions of  $X$  to  $W$ , we put  $\partial W = X \amalg X'$ , where  $X'$  is a copy of  $X$ . Let  $i: X \rightarrow W$  and  $i': X' \rightarrow W$  be the natural inclusion maps. Let  $r$  be a  $G$ -homotopy inverse of  $i$ . Since the order of  $G$  is greater than or equal to 3 and  $G$  acts freely on a homotopy sphere  $X$  with  $\dim X \geq 5$ , any  $G$ -self-homotopy equivalence of  $X$  is  $G$ -homotopic to the identity map. Hence, we have

$$\tau(r \circ i') = \tau(id) = 0.$$

On the other hand,

$$\begin{aligned} \tau(r \circ i') &= \tau(r) + r_* \tau(i') \\ &= -r_* \tau(i) + r_* \tau(i') \\ &= r_*(\tau(i') - \tau(i)). \end{aligned}$$

Thus we have  $\tau(i') = \tau(i)$ , that is,

$$\tau(W, X) = \tau(W, X').$$

By the duality theorem ([6; p. 394]), we also get

$$\tau(W, X') = -\overline{\tau(W, X)}.$$

Hence by these formulae, we see that  $\tau = -\bar{\tau}$ , that is,  $\tau$  is an element of  $\tilde{A}_{2n+1}(G)$ .

Since  $G$  has periodic cohomology,  $\tilde{A}_{2n+1}(G)$  is isomorphic to  $SK_1(\mathbb{Z}[G])$  by [9; Theorem 3]. Hence we have  $\tilde{A}_{2n+1}(G) = 0$ , thereby  $\tau = 0$ .

(2) Let  $C$  be a cyclic subgroup of  $G$ . By Theorem 1.2,  $\text{res}_c V = \text{res}_c V'$  as real  $C$ -modules. Thus  $V = V'$  as real  $G$ -modules, and then  $S(V')$  is  $G$ -diffeomorphic to  $S(V)$ . Since  $SK_1(\mathbb{Z}[G]) = 0$ , the conclusion now follows from (1) of this proposition.  $\square$

**Proof of Examples.** Example B follows from Theorem A and Proposition 3.1 immediately. By [10], if a finite group  $G$  whose 2-Sylow subgroups are quaternionic acts freely on spheres, its dimension must be  $4n - 1 (n \in \mathbb{N})$ . Hence, if a finite group  $G$  can act freely on spheres of dimension  $4n + 1$ , the 2-Sylow subgroups of  $G$  are cyclic. Thus  $G$  must be of Type I in Appendix, thereby  $SK_1(\mathbb{Z}[G]) = 0$ , which proves Example C.

In [13], we studied  $G$ - $h$ -cobordisms between  $G$ -homotopy spheres and obtained the following results:

**Theorem 3.2** ([13; Theorem A]). *Let  $G$  be a finite group, and  $X$  a free  $G$ -homotopy sphere of dimension  $2n - 1 \geq 5$ . Then the following (1) and (2) are equivalent.*

- (1) *Any  $G$ - $h$ -cobordism  $W$  between  $X$  and itself must be  $G$ -diffeomorphic to  $X \times I$ .*
- (2)  *$\ker \tilde{c}$  is trivial.*

**Corollary 3.3** ([13; Corollary B]). *Suppose  $\ker \tilde{c} = 0$ . Let  $S(V)$  and  $S(V')$  be free linear  $G$ -spheres of dimension  $2n - 1 \geq 5$ . Then a  $G$ - $h$ -cobordism  $W$  between  $S(V)$  and  $S(V')$  must be  $G$ -diffeomorphic to  $S(V) \times I$ .*

Theorem 3.2 is shown by using surgery theory. Corollary 3.3 is an immediate consequence of Theorem 3.2. Since by [9; Theorem 3]  $SK_1(\mathbb{Z}[G]) \cong \tilde{A}_{2n+1}(G)$  for a periodic group  $G$ , Proposition 3.1 is a special case of Theorem 3.2 and Corollary 3.3. Moreover, as in [13], there exists a finite group  $G$  such that  $SK_1(\mathbb{Z}[G]) \neq 0$  and  $\ker \tilde{c} = 0$ . For example, let  $p$  be an odd prime,  $q$  a prime such that  $q \geq 5$ . Let  $G$  be  $\mathbb{Q}_8 \times \mathbb{Z}_p$ ,  $\mathbb{T}^* \times \mathbb{Z}_q$ , or  $\mathbb{O}^* \times \mathbb{Z}_q$ , where  $\mathbb{Q}_8$ ,  $\mathbb{T}^*$ , and  $\mathbb{O}^*$  denote the quaternionic group, the binary tetrahedral group, and the binary octahedral group

respectively. Then we see that  $SK_1(\mathbf{Z}[G]) \cong \mathbf{Z}_2$  and any  $G$ - $h$ -cobordism  $W$  between a free  $G$ -homotopy sphere  $X$  of dimension  $4n-1 \geq 7$  and itself must be  $G$ -diffeomorphic to  $X \times I$ , because  $\ker \tilde{c} = 0$ .

**4. Proof of Theorem D**

Let  $G$  be a compact Lie group of positive dimension which can act freely on a sphere. Then by [3; p. 153, Theorem 8.5],  $G$  must be isomorphic to  $S^1$ ,  $S^3$  or  $NS^1$  the normalizer of  $S^1$  in  $S^3$ . If  $G$  is  $S^1$ , the dimension of a sphere on which  $G$  acts freely is  $2n-1 (n \geq 1)$ . If  $G$  is  $NS^1$  or  $S^3$ , it is  $4n-1 (n \geq 1)$  because  $G$  has a subgroup which is isomorphic to  $\mathbf{Q}_8$ . Now we recall the equivariant Whitehead group which is defined by S. Illman. By [4; Corollary 2,8],

$$\begin{aligned} Wh_{S^1}(X^m) \cong Wh(1) = 0 & \quad \text{where} \quad m = 2n - 1 \geq 7, \\ Wh_{NS^1}(X^m) \cong Wh(\mathbf{Z}_2) = 0 & \quad \text{where} \quad m = 4n - 1 \geq 7, \\ Wh_{S^3}(X^m) \cong Wh(1) = 0 & \quad \text{where} \quad m = 4n - 1 \geq 11. \end{aligned}$$

Thus  $(W; X, X')$  is a  $G$ - $s$ -cobordism in the sense of [1]. The conclusion now follows from the conditions about the dimension of the homotopy sphere by using [1; Theorem 1].

**5. Appendix**

Let  $G$  be a finite solvable group. Then  $G$  has a fixed point free complex representation if and only if  $G$  is of type I, II, III, IV below, with the additional condition: if  $d$  is the order of  $r$  in the multiplicative group of residues modulo  $m$ , of integers prime to  $m$ , then  $n/d$  is divisible by every prime divisor of  $d$ .

**TYPE I.** A group of order  $mn$  that is generated by the elements of the form  $A$  and  $B$ , and that has relations:

$$A^m = B^n = 1, BAB^{-1} = A^r,$$

where  $m, n$  and  $r$  satisfy the following conditions:

$$m \geq 1, n \geq 1, (n(r-1), m) = 1, r^n \equiv 1(m).$$

**TYPE II.** A group of order  $2mn$  that is generated by the elements of the form  $A$ ,  $B$  and  $R$ , and that has relations:

$$R^2 = B^{n/2}, RAR^{-1} = A^l, RBR^{-1} = B^k$$

in addition to the relations in I, where  $m, n, r, l$  and  $k$  satisfy the following conditions:

$$l^2 \equiv r^{k-1} \equiv 1(m), \quad k \equiv -1(2^u), \\ n = 2^u v (u \geq 2, (v, 2) = 1), \quad k^2 \equiv 1(n)$$

in addition to the conditions in I.

TYPE III. A group of order  $8mn$  that is generated by the elements of the form  $A, B, P$  and  $Q$ , and that has relations:

$$P^2 = Q^2 = (PQ)^2, \quad AP = PA, \quad AQ = QA, \\ BPB^{-1} = Q, \quad BQB^{-1} = PQ$$

in addition to the relations in I, where  $m, n$  and  $r$  satisfy the following conditions:

$$n \equiv 1(2), \quad n \equiv 0(3)$$

in addition to the conditions in I.

TYPE IV. A group of order  $16mn$  that is generated by the elements of the form  $A, B, P, Q$  and  $R$ , and that has relations:

$$R^2 = P^2, \quad RPR^{-1} = QP, \quad RQR^{-1} = Q^{-1}, \\ RAR^{-1} = A^l, \quad RBR^{-1} = B^k$$

in addition to the relations in III, where  $m, n, r, k$  and  $l$  satisfy the following conditions:

$$l^2 \equiv r^{k-1} \equiv 1(m), \quad k \equiv -1(3), \quad k^2 \equiv 1(n)$$

in addition to the conditions in III.

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