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ASYMPTOTIC SUFFICIENCY UP TO HIGHER ORDERS AND ITS APPLICATIONS TO STATISTICAL TESTS AND ESTIMATES

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1. Introduction. Suppose that n-dimensional random variable $z_n=(x_1, x_2, \cdots, x_n)$ is distributed according to a probability distribution $P_{\theta,n}$ parameterised by $\theta\in\Theta\subset R^1$, and each x_i is independently and identically distributed. In LeCam [1] it was shown that every estimator t_n with the form $t_n=\hat{\theta}_n+n^{-1}I^{-1}(\hat{\theta}_n)\cdot\Phi_n^{(1)}(z_n, \hat{\theta}_n)$ ($I(\theta)$ means Fisher information number), which is constructed using a reasonable estimator $\hat{\theta}_n$ and the logarithmic derivative $\Phi_n^{(1)}(z_n, \hat{\theta}_n)$ relative to θ of density of $P_{\theta,n}$, is asymptotically sufficient in the following sense; t_n is sufficient for a family $\{Q_{\theta,n};\ \theta\in\Theta\}$ of probability distributions and that

$$\lim_{n\to\infty} ||P_{\theta,n}-Q_{\theta,n}|| = 0$$

uniformly on any compact set in Θ (where $||\cdot||$ means the totally variation of a measure). This implies that the statistic $(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n))$ is asymptotically sufficient up to order $o(1)$. As a refinement of this result it will be shown in this paper that for $k\geqq 1$ a statistic $t_n^*=(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n))$, where $\Phi_n^{(i)}(z_n, \theta)$ means the $(i-1)$th derivative relative to θ of $\Phi_n^{(1)}(z_n, \theta)$, is asymptotically sufficient up to order $o(n^{-(k-1)/2})$ in the following sense; t_n^* is sufficient for a family $\{Q_{\theta,n};\ \theta\in\Theta\}$ and

$$\lim_{n\to\infty} n^{(k-1)/2}||P_{\theta,n}-Q_{\theta,n}|| = o$$

uniformly on any compact subset of Θ. From our result it follows that if we use the maximum likelihood estimator $\hat{\theta}_n^*$ as the initial estimator $\hat{\theta}_n$ then the statistic $(\hat{\theta}_n^*, \Phi_n^{(2)}(z_n, \hat{\theta}_n^*), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n^*))$ is asymptotically sufficient up to order $o(n^{-(k-1)/2})$. In Ghosh and Subramanyam [4] it was mentioned that for exponential family of distributions $(\hat{\theta}_n^*, \Phi_n^{(2)}(z_n, \hat{\theta}_n^*), \Phi_n^{(3)}(z_n, \hat{\theta}_n^*), \Phi_n^{(4)}(z_n, \hat{\theta}_n^*))$ is asymptotically sufficient up to order $o(n^{-1})$ in pointwise sense relative to θ. Our result is more general and accurate one.

As an application of our result we try to improve arbitrarily given statistical tests or estimators. It will be shown that for arbitrarily given test sequence $\{\phi_n\}$, which is asymptotically similar of size α up to order $o(n^{-(k-1)/2})$ uniformly on compacts in hypothesis, there exists a test sequence $\{\psi_n\}$ such that ψ_n is a function of t_n^* and the difference between the powers of ϕ_n and ψ_n is asymptotically up to order $o(n^{-(k-1)/2})$ uniformly on compacts in alternative hypothesis. It is also shown that for any estimators $\{\tilde{\theta}_n\}$ belonging to a class $\boldsymbol{D}$ (or $\boldsymbol{D}'$) (See Section 5 for the precise definition of $\boldsymbol{D}$ and $\boldsymbol{D}'$) there exists a sequence $\{\tilde{\theta}_n^*\}$ of estimators such that $\tilde{\theta}_n^*$ is a function of t_n^* and for every compact subset K of Θ

$$\limsup_{n\to\infty} \sup_{\theta\in K} \{E[(\tilde{\theta}_n^*-\theta)^2; P_{\theta,n}]/E(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]\} \leqq 1 .$$

It seems to the author that the class $\boldsymbol{D}'$ is so wide that such reasonable estimators as maximum likelihood estimators, minimum contrast estimators and BAN-estimators are all contained in it.

A lemma is proved in Section 2, which is a refinement of the inequality concerning large deviation probabilities. In Section 3 main theorem, Theorem 2, will be proved. Section 4 and 5 are devoted to the applications of Theorem 2 to statistical tests and estimates.

2. A lemma. Let $\Theta(\neq\phi)$ be an open set in R^1. Suppose that for each $\theta\in\Theta$ there corresponds a probability measure P_θ defined on a measurable space $(\boldsymbol{X}, \boldsymbol{A})$. For each $n\in N=\{1, 2, \cdots\}$ let $(\boldsymbol{X}^{(n)}, \boldsymbol{A}^{(n)})$ be the cartesian product of n copies of $(\boldsymbol{X}, \boldsymbol{A})$, and $P_{\theta,n}$ the product measure of n copies of P_θ. For a function h and a probability measure P, $E[h; P]$ stands for the expectation of h under P.

Lemma. *Let $\Theta_0(\neq\phi)$ be a subset of Θ and ε_0 be a positive number. For each $(\theta, \varepsilon)\in\Theta_0\times(0, \varepsilon_0]$ let $\{Z_\nu(\varepsilon, \theta)\}_{\nu=1,2,\dots}$ be a sequence of random variables each of which is independently and identically distributed according to P_θ. Suppose that; (1) There exist positive numbers ρ_1 and $\varepsilon_1(\leqq\varepsilon_0)$ such that for every $(t, \varepsilon, \theta)\in(-\rho_1, \rho_1)\times(0, \varepsilon_1]\times\Theta_0$ the moment generating function (m.g.f.) of $Z_1(\varepsilon, \theta)$, which is denoted by $\phi(t; \varepsilon, \theta)=E[\exp(tZ_1(\varepsilon, \theta)); P_\theta]$, converges uniformly with respect to (ε, θ) in $(0, \varepsilon_1]\times\Theta_0$. (2) $0<\alpha_0=\liminf_{\varepsilon\to 0}\inf_{\theta\in\Theta_0} E[(Z_1(\varepsilon, \theta))^2; P_\theta]\leqq\limsup_{\varepsilon\to 0}\sup_{\theta\in\Theta_0} E[(Z_1(\varepsilon, \theta))^2; P_\theta]=\alpha_1<\infty$ (3) There exists $\alpha_2(0<\alpha_2<1)$ and $\varepsilon_2(0<\varepsilon_2\leqq\varepsilon_0)$ such that $\sup_{\theta\in\Theta_0} |E[Z_1(\varepsilon, \theta); P_\theta]|\leqq\alpha_2\varepsilon$ for every ε satisfying $0<\varepsilon\leqq\varepsilon_2$. Then, for any β satisfing $0<\beta<(1-\alpha_2)^2/(2\alpha_1)$ there exists a positive number ε_3 such that for every $n\in N$ and every ε with $0<\varepsilon\leqq\varepsilon_3$*

$$\sup_{\theta\in\Theta_0} P_{\theta,n}\Big(\sum_{\nu=1}^{n} Z_\nu(\varepsilon, \theta)\geqq n\varepsilon\Big)\leqq(1-\beta\varepsilon^2)^n .$$

Proof. Let $\varepsilon^*=\min\{\varepsilon_1, \varepsilon_2\}$ and ε be any number satisfying $0<\varepsilon\leqq\varepsilon^*$. We have

$$(J=)\inf_{t\geqq 0} \exp(-t\varepsilon)\phi(t; \varepsilon, \theta) = \inf_{t\geqq 0} E[\exp(t(Z_1(\varepsilon, \theta)-\varepsilon)); P_\theta]$$

$$\leqq \inf_{\rho_1 > t \geqq 0} \{1-(1/2)\,[E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]]^2/E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta] \\ +(1/2)E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta]\,(t+E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]/E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta])^2 \\ +R(t,\varepsilon,\theta)\}$$

where $\lim_{t\to 0}\sup_{0<\varepsilon\leqq\varepsilon^*}\sup_{\theta\in\Theta_0}|R(t,\varepsilon,\theta)/t^2|=0$ (This follows from the assumption (1)). By the assumption (3) we have $E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]\leqq 0$ and $t_\varepsilon\to 0$ as $\varepsilon\to 0$ where $t_\varepsilon=-E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]/E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta]$. Hence for sufficiently small $\varepsilon>0$,

$$J\leqq 1-(1/2)[E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]]^2/E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta]+R(t_\varepsilon,\varepsilon,\theta)\,. \tag{2.1}$$

From assumption (2) and (3) we have

$$\liminf_{\varepsilon\to 0}\inf_{\theta\in\Theta_0}\{[E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]]^2/(\varepsilon^2E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta])\} \\ \geqq\{\liminf_{\varepsilon\to 0}\inf_{\theta\in\Theta_0}(1-E[Z_1(\varepsilon,\theta);P_\theta]/\varepsilon)^2\}/\{\limsup_{\varepsilon\to 0}\sup_{\theta\in\Theta_0}E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta]\} \\ \geqq(1-\alpha_2)^2/\alpha_1\,. \tag{2.2}$$

Let β be any number such that $0<\beta<(1-\alpha_2)^2/(2\alpha_1)$ and β' be a number satisfying $\beta<\beta'<(1-\alpha_2)^2/(2\alpha_1)$. By (2.2) there exists a positive number $\varepsilon'(\leqq\varepsilon^*)$ such that for every $\varepsilon\leqq\varepsilon'$

$$\inf_{\theta\in\Theta_0}[\{E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]\}^2/E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta]]\geqq 2\beta'\varepsilon^2\,. \tag{2.3}$$

By assumption (2) and (3)

$$\limsup_{\varepsilon\to 0}\sup_{\theta\in\Theta_0}|t_\varepsilon|/\varepsilon \\ \leqq[\limsup_{\varepsilon\to 0}\{\sup_{\theta\in\Theta_0}|E[Z_1(\varepsilon,\theta);P_\theta]|+\varepsilon\}/\varepsilon]/[\liminf_{\varepsilon\to 0}\inf_{\theta\in\Theta_0}\{E[(Z_1(\varepsilon,\theta)^2;P_\theta]-\varepsilon^2\}] \\ \leqq\limsup_{\varepsilon\to 0}[(2\varepsilon)/\varepsilon]/\alpha_0 \\ =2/\alpha_0$$

Hence it follows that

$$\limsup_{\varepsilon\to 0}\sup_{\theta\in\Theta_0}|R(t_\varepsilon,\varepsilon,\theta)|/\varepsilon^2 \\ \leqq\{\limsup_{\varepsilon\to 0}\sup_{\theta\in\Theta_0}(t_\varepsilon^2/\varepsilon^2)\}\times\{\limsup_{\varepsilon\to 0}\sup_{\theta\in\Theta_0}|R(t_\varepsilon,\varepsilon,\theta)|^2/t_\varepsilon^2 \\ =(4/\alpha_0^2)\cdot 0 \\ =0\,.$$

Thus there exists a positive number ε'' such that

$$\sup_{\theta\in\Theta_0}|R(t_\varepsilon,\varepsilon,\theta)|/\varepsilon^2\leqq\beta'-\beta$$

for every ε satisfying $0<\varepsilon\leqq\varepsilon''$. Hence from (2.1) and (2.3) for every ε satisfying $0<\varepsilon\leqq\min\{\varepsilon',\varepsilon''\}$ and every $\theta\in\Theta_0$,

$$J \leqq 1-\beta\varepsilon^2+\varepsilon^2\{\beta-(1/2)\varepsilon^{-2}[E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]]^2/E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta] + R(t_\varepsilon,\varepsilon,\theta)/\varepsilon^2\}$$
$$\leqq 1-\beta\varepsilon^2+\varepsilon^2(\beta'-(1/2)\varepsilon^{-2}[E[Z_1(\varepsilon,\theta)-\varepsilon;P_\theta]]^2/E[(Z_1(\varepsilon,\theta)-\varepsilon)^2;P_\theta])$$
$$\leqq 1-\beta\varepsilon^2 .$$

Thus we have for every $\varepsilon(0<\varepsilon\leqq\min\{\varepsilon',\varepsilon''\})$, every $n\in N$ and every $\theta\in\Theta_0$

$$P_\theta(\sum_{\nu=1}^n Z_\nu(\varepsilon,\theta)\geqq n\varepsilon)\leqq\inf_{t\geqq 0}[\exp(-t\varepsilon)\phi(t;\varepsilon,\theta)]^n \leqq(1-\beta\cdot\varepsilon^2)^n .$$

This completes the proof.

3. Asymptotically sufficient statistics up to higher orders. Assume that the map: $\theta\rightarrow P_\theta$ is one to one, and that for each $\theta\in\Theta$ P_θ has a density $f(\cdot,\theta)$ relative to a sigma-finite measure μ on $(\boldsymbol{X},\boldsymbol{A})$. We assume that $f(x,\theta)>0$ for every $x\in\boldsymbol{X}$ and every $\theta\in\Theta$. We denote by μ_n the product measure of n copies of the same component μ. We define $\Phi(x,\theta)=\log f(x,\theta)$ for each $x\in\boldsymbol{X}$ and $\theta\in\Theta$, and $\Phi_n(z_n,\theta)=\sum_{\nu=1}^n\Phi(x_\nu,\theta)$ for each $n\in N$, each $z_n=(x_1,x_2,\cdots,x_n)\in\boldsymbol{X}^{(n)}$ and $\theta\in\Theta$. Let k be a positive integer which would be fixed throughout this paper.

Condition R. (1). $\Phi(x,\theta)$ is $(k+2)$-times continuously differentiable with respect to θ in Θ for each $x\in\boldsymbol{X}$. For each j $(1\leqq j\leqq k+2)$ we define $\Phi^{(j)}(x,\theta)=\partial^j\Phi(x,\theta)/\partial\theta^j$ and $\Phi_n^{(j)}(z_n,\theta)=\sum_{\nu=1}^n\Phi^{(j)}(x_\nu,\theta)$.

(2). For each $\theta\in\Theta$ there exists a positive number ε such that

a. $\sup_{|\tau-\theta|\leqq\varepsilon}E[\sup_{|\sigma-\theta|\leqq\varepsilon}|\Phi^{(k+2)}(x,\sigma)|^2;P_\tau]<\infty$

b. $\sup_{|\tau-\theta|\leqq\varepsilon}E[|\Phi^{(k+1)}(x,\tau)|\cdot u_\varepsilon(x,\tau);P_\tau]<\infty$ and $E[u_\varepsilon(x,\theta);P_\theta]<\infty$

where $u_\varepsilon(x,\tau)=\sup_{|\sigma-\tau|\leqq\varepsilon}|f'(x,\sigma)/f(x,\tau)|$

c. $\mathrm{Var}(\Phi^{(k+1)}(x,\tau);P_\tau)$ (i.e., variance of $\Phi^{(k+1)}(x,\tau)$ under P_τ) are positive and finite uniformly for every τ satisfying $|\tau-\theta|\leqq\varepsilon$.

(3). Define $\bar{Z}(x;\varepsilon',\sigma)=\sup\{\Phi^{(k+1)}(x,\tau)-E[\Phi^{(k+1)}(x,\tau);P_\tau];\tau\in\Theta,|\tau-\sigma|\leqq\varepsilon'\}$ and $Z^*(x;\varepsilon',\sigma)=-\inf\{\Phi^{(k+1)}(x,\tau)-E[\Phi^{(k+1)}(x,\tau);P_\tau];\tau\in\Theta,|\tau-\sigma|\leqq\varepsilon'\}$ for each $\varepsilon'>0$ and $\sigma\in\Theta$. For each $\theta\in\Theta$ there exist positive numbers η and ρ such that for every $(t,\varepsilon',\sigma)\in(-\rho,\rho)\times(0,\eta]\times(\theta-\eta,\theta+\eta)$ the m.g.f.'s of $\bar{Z}(x;\varepsilon',\sigma)$ and $Z^*(x;\varepsilon',\sigma)$ exist and converge uniformly with respect to σ in $(\theta-\eta,\theta+\eta)$.

Remark 1. An example satisfying Condition R is the following one. Let μ be a sigma-finite measure on $(\boldsymbol{X},\boldsymbol{A})$ and the density function $f(x,\theta)$ of P_θ relative to μ be given by

$$f(x, \theta) = h(x)c(\theta)\exp[\sum_{i=1}^{m} s_i(\theta)t_i(x)]$$

where $c(\theta)$, $s_i(\theta)$ $(1 \leqq i \leqq m)$ are $(k+2)$-times continuously differentiable real valued functions of θ only, and $h(x)$, $t_i(x)$ $(1 \leqq i \leqq m)$ are θ-independent $\boldsymbol{A}$-measurable functions of x. Let $S = \{(s_1, s_2, \cdots, s_m) \in R^m; \int_X \exp[\sum_{i=1}^{m} s_i t_i(x)]h(x)d\mu(x) < \infty\}$ and $S(\Theta) = \{(s_i(\theta), \cdots, s_m(\theta)); \theta \in \Theta\}$. If $S(\Theta) \subset \text{int}\, S$ (interior of S) and if $\sum_i \sum_j s_i^{(k+1)}(\theta)s_j^{(k+1)}(\theta) \cdot \text{Cov}(t_i, t_j; P_\theta) > 0$ for every $\theta \in \Theta$, then Condition R is satisfied with the family $\{P_\theta; \theta \in \Theta\}$. Here for each i $s_i^{(k+1)}$ means $(k+1)$-th derivative of s_i, and $\text{Cov}(t_i, t_j; P_\theta)$ means the covariance of (t_i, t_j) under P_θ for each (i, j).

An *estimator* of θ depending on $z_n = (x_1, x_2, \cdots, x_n) \in \boldsymbol{X}^{(n)}$ is an $\boldsymbol{A}^{(n)}$-measurable function from $\boldsymbol{X}^{(n)}$ to R^1. Such estimator will be called *strict* if its range is a subset of Θ. In [2] Pfanzagl has shown the existence of a 'reasonable estimator' of θ. We quote here his result without proof.

Theorem 1. (Pfanzagl [2]) *Suppose that Condition R is satisfied, then for any sequence $\{\alpha_n\}$ of positive numbers satisfying $n^{-1/2}\alpha_n \to 0$ and $\alpha_n \to \infty$ as $n \to \infty$ there exists a sequence $\{\hat{\theta}_n\}_{n \in N}$ of strict estimators with the following property: For every compact set $K \subset \Theta$ there exists a constant $c_K > 0$ such that*

$$P_{\theta,n}(\{z_n \in \boldsymbol{X}^{(n)}; n^{1/2}|\hat{\theta}_n(z_n) - \theta| > \alpha_n\}) \leqq c_K \cdot \exp(-\alpha_n)$$

for all $\theta \in K$ and $n \in N$.

For each δ satisfying $0 < \delta < 1/2$ we denote by $\boldsymbol{C}_k(\delta)$ the class of all sequences of strict estimators $\hat{\theta}_n$ of θ such that for every compact subset K of Θ

$$\sup_{\theta \in K} P_{\theta,n}(n^{1/2}|\hat{\theta}_n - \theta| > n^\delta) = o(n^{-(k-1)/2}) .$$

The notation $o(a_n)$ means that $\lim_{n \to \infty} o(a_n)/a_n = 0$.

REMARK 2. By Theorem 1 for every $\delta(0 < \delta < 1/2)\boldsymbol{C}_k(\delta)$ does not empty. The maximum likelihood estimator (or more generally minimum contrast estimator) is contained in $\bigcap_{\delta > 0} \boldsymbol{C}_k(\delta)$ under suitable regularity conditions (cf. Pfanzagle [3], Lemma 3).

Let $\delta_0 = 1/[2(k+2)]$ and $\boldsymbol{C}_k = \bigcup_{0 < \delta < \delta_0} \boldsymbol{C}_k(\delta)$.

Theorem 2. *Suppose that Condition R is satisfied, and that $\{\hat{\theta}_n\} \in \boldsymbol{C}_k$ then there exists a sequence $\{Q_{\theta,n}; \theta \in \Theta\}$, $n \in N$, of families of probability measures on $(\boldsymbol{X}^{(n)}, \boldsymbol{A}^{(n)})$ with the following property: (1) For each $n \in N$, the statistic $t_n^* = (\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n))$ is sufficient for $\{Q_{\theta,n}; \theta \in \Theta\}$. (2). For every compact set $K \subset \Theta$,*

$$\sup_{\theta\in K} ||P_{\theta,n}-Q_{\theta,n}|| = o(n^{-(k-1)/2}) .$$

Proof. We shall divide the proof into several steps.

The first step. Suppose that Condition R is satisfied, and that $\{\hat{\theta}_n\}\in \boldsymbol{C}_k(\delta_1)$ where δ_1 satisfies $0<\delta_1<\delta_0$. Let δ and γ be two numbers satisfying $\delta_1<\delta<\delta_0$ and $\delta<\gamma<1/2-(k+1)\delta$, and let $\varepsilon_n=n^{\delta-(1/2)}$ and $\varepsilon'_n=n^{\gamma-(1/2)}$. Define

$$W_n^1 = \{z_n\in \boldsymbol{X}^{(n)};\ |\theta-\hat{\theta}_n(z_n)|\leqq \varepsilon_n,\ [\theta:\hat{\theta}_n]\subset\Theta\}$$
$$W_n^2 = \{z_n\in \boldsymbol{X}^{(n)};\ \gamma_n(z_n)\leqq \varepsilon'_n\}$$

where $[\theta:\hat{\theta}_n]=\{t\theta+(1-t)\hat{\theta}_n;\ 0\leqq t\leqq 1\}$ and

$$\gamma_n(z_n) = \sup\{|\Phi_n^{(k+1)}(z_n,\tau)/n-E[\Phi^{(k+1)}(x,\tau);P_\tau]|\,;\,\tau\in\Theta,\ |\tau-\hat{\theta}_n|\leqq 2\varepsilon_n\} .$$

By a Taylor expansion of $\Phi_n(z_n,\theta)$ around $\theta=\hat{\theta}_n$ we have

$$\Phi_n(z_n,\theta) = \sum_{m=0}^{k}\Phi_n^{(m)}(z_n,\hat{\theta}_n)\,(\theta-\hat{\theta}_n)^m/m!+s_n(\hat{\theta}_n,\theta)+R_n(z_n,\theta) \tag{3.1}$$

where $s_n(\hat{\theta}_n,\theta)=n(\theta-\hat{\theta}_n)^{k+1}\cdot E[\Phi^{(k+1)}(x,\theta);P_\theta]/(k+1)!$, $R_n(z_n,\theta)=0$ (if $[\theta:\hat{\theta}_n]\not\subset\Theta$),

$$R_n(z_n,\theta) = n(\theta-\hat{\theta}_n)^{k+1}\Big[\int_0^1(1-\lambda)^k\{\Phi_n^{(k+1)}(z_n,\hat{\theta}_n+\lambda(\theta-\hat{\theta}_n))/n- E[\Phi^{(k+1)}(x,\theta);P_\theta]\}d\lambda\Big]/k!\ (\text{if } [\theta:\hat{\theta}_n]\subset\Theta),$$
$$\Phi_n^{(0)}(z_n,\hat{\theta}_n) = \Phi_n(z_n,\hat{\theta}_n) .$$

We define $q_n(z_n,\theta)=c_n(\theta)I_{W_n^1}(z_n)I_{W_n^2}(z_n)\exp[\Phi_n(z_n,\hat{\theta}_n)+\Psi_n(t_n^*,\theta)]$ where $I_{W_n^i}$ designate the indicator functions of W_n^i, $\Psi_n(t_n^*,\theta)=\sum_{m=1}^{k}\Phi_n^{(m)}(z_n,\hat{\theta}_n)\,(\theta-\hat{\theta}_n)^m/m! +s_n(\hat{\theta}_n,\theta)$ and $c_n(\theta)^{-1}=\int_{X^{(n)}}I_{W_n^1}I_{W_n^2}\exp[\Phi_n(z_n,\hat{\theta}_n)+\Psi_n(t_n^*,\theta)]d\mu_n$. Let $Q_{\theta,n}(A) =\int_A q_n(z_n,\theta)d\mu_n$, and $Q^*_{\theta,n}(A)=\int_A q_n^*(z_n,\theta)d\mu_n$ for every $A\in \boldsymbol{A}^{(n)}$ where $q_n^*(z_n,\theta) =c_n(\theta)^{-1}\cdot q_n(z_n,\theta)$. According to the factorization theorem for each $n\in N$ t_n^* is sufficient for $\{Q_{\theta,n};\theta\in\Theta\}$.

By (3.1) we have

$$\begin{aligned}2||P_{\theta,n}-Q^*_{\theta,n}|| &= \int_{X^{(n)}}|p_n(z_n,\theta)-q_n^*(z_n,\theta)|\,d\mu_n\\ &= \int_{W_n\cap W_n^2}|1-\exp[-R_n(z_n,\theta)]|\,p_n(z_n,\theta)d\mu_n\\ &\quad +P_{\theta,n}((W_n^1)^c)+P_{\theta,n}(W_n^1\cap(W_n^2)^c)\\ &= T_n^1(\theta)+T_n^2(\theta)+T_n^3(\theta)\end{aligned} \tag{3.2}$$

where
$$p_n(z_n,\theta) = dP_{\theta,n}/d\mu_n ,$$
$$T_n^1(\theta) = \int_{W_n^1\cap W_n}|1-\exp[-R_n(z_n,\theta)]|\,p_n(z_n,\theta)d\mu_n ,$$
$$T_n^2(\theta) = P_{\theta,n}((W_n^1)^c) \text{ and } T_n^3(\theta) = P_{\theta,n}(W_n^1\cap(W_n^2)^c) .$$

The second step. Let K be a compact subset of Θ. From Condition R it implies that there exist positive numbers ε^*, ρ^* and η^* which depending only on K, such that

$$
\begin{aligned}
&M_1 = \sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon^*} E[\sup_{|\sigma-\theta|\leqq\varepsilon^*} \{\Phi^{(k+2)}(x, \sigma)\}^2; P_\tau] < \infty \\
&M_2 = \sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon^*} E[|\Phi^{(k+1)}(x, \tau)\cdot u_{\varepsilon^*}(x, \tau); P_\theta] < \infty \\
&0 < \inf_{\tau\in K} \mathrm{Var}(\Phi^{(k+1)}(x, \tau); P_\tau) \leqq \sup_{\tau\in K} \mathrm{Var}(\Phi^{(k+1)}(x, \tau); P_\tau) < +\infty
\end{aligned}
\tag{3.3}
$$

and that for every $\theta\in K$ and every $(t, \varepsilon', \sigma)\in(-\rho^*, \rho^*)\times(0, \eta^*]\times(\theta-\eta^*, \theta+\eta^*)$ the m.g.f.'s of $\bar{Z}(x; \varepsilon', \sigma)$ and $Z^*(x; \varepsilon', \sigma)$ exist and converge uniformly with respect to σ in $(\theta-\eta^*, \theta+\eta^*)$.
For any ε such that $\varepsilon\leqq\varepsilon^*$ we have

$$
\begin{aligned}
&\sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon} |E[\Phi^{(k+1)}(x, \tau); P_\tau] - E[\Phi^{(k+1)}(x, \theta); P_\tau] \\
&\quad \leqq \sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon} E[\sup_{|\sigma-\theta|\leqq\varepsilon} |\Phi^{(k+2)}(x, \sigma)|; P_\tau]\cdot\varepsilon \\
&\quad \leqq \sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon^*} [E[\sup_{|\sigma-\theta|\leqq\varepsilon^*} (\Phi^{(k+2)}(x, \sigma))^2; P_\tau]]^{1/2}\cdot\varepsilon \\
&\quad = M_1^{1/2}\varepsilon\,.
\end{aligned}
$$

Also we have

$$
\begin{aligned}
&\sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon} |E[\Phi^{(k+1)}(x, \theta); P_\tau] - E[\Phi^{(k+1)}(x, \theta); P_\theta]| \\
&\quad \leqq \sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon} E[|\Phi^{(k+1)}(x, \theta)|\cdot u_{\varepsilon^*}(x, \theta); P_\theta]\cdot\varepsilon \\
&\quad = M_2\cdot\varepsilon\,.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon} |E[\Phi^{(k+1)}(x, \tau); P_\tau] - E[\Phi^{(k+1)}(x, \theta); P_\theta]| \\
&\quad \leqq \sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon} |E[\Phi^{(k+1)}(x, \tau); P_\tau] - E[\Phi^{(k+1)}(x, \theta); P_\tau]| \\
&\qquad + \sup_{\theta\in K} \sup_{|\tau-\theta|\leqq\varepsilon} |E[\Phi^{(k+1)}(x, \theta); P_\tau] - E[\Phi^{(k+1)}(x, \theta); P_\theta]| \\
&\quad \leqq (M_1^{1/2}+M_2)\varepsilon\,.
\end{aligned}
\tag{3.4}
$$

Since for sufficiently large number $n\in N$ $\varepsilon_n'\geqq(M_1^{1/2}+M_2)\varepsilon_n$, there exists a number n_1 such that for every $n\geqq n_1$, every $\theta\in K$ and every $z_n\in W_n^1\cap W_n^2$,

$$
\begin{aligned}
&\sup_{|\tau-\theta|\in\varepsilon_n} |\Phi_n^{(k+1)}(z_n, \tau)/n - E[\Phi^{(k+1)}(x, \theta); P_\theta]| \\
&\quad \leqq \sup_{|\tau-\theta|\leqq\varepsilon_n} |\Phi_n^{(k+1)}(z_n, \tau)/n - E[\Phi^{(k+1)}(x, \tau); P_\tau]| \\
&\qquad + \sup_{|\tau-\theta|\leqq\varepsilon_n} |E[\Phi^{(k+1)}(x, \tau); P_\tau] - E[\Phi^{(k+1)}(x, \theta); P_\theta]|
\end{aligned}
$$

$$\leqq \sup_{|\tau-\hat{\theta}_n|\leqq 2\varepsilon_n} |\Phi_n^{(k+1)}(z_n, \tau)/n - E[(\Phi^{(k+1)}(x, \tau); P_\tau]| + (M_1^{1/2}+M_2)\varepsilon_n$$
$$\leqq \gamma_n(z_n)+\varepsilon'_n$$
$$\leqq 2\varepsilon'_n\,.$$

Thus we have for every $n \geqq n_1$, every $z_n \in W_n^1 \cap W_n^2$ and every $\theta \in K$

$$|R_n(z_n, \theta)| \leqq n|\theta-\hat{\theta}_n|^{k+1}2\varepsilon'_n/(k+1)!$$
$$\leqq 2n\varepsilon_n^{k+1}\varepsilon'_n/(k+1)!$$
$$= 2n^{(k+1)(\delta-(1/2))+\gamma+(1/2)}/(k+1)!\,.$$

Hence there exists a number n_2 such that

$$\sup_{\theta\in K} T_n^1(\theta) \leqq \sup_{\theta\in K} \int_{X^{(n)}} |R_n(z_n, \theta)| \exp(|R_n|) dP_{\theta,n}$$
$$\leqq 4n^{-(k-1)/2} n^{(k+1)\delta+\gamma-1/2}/(k+1)!$$

for all $n \geqq n_2$. Thus we have

$$\sup_{\theta\in K} T_n^1(\theta) = o(n^{-(k-1)/2})\,. \tag{3.5}$$

The third step. Since $\{\hat{\theta}_n\} \in C_k(\delta_1)$ it follows that

$$\begin{aligned}\sup_{\theta\in K} T_n^2(\theta) &\leqq \sup_{\theta\in K} P_{\theta,n}(|\theta-\hat{\theta}_n|>\varepsilon_n) + \sup_{\theta\in K} P_{\theta,n}(|\theta-\hat{\theta}_n|\leqq\varepsilon_n, [\theta:\hat{\theta}_n]\not\subset\Theta)\\ &= \sup_{\theta\in K} P_{\theta,n}(n^{1/2}|\theta-\hat{\theta}_n|>n^{\delta}) \text{ (for suff. large } n)\\ &\leqq \sup_{\theta\in K} P_{\theta,n}(n^{1/2}|\theta-\hat{\theta}_n|>n^{\delta_1})\\ &= o(n^{-(k-1)/2})\,.\end{aligned} \tag{3.6}$$

The fourth step.

$$\begin{aligned}&\sup_{\theta\in K} T_n^3(\theta) = \sup_{\theta\in K} P_{\theta,n}(|\theta-\hat{\theta}_n|\leqq\varepsilon_n, \gamma_n(z_n)>\varepsilon'_n)\\ &\leqq \sup_{\theta\in K} P_{\theta,n}(\sup_{|\tau-\theta|\leqq 3\varepsilon_n} |\Phi_n^{(k+1)}(z_n, \tau)/n - E[\Phi^{(k+1)}(x, \tau); P_\tau]| > \varepsilon'_n)\\ &\leqq \sup_{\theta\in K} P_{\theta,n}(\sum_{\nu=1}^{n} \sup_{|\tau-\theta|\leqq 3\varepsilon_n} \{\Phi^{(k+1)}(x_\nu, \tau) - E[\Phi^{(k+1)}(x, \tau); P_\tau]\} > n\varepsilon'_n)\\ &+ \sup_{\theta\in K} P_{\theta,n}(\sum_{\nu=1}^{n} \inf_{|\tau-\theta|\leqq 3\varepsilon_n} \{\Phi^{(k+1)}(x_\nu, \tau) - E[\Phi^{(k+1)}(x, \tau); P_\tau]\} < -n\varepsilon'_n)\end{aligned} \tag{3.7}$$

Let $a(\varepsilon)=\varepsilon/(4M_1^{1/2}+2M_2)$ and let $Z_\nu(\varepsilon, \theta)=\bar{Z}(x_\nu; a(\varepsilon), \theta)$ $(\nu=1, 2, \cdots, n)$. For any ε such that $a(\varepsilon)\leqq\varepsilon^*$ we have

$$\sup_{\theta\in K} |E[Z_1(\varepsilon, \theta); P_\theta]| \leqq \sup_{\theta\in K} |E[\sup_{|\tau-\theta|\leqq a(\varepsilon)} \{\Phi^{(k+1)}(x, \tau) - E[\Phi^{(k+1)}(x, \tau); P_\tau]\}; P_\theta]|$$
$$\leqq \sup_{\theta\in K} [E[\sup_{|\tau-\theta|\leqq a(\varepsilon)} |\Phi^{(k+2)}(x, \tau)|; P_\theta]]\cdot a(\varepsilon)$$

$$+\sup_{\theta\in K}\sup_{|\tau-\theta|\leqq a(\varepsilon)}|E[\Phi^{(k+1)}(x,\tau);P_\tau]-E[\Phi^{(k+1)}(x,\theta);P_\theta]|$$
$$\leqq (2M_1^{1/2}+M_2)\cdot a(\varepsilon) \text{ (This follows from (3.4))}$$
$$=\varepsilon/2\,.$$

Define $Z_1(0,\theta)=\Phi^{(k+1)}(x,\theta)-E[\Phi^{(k+1)}(x,\theta);P_\theta]$. We note that $\mathrm{Var}(\Phi^{(k+1)}(x,\theta);P_\theta)=E[(Z_1(0,\theta))^2;P_\theta]$. For every sufficiently small $\varepsilon>0$, we have

$$\sup_{\theta\in K}|(E[Z_1(\varepsilon,\theta)^2;P_\theta])^{1/2}-(E[Z_1(0,\theta)^2;P_\theta])^{1/2}|^2$$
$$\leqq \sup_{\theta\in K}E[|Z_1(\varepsilon,\theta)-Z_1(0,\theta)|^2;P_\theta]$$
$$\leqq 2(\sup_{\theta\in K}E[\sup_{|\tau-\theta|\leqq a(\varepsilon)}|\Phi^{(k+1)}(x,\tau)-\Phi^{(k+1)}(x,\theta)|^2;P_\theta]$$
$$+\sup_{\theta\in K}\sup_{|\tau-\theta|\leqq a(\varepsilon)}|E[\Phi^{(k+1)}(x,\tau);P_\tau]-E[\Phi^{(k+1)}(x,\theta);P_\theta]|^2)$$
$$\leqq 2\cdot a(\varepsilon)^2[M_1+(M_1^{1/2}+M_2)^2]\,.$$

From this it follows that

$$\lim_{\varepsilon\to 0}\sup_{\theta\in K}|E[(Z_1(\varepsilon,\theta))^2;P_\theta]-E[(Z_1(0,\theta))^2;P_\theta]|=0\,.$$

Thus taking account of (3.3) we have

$$0<\inf_{\theta\in K}E[(Z_1(\varepsilon,\theta))^2;P_\theta]\leqq\sup_{\theta\in K}E[(Z_1(\varepsilon,\theta))^2;P_\theta]<+\infty$$

for every sufficiently small ε.
According to the lemma, there exist a constant $\beta>0$ and $\varepsilon^{**}>0$ such that

$$\sup_{\theta\in K}P_{\theta,n}(\sum_{\nu=1}^{n}Z_\nu(\varepsilon,\theta)\geqq n\varepsilon)\leqq(1-\beta\cdot\varepsilon^2)^n$$

for all $n\in N$ and all ε satisfying $0<\varepsilon\leqq\varepsilon^{**}$. Hence for all sufficiently large number n we have

$$\sup_{\theta\in K}P_{\theta,n}(\sum_{\nu=1}^{n}Z_\nu(\varepsilon_n',\theta)\geqq n\varepsilon_n')\leqq(1-\beta\cdot(\varepsilon_n')^2)^n\,.$$

Furthermore it holds that

$$(1-\beta(\varepsilon_n')^2)^n=(1-\beta n^{2\gamma-1})^n$$
$$\leqq 2\cdot\exp(-\beta n^{2\gamma})$$

for all sufficiently large number n. Thus we have

$$\sup_{\theta\in K}P_{\theta,n}(\sum_{\nu=1}^{n}Z_\nu(\varepsilon_n',\theta)\geqq n\varepsilon_n')=o(n^{-(k-1)/2})\,. \tag{3.8}$$

Since $a(\varepsilon_n')\geqq 3\varepsilon_n$ for all sufficiently large number n it follows from the definition of $Z_\nu(\varepsilon_n',\theta)$ that

$$(3.9)\qquad \sup_{\theta\in K} P_{\theta,n}\Big(\sum_{\nu=1}^{n} \sup_{|\tau-\theta|\leqq 3\varepsilon_n}\{\Phi^{(k+1)}(x_\nu,\tau)-E[\Phi^{(k+1)}(x,\tau);P_\tau]\}>n\varepsilon_n'\Big)$$

$$\leqq \sup_{\theta\in K} P_{\theta,n}\Big(\sum_{\nu=1}^{n} Z_\nu(\varepsilon_n',\theta)\geqq n\varepsilon_n'\Big).$$

Similarly, taking $Z^*(x; a(\varepsilon), \theta)$ instead of $\bar{Z}(x; a(\varepsilon), \theta)$, we have

$$(3.10)\quad \sup_{\theta\in K} P_{\theta,n}\Big(\sum_{\nu=1}^{n} \inf_{|\tau-\theta|\leqq 3\varepsilon_n}\{\Phi^{(k+1)}(x_\nu,\tau)-E[\Phi^{(k+1)}(x,\tau);P_\tau]\}<-n\varepsilon_n'\Big)=o(n^{-(k-1)/2}).$$

From (3.7), (3.8), (3.9) and (3.10) we have

$$(3.11)\qquad \sup_{\theta\in K} T_n^3(\theta)=o(n^{-(k-1)/2}).$$

The fifth step. From (3.2), (3.5), (3.6) and (3.11) it follows that

$$\sup_{\theta\in K} \|P_{\theta,n}-Q_{\theta,n}^*\|=o(n^{-(k-1)/2}).$$

Furthermore it holds that

$$\begin{aligned}\sup_{\theta\in K} |1-c_n(\theta)^{-1}| &= \sup_{\theta\in K} |1-c_n(\theta)^{-1}Q_{\theta,n}(\boldsymbol{X}^{(n)})| \\ &= \sup_{\theta\in K} |P_{\theta,n}(\boldsymbol{X}^{(n)})-Q_{\theta,n}^*(\boldsymbol{X}^{(n)})| \\ &= o(n^{-(k-1)/2}).\end{aligned}$$

Therefore we have

$$\begin{aligned}\sup_{\theta\in K} \|P_{\theta,n}-Q_{\theta,n}\| &\leqq \sup_{\theta\in K} \|P_{\theta,n}-Q_{\theta,n}^*\|+\sup_{\theta\in K} \|Q_{\theta,n}^*-Q_{\theta,n}\| \\ &\leqq \sup_{\theta\in K} \|P_{\theta,n}-Q_{\theta,n}^*\|+\sup_{\theta\in K} |c_n(\theta)^{-1}-1|/2 \\ &= o(n^{-(k-1)/2}).\end{aligned}$$

Thus the proof is completed.

4. Tetsts based on asymptotically sufficient statistics. Let $\omega(\neq\phi)$ be a subset of Θ. Suppose that it is desired to test the null hypothesis that $\theta\in\Theta$ against the alternative that $\theta\in\Theta-\omega$. For a statistical test ϕ_n based on $z_n\in\boldsymbol{X}^{(n)}$ we denote by $\beta_n(\theta;\phi_n)$ the power function of ϕ_n, i.e., $\beta_n(\theta;\phi_n)=E[\phi_n;P_{\theta,n}]$. Let $\Phi(\alpha)$ be the class of all test sequences $\{\phi_n\}$ such that for every compact subset K of ω, $\sup_{\theta\in K}|\beta_n(\theta;\phi_n)-\alpha|=o(n^{-(k-1)/2})$. (In LeCam [1] such a test sequence, in the case of $k=1$, is called *asymptotically similar of size α uniformly on compacts.*)

Theorem 3. *Suppose that Condition R is satisfied and that $\{\hat{\theta}_n\}_{n\in N}$ is a sequence of estimators belonging to $\boldsymbol{C}_k$. Then, for any sequence $\{\phi_n; n=1, 2, \cdots\}$ of statistical tests contained in $\Phi(\alpha)$ there exists a sequence $\{\Psi_n; n=1, 2, \cdots\}$ of stati-*

stical tests contained in $\Phi(\alpha)$ *with the following properties*: (1) *For each* $n\in N$, Ψ_n *is a function of* $t_n^*=(\hat{\theta}_n, \Phi_n^{(n)}(z_n, \hat{\theta}_n), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n))$ only. (2) *For every compact subset* K *of* $\Theta-\omega$

$$\sup_{\theta\in K} |\beta_n(\theta; \phi_n)-\beta_n(\theta; \Psi_n)| = o(n^{-(k-1)/2}) .$$

Proof. Suppose that Condition R is satisfied, and that $\{\phi_n\}\in\Phi(\alpha)$. Let $\{\hat{\theta}_n\}\in \boldsymbol{C}_k$ and let $t_n^*=(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \cdots, \Phi_n^{(k)}(z_n, \hat{\theta}_n))$. According to Theorem 2 there exists a family $\{Q_{\theta,n}; \theta\in\Theta\}$ of probability measures such that t_n^* is sufficient for $\{Q_{\theta,n}; \theta\in\Theta\}$, and that for every compact subset K of Θ

$$\sup_{\theta\in K} \|P_{\theta,n}-Q_{\theta,n}\| = o(n^{-(k-1/2)}) . \tag{4.1}$$

Define $\Psi_n=E[\phi_n | t_n^*; Q_{\theta,n}]$, which is the conditional expectation of ϕ_n given t_n^* with respect to $Q_{\theta,n}$. For every compact subset K of Θ we have

$$\begin{aligned}\sup_{\theta\in K} |\beta_n(\theta; \phi_n)-\beta_n(\theta; \Psi_n)| &= \sup_{\theta\in K} |E[\phi_n; P_{\theta,n}]-E[E[\phi_n | t_n^*; Q_{\theta,n}]; P_{\theta,n}]| \\ &\leqq \sup_{\theta\in K} |E[\phi_n; P_{\theta,n}]-E[\phi_n; Q_{\theta,n}]| \\ &\quad +\sup_{\theta\in K} |E[E[\phi_n | t_n^*; Q_{\theta,n}]; P_{\theta,n}]-E[E[\phi_n | t_n^*; Q_{\theta,n}]; Q_{\theta,n}]| \\ &\leqq 2 \sup_{\theta\in K} \|P_{\theta,n}-Q_{\theta,n}\| .\end{aligned}$$

From this and (4.1) it follows that the test sequence $\{\Psi_n\}$ is a required one.
This completes the proof.

5. Estimators based on asymptotically sufficient statistics.

Let $\boldsymbol{D}$ be the class of all sequences $\{\tilde{\theta}_n\}$ of estimators of θ satisfying the following properties (1) and (2).

(1) $\{\tilde{\theta}_n\}$ is *locally uniformly consistent* in the sense that for every compact subset K of Θ and every $\varepsilon>0$,

$$\lim_{n\to\infty} \sup_{\theta\in K} P_{\theta,n}(|\tilde{\theta}_n-\theta|>\varepsilon) = 0$$

(2) For each $\theta\in\Theta$ there exists a probability measure λ_θ on R^1 which is weakly continuous relative to θ such that for any compact subset K of Θ the distribution of $n^{1/2}(\tilde{\theta}_n-\theta)$ converges weakly to λ_θ uniformly relative to θ in K and that $\lambda_\theta(\{0\})\neq 1$ for every $\theta\in\Theta$.

REMARK 3. If $\{\tilde{\theta}_n\}$ satisfies the property (2) then it follows that for any compact subset K of Θ

$$\liminf_{n\to\infty} \inf_{\theta\in K} nE[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]>0 .$$

We denote by $\boldsymbol{D}'$ the class of all sequences $\{\tilde{\theta}_n\}$ with the property (2) such that for every $\rho>0$ and every compact subset K of Θ

(3) $\sup_{\theta\in K} P_{\theta,n}(|\tilde{\theta}_n-\theta|>\rho)=o(n^{-1})$.

REMARK 4. From Theorem 1 and Remark 2 following it, every sequence $\{\tilde{\theta}_n\}$ in $\boldsymbol{C}_3$ has property (3).

Theorem 4. *Suppose that* $\Theta=R^1$ *and that Condition R is satisfied with* $k=3$. *Let* $\{\hat{\theta}_n\}_{n\in N}$ *be an element of* $\boldsymbol{C}_3$ *and* $t_n^*=(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \Phi_n^{(2)}(z_n, \hat{\theta}_n), \Phi_n^{(3)}(z_n, \hat{\theta}_n))$.

Then for any sequence $\{\tilde{\theta}_n\}$ in $\boldsymbol{D}$ there exists a sequence $\{\tilde{\theta}_n^*\}$ of estimators of θ such that

(1) $\{\tilde{\theta}_n^*\}$ is locally uniformly consistent.

(2) For each $n\in N$ $\tilde{\theta}_n^*$ is a function of t_n^* only,

(3) $\lim_{n\to\infty}\sup \sup_{\theta\in K} \{E[(\tilde{\theta}_n^*-\theta)^2; P_{\theta,n}]/E[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]\}\leqq 1$ for every compact subset K of Θ.

Proof. Suppose that Condition R is satisfied, and that $\{\hat{\theta}_n\}\in\boldsymbol{C}_3$. Let $t_n^*=(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \Phi_n^{(2)}(z_n, \hat{\theta}_n), \Phi_n^{(3)}(z_n, \hat{\theta}_n))$. According to Theorem 2 there exists a sequence $\{Q_{\theta,n}; \theta\in\Theta\}_{n\in N}$ of families of probability measures such that for every compact subset K of Θ

$$\sup_{\theta\in K} ||P_{\theta,n}-Q_{\theta,n}|| = o(n^{-1}) .$$

Let $\{\tilde{\theta}_n\}_{n\in N}$ be a sequence of locally uniformly consistent estimators of θ. Let $K_j=[-j, j]$. For each $n\in N$ and $j\in N$, define

$$\tilde{\theta}_{n,j} = 0 \text{ if } \tilde{\theta}_n\notin K_j, \text{ and } \tilde{\theta}_{n,j} = \tilde{\theta}_n \text{ if } \tilde{\theta}_n\in K_j . \tag{5.1}$$

We define

$$\tilde{\theta}_{n,j}^* = E[\tilde{\theta}_{n,j}|t_n^*; Q_{\theta,n}] . \tag{5.2}$$

For every $\varepsilon>0$ and every $j\geqq 1$, we have

$$\begin{aligned}\sup_{\theta\in K_j} Q_{\theta,n}(|\tilde{\theta}_{n,j+1}^*-\theta|>\varepsilon) &\leqq \sup_{\theta\in K_j} E[|\tilde{\theta}_{n,j+1}^*-\theta|; Q_{\theta,n}]/\varepsilon \\ &\leqq \sup_{\theta\in K_j} E[|\tilde{\theta}_{n,j+1}-\theta|; Q_{\theta,n}]/\varepsilon \\ &\leqq 2(j+1)\cdot\sup_{\theta\in K_j} Q_{\theta,n}(|\tilde{\theta}_{n,j+1}-\theta|>\varepsilon^2)/\varepsilon+\varepsilon\end{aligned} \tag{5.3}$$

On the other hand

$$\begin{aligned}\sup_{\theta\in K_j} P_{\theta,n}(|\tilde{\theta}_{n,j+1}-\theta|>\varepsilon^2) &\leqq \sup_{\theta\in K_j} P_{\theta,n}(|\tilde{\theta}_n-\theta|>\varepsilon^2) \\ &\quad +\sup_{\theta\in K_j} P_{\theta,n}(|\tilde{\theta}_n-\theta|>1) .\end{aligned} \tag{5.4}$$

Since $\{\tilde{\theta}_n\}$ is locally uniformly consistent, each term of the right hand side of

(5.4) tends to zero as $n\to\infty$. Hence we have

$$\lim_{n\to\infty}\sup_{\theta\in K_j} P_{\theta,n}(|\tilde{\theta}_{n,j+1}-\theta|>\varepsilon^2)=0\,.$$

Thus we have

$$\lim_{n\to\infty}\sup_{\theta\in K_j} Q_{\theta,n}(|\tilde{\theta}_{n,j+1}-\theta|>\varepsilon^2)=0\,.$$

From this and (5.3) we have

$$\limsup_{n\to\infty}\sup_{\theta\in K_j} Q_{\theta,n}(|\tilde{\theta}^*_{j+1}-\theta|>\varepsilon)\leqq\varepsilon\,.$$

Thus it follows that there exists a subsequence $\{N(j); j=1,2,\cdots\}$ of N such that for every $j\geqq 1$ and every $n\geqq N(j)$

$$\sup_{\theta\in K_j} P_{\theta,n}(|\tilde{\theta}^*_{n,j+1}-\theta|>1/j)<1/j\,. \tag{5.5}$$

Suppose that $\{\tilde{\theta}_n\}_{n\in N}$ is an element of $\boldsymbol{D}$. For each $j\geqq 1$, let $\{\tilde{\theta}_{n,j}\}$ be a sequence defined by (5.1). For each j let $\{\tilde{\theta}^*_{n,j}\}$ be a sequence of estimators constructed by (5.2) from $\{\tilde{\theta}_{n,j}\}$. For each $j\geqq 1$ and $l(1\leqq l\leqq j)$ we have uniformly with respect to $\theta\in K_l$

$$\begin{aligned}
&\limsup_{n\to\infty}\ \{E[(\tilde{\theta}^*_{n,j+1}-\theta)^2; P_{\theta,n}]/E[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]\}\\
&\quad=\limsup_{n\to\infty}\ \{nE[(\tilde{\theta}^*_{n,j+1}-\theta)^2; P_{\theta,n}]/[nE[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]]\}\\
&\quad\leqq\limsup_{n\to\infty}\ \{[4n(j+1)^2\sup_{\tau\in K_j}\|P_{\tau,n}-Q_{\tau,n}\|+nE[(\tilde{\theta}_{n,j+1}-\theta)^2; Q_{\theta,n}]]/\\
&\qquad nE[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]\}\\
&\quad\leqq\limsup_{n\to\infty}\ \{[8n(j+1)^2\sup_{\tau\in K_j}\|P_{\tau,n}-Q_{\tau,n}\|+nE[(\tilde{\theta}_{n,j+1}-\theta)^2; P_{\theta,n}]]/\\
&\qquad nE[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]\}\\
(5.6)\quad&\quad\leqq\limsup_{n\to\infty}\ [nE[(\tilde{\theta}_{n,j+1}-\theta)^2; P_{\theta,n}]/[nE[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]]\\
&\quad\leqq\limsup_{n\to\infty}\ \{[n\int_{\tilde{\theta}_n\in K_{j+1}}(\tilde{\theta}_n-\theta)^2dP_{\theta,n}+nl^2P_{\theta,n}(\tilde{\theta}_n\notin K_{j+1})]/\\
&\qquad nE[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]\}\\
&\quad\leqq 1+\limsup_{n\to\infty}\ \{[nl^2E(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]/(j-l+1)^2]/\\
&\qquad nE[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]\}\\
&\quad=1+[l^2/(j-l+1)^2]\,.
\end{aligned}$$

Hence there exists a subsequence $\{N'(j)\}$ of N such that for every $j\geqq 1$, every $n\geqq N'(j)$ and every $\theta\in K_{l(j)}$ (where $l(j)$ means the maximum integer not greater than $j^{1/2}$)

$$E[(\tilde{\theta}^*_{n,j+1}-\theta)^2; P_{\theta,n}]/E[(\tilde{\theta}_n-\theta)^2; P_{\theta,n}]\leqq 1+[5/j]\,. \tag{5.7}$$

Let $N^*(j)=\max\{N(j), N'(j)\}$ for each $j\geqq 1$, and for each $n\in N$ $m(n)$ be the number such that $N^*(m(n))\leqq n<N^*(m(n)+1)$. If we define $\tilde{\theta}_n^*=\tilde{\theta}_{n,m(n)+1}^*$ for each $n\in N$ then it can be seen from (5.5) and (5.7) that the sequence $\{\tilde{\theta}_n^*\}_{n\in N}$ is a required one.

Thus the proof is completed.

REMARK 5. If Θ is any open set in R^1 and if we take $t_n^*=(\hat{\theta}_n, \Phi_n^{(1)}(z_n, \hat{\theta}_n), \Phi_n^{(2)}(z_n, \hat{\theta}_n), \Phi_n^{(3)}(z_n, \hat{\theta}_n))$, then under Condition R with $k=3$ for any $\{\tilde{\theta}_n\}$ in $\boldsymbol{D}'$ there exists a sequence $\{\tilde{\theta}_n^*\}$ of estimators of θ having the properties (1), (2) and (3) in Theorem 4. This can be shown by a similar method developed in the above proof.

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