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THE K_*-LOCALIZATIONS OF WOOD AND ANDERSON SPECTRA AND THE REAL PROJECTIVE SPACES

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0. Introduction

Let E be an associative ring spectrum with unit. For any CW-spectra X and Y we say that X is *quasi E_*-equivalent* to Y (see [15] or [16]) if there exists a map $f: Y \to E_\wedge X$ such that the composite map $(\mu_\wedge 1)(1_\wedge f): E_\wedge Y \to E_\wedge X$ is an equivalence where $\mu: E_\wedge E \to E$ denotes the multiplication of E. Let KO, KU and KT be the real, the complex and the self-conjugate K-spectrum respectively (see [3] or [7]). It is known that there is no difference among the KO_*-, KU_*- and KT_*-localizations ([11], [5] or [13]). So we denote by S_K the K_*-localization of the sphere spectrum $S=\Sigma^0$. These spectra KO, KU, KT and S_K are all associative ring spectra with unit.

In [15] we studied the quasi K_*-equivalences, especially the quasi KO_*-equivalence, and in [16] and [17] we determined the quasi KO_*-types of the real projective spaces RP^n and the stunted real projective spaces RP^n/RP^m. In this note we will be interested in the quasi S_{K*}-equivalence in advance of the quasi KO_*-equivalence. According to the smashing theorem [6, Corollary 4.7] (or [13]), for any CW-spectrum X the smash product $S_{K\wedge}X$ is actually the K_*-localization of X. Hence we notice that two CW-spectra X and Y have the same K_*-local type if and only if X is quasi S_{K*}-equivalent to Y.

For any map $f: X \to Y$ its cofiber is usually denoted by $C(f)$. Let $\eta: \Sigma^1 \to \Sigma^0$ be the stable Hopf map of order 2. The KO-homologies of the cofibers $C(\eta)$ and $C(\eta^2)$ are well known as follows: $KO_iC(\eta) \cong \pi_iKU \cong Z$ or 0 according as i is even or odd, and $KO_iC(\eta^2) \cong \pi_iKT \cong Z$, $Z/2$, 0 or Z according as $i \equiv 0, 1, 2$ or 3 mod 4. A CW-spectrum X is said to be *a Wood spectrum* if it is quasi KO_*-equivalent to the cofiber $C(\eta)$, and *an Anderson spectrum* if it is quasi KO_*-equivalent to the cofiber $C(\eta^2)$ (see [12], [15] or [18]).

Let $\bar\eta: \Sigma^1SZ/2 \to \Sigma^0$ and $\tilde\eta: \Sigma^2 \to SZ/2$ be an extension and a coextension of η with $\bar\eta i = \eta$ and $j\tilde\eta = \eta$, where $SZ/2$ denotes the Moore spectrum of type $Z/2$ constructed by the cofiber sequence $\Sigma^0 \xrightarrow{2} \Sigma^0 \xrightarrow{i} SZ/2 \xrightarrow{j} \Sigma^1$. Choose two maps $\bar h$: $\Sigma^3SZ/2 \to C(\bar\eta)$ and $\bar k$: $\Sigma^5SZ/2 \to C(\bar\eta)$ with $\bar j\bar h = \tilde\eta j$ and $\bar j\bar k = \tilde\eta\bar\eta$ where $\bar j$: $C(\bar\eta) \to \Sigma^2SZ/2$ denotes the bottom cell collapsing. Using a fixed Adams' K_*-equivalence A_2: $\Sigma^8 SZ/2 \to SZ/2$ in [2] we can introduce four kinds of maps f_t ($t \geqq 1$) as follows:

$$\alpha_{4r} = jA_2^r i: \Sigma^{8r-1} \to \Sigma^0, \qquad \mu_{4r+1} = \bar{\eta} A_2^r i: \Sigma^{8r+1} \to \Sigma^0,$$
$$a_{4r+2} = \bar{h} A_2^r i: \Sigma^{8r+3} \to C(\bar{\eta}) \quad \text{and} \quad m_{4r+3} = \bar{k} A_2^r i: \Sigma^{8r+5} \to C(\bar{\eta}).$$

Setting $\bar{\alpha}_{4r} = jA_2^r$, $\bar{\mu}_{4r+1} = \bar{\eta} A_2^r$, $\bar{a}_{4r+2} = \bar{h} A_2^r$ and $\bar{m}_{4r+3} = \bar{k} A_2^r$, we can also introduce four kinds of maps f_{-t} ($t \geqq 1$) as follows:

$$\alpha_{-4r}: \Sigma^{-8r-1} C(\bar{\alpha}_{4r}) \to \Sigma^0, \qquad \mu_{-4r-1}: \Sigma^{-8r-3} C(\bar{\mu}_{4r+1}) \to \Sigma^0,$$
$$a_{-4r-2}: \Sigma^{-8r-5} C(\bar{a}_{4r+2}) \to \Sigma^0 \quad \text{and} \quad m_{-4r-3}: \Sigma^{-8r-7} C(\bar{m}_{4r+3}) \to \Sigma^0$$

of which each cofiber $C(f_{-t})$ coincides with $\Sigma^{-2t} C(f_t)$.

In §1 and §3 we will determine the K_*-local types of Wood and Anderson spectra as our results (Theorems 1.7 iii) and 3.4 ii)):

Theorem 1. *Let X be a Wood spectrum whose rationalization $X_\wedge SQ$ is $(\Sigma^0 \vee \Sigma^{2t})_\wedge SQ$ for some odd integer $t \geqq 1$. Then X has the same K_*-local type as the following cofiber $C(\mu_t)$ or $C(m_t)$ according as $t = 4r+1$ or $4r+3$.*

Theorem 2. *Let X be an Anderson spectrum whose rationalization $X_\wedge SQ$ is $(\Sigma^0 \vee \Sigma^{2t+1})_\wedge SQ$ for some odd integer t. Assume that $t \neq -1$. Then X has the same K_*-local type as the following cofiber $C(\eta\mu_t)$ or $C(\eta m_t)$ according as $t = \pm(4r+1)$ or $\pm(4r+3)$.*

For the Moore spectrum $SZ/2^t$ of type $Z/2^t$ we denote by i_t: $\Sigma^0 \to SZ/2^t$ and j_t: $SZ/2^t \to \Sigma^1$ with the subscript "t" the bottom cell inclusion and the top cell projection. Abbreviating the cofiber $C(i_{t-1}\bar{\eta})$ to be V_{2^t} we have a cofiber sequence $\Sigma^0 \xrightarrow{2^{t-1}\bar{i}} C(\bar{\eta}) \xrightarrow{i_{V,t}} V_{2^t} \xrightarrow{j_{V,t}} \Sigma^1$. In §4 the K_*-local types of the real projective spaces RP^n ($2 \leqq n \leqq \infty$) will be determined as our main result (Theorem 4.6 ii)):

Theorem 3. *The real projective space $\Sigma^1 RP^n$ has the same K_*-local type as the following elementary spectrum: $SZ/2^{4r}$, $C(i_{4r}\mu_{4r+1})$, $V_{2^{4r+1}}$, $C(i_{V,4r+1}a_{4r+2})$, $V_{2^{4r+2}}$, $C(i_{V,4r+2}m_{4r+3})$, $SZ/2^{4r+3}$, $C(i_{4r+3}\alpha_{4r+4})$ according as $n = 8r, 8r+1, \cdots, 8r+7$. In addition, $\Sigma^1 RP^\infty$ has the same K_*-local type as $SZ/2^\infty$* (cf. [8, Theorem 4.2] or [13, Theorem 9.1]).

In order to prove the above theorems we will need the following powerful tool due to Bousfield [7, Theorems 7.11 and 7.12].

Theorem 4. *Let Y be a certain CW-spectrum satisfying either of the following two conditions:* i) *KU_*Y is either free or divisible and* $\mathrm{Hom}(\pi_i Y \otimes Q, \pi_{i+1} Y \otimes Q) = 0$ *for each i;* ii) *$KU_1 Y = 0$ (or $KU_0 Y = 0$). Assume that a CW-spectrum X is quasi KO_*-equivalent to Y, and the real Adams operations ψ_R^k in*

KO_*X and KO_*Y behave as the same action for each $k\neq 0$ when KO_*X is identified with KO_*Y as a KO_*-module. Then X is quasi S_{K_*}-equivalent to Y, thus X has the same K_*-local type as Y (cf. [7, 9.8]).

In §1 we will mainly deal with CW-spectra X satisfying the following property:

(I) $KU_0X\cong Z$ with $\psi_C^k=1$ and $KU_1X=0$;
(I_{2m}) $KU_0X\cong Z/2m$ with $\psi_C^k=1$ and $KU_1X=0$; or
$(II)_t$ $KU_0X\cong Z\oplus Z$ with $\psi_C^k=A_{k,t}$ and $KU_1X=0$.

Here $A_{k,t}=\begin{pmatrix} 1/k^t & 0 \\ 1-k^t/2k^t & 1\end{pmatrix}$, which operates on $(Z\oplus Z)\otimes Z[1/k]$ as left action. After investigating the behavior of the real Adams operation ψ_R^k for CW-spectra X with the above property we will determine their K_*-local types (Theorems 1.2 and 1.7). In §2 and §3 we will next deal with CW-spectra X satisfying the following property:

$(II_{2m})_t$ $KU_0X\cong Z\oplus Z/2m$ with $\psi_C^k=A_{k,t}$ and $KU_1X=0$; or
$(III)_t$ $KU_0X\cong Z$ with $\psi_C^k=1$ and $KU_1X\cong Z$ with $\psi_C^k=1/k^t$.

As in §1 we will also determine the K_*-local types of such CW-spectra X (Theorems 2.6 and 3.4). In §4 we will finally deal with the symmetric squares SP^2S^n of the n-spheres and the real projective n-spaces RP^n. After investigating the behavior of the Adams operations ψ_C^k and ψ_R^k for the spaces SP^2S^n and RP^n, we will determine their K_*-local types (Theorem 4.6) by applying Theorems 1.2, 1.7 and 2.6.

In the forthcoming paper [19] we will completely determine the K_*-local types of the stunted real projective spaces RP^n/RP^m $(0\leqq m<n\leqq\infty)$ along our line.

1. K_*-local types of Wood spectra

1.1. Let X be a CW-spectrum with $KU_0X\cong Z$ and $KU_1X=0$. For such a CW-spectrum X we may assume that the stable complex Adams operation ψ_C^k acts identically on $KU_0X\otimes Z[1/k]$ for each $k\neq 0$. Thus X satisfies the following property:

(I) $KU_0X\cong Z$ in which $\psi_C^k=1$ and $KU_1X=0$.

Whenever a CW-spectrum X satisfies the property (I), it is quasi KO_*-equivalent to either of Σ^0 and Σ^4 (see [7, Theorem 3.2] or [15, Theorem I.2.4]). In this case it is easily seen that the stable real Adams operation ψ_R^k acts always on $KO_iX\otimes Z[1/k]$ $(0\leqq i\leqq 7)$ for each $k\neq 0$ as follows:

(1.1) $\psi_R^k=k^2$ or 1 according as $i=4$ or otherwise.

The Moore spectrum $SZ/2m$ of type $Z/2m$ is constructed as the cofiber of multiplication by $2m$ on Σ^0. Thus we have a cofiber sequence $\Sigma^0 \xrightarrow{2m} \Sigma^0 \xrightarrow{i} SZ/2m \xrightarrow{j} \Sigma^1$. Let $\bar{\eta}_{2m}$: $\Sigma^1 SZ/2m \to \Sigma^0$ and $\tilde{\eta}_{2m}$: $\Sigma^2 \to SZ/2m$ be an extension and a coextension of η satisfying $\bar{\eta}_{2m} i = \eta$ and $j\tilde{\eta}_{2m} = \eta$ respectively, where η: $\Sigma^1 \to \Sigma^0$ denotes the stable Hopf map of order 2. The maps $\bar{\eta}_2$ and $\tilde{\eta}_2$ are often abbreviated to be $\bar{\eta}$ and $\tilde{\eta}$. Consider the two cofiber sequences

$$\Sigma^1 SZ/2 \xrightarrow{\bar{\eta}} \Sigma^0 \xrightarrow{\bar{i}} C(\bar{\eta}) \xrightarrow{\bar{j}} \Sigma^2 SZ/2 \quad \text{and} \quad \Sigma^2 \xrightarrow{\tilde{\eta}} SZ/2 \xrightarrow{\tilde{i}} C(\tilde{\eta}) \xrightarrow{\tilde{j}} \Sigma^3$$

in which the cofibers $C(\bar{\eta})$ and $C(\tilde{\eta})$ are denoted by P_2' and P_2 respectively in [15, I.4.1]. Between these cofibers there holds a Spanier-Whitehead duality as $C(\tilde{\eta}) = \Sigma^3 DC(\bar{\eta})$. By observing [15, Propositions I.4.1 and I.4.2] we verify that

(1.2) both $C(\bar{\eta})$ and $\Sigma^{-3}C(\tilde{\eta})$ satisfy the property (I), and they are quasi KO_*-equivalent to Σ^4.

Let X be a CW-spectrum with $KU_0 X \cong Z/2m$ and $KU_1 X = 0$. In this case we assume that the Adams operation ψ_C^k acts identically in $KU_0 X$ for each $k \neq 0$. Thus we here deal with a CW-spectrum X satisfying the following property:

(I_{2m}) $KU_0 X \cong Z/2m$ in which $\psi_C^k = 1$ and $KU_1 X = 0$.

Consider the cofibers $C(i\bar{\eta})$ and $C(\tilde{\eta} j)$ of the composite maps $i\bar{\eta}$: $\Sigma^1 SZ/2 \to SZ/m$ and $\tilde{\eta} j$: $\Sigma^1 SZ/m \to SZ/2$, which are denoted by V_{2m} and V_{2m}' respectively as in [15, I.4.4]). Between them we have a Spanier-Whitehead duality as $V_{2m}' = \Sigma^3 DV_{2m}$. Since there exist cofiber sequences

$$\Sigma^0 \xrightarrow{m\bar{i}} C(\bar{\eta}) \xrightarrow{i_V} V_{2m} \xrightarrow{j_V} \Sigma^1 \quad \text{and} \quad \Sigma^2 \xrightarrow{i_V'} V_{2m}' \xrightarrow{j_V'} C(\tilde{\eta}) \xrightarrow{m\tilde{j}} \Sigma^3 ,$$

it follows from [15, Corollaries I.4.6 and I.5.4] that

(1.3) both V_{2m} and $\Sigma^{-2} V_{2m}'$ satisfy the property (I_{2m}), and $\Sigma^2 V_{2m}'$ is quasi KO_*-equivalent to V_{2m}, whose KO-homology $KO_i V_{2m} \cong Z/m$, 0, $Z/2$, $Z/2$, $Z/4m$, $Z/2$, $Z/2$, 0 according as $i = 0, 1, \cdots, 7$.

Notice that a CW-spectrum X is quasi KO_*-equivalent to one of the four elementary spectra $SZ/2m$, $\Sigma^4 SZ/2m$, V_{2m} and $\Sigma^4 V_{2m}$ whenever it satisfies the property (I_{2m}) (see [15, Theorem II.2 or Theorem I.5.2]).

Lemma 1.1. *Let W and Y be CW-spectra satisfying the property* (I), *and g: $W \to Y$ be a map whose cofiber $C(g)$ satisfies the property* (I_{2m}). *Then the cofiber $C(g)$ is quasi KO_*-equivalent to $W \wedge SZ/2m$ or $W \wedge V_{2m}$ according as W is* *quasi* KO_**-equivalent to* Y *or not. In the latter case the Adams operation* ψ_R^k *acts normally in* $KO_iC(g)\cong KO_iW_\wedge V_{2m}$ $(0\leqq i\leqq 7)$ *for each* $k\neq 0$ *as follows:* $\psi_R^k=k^2$ *or* 1 *according as* $i=4$ *or otherwise.*

Proof. The induced homomorphism $g_*\colon KO_iW\to KO_iY$ is trivial in dimension $i=1, 2, 5$ or 6 because $g_*\colon KU_0W\to KU_0Y$ is multiplication by $2m$ on Z. Therefore it is immediate that $KO_6C(g)=0$ if both W and Y are quasi KO_*-equivalent to Σ^0, and $KO_2C(g)\cong Z/2$ and $KO_1C(g)=0$ if W and Y are quasi KO_*-equivalent to Σ^0 and Σ^4 respectively. Thus $C(g)$ is quasi KO_*-equivalent to $SZ/2m$ in the first case, and it is quasi KO_*-equivalent to V_{2m} in the second case. In the other two cases we can similarly observe the quasi KO_*-type of $C(g)$. When $C(g)$ is quasi KO_*-equivalent to either of V_{2m} and Σ^4V_{2m}, it is easily checked that $\psi_R^k=1$ or k^2 in $KO_i\,C(g)$ for each $k\neq 0$ according as $i=0$ or 4.

Since the maps $\bar{\eta}\colon \Sigma^1SZ/2\to\Sigma^0$ and $\tilde{\eta}\colon \Sigma^2\to SZ/2$ have order 4 [4, (4.2)], we can choose maps

$$\bar{\eta}_{4m/2}\colon \Sigma^2SZ/2 \to SZ/4m \quad \text{and} \quad \tilde{\eta}_{4m/2}\colon \Sigma^2SZ/4m \to SZ/2$$

with $j\bar{\eta}_{4m/2}=\bar{\eta}$ and $\tilde{\eta}_{4m/2}\,i=\tilde{\eta}$. Denote by U_{2m} and U'_{2m} their cofibers $C(\bar{\eta}_{4m/2})$ and $C(\tilde{\eta}_{4m/2})$ respectively. Between them there holds a Spanier-Whitehead duality as $U'_{2m}=\Sigma^4DU_{2m}$. Using the cofiber sequences

$$C(\bar{\eta}) \xrightarrow{m\bar{\lambda}} \Sigma^0 \xrightarrow{i_U} U_{2m} \xrightarrow{j_U} \Sigma^1C(\bar{\eta}) \quad \text{and} \quad \Sigma^3 \xrightarrow{m\tilde{\lambda}} C(\tilde{\eta}) \xrightarrow{i'_U} U'_{2m} \xrightarrow{j'_U} \Sigma^4$$

with $\bar{\lambda}\bar{i}=4$ and $\tilde{j}\tilde{\lambda}=4$, we can easily show by the aid of Lemma 1.1 that

(1.4) both U_{2m} and $\Sigma^1U'_{2m}$ satisfy the property (I_{2m}), and they are quasi KO_*-equivalent to $\Sigma^4\,V_{2m}$.

If a CW-spectrum X satisfies the property (I_{2m}), then the smash product $X_\wedge C(\bar{\eta})$ does the same property, but it is quasi KO_*-equivalent to Σ^4X because of (1.2). Whenever $X=SZ/2m$, V_{2m}, $\Sigma^{-2}V'_{2m}$, U_{2m} or $\Sigma^{-3}U'_{2m}$, the Adams operation ψ_R^k behaves normally in KO_iX and $KO_iX_\wedge C(\bar{\eta})$ $(0\leqq i\leqq 7)$ for each $k\neq 0$ as follows:

(1.5) $\psi_R^k=k^2$ or 1 according as $i=4$ or otherwise.

Because the $X=SZ/2m$ case is well known, and the other four cases are immediately shown by Lemma 1.1.

Let X be a CW-spectrum satisfying the property:

(I_{2^∞}) $KU_0X\cong Z/2^\infty$ in which $\psi_C^k=1$ and $KU_1\,X=0$.

Such a CW-spectrum X is quasi KO_*-equivalent to either of $SZ/2^\infty$ and

$\Sigma^4 SZ/2^\infty$ (see [7, Theorem 3.3]). In this case it is easily seen that the Adams operation ψ_R^k behaves always in KO_iX $(0\leqq i\leqq 7)$ for each $k\neq 0$ as follows:

(1.6) $\psi_R^k=k^2$ or 1 according as $i=4$ or otherwise.

Use (1.1), (1.5) or (1.6) to apply Theorem 4 for CW-spectra X with the property (I), (I_{2m}) or (I_{2^∞}). Then we obtain

Theorem 1.2. i) *Let X be a CW-spectrum satisfying the property* (I). *Then it has the same K_*-local type as either of Σ^0 and $C(\bar{\eta})$.*

ii) *Let X be a CW-spectrum satisfying the property* (I_{2m}). *Assume that the real Adams operation ψ_R^k behaves normally in KO_*X in the sense of* (1.5). *Then X has the same K_*-local type as one of the following spectra $SZ/2m$, $SZ/2m_\wedge C(\bar{\eta})$, V_{2m} and U_{2m}.*

iii) *Let X be a CW-spectrum satisfying the property* (I_{2^∞}). *Then X has the same K_*-local type as either of $SZ/2^\infty$ and $SZ/2^\infty{}_\wedge C(\bar{\eta})$.*

1.2. Let X be a CW-spectrum with $KU_0X\cong Z\oplus Z$ and $KU_1X=0$. For such a CW-spectrum X we may assume that $X_\wedge SQ=(\Sigma^{2t}\vee\Sigma^0)_\wedge SQ$ for some integer $t\geqq 0$. In this case the complex Adams operation ψ_C^k on $KU_0X\otimes Z[1/k]$ is represented as the matrix $C^{-1}A_{k,t,0}C$ for each $k\neq 0$ where the matrix $C\in GL(2, Q)$ associated with the Chern character is independent of k and $A_{k,t,0}=\begin{pmatrix}1/k^t & 0\\ 0 & 1\end{pmatrix}$. When t is odd, we may regard that the conjugation ψ_C^{-1}on $KU_0X\cong Z\oplus Z$ is expressed by either of the matrices $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ and $\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}$ (see [7, Proposition 3.7] or [15, I.2.1]). This observation implies easily that the Adams operation ψ_C^k in KU_0X for each $k\neq 0$ can be expressed by the following matrix

$$A_{k,t,0}=\begin{pmatrix}1/k^t & 0\\ 0 & 1\end{pmatrix} \quad\text{or}\quad A_{k,t}=\begin{pmatrix}1/k^t & 0\\ 1-k^t/2k^t & 1\end{pmatrix}$$

according as $\psi_C^{-1}=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}$, whenever t is odd.

Let X be a CW-spectrum satisfying the following property:

$(\mathrm{II})_{t,0}$ $KU_0X\cong Z\oplus Z$ in which $\psi_C^k=A_{k,t,0}$ and $KU_1X=0$.

Then X is quasi KO_*-equivalent to one of the wedge sums $\Sigma^0\vee\Sigma^0$, $\Sigma^0\vee\Sigma^4$ and $\Sigma^4\vee\Sigma^4$ when t is even, and it is quasi KO_*-equivalent to one of the wedge sums $\Sigma^2\vee\Sigma^0$, $\Sigma^2\vee\Sigma^4$, $\Sigma^6\vee\Sigma^0$ and $\Sigma^6\vee\Sigma^4$ when t is odd (see [7, Theorem 3.2] or [15, Theorem I.2.4]). By an easy argument using the long exact sequence induced by the Bott cofiber sequence $\Sigma^1KO\to KO\to KU\to\Sigma^2KO$ we can show that in the case when t is even the Adams operation ψ_R^k behaves always in KO_iX

$(0 \leqq i \leqq 7)$ for each $k \neq 0$ as follows (cf. [2, Proposition 7.14]):

(1.7) i) If X is quasi KO_*-equivalent to either of $\Sigma^0 \vee \Sigma^0$ and $\Sigma^4 \vee \Sigma^4$, then $\psi_R^k = A_{k,t,0}$, $k^2 A_{k,t,0}$ or 1 according as $i=0, 4$ or otherwise.

ii) If X is quasi KO_*-equivalent to $\Sigma^0 \vee \Sigma^4$, then $\psi_R^k = A_{k,t,\varepsilon}$, $k^2 A_{k,t,\varepsilon'}$ or 1 according as $i=0, 4$ or otherwise where $(\varepsilon, \varepsilon') = (0, 0)$, $(0, 1)$ or $(1, 0)$ and $A_{k,t,1} = A_{k,t}$.

Let X be a CW-spectrum satisfying the following property:

$(\mathrm{II})_t$ $KU_0 X \cong Z \oplus Z$ in which $\psi_C^k = A_{k,t}$ and $KU_1 X = 0$.

Then X is quasi KO_*-equivalent to one of the wedge sums $\Sigma^0 \vee \Sigma^0$, $\Sigma^0 \vee \Sigma^4$ and $\Sigma^4 \vee \Sigma^4$ when t is even, but it is only quasi KO_*-equivalent to the cofiber $C(\eta)$ when t is odd (see [7, Theorem 3.2] or [15, Theorem I.2.4]). Thus X is always a Wood spectrum in the case when t is odd. By a similar argument to (1.7) we can also show that the Adams operation ψ_R^k behaves always in $KO_i X$ $(0 \leqq i \leqq 7)$ for each $k \neq 0$ as follows:

(1.8) i) If X is quasi KO_*-equivalent to either of $\Sigma^0 \vee \Sigma^0$ and $\Sigma^4 \vee \Sigma^4$, then $\psi_R^k = A_{k,t}$, $k^2 A_{k,t}$ or 1 according as $i=0, 4$ or otherwise.

ii) If X is quasi KO_*-equivalent to $\Sigma^0 \vee \Sigma^4$, then $\psi_R^k = A_{k,t,\varepsilon}$, $k^2 A_{k,t,2-\varepsilon}$ or 1 according as $i=0, 4$ or otherwise where $\varepsilon = 0$ or 2 and $A_{k,t,j} = \begin{pmatrix} 1/k^t & 0 \\ 1-k^t/2^j k^t & 1 \end{pmatrix}$.

iii) If X is a Wood spectrum, then $\psi_R^k = 1$, $1/k^{t-1}$, k^2 or $1/k^{t-3}$ according as $i=0, 2, 4$ or 6.

For any map $\alpha_{2s/j}\colon \Sigma^{4s-1} \to \Sigma^0$ whose e_C-invariant $e_C(\alpha_{2s/j}) \equiv 1/2^j \bmod 1$, we notice that the Adams operation ψ_C^k in $KU_0\, C(\alpha_{2s/j}) \cong Z \oplus Z$ is represented by the matrix $A_{k,2s,j}$ as given in (1.8) ii) for each $k \neq 0$ [2, Proposition 7.5]. Consider the maps

(1.9) $\alpha_{2s}\colon \Sigma^{4s-1} \to \Sigma^0$, $\bar{i}\alpha_{2s/2}\colon \Sigma^{4s-1} \to C(\bar{\eta})$ and $\alpha_{2s/2}\, \tilde{j}\colon \Sigma^{4s-4} C(\tilde{\eta}) \to \Sigma^0$

where $s \geqq 1$ and $\alpha_{2s/1}$ is abbreviated as α_{2s}.

Proposition 1.3. *The cofibers $C(\mu_{2s})$, $C(\bar{i}\alpha_{2s/2})$ and $C(\alpha_{2s/2}\tilde{j})$ satisfy the property $(\mathrm{II})_{2s}$, and they are quasi KO_*-equivalent to the wedge sum $\Sigma^{4s} \vee \Sigma^0$, $\Sigma^{4s} \vee \Sigma^4$ and $\Sigma^{4s-4} \vee \Sigma^0$ respectively.*

Proof. The fiist half is easy, and the latter half is immediate bacause $\pi_{4s-1} KO = 0$.

1.3. Let us fix an Adams' K_*-equivalence $A_2\colon \Sigma^8 SZ/2 \to SZ/2$ [2]. We first consider the composite maps $A_2^r\, i\colon \Sigma^{8r} \to SZ/2$ and $j A_2^r\colon \Sigma^{8r-1} SZ/2 \to \Sigma^0$ $(r \geqq 0)$.

Lemma 1.4. *The cofibers $\Sigma^{-8r-1}C(A_2^r i)$ and $C(jA_2^r)$ satisfy the property* (I), *and they are quasi KO_*-equivalent to Σ^0.*

Proof. Since the Adams' K_*-equivalence A_2: $\Sigma^8 SZ/2 \to SZ/2$ induces an isomorphism in KU-homology, we obtain that $KU_1\, C(A_2^r i) \cong KU_1\, \Sigma^{8r+1} \cong Z$, $KU_0\, C(jA_2^r) \cong KU_0\, \Sigma^0 \cong Z$ and $KU_0\, C(A_2^r i) = 0 = KU_1\, C(jA_2^r)$. Moreover it follows that $\Sigma^{-1}C(A_2^r i)$ and $C(jA_2^r)$ are both quasi KO_*-equivalent to Σ^0 but not to Σ^4 because $KO_6\, C(A_2^r i) = 0 = KO_5\, C(jA_2^r)$.

Lemma 1.5. *Let X be a CW-spectrum satisfying the property* (I).

i) *Let f: $\Sigma^{2t-1}SZ/2 \to X$ be a map whose cofiber $C(f)$ satisfies the property* (I). *For the composite map $fA_2^r i$: $\Sigma^{8r+2t-1} \to X$ its cofiber $C(fA_2^r i)$ satisfies the property* $(\mathrm{II})_{4r+t}$, *and it is quasi KO_*-equivalent to $\Sigma^{2t} \vee C(f)$ or $C(\eta)$ according as t is even or odd.*

ii) *Let g: $\Sigma^{2t}X \to SZ/2$ be a map whose cofiber $\Sigma^{2t-1}C(g)$ satisfies the property* (I). *For the composite map $jA_2^r g$: $\Sigma^{8r+2t-1}X \to \Sigma^0$ its cofiber $C(jA_2^r g)$ satisfies the property* $(\mathrm{II})_{4r+t}$, *and it is quasi KO_*-equivalent to $\Sigma^{-1}C(g) \vee \Sigma^0$ or $C(\eta)$ according as t is even or odd.*

Proof. i) Consider the commutative diagram

$$\begin{array}{ccccccc}
\Sigma^{8r+2t-1} & \xrightarrow{A_2^r i} & \Sigma^{2t-1}SZ/2 & \to & \Sigma^{2t-1}C(A_2^r i) & \to & \Sigma^{8r+2t} \\
\| & & \downarrow f & & \downarrow F & & \| \\
\Sigma^{8r+2t-1} & \xrightarrow{fA_2^r i} & X & \to & C(fA_2^r i) & \to & \Sigma^{8r+2t} \\
 & & \downarrow i_f & & \downarrow i_F & & \\
 & & C(f) & = & C(f) & &
\end{array}$$

involving four cofiber sequences. It is obvious that $KU_0\, C(fA_2^r i) \cong KU_0\, \Sigma^{8r+2t} \oplus KU_0\, X \cong Z \oplus Z$ and $KU_1\, C(fA_2^r i) = 0$. Observe that the induced homomorphism F_*: $KU_0\, \Sigma^{2t-1}C(A_2^r i) \to KU_0\, C(fA_2^r i)$ is given by $F_*(1) = (2, a)$ for some integer a. Since the integer a must be odd, we may take a to be 1. By an easy argument we can then show that $\psi_C^k = A_{k,4r+t}$ in $KU_0\, C(fA_2^r i)$ for each $k \neq 0$. Since $\Sigma^{-1}C(A_2^r i)$ is quasi KO_*-equivalent to Σ^0 by Lemma 1.4 and $C(f)$ is quasi KO_*-equivalent to either of Σ^0 and Σ^4, the cofiber $C(fA_2^r i)$ becomes quasi KO_*-equivalent to the wedge sum $C(f) \vee \Sigma^{2t}$ in the case when t is even. On the other hand, it is exactly a Wood spectrum in the case when t is odd, because $\psi_C^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ on $KU_0\, C(fA_2^r i)$.

ii) is similarly shown by a dual argument.

Consider the composite map $\tilde{\eta}\eta$: $\Sigma^3 SZ/2 \to SZ/2$. Since $KO_7\, C(\tilde{\eta}\eta) \cong KO_3$

$SZ/2\cong Z/2$ and $KO_6\,C(\tilde\eta\bar\eta)\cong KO_2\,SZ/2\cong Z/4$, a routine argument with (1.2) shows that

(1.10) $\Sigma^{-2}C(\tilde\eta\bar\eta)$ satisfies the property (I_4), and it is quasi KO_*-equivalent to $\Sigma^4 SZ/4$.

Since the composite maps $\bar\eta\tilde\eta j\colon \Sigma^3SZ/2\to\Sigma^1$, $i\bar\eta\tilde\eta\colon \Sigma^3\to SZ/2$, $\bar\eta\tilde\eta\bar\eta\colon \Sigma^5SZ/2\to\Sigma^1$ and $\tilde\eta\bar\eta\tilde\eta\colon \Sigma^5\to SZ/2$ are all trivial [4, §4], we can choose the following maps

$$(1.11)\qquad \begin{aligned} &\bar h\colon \Sigma^3SZ/2\to C(\bar\eta), && \tilde h\colon \Sigma^1C(\tilde\eta)\to SZ/2,\\ &\bar k\colon \Sigma^5SZ/2\to C(\bar\eta) \quad\text{and} && \tilde k\colon \Sigma^3C(\tilde\eta)\to SZ/2 \end{aligned}$$

such that $\bar j\bar h=\tilde\eta j$, $\tilde h\tilde i=i\bar\eta$, $\bar j\bar k=\tilde\eta\bar\eta$ and $\tilde k\tilde i=\tilde\eta\bar\eta$. Among their cofibers there hold Spanier-Whitehead dualities as $C(\tilde h)=\Sigma^5DC(\bar h)$ and $C(\tilde k)=\Sigma^7DC(\bar k)$. Since $KU_0\,C(\tilde\eta j)\cong KU_0\,C(i\bar\eta)\cong KU_0\,C(\tilde\eta\bar\eta)\cong Z/4$ by (1.3) and (1.10), we can easily observe that

(1.12) the cofibers $C(\bar h)$, $\Sigma^{-5}C(\tilde h)$, $C(\bar k)$ and $\Sigma^{-7}C(\tilde k)$ satisfy the property (I), and the first two and the last two are respectively quasi KO_*-equivalent to Σ^4 and Σ^0,

because $KO_1C(\bar h)=KO_7C(\tilde h)=KO_5C(\bar k)=0$ and $KO_1C(\bar k)\cong KO_3\,SZ/2\cong Z/2$.

By taking f in Lemma 1.5 i) as the map j, $\bar\eta$, $\bar h$ or $\bar k$, and g in Lemma 1.5 ii) as the map i, $\tilde\eta$, $\tilde h$ or $\tilde k$, we can now introduce the following maps of order 2:

$$(1.13)\qquad \begin{aligned} \alpha_{4r} &= jA_2^r i\colon \Sigma^{8r-1}\to\Sigma^0, \\ \mu_{4r+1} &= \bar\eta A_2^r i\colon \Sigma^{8r+1}\to\Sigma^0, & \mu'_{4r+1} &= jA_2^r\tilde\eta\colon \Sigma^{8r+1}\to\Sigma^0\\ a_{4r+2} &= \bar hA_2^r i\colon \Sigma^{8r+3}\to C(\bar\eta), & a'_{4r+2} &= jA_2^r\tilde h\colon \Sigma^{8r}C(\tilde\eta)\to\Sigma^0\\ m_{4r+3} &= \bar kA_2^r i\colon \Sigma^{8r+5}\to C(\bar\eta), & m'_{4r+3} &= jA_2^r\tilde k\colon \Sigma^{8r+2}C(\tilde\eta)\to\Sigma^0. \end{aligned}$$

Among their cofibers we may regard that there hold Spanier-Whitehead dualities as $C(f'_t)=\Sigma^{2t}DC(f_t)$ for $f_t=\alpha_{4r}$, μ_{4r+1}, a_{4r+2} or m_{4r+3} where $r\geqq 0$ and $\alpha'_{4r}=\alpha_{4r}$.

Combining Lemma 1.5 with (1.2) and (1.12) we obtain

Proposition 1.6. *Set $f_t=\alpha_{4r}$, μ_{4r+1}, μ'_{4r+1}, a_{4r+2}, a'_{4r+2}, m_{4r+3} or m'_{4r+3} ($r\geqq 0$). Then each cofiber $C(f_t)$ satisfies the property* $(\mathrm{II})_t$. *Moreover $C(\alpha_{4r})$, $C(a'_{4r+2})$ and $\Sigma^4C(a_{4r+2})$ are quasi KO_*-equivalent to $\Sigma^0\vee\Sigma^0$, and $C(\mu_{4r+1})$, $C(\mu'_{4r+1})$, $C(m_{4r+3})$ and $C(m'_{4r+3})$ are all Wood spectra.*

Use Proposition 1.6 combined with (1.8) to apply Theorem 4. Then we obtain the following result, which contains Theorem 1.

Theorem 1.7. *Let X be a CW-spectrum satisfying the property* $(\mathrm{II})_t$ *with $t\geqq 0$.*

i) *If X is quasi KO_*-equivalent to $\Sigma^0\vee\Sigma^0$, then it has the same K_*-local* *type as* $C(\alpha_{4r})$ *or* $C(a'_{4r+2})$ *according as* $t=4r$ *or* $4r+2$.

ii) *If* X *is quasi* KO_**-equivalent to* $\Sigma^4\vee\Sigma^4$, *then it has the same* K_**-local type as* $C(\alpha_{4r})_\wedge C(\bar{\eta})$ *or* $C(a_{4r+2})$ *according as* $t=4r$ *or* $4r+2$.

iii) *If* X *is a Wood spectrum, then it has the same* K_**-local type as* $C(\mu_{4r+1})$ *or* $C(m_{4r+3})$ *according as* $t=4r+1$ *or* $4r+3$.

2. K_*-local types of spectra with the property $(\mathrm{II}_{2m})_t$

2.1. Consider the cofibers $C(i\eta)$, $C(\bar{\eta}_{2m})$ and $C(\eta^2\bar{\eta}_{2m})$ of the maps $i\eta$: $\Sigma^1\to SZ/2$, $\bar{\eta}_{2m}$: $\Sigma^1 SZ/2m\to\Sigma^0$ and $\eta^2\bar{\eta}_{2m}$: $\Sigma^3 SZ/2m\to\Sigma^0$, which are denoted by M_{2m}, P'_{2m} and R'_{2m} respectively in [15, I.4.1]. Recall that $KU_0M_{2m}\cong Z\oplus Z/2m$ on which $\psi_C^{-1}=\begin{pmatrix}-1 & 0\\ -1 & 1\end{pmatrix}$, $KU_0P'_{2m}\cong Z\oplus Z/m$ on which $\psi_C^{-1}=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$, $KU_0R'_{2m}\cong Z\oplus Z/2m$ on which $\psi_C^{-1}=1$, and $KU_1M_{2m}=KU_1P'_{2m}=KU_1R'_{2m}=0$ [15, Proposition I.4.1]. Note that $\Sigma^{-2}P'_{4m}$ is quasi KO_*-equivalent to M_{2m}, whose KO-homology $KO_i\,M_{2m}\cong Z/2m$, 0, $Z\oplus Z/2$, $Z/2$, $Z/4m$, 0, Z, 0 according as $i=0, 1, \cdots, 7$ (see [15, Proposition I.4.2 and Corollary I.5.4]).

Let X be a CW-spectrum satisfying the following property:

$(\mathrm{II}_{2m})_t$ $KU_0X\cong Z\oplus Z/2m$ in which $\psi_C^k=A_{k,t}$ and $KU_1\,X=0$.

Then X is quasi KO_*-equivalent to one of the following elementary spectra $\Sigma^{4i}\vee\Sigma^{4j}SZ/2m$, $\Sigma^{4i}\vee\Sigma^{4j}V_{2m}$ and $\Sigma^{4j}R'_{2m}$ for $i, j=0$ or 1 when t is even, and it is quasi KO_*-equivalent to either of M_{2m} and Σ^4M_{2m} when t is odd.

Lemma 2.1. *Let* X, Y *and* W *be* CW*-spectra satisfying the property* (I). *Let* f: $\Sigma^{2t-1}X\to Y$ *and* g: $W\to Y$ *be maps whose cofibers* $C(f)$ *and* $C(g)$ *satisfy the properties* $(\mathrm{II})_t$ *and* (I_{2m}) *respectively. Then the cofiber* $C(i_gf)$ *of the composite map* i_gf: $\Sigma^{2t-1}X\to Y\to C(g)$ *satisfies the property* $(\mathrm{II}_{2m})_t$. *Moreover it is quasi* KO_**-equivalent to* $\Sigma^{2t}X\vee C(g)$ *when* t *is even, and it is quasi* KO_**-equivalent to* M_{2m} *or* Σ^4M_{2m} *according as* W *is quasi* KO_**-equivalent to* Σ^0 *or* Σ^4 *when* t *is odd.*

Proof. Use the commutative diagram

$$\begin{array}{ccccccc}
 & & W & = & W & & \\
 & & \downarrow g & & \downarrow G & & \\
\Sigma^{2t-1}X & \xrightarrow{f} & Y & \to & C(f) & \to & \Sigma^{2t}X \\
\| & & \downarrow i_g & & \downarrow i_G & & \| \\
\Sigma^{2t-1}X & \to & C(g) & \to & C(i_gf) & \to & \Sigma^{2t}X
\end{array}$$

involving four cofiber sequences. Obviously $KU_0\,C(i_gf)\cong KU_0\,\Sigma^{2t}X\oplus KU_0\,C(g)$ in which $\psi_C^k=A_{k,t}$ and $KU_1\,C(i_gf)=0$. When t is even, $C(i_gf)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^{2t}X\vee C(g)$ since $\Sigma^{2t-1}X$ is quasi KO_*-equivalent to Σ^3 or Σ^7 and Y is quasi KO_*-equivalent to Σ^0 or Σ^4. On the other hand, $C(i_g f)$ is quasi KO_*-equivalent to either of M_{2m} and $\Sigma^4 M_{2m}$ when t is odd. However we notice that $KO_3\, C(i_g f) \cong KO_2\, W$ because $C(f)$ is a Wood spectrum in the case when t is odd.

Let X be a CW-spectrum with $(\mathrm{II}_{2m})_{2s+1}$, which is quasi KO_*-equivalent to either M_{2m} or $\Sigma^4 M_{2m}$. Using the long exact sequence induced by the Bott cofiber sequence $\Sigma^1 KO \to KO \to KU \to \Sigma^2 KO$ we can easily show that the Adams operation ψ^k_R behaves always in $KO_i X$ $(0 \leqq i \leqq 7)$ for each $k \neq 0$ as follows:

(2.1) $\psi^k_R = 1/k^{2s},\ k^2,\ 1/k^{2s-2}$ or 1 according as $i = 2, 4, 6$ or otherwise.

Lemma 2.2. *Let X, Y and W be CW-spectra satisfying the property* (I). *Let $f\colon \Sigma^{4s-1} X \to Y$ and $g\colon W \to Y$ be maps whose cofibers $C(f)$ and $C(g)$ satisfy the properties $(\mathrm{II})_{2s}$ and (I_{2m}) respectively. Assume that the Adams operation ψ^k_R behaves normally in $KO_* C(g)$ in the sense of* (1.5). *Then the Adams operation ψ^k_R acts normally in $KO_i C(i_g f)$ $(0 \leqq i \leqq 7)$ for each $k \neq 0$ as follows:*

i) *If both $\Sigma^{4s} X$ and Y are quasi KO_*-equivalent to either of Σ^0 and Σ^4, then $\psi^k_R = A_{k,2s},\ k^2 A_{k,2s}$ or 1 according as $i = 0, 4$ or otherwise.*

ii) *If $\Sigma^{4s} X$ and Y are respectively quasi KO_*-equivalent to Σ^0 and Σ^4, then $\psi^k_R = A_{k,2s,2},\ k^2 A_{k,2s,0}$ or 1 according as $i = 0, 4$ or otherwise.*

iii) *If $\Sigma^{4s} X$ and Y are respectively quasi KO_*-equivalent to Σ^4 and Σ^0, then $\psi^k_R = A_{k,2s,0},\ k^2 A_{k,2s,2}$ or 1 according as $i = 0, 4$ or otherwise.*

Proof. Use the cofiber sequence $W \xrightarrow{G} C(f) \xrightarrow{i_G} C(i_g f) \xrightarrow{j_G} \Sigma^1 W$ appeared in the proof of Lemma 2.1 where $C(f)$ and $C(i_g f)$ are quasi KO_*-equivalent to $\Sigma^{4s} X \vee Y$ and $\Sigma^{4s} X \vee C(g)$ respectively. Since W is quasi KO_*-equivalent to either of Σ^0 and Σ^4, the map i_G induces epimorphisms $i_{G*}\colon KO_i C(f) \to KO_i C(i_g f)$ in dimensions $i = 0, 1, 4$ and 5. By using (1.8) i) and ii) we can immediately observe the behavior of ψ^k_R in $KO_i C(i_g f)$ for $i = 0, 1, 4$ or 5. We will next show that $\psi^k_R = 1$ in $KO_i C(i_g f)$ for $i = 2$ or 6. It is obvious that $KO_2 C(i_g f)$ is isomorphic to $KO_2 C(g)$, $KO_2 C(f)$ or $KO_2 C(f) \oplus KO_2 C(g)$ according as $\Sigma^{4s} X$, W or Y is quasi KO_*-equivalent to Σ^4. Therefore it is easy to see that $\psi^k_R = 1$ in $KO_2 C(i_g f)$ in these three cases. Assume that $\Sigma^{4s} X$, W and Y are all quasi KO_*-equivalent to Σ^0. Then we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & KO_2 Y & \to & KO_2 C(f) & \to & KO_2\, \Sigma^{4s} X & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & KO_2 C(g) & \to & KO_2 C(i_g f) & \to & KO_2\, \Sigma^{4s} X & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & KO_1 W & = & KO_1 W & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact rows and columns, where $C(g)$ is quasi KO_*-equivalent to $SZ/2m$ by Lemma 1.1. Then a routine computation shows that $\psi_R^k=1$ in $KO_2C(i_g f)$ as desired, because $\psi_R^k=1$ in $KO_2C(f)$ and $KO_2C(g)$. Similarly as to $KO_6 C(i_g f)$.

We remark that the Adams operation ψ_R^k acts normally in $KO_*C(i_g f)_\wedge C(\bar{\eta})$ as stated in the above lemma if it behaves normally in $KO_*C(g)_\wedge C(\bar{\eta})$ in the sense of (1.5).

Take f in Lemma 2.1 as the map α_{4r}, μ_{4r+1}, a_{4r+2}, a'_{4r+2} or m_{4r+3} given in (1.13) and g in Lemma 2.1 as the map $2m$: $\Sigma^0\to\Sigma^0$, $m\bar{\lambda}$: $C(\bar{\eta})\to\Sigma^0$, $2m$: $C(\bar{\eta})\to C(\bar{\eta})$ or $m\bar{i}$: $\Sigma^0\to C(\bar{\eta})$ whose cofiber is $SZ/2m$, U_{2m}, $SZ/2m_\wedge C(\bar{\eta})$ or V_{2m}. Then we can introduce the composite maps $i_g f_t$ $(t\geqq 0)$ as follows:

$$
(2.2)\qquad
\begin{array}{ll}
i\alpha_{4r}\colon \Sigma^{8r-1}\to SZ/2m\,, & i_U\alpha_{4r}\colon \Sigma^{8r-1}\to U_{2m}\,,\\
i\mu_{4r+1}\colon \Sigma^{8r+1}\to SZ/2m\,, & i_U\mu_{4r+1}\colon \Sigma^{8r+1}\to U_{2m}\,,\\
ia'_{4r+2}\colon \Sigma^{8r}C(\bar{\eta})\to SZ/2m\,, & i_U a'_{4r+2}\colon \Sigma^{8r}C(\bar{\eta})\to U_{2m}\,,\\
(i_\wedge 1)\,a_{4r+2}\colon \Sigma^{8r+3}\to SZ/2m_\wedge C(\bar{\eta})\,, & i_V a_{4r+2}\colon \Sigma^{8r+3}\to V_{2m}\,,\\
(i_\wedge 1)\,m_{4r+3}\colon \Sigma^{8r+5}\to SZ/2m_\wedge C(\bar{\eta})\,, & i_V m_{4r+3}\colon \Sigma^{8r+5}\to V_{2m}\,.
\end{array}
$$

Applying Lemmas 2.1 and 2.2 and (2.1) with the aid of Proposition 1.6, (1.3), (1.4) and (1.5), we obtain

Proposition 2.3. *For each composite map $i_g f_t$ $(t\geqq 0)$ given in* (2.2), *its cofiber $C(i_g f_t)$ satisfies the property $(\mathrm{II}_{2m})_t$, and the Adams operation ψ_R^k behaves normally in $KO_*C(i_g f_t)$ as stated in Lemma* 2.2 i) *when t is even, or as stated in* (2.1) *when t is odd. Moreover $C(i\alpha_{4r})$, $C(ia'_{4r+2})$ and $\Sigma^4C((i_\wedge 1)\,a_{4r+2})$ are quasi KO_*-equivalent to $\Sigma^0\vee SZ/2m$, and $C(i_U\alpha_{4r})$, $C(i_U a'_{4r+2})$ and $\Sigma^4C(i_V a_{4r+2})$ are quasi KO_*-equivalent to $\Sigma^0\vee\Sigma^4V_{2m}$. On the other hand, $C(i\mu_{4r+1})$, $C(i_V m_{4r+3})$, $\Sigma^4C(i_U\mu_{4r+1})$ and $\Sigma^4C((i_\wedge 1)\,m_{4r+3})$ are all quasi KO_*-equivalent to M_{2m}.*

2.2. Let f: $\Sigma^{2t-1}X\to Y$ be a map of order 2. Then we have extensions

$$
\bar{f}_{2m}\colon \Sigma^{2t-1}X_\wedge SZ/2m\to Y\,,\quad \bar{f}_{U,4m}\colon \Sigma^{2t-1}X_\wedge U_{4m}\to Y\quad\text{and}
$$
$$
\bar{f}_{V,4m}\colon \Sigma^{2t-1}X'_\wedge V_{4m}\to Y\quad\text{when}\quad X=X'_\wedge C(\bar{\eta})
$$

such that $\bar{f}_{2m}(1_\wedge i)=f$, $\bar{f}_{U,4m}(1_\wedge i_U)=f$ and $\bar{f}_{V,4m}(1_\wedge i_V)=f$ because U_{4m} and V_{4m} are constructed as the cofibers of the maps $2m\bar{\lambda}$: $C(\bar{\eta})\to\Sigma^0$ and $2m\bar{i}$: $\Sigma^0\to C(\bar{\eta})$ respectively.

Lemma 2.4. *Let X and Y be CW-spectra satisfying the property* (I), *and f: $\Sigma^{2t-1}X\to Y$ be a map of order* 2 *whose cofiber $C(f)$ satisfies the property $(\mathrm{II})_t$ $(t\neq 0)$.*

(i) *The cofiber $C(\bar{f}_2)$ satisfies the property* (I), *and it is quasi KO_*-equivalent* *to Y or $\Sigma^4 Y$ according as t is even or odd.*

ii) *For $\bar{\varphi}_{4m}=\bar{f}_{4m}, \bar{f}_{U,4m}$ or $\bar{f}_{V,4m}$ each cofiber $\Sigma^{-2t}C(\bar{\varphi}_{4m})$ satisfies the property $(\mathrm{II}_{2m})_{-t}$. Whenever t is odd, all of $C(\bar{f}_{4m})$, $\Sigma^4 C(\bar{f}_{U,4m})$ and $C(\bar{f}_{V,4m})$ are quasi KO_*-equivalent to P'_{4m} or $\Sigma^4 P'_{4m}$ according as $\Sigma^{2t}X$ is quasi KO_*-equivalent to Σ^2 or Σ^6.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & \Sigma^{2t}X & = & \Sigma^{2t}X \\
 & & & & \downarrow\lambda & & \downarrow 2m \\
\Sigma^{2t-1}X & \xrightarrow{f} & Y & \to & C(f) & \to & \Sigma^{2t}X \\
\downarrow 1_\wedge i & & \| & & \downarrow & & \downarrow \\
\Sigma^{2t-1}X_\wedge SZ/2m & \xrightarrow[\bar{f}_{2m}]{} & Y & \to & C(\bar{f}_{2m}) & \to & \Sigma^{2t}X_\wedge SZ/2m
\end{array}$$

involving four cofiber sequences. The induced homomorphism $\lambda_*: KU_0\Sigma^{2t}X \to KU_0C(f)$ is given by $\lambda_*(1)=(2m, m)\in KU_0C(f)\cong KU_0\Sigma^{2t}X\oplus KU_0Y\cong Z\oplus Z$, since $\psi_C^k=A_{k,t}$ in $KU_0C(f)$. Hence it is immediate that $KU_0C(\bar{f}_{2m})\cong Z\oplus Z/m$ and $KU_1C(\bar{f}_{2m})=0$. Moreover the Adams operation ψ_C^k in $KU_0C(\bar{f}_{2m})$ is represented as the matrix $\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}A_{k,t}\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}=k^{-t}A_{k,-t}$. In other words, $\psi_C^k=1$ in $KU_0C(\bar{f}_2)\cong Z$ and $\psi_C^k=A_{k,-t}$ in $KU_0\,\Sigma^{-2t}C(\bar{f}_{2m})\cong Z\oplus Z/m$ unless $m=1$.

Assume that Y is quasi KO_*-equivalent to Σ^0. Then it is obvious that $KO_7\,\Sigma^{2t}X_\wedge SZ/2=0$ because $KO_7\,C(\bar{f}_2)=0=KO_6Y$. Therefore $\Sigma^{2t}X_\wedge SZ/2$ becomes quasi KO_*-equivalent to $SZ/2$ or $\Sigma^2 SZ/2$ according as t is even or odd. This implies easily that $C(\bar{f}_2)$ is quasi KO_*-equivalent to Σ^0 or Σ^4 according as t is even or odd. When Y is quasi KO_*-equivalent to Σ^4, a similar result can be shown. Since $C(f)$ is a Wood spectrum when t is odd, it is immediate that $KO_1C(\bar{f}_{2m})\cong KO_0\Sigma^{2t}X$. Hence $C(\bar{f}_{2m})$ is quasi KO_*-equivalent to P'_{2m} or $\Sigma^4P'_{2m}$ according as $\Sigma^{2t}X$ is quasi KO_*-equivalent to Σ^2 or Σ^6.

We can similarly prove as for $C(\bar{f}_{U,4m})$ and $C(\bar{f}_{V,4m})$.

Since $[\Sigma^3SZ/2, C(\eta)]\cong[\Sigma^1C(\tilde{\eta}), SZ/2]\cong Z/2$ (use [4, §4]), the maps $\bar{h}$: $\Sigma^3SZ/2\to C(\eta)$ and $\tilde{h}$: $\Sigma^1C(\tilde{\eta})\to SZ/2$ have order 2. So there exist maps

$$\bar{h}_{2m/2}\colon \Sigma^4SZ/2 \to SZ/2m_\wedge C(\eta) \quad\text{and}\quad \tilde{h}_{2m/2}\colon \Sigma^1SZ/2m_\wedge C(\tilde{\eta}) \to SZ/2$$

satisfying $(j_\wedge 1)\,\bar{h}_{2m/2}=\bar{h}$ and $\tilde{h}_{2m/2}(i_\wedge 1)=\tilde{h}$. We now set

$$\begin{aligned}
&\bar{\alpha}_{4r}=jA_2^r\colon \Sigma^{8r-1}SZ/2\to\Sigma^0, &&\bar{\mu}_{4r+1}=\eta A_2^r\colon \Sigma^{8r+1}SZ/2\to\Sigma^0,\\
(2.3)\quad &\bar{a}_{4r+2}=\bar{h}A_2^r\colon \Sigma^{8r+3}SZ/2\to C(\eta), &&\bar{m}_{4r+3}=\bar{k}A_2^r\colon \Sigma^{8r+5}SZ/2\to C(\eta),\\
&\bar{a}'_{4r+2}=jA_2^r\,\tilde{h}_{2/2}\colon \Sigma^{8r}SZ/2_\wedge C(\tilde{\eta})\to\Sigma^0.
\end{aligned}$$

Then Lemma 2.4 i) combined with Proposition 1.6 shows that

(2.4) the cofibers $C(\bar{\alpha}_{4r})$, $C(\bar{\mu}_{4r+1})$, $C(\bar{a}_{4r+2})$, $C(\bar{m}_{4r+3})$ and $C(\bar{a}'_{4r+2})$ satisfy the property (I), and the first, the forth and the last are quasi KO_*-equivalent to Σ^0 and the other two are quasi KO_*-equivalent to Σ^4.

Let f: $\Sigma^{2t-1}X \to Y$ be a map of order 2 and $\bar{f}$: $\Sigma^{2t-1}X_\wedge SZ/2 \to Y$ be its extension with $\bar{f}(1_\wedge i)=f$. Then there exists a map φ: $\Sigma^{-2t-1}C(\bar{f}) \to X$ of order 2 whose cofiber $C(\varphi)$ coincides with $\Sigma^{-2t}C(f)$. Hence we can choose the following maps of order 2:

$$
(2.5)\quad \begin{array}{ll}
\alpha_{-4r}\colon \Sigma^{-8r-1}C(\bar{\alpha}_{4r}) \to \Sigma^0\,, & \mu_{-4r-1}\colon \Sigma^{-8r-3}C(\bar{\mu}_{4r+1}) \to \Sigma^0\,,\\
a_{-4r-2}\colon \Sigma^{-8r-5}C(\bar{a}_{4r+2}) \to \Sigma^0\,, & m_{-4r-3}\colon \Sigma^{-8r-7}C(\bar{m}_{4r+3}) \to \Sigma^0\,,\\
b_{-4r-2}\colon \Sigma^{-8r-5}C(\bar{a}'_{4r+2}) \to \Sigma^{-3}C(\bar{\eta}) &
\end{array}
$$

of which each cofiber $C(f_{-t})$ coincides with $\Sigma^{-2t}C(f_t)$ where $f_t=\alpha_{4r}$, μ_{4r+1}, a_{4r+2}, m_{4r+3} or b_{4r+2} $(r\geqq 0)$ with $b_{4r+2}=a'_{4r+2}$.

Take f in Lemma 2.1 as the above map α_{-4r}, μ_{-4r-1}, a_{-4r-2}, m_{-4r-3} or b_{-4r-2}, and g in Lemma 2.1 as the map $2m$: $\Sigma^0\to\Sigma^0$, $m\bar{\lambda}$: $C(\bar{\eta})\to\Sigma^0$, $2m$: $C(\bar{\eta})\to C(\bar{\eta})$ or $m\bar{\lambda}$: $\Sigma^3\to C(\bar{\eta})$. Then we obtain the following composite maps $i_g f_{-t}$ $(t\geqq 0)$:

$$
(2.6)\quad \begin{array}{ll}
i\alpha_{-4r}\colon \Sigma^{-8r-1}C(\bar{\alpha}_{4r}) \to SZ/2m\,, & i_U\alpha_{-4r}\colon \Sigma^{-8r-1}C(\bar{\alpha}_{4r}) \to U_{2m}\,,\\
i\mu_{-4r-1}\colon \Sigma^{-8r-3}C(\bar{\mu}_{4r+1}) \to SZ/2m\,, & i_U\mu_{-4r-1}\colon \Sigma^{-8r-3}C(\bar{\mu}_{4r+1}) \to U_{2m}\,,\\
ia_{-4r-2}\colon \Sigma^{-8r-5}C(\bar{a}_{4r+2}) \to SZ/2m\,, & i_U a_{-4r-2}\colon \Sigma^{-8r-5}C(\bar{a}_{4r+2}) \to U_{2m}\,,\\
im_{-4r-3}\colon \Sigma^{-8r-7}C(\bar{m}_{4r+3}) \to SZ/2m\,, & i_U m_{-4r-3}\colon \Sigma^{-8r-7}C(\bar{m}_{4r+3}) \to U_{2m}\,,\\
(i_\wedge 1)\,b_{-4r-2}\colon \Sigma^{-8r-5}C(\bar{a}'_{4r+2}) \to \Sigma^{-3}SZ/2m_\wedge C(\bar{\eta}) \quad\text{and} & \\
i'_U b_{-4r-2}\colon \Sigma^{-8r-5}C(\bar{a}'_{4r+2}) \to \Sigma^{-3}U'_{2m}\,. &
\end{array}
$$

By making use of Lemmas 2.1 and 2.2 and (2.1) we obtain

Proposition 2.5. *For each composite map $i_g f_{-t}$ $(t\geqq 0)$ given in* (2.6), *its cofiber $C(i_g f_{-t})$ satisfies the property* $(\mathrm{II}_{2m})_{-t}$, *and the Adams operation ψ_R^k behaves normally in $KO_*C(i_g f_{-t})$ as stated in Lemma* 2.2 i) *when t is even, or as stated in* (2.1) *when t is odd. Moreover $C(i\alpha_{-4r})$, $C(ia_{-4r-2})$ and $\Sigma^4C((i_\wedge 1)\,b_{-4r-2})$ are quasi KO_*-equivalent to $\Sigma^0\vee SZ/2m$, and $C(i_U\alpha_{-4r})$, $C(i_U a_{-4r-2})$ and $\Sigma^4C(i'_U b_{-4r-2})$ are quasi KO_*-equivalent to $\Sigma^0\vee\Sigma^4V_{2m}$. On the other hand, $C(i\mu_{-4r-1})$, $C(im_{-4r-3})$, $\Sigma^4C(i_U\mu_{-4r-1})$ and $\Sigma^4C(i_U m_{-4r-3})$ are all quasi KO_*-equivalent to M_{2m}.*

By virtue of Propositions 2.3 and 2.5 we can apply Theorem 4 to show the followong result.

Theorem 2.6. *Let X be a CW-spectrum satisfying the property* $(\mathrm{II}_{2m})_t$.

i) *Assume that X is quasi KO_*-equivalent to $\Sigma^0\vee SZ/2m$. If the Adams operation ψ_R^k behaves normally in KO_*X for each $k\neq 0$ as stated in Lemma* 2.2

i), *then* X *has the same* K_*-*local type as* $C(i\alpha_{4r})$, $C(ia'_{4r+2})$, $C(i\alpha_{-4r})$ *or* $C(ia_{-4r-2})$ *according as* $t=4r$, $4r+2$, $-4r$ *or* $-4r-2$ $(r\geqq 0)$.

ii) *Assume that* X *is quasi* KO_*-*equivalent to* $\Sigma^0\vee\Sigma^4V_{2m}$. *If the Adams operation* ψ_R^k *behaves normally in* KO_*X *for each* $k\neq 0$ *as stated in Lemma* 2.2 i), *then* X *has the same* K_*-*local type as* $C(i_U\alpha_{4r})$, $C(i_Ua'_{4r+2})$, $C(i_U\alpha_{-4r})$ *or* $C(i_Ua_{-4r-2})$ *according as* $t=4r$, $4r+2$, $-4r$ *or* $-4r-2$ $(r\geqq 0)$.

iii) *Assume that* X *is quasi* KO_*-*equivalent to* M_{2m}. *Then* X *has the same* K_*-*local type as* $C(i\mu_{4r+1})$, $C(i_Vm_{4r+3})$, $C(i\mu_{-4r-1})$ *or* $C(im_{-4r-3})$ *according as* $t=4r+1, 4r+3, -4r-1, -4r-3$ $(r\geqq 0)$.

iv) *Assume that* X *is quasi* KO_*-*equivalent to* Σ^4M_{2m}. Then X has the same K_*-local type as $C(i_U\mu_{4r+1})$, $C((i_\wedge 1)m_{4r+3})$, $C(i_U\mu_{-4r-1})$ *or* $C(i_Um_{-4r-3})$ *according as* $t=4r+1, 4r+3, -4r-1$ *or* $-4r-3$ $(r\geqq 0)$.

In the above theorem we may replace the map i_U: $\Sigma^0\to U_{2m}$ by the map i'_V: $\Sigma^0\to\Sigma^{-2}V'_{2m}$, and also the maps μ_{4r+1}: $\Sigma^{8r+1}\to\Sigma^0$, i_Vm_{4r+3}: $\Sigma^{8r+5}\to V_{2m}$ and $(i_\wedge 1)m_{4r+3}$: $\Sigma^{8r+5}\to SZ/2m_\wedge C(\bar\eta)$ by μ'_{4r+1}: $\Sigma^{8r+1}\to\Sigma^0$, im'_{4r+3}: $\Sigma^{8r+2}C(\bar\eta)\to SZ/2m$ and $i'_Vm'_{4r+3}$: $\Sigma^{8r+2}C(\bar\eta)\to\Sigma^{-2}V'_{2m}$ respectively. Thus

(2.7) i) $C(i'_Vf_t)$ has the same K_*-local type as $C(i_Uf_t)$ for $f_t=\alpha_{\pm 4r}$, $\mu_{\pm(4r+1)}$, a'_{4r+2}, a_{-4r-2}, m'_{4r+3} or m_{-4r-3}.

ii) $C(i\mu'_{4r+1})$ and $C(i_U\mu'_{4r+1})$ have the same K_*-local types as $C(i\mu_{4r+1})$ and $C(i_U\mu_{4r+1})$ respectively.

iii) $C(im'_{4r+3})$ and $C(i'_Vm'_{4r+3})$ have the same K_*-local types as $C(i_Vm_{4r+3})$ and $C((i_\wedge 1)m_{4r+3})$ respectively.

When X is quasi KO_*-equivalent to $\Sigma^4\vee\Sigma^4SZ/2m$ or $\Sigma^4\vee V_{2m}$, we can obtain a similar result corresponding to the above theorem i) or ii). In fact, if the Adams operation ψ_R^k behaves normally in KO_*X for each $k\neq 0$ as stated in Lemma 2.2 i), then X has the same K_*-local type as the cofiber appeared in Theorem 2.6 i) or ii) smashed with $C(\bar\eta)$ (see the remark following Lemma 2.2). In particular, by means of Propositions 2.3 and 2.5 again we obtain that

(2.8) $C((i_\wedge 1)a_{4r+2})$, $C(i_Va_{4r+2})$, $C((i_\wedge 1)b_{-4r-2})$ and $C(i'_Ub_{-4r-2})$ have the same K_*-local types as $C(ia'_{4r+2})_\wedge C(\bar\eta)$, $C(i_Ua'_{4r+2})_\wedge C(\bar\eta)$, $C(ia_{-4r-2})_\wedge C(\bar\eta)$ and $C(i_Ua_{-4r-2})_\wedge C(\bar\eta)$ respectively.

3. K_*-local types of Anderson spectra

3.1. Let X be a CW-spectrum with $KU_0X\cong KU_1X\cong Z$. For such a CW-spectrum X we may assume that $X_\wedge SQ=(\Sigma^0\vee\Sigma^{2t+1})_\wedge SQ$ for some integer t. In this case X satisfies the following property:

$(\mathrm{III})_t$ $KU_0X\cong Z$ with $\psi_C^k=1$ and $KU_1X\cong Z$ with $\psi_C^k=1/k^t$.

If X satisfies the property $(\mathrm{III})_{2s+1}$, then it is quasi KO_*-equivalent to one of the following spectra $\Sigma^0\vee\Sigma^3$, $\Sigma^0\vee\Sigma^7$, $\Sigma^4\vee\Sigma^3$, $\Sigma^4\vee\Sigma^7$ or $C(\eta^2)$ (see [7, Theorem 3.2] or [15, Theorem I.3.4]).

Lemma 3.1. *Let X and Y be CW-spectra satisfying the property* (I) *and $f\colon \Sigma^{2t-1}X\to Y$ be a map whose cofiber $C(f)$ satisfies the property* $(\mathrm{II})_t$. *Then the cofiber $C(\eta f)$ of the composite map $\eta f\colon \Sigma^{2t}X\to Y$ satisfies the property* $(\mathrm{III})_t$, *and it is quasi KO_*-equivalent to $Y\vee\Sigma^{2t+1}X$ or $C(\eta^2)$ according as t is even or odd.*

Proof. Obviosuly $KU_0C(\eta f)\cong KU_0Y\cong Z$ and $KU_1C(\eta f)\cong KU_1\Sigma^{2t+1}X\cong Z$. In the case when t is even, $C(\eta f)$ is quasi KO_*-equivalent to the wedge sum $Y\vee\Sigma^{2t+1}X$ since $C(f)$ is quasi KO_*-equivalent to $Y\vee\Sigma^{2t}X$. On the other hand, $C(\eta f)$ is just an Anderson spectrum in the case when t is odd, because $KO_2C(\eta f)=0=KO_6C(\eta f)$.

Let X be an Anderson spectrum satisfying the property $(\mathrm{III})_{2s+1}$. Then we can easily observe that the Adams operation ψ_R^k behaves always in KO_iX $(0\leqq i\leqq 7)$ for each $k\neq 0$ as follows:

(3.1) $\psi_R^k=1/k^{2s}$, k^2, $1/k^{2s-2}$ or 1 according as $i=3, 4, 7$ or otherwise.

Lemma 3.2. *Let X and Y be CW-spectra satisfying the property* (I) *and $f\colon \Sigma^{4s-1}X\to Y$ be a map whose cofiber $C(f)$ satisfies the property* $(\mathrm{II})_{2s}$. *Then the Adams operation ψ_R^k acts normally in $KO_iC(\eta f)$ $(0\leqq i\leqq 7)$ for each $k\neq 0$ as follows: $\psi_R^k=1/k^{2s}$, k^2, $1/k^{2s-2}$ or 1 according as $i=1, 4, 5$ or otherwise.*

Proof. Use the cofiber sequence $\Sigma^1C(f)\to C(\eta f)\to C(\eta)_\wedge Y\to\Sigma^2C(f)$ where $C(f)$, $C(\eta f)$ and $C(\eta)_\wedge Y$ are quasi KO_*-equivalent to $Y\vee\Sigma^{4s}X$, $Y\vee\Sigma^{4s+1}X$ and $C(\eta)$ respectively. Then the result follows immediately from (1.8) i) and ii).

Take f in Lemma 3.1 as the map $\alpha_{\pm 4r}$, $\mu_{\pm(4r+1)}$, $a_{\pm(4r+2)}$, $m_{\pm(4r+3)}$, a'_{4r+2} or b_{-4r-2} given in (1.13) or (2.5). Using Lemmas 3.1 and 3.2 and (3.1) by virtue of Proposition 1.6 we obtain

Proposition 3.3. *Set $f_t=\alpha_{\pm 4r}$, $\mu_{\pm(4r+1)}$, $a_{\pm(4r+2)}$, $m_{\pm(4r+3)}$, a'_{4r+2} or b_{-4r-2} $(r\geqq 0)$. Then each cofiber $C(\eta f_t)$ satisfies the property* $(\mathrm{III})_t$, *and the Adams operation ψ_R^k behaves normally in $KO_*C(\eta f_t)$ as stated in Lemma* 3.2 *when t is even, or as stated in* (3.1) *when t is odd. Moreover the cofibers $C(\eta f_t)$ for $f_t=\alpha_{\pm 4r}$, a'_{4r+2} and a_{-4r-2} are quasi KO_*-equivalent to $\Sigma^0\vee\Sigma^1$, but $C(\eta a_{4r+2})$ and $C(\eta b_{-4r-2})$ are quasi KO_*-equivalent to $\Sigma^4\vee\Sigma^5$. On the other hand, the cofibers $C(\eta f_t)$ for $f_t=\mu_{\pm(4r+1)}$ and $m_{\pm(4r+3)}$ are Anderson spectra.*

By applying Theorem 4 combined with Proposition 3.3 we can show the following result, which contains Theorem 2.

Theorem 3.4. *Let X be a CW-spectrum satisfying the property* $(\mathrm{III})_t$ *with*

$t \neq -1$.

i) *Assume that X is quasi KO_*-equivalent to $\Sigma^0 \vee \Sigma^1$. If the Adams operation ψ_R^k behaves normally in KO_*X for each $k \neq 0$ as stated in Lemma* 3.2, *then X has the same K_*-local type as $C(\eta\alpha_{4r})$, $C(\eta a'_{4r+2})$, $C(\eta\alpha_{-4r})$ or $C(\eta a_{-4r-2})$ according as $t=4r$, $4r+2$, $-4r$ or $-4r-2$ $(r \geqq 0)$.*

ii) *When X is an Anderson spectrum, then it has the same K_*-local type as $C(\eta\mu_{4r+1})$, $C(\eta m_{4r+3})$, $C(\eta\mu_{-4r-1})$ or $C(\eta m_{-4r-3})$ according as $t=4r+1$, $4r+3$, $-4r-1$ or $-4r-3$ $(r \geqq 0)$ where $t \neq -1$.*

3.3. As duals of M_{2m}, P'_{2m} and R'_{2m} appeared in §2 we next consider the cofibers $C(\eta j)$, $C(\tilde{\eta}_{2m})$ and $C(\tilde{\eta}_{2m}\eta^2)$ of the maps $\eta j: SZ/2m \to \Sigma^0$, $\tilde{\eta}_{2m}: \Sigma^2 \to SZ/2m$ and $\tilde{\eta}_{2m}\eta^2: \Sigma^4 \to SZ/2m$, which are denoted by M'_{2m}, P_{2m} and R_{2m} respectively in [15, I.4.1]. Then there hold Spanier-Whitehead dualities as $M'_{2m} = \Sigma^2 DM_{2m}$, $P'_{2m} = \Sigma^3 DP_{2m}$ and $R'_{2m} = \Sigma^5 DR_{2m}$. Hence $KU^0 M'_{2m} \cong Z \oplus Z/2m$ on which $\psi_C^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, $KU^1 P_{2m} \cong Z \oplus Z/m$ on which $\psi_C^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$, $KU^1 R_{2m} \cong Z \oplus Z/2m$ on which $\psi_C^{-1} = 1$, and $KU^1 M'_{2m} = KU^0 P_{2m} = KU^0 R_{2m} = 0$ (cf. [15, Proposition I.4.1]). Note that $\Sigma^1 P_{4m}$ is quasi KO_*-equivalent to M'_{2m}, whose KO-homology $KO_i M'_{2m} \cong Z$, $Z/4m$, $Z/2$, $Z/2$, Z, $Z/2m$, 0, 0 according as $i = 0, 1, \cdots, 7$ (see [15, Proposition I.4.2 and Corollary I.5.4]).

Let X be a CW-spectrum satisfying the following property:

$(\mathrm{II}_{2m})_t^*$ $\quad KU^0 X \cong Z \oplus Z/2m$ in which $\psi_C^k = A_{k,t}$ and $KU^1 X = 0$.

If $KU_i X$ is finitely generated for each i, then the property $(\mathrm{II}_{2m})_t^*$ implies that $KU_0 X \cong Z$ with $\psi_C^k = k^t$ and $KU_{-1} X \cong Z/2m$ with $\psi_C^k = 1$. Under the assumption that X is finite, we note that X satisfies the property $(\mathrm{II}_{2m})_t^*$ if and only if its Spanier-Whitehead dual DX does the property $(\mathrm{II}_{2m})_t$. As a dual of Lemma 2.1 we have

Lemma 3.5. *Let X, Y and W be CW-spectra satisfying the property* (I). *Let $f: \Sigma^{2t-1} X \to Y$ and $g: X \to W$ be maps whose cofibers $C(f)$ and $C(g)$ satisfy the properties $(\mathrm{II})_t$ and (I_{2m}) respectively. Then for the composite map $fj_g: \Sigma^{2t-2} C(g) \to \Sigma^{2t-1} X \to Y$ the cofiber $\Sigma^{-2t} C(fj_g)$ satisfies the property $(\mathrm{II}_{2m})_t^*$. Moreover $C(fj_g)$ is quasi KO_*-equivalent to the wedge sum $Y \vee \Sigma^{2t-1} C(g)$ when t is even. On the other hand, under the assumption that $C(fj_g)$ is finite, it is quasi KO_*-equivalent to M'_{2m} or $\Sigma^4 M'_{2m}$ according as $\Sigma^{2t} W$ is quasi KO_*-equivalent to Σ^2 or Σ^6 when t is odd.*

Let $f: \Sigma^{2t-1} X \to Y$ be a map of order 2. Then we have coextensions

$$\tilde{f}_{2m}: \Sigma^{2t} X \to Y \wedge SZ/2m, \quad \tilde{f}_{V,4m}: \Sigma^{2t} X \to Y \wedge V_{4m} \quad \text{and}$$
$$\tilde{f}_{U,4m}: \Sigma^{2t} X \to Y' \wedge U_{4m} \quad \text{when} \quad Y = Y' \wedge C(\bar{\eta})$$

such that $(1_\wedge j)\tilde{f}_{2m}=f$, $(1_\wedge j_V)\tilde{f}_{V,4m}=f$ and $(1_\wedge j_U)\tilde{f}_{U,4m}=f$. As a dual of Lemma 2.4 we have

Lemma 3.6. *Let X and Y be CW-spectra satisfying the property* (I), *and $f\colon \Sigma^{2t-1}X\to Y$ be a map of order 2 whose cofiber $C(f)$ satisfies the property* $(\mathrm{II})_t$ $(t\neq 0)$.

i) *The cofiber $\Sigma^{-2t-1}C(\tilde{f}_2)$ satisfies the property* (I), *and it is quasi KO_*-equivalent to X or Σ^4X according as t is even or odd.*

ii) *For $\tilde{\varphi}_{4m}=\tilde{f}_{4m}, \tilde{f}_{V,4m}$ or $\tilde{f}_{U,4m}$ each cofiber $\Sigma^{-1}C(\tilde{\varphi}_{4m})$ satisfies the property $(\mathrm{II}_{2m})^*_{-t}$. Under the assumption that these cofibers are finite, all of $C(\tilde{f}_{4m})$, $\Sigma^4C(\tilde{f}_{V,4m})$ and $C(\tilde{f}_{U,4m})$ are quasi KO_*-equivalent to P_{4m} or Σ^4P_{4m} according as Y is quasi KO_*-equivalent to Σ^0 or Σ^4 whenever t is odd.*

As a dual of (2.3) we set

$$(3.2)\quad \begin{aligned} &\tilde{\alpha}_{4r}=A_2^r i\colon \Sigma^{8r}\to SZ/2\,, && \tilde{\mu}'_{4r+1}=A_2^r\tilde{\eta}\colon \Sigma^{8r+2}\to SZ/2\,,\\ &\tilde{a}'_{4r+2}=A_2^r\tilde{h}\colon \Sigma^{8r+1}C(\tilde{\eta})\to SZ/2\,, && \tilde{m}'_{4r+3}=A_2^r\tilde{k}\colon \Sigma^{8r+3}C(\tilde{\eta})\to SZ/2\,,\\ &\tilde{a}_{4r+2}=\tilde{h}_{2/2}A_2^r i\colon \Sigma^{8r+4}\to SZ/2_\wedge C(\tilde{\eta})\,. \end{aligned}$$

Since $\Sigma^{-2t-1}C(\tilde{f}'_t)=DC(\tilde{f}_t)$ for $f_t=\alpha_{4r}, \mu_{4r+1}, a_{4r+2}, m_{4r+3}$ or a'_{4r+2} $(r\geqq 0)$ with $\alpha'_{4r}=\alpha_{4r}$ and $a''_{4r+2}=a_{4r+2}$, (2.5) implies that

(3.3) each cofiber $\Sigma^{-2t-1}C(\tilde{f}'_t)$ satisfies the property (I) for $\tilde{f}'_t$ given in (3.2), and $C(\tilde{\alpha}_{4r})$, $\Sigma^2C(\tilde{\mu}'_{4r+1})$, $C(\tilde{a}'_{4r+2})$, $\Sigma^2C(\tilde{m}'_{4r+3})$ and $\Sigma^4C(\tilde{a}_{4r+2})$ are all quasi KO_*-equivalent to Σ^1.

Let $f\colon \Sigma^{2t-1}X\to Y$ be a map of order 2 and $\tilde{f}\colon \Sigma^{2t}X\to Y_\wedge SZ/2$ be its coextension with $(1_\wedge j)\tilde{f}=f$. Then there exists a map $\psi\colon Y\to C(\tilde{f})$ of order 2 whose cofiber $C(\psi)$ coincides with $\Sigma^1C(f)$. So we can choose the following maps of order 2:

$$(3.4)\quad \begin{aligned} &\alpha'_{-4r}\colon \Sigma^0\to C(\tilde{\alpha}_{4r})\,, && \mu'_{-4r-1}\colon \Sigma^0\to C(\tilde{\mu}'_{4r+1})\,,\\ &a'_{-4r-2}\colon \Sigma^0\to C(\tilde{a}'_{4r+2})\,, && m'_{-4r-3}\colon \Sigma^0\to C(\tilde{m}'_{4r+3})\quad\text{and}\\ &b'_{-4r-2}\colon C(\tilde{\eta})\to C(\tilde{a}_{4r+2}) \end{aligned}$$

of which each cofiber $C(f'_{-t})$ coincides with $\Sigma^1C(f'_t)$ where $f'_t=\alpha_{4r}, \mu'_{4r+1}, a'_{4r+2}, m'_{4r+3}$ and b'_{4r+2} $(r\geqq 0)$ with $b'_{4r+2}=a_{4r+2}$. Since the maps f'_{-t} given in (3.4) are respectively dual to those f_{-t} given in (2.5), we have Spanier-Whitehead dualities as $C(f'_{-t})=\Sigma^1DC(f_{-t})$ for $f_{-t}=\alpha_{-4r}, \mu_{-4r-1}, a_{-4r-2}, m_{-4r-3}$ or b_{-4r-2} $(r\geqq 0)$.

Dually to (2.2) and (2.6) we obtain the composite maps f_tj_g and $f_{-t}j_g$ $(t\geqq 0)$ as follows:

$$(3.5)\quad \begin{aligned} &\alpha_{4r}j\colon \Sigma^{8r-2}SZ/2m\to\Sigma^0\,, && \alpha_{4r}j_V\colon \Sigma^{8r-2}V_{2m}\to\Sigma^0\,,\\ &\mu'_{4r+1}j\colon \Sigma^{8r}SZ/2m\to\Sigma^0\,, && \mu'_{4r+1}j_V\colon \Sigma^{8r}V_{2m}\to\Sigma^0\,, \end{aligned}$$

$$a_{4r+2}j\colon \Sigma^{8r+2}SZ/2m \to C(\bar\eta)\,, \qquad a_{4r+2}j_V\colon \Sigma^{8r+2}V_{2m} \to C(\bar\eta)\,,$$
$$a'_{4r+2}(j_\wedge 1)\colon \Sigma^{8r-1}SZ/2m_\wedge C(\tilde\eta) \to \Sigma^0\,, \quad a'_{4r+2}j'_V\colon \Sigma^{8r}V'_{2m} \to \Sigma^0\,,$$
$$m'_{4r+3}(j_\wedge 1)\colon \Sigma^{8r+1}SZ/2m_\wedge C(\tilde\eta) \to \Sigma^0\,, \quad m'_{4r+3}j'_V\colon \Sigma^{8r+2}V'_{2m} \to \Sigma^0\,.$$

$$\begin{aligned}
&\alpha'_{-4r}j\colon \Sigma^{-1}SZ/2m \to C(\tilde\alpha_{4r})\,, && \alpha'_{-4r}j_V\colon \Sigma^{-1}V_{2m} \to C(\tilde\alpha_{4r})\,,\\
&\tilde\mu'_{-4r-1}j\colon \Sigma^{-1}SZ/2m \to C(\tilde\mu_{4r+1})\,, && \mu'_{-4r-1}j_V\colon \Sigma^{-1}V_{2m} \to C(\tilde\mu_{4r+1})\,,\\
(3.6)\quad &a'_{-4r-2}j\colon \Sigma^{-1}SZ/2m \to C(\tilde a_{4r+2})\,, && a'_{-4r-2}j_V\colon \Sigma^{-1}V_{2m} \to C(\tilde a_{4r+2})\,,\\
&m'_{-4r-3}j\colon \Sigma^{-1}SZ/2m \to C(\tilde m_{4r+3})\,, && m'_{-4r-3}j_V\colon \Sigma^{-1}V_{2m} \to C(\tilde m_{4r+3})\,,\\
&b'_{-4r-2}(j_\wedge 1)\colon \Sigma^{-1}SZ/2m_\wedge C(\bar\eta) \to C(\tilde a_{4r+2}) \text{ and} &&\\
&b'_{-4r-2}j_U\colon \Sigma^{-1}U_{2m} \to C(\tilde a_{4r+2})\,. &&
\end{aligned}$$

Then there hold Spanier-Whitehead dualities as

(3.7) i) $C(f'_t j)=\Sigma^{2t}DC(if_t)$ and $C(f'_t j_V)=\Sigma^{2t}DC(i'_V f_t)$ for $f_t=\alpha_{4r}$, μ_{4r+1} or $a'_{4r+2}(r\geqq 0)$ where $\alpha'_{4r}=\alpha_{4r}$ and $a''_{4r+2}=a_{4r+2}$.

ii) $C(f'_t j'_V)=\Sigma^{2t}DC(i_V f_t)$ and $C(f'_t(j_\wedge 1))=\Sigma^{2t}DC((i_\wedge 1)f_t)$ for $f_t=a_{4r+2}$ or $m_{4r+3}(r\geqq 0)$.

iii) $C(f'_{-t}j)=\Sigma^1 DC(if_{-t})$ and $C(f'_{-t}j_V)=\Sigma^1 DC(i'_V f_{-t})$ for $f_{-t}=\alpha_{-4r}$, μ_{-4r-1}, a_{-4r-2} or $m_{-4r-3}(r\geqq 0)$.

iv) $C(b'_{-4r-2}(j_\wedge 1))=\Sigma^1 DC((i_\wedge 1)b_{-4r-2})$ and $C(b'_{-4r-2}j_U)=\Sigma^1 DC(i'_U b_{-4r-2})$ $(r\geqq 0)$.

By making use of Lemma 3.5 we obtain the following result, which is a dual of Propositions 2.3 and 2.5.

Proposition 3.7. i) *For each composite map $f_t j_g (t\geqq 0)$ given in* (3.5) *the cofiber $\Sigma^{-2t}C(f_t j_g)$ satisfies the property* $(\mathrm{II}_{2m})^*_t$.

ii) *For each composite map $f_{-t}j_g(t\geqq 0)$ given in* (3.6) *the cofiber $\Sigma^{-1}C(f_{-t}j_g)$ satisfies the property* $(\mathrm{II}_{2m})^*_{-t}$.

iii) $C(\alpha_{4r}j)$, $\Sigma^4 C(a_{4r+2}j)$, $C(a'_{4r+2}(j_\wedge 1))$, $\Sigma^{-1}C(\alpha'_{-4r}j)$, $\Sigma^{-1}C(a'_{-4r-2}j)$ *and* $\Sigma^3 C(b'_{-4r-2}(j_\wedge 1))$ *are quasi KO_*-equivalent to $\Sigma^0\vee\Sigma^{-1}SZ/2m$, and* $C(\alpha_{4r}j_V)$, $\Sigma^4 C(a_{4r+2}j_V)$, $C(a'_{4r+2}j'_V)$, $\Sigma^{-1}C(\alpha'_{-4r}j_V)$, $\Sigma^{-1}C(a'_{-4r-2}j_V)$ *and* $\Sigma^3 C(b'_{-4r-2}j_U)$ *are quasi KO_*-equivalent to* $\Sigma^0\vee\Sigma^{-1}V_{2m}$.

iv) $C(\mu'_{4r+1}j)$, $\Sigma^4 C(\mu'_{4r+1}j_V)$, $\Sigma^4 C(m'_{4r+3}j'_V)$, $C(m'_{4r+3}(j_\wedge 1))$, $\Sigma^1 C(\mu'_{-4r-1}j)$, $\Sigma^5 C(\mu'_{-4r-1}j_V)$, $\Sigma^1 C(m'_{-4r-3}j)$ *and* $\Sigma^5 C(m'_{-4r-3}j_V)$ *are all quasi KO_*-equivalent to* M'_{2m}.

4. K_*-local types of the real projective spaces

4.1. Let RP^n be the real projective n-space and X_n denote the suspension spectrum $\Sigma^{-n}SP^2S^n$ whose n-th term is the symmetric square SP^2S^n of the n-sphere as in [16, §2]. The suspension spectra X_n and RP^n are related by the following commutative diagram

$$
(4.1)\qquad
\begin{array}{ccccccc}
 & & & & \Sigma^n & = & \Sigma^n \\
 & & & & \downarrow & & \downarrow \\
RP^{n-1} & \to & \Sigma^0 & \to & X_n & \to & \Sigma^1 RP^{n-1} \\
\downarrow & & \| & & \downarrow & & \downarrow \\
RP^n & \to & \Sigma^0 & \to & X_{n+1} & \to & \Sigma^1 RP^n \\
 & & & & \downarrow & & \downarrow \\
 & & & & \Sigma^{n+1} & = & \Sigma^{n+1}
\end{array}
$$

involving four cofiber sequences [10]. Their KU-homologies and KU-cohomologies are well known ([1, Theorem 7.3] and [14, Theorem 3.3]):

(4.2) i) $KU_0X_{n+1}\cong Z$ or $Z\oplus Z$ and $KU_{-1}RP^n\cong Z/2^t$ or $Z\oplus Z/2^t$ according as $n=2t$ or $2t+1$, and $KU_1X_{n+1}=0=KU_0RP^n$.

ii) $KU^0X_{n+1}\cong Z$ or $Z\oplus Z$ and $KU^{-1}RP^n\cong 0$ or Z according as n is even or odd, and $KU^1X_{n+1}=0$ and $KU^0RP^n\cong Z/2^t$ when $n=2t$ or $2t+1$.

We here investigate the behavior of the Adams operation ψ_C^k for X_{n+1} and RP^n.

Lemma 4.1. i) *$X_{n+1}=\Sigma^{-n-1}SP^2S^{n+1}$ satisfies the property* (I) *or* $(\mathrm{II})_{t+1}$ *according as $n=2t$ or $2t+1$.*

ii) *Σ^1RP^n satisfies the property* (I_{2^t}) *or* $(\mathrm{II}_{2^t})_{t+1}$ *according as $n=2t$ or $2t+1$. In addition, Σ^1RP^∞ satisfies the property* (I_{2^∞}).

Proof. It is sufficient to show that in both KU_0X_{n+1} and $KU_0\Sigma^1RP^n$, $\psi_C^k=1$ or $A_{k,t+1}$ according as $n=2t$ or $2t+1$. The $n=0$ case is evident because $X_1=\Sigma^0$ and $RP^0=\{pt\}$. Assume that $\psi_C^k=1$ in $KU_0X_{2t-1}\cong Z$ and $KU_{-1}RP^{2t-2}\cong Z/2^{t-1} (t\geqq 1)$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & KU_0\Sigma^0 & \to & KU_0X_{2t-1} & \to & KU_{-1}RP^{2t-2} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & KU_0\Sigma^0 & \to & KU_0X_{2t} & \to & KU_{-1}RP^{2t-1} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & KU_0\Sigma^{2t} & = & KU_0\Sigma^{2t} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

with exact rows and columns. Then $KU_0X_{2t}\cong KU_0\Sigma^{2t}\oplus KU_0X_{2t-1}\cong Z\oplus Z$ and $KU_{-1}RP^{2t-1}\cong KU_0\Sigma^{2t}\oplus KU_{-1}RP^{2t-2}\cong Z\oplus Z/2^{t-1}$, in both of which ψ_C^k is expressed by a matrix $\begin{pmatrix}1/k^t & 0\\ c_{k,t} & 1\end{pmatrix}$ for some rational number $c_{k,t}$. We here use another commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & KU_0\Sigma^{2t} & = & KU_0\Sigma^{2t} & & \\
 & & & & h_{2t*}\downarrow & & \downarrow & & \\
0 & \to & KU_0\Sigma^0 & \to & KU_0X_{2t} & \to & KU_{-1}RP^{2t-1} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & KU_0\Sigma^0 & \to & KU_0X_{2t+1} & \to & KU_{-1}RP^{2t} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Since the right vertical sequence is expressed into the form of $0\to Z\to Z\oplus Z/2^{t-1}\to Z/2^t\to 0$, we may regard that the induced homomorphism h_{2t*}: $KU_0\Sigma^{2t}\to KU_0X_{2t}$ is given by $h_{2t*}(1)=(2, 1)$ where $KU_0X_{2t}\cong KU_0\Sigma^{2t}\oplus KU_0X_{2t-1}$. Since the Adams operation ψ_C^k commutes with h_{2t*}, it is easily computed that $c_{k,t}=1-k^t/2k^t$. Thus $\psi_C^k=A_{k,t}$ in both $KU_0X_{2t}\cong Z\oplus Z$ and $KU_{-1}RP^{2t-1}\cong Z\oplus Z/2^{t-1}$. Further it is immediate that $\psi_C^k=1$ in both $KU_0X_{2t+1}\cong Z$ and $KU_{-1}RP^{2t}\cong Z/2^t$.

As a dual of Lemma 4.1 we have

Corollary 4.2. i) *The Spanier-Whitehead dual DX_{n+1} satisfies the property* (I) *or* $(\mathrm{II})_{-t-1}$ *according as $n=2t$ or $2t+1$. Thus $\psi_C^k=1$ or $A_{k,-t-1}$ in $KU^0X_{n+1}\cong Z$ or $Z\oplus Z$ according as $n=2t$ or $2t+1$.*

ii) *The Spanier-Whitehead dual DRP^{2t} satisfies the property (I_{2^t}) and $\Sigma^{-1}DRP^{2t+1}$ does the property $(\mathrm{II}_{2^t})^*_{t+1}$. Thus $\psi_C^k=1$ in $KU^0RP^{2t}\cong KU^0RP^{2t+1}\cong Z/2^t$ and $\psi_C^k=k^{t+1}$ in $KU^{-1}RP^{2t+1}\cong Z$.*

4.2. In [16, Theorem 2.7] we have determined the quasi KO_*-types of the symmetric square $X_n=\Sigma^{-n}SP^2S^n$ of the n-sphere and the real projective n-space RP^n.

Theorem 4.3. i) *X_{n+1} is quasi KO_*-equivalent to the following elementary spectrum: $\Sigma^0, C(\eta), \Sigma^4, \Sigma^4\vee\Sigma^4, \Sigma^4, C(\eta), \Sigma^0, \Sigma^0\vee\Sigma^0$ according as $n\equiv 0, 1, \cdots, 7$ mod 8.*

ii) *Σ^1RP^n is quasi KO_*-equivalent to the following elementary spectrum: $SZ/2^{4r}, M_{2^{4r}}, V_{2^{4r+1}}, \Sigma^4\vee V_{2^{4r+1}}, V_{2^{4r+2}}, M_{2^{4r+2}}, SZ/2^{4r+3}, \Sigma^0\vee SZ/2^{4r+3}$ according as $n=8r, 8r+1, \cdots, 8r+7$.*

By virtue of Lemma 4.1 we can easily observe the behavior of the Adams operation ψ_R^k for X_{n+1}. In fact, (1.1) and (1.8) i) and iii) assert that the Adams operation ψ_R^k behaves in KO_iX_{n+1} $(0\leqq i\leqq 7)$ for each $k\neq 0$ as follows:

(4.3) i) When n is even, $\psi_R^k=k^2$ or 1 according as $i=4$ or otherwise.

ii) When $n=4s+1$, $\psi_R^k=1$, $1/k^{2s}$, k^2 or $1/k^{2s-2}$ according as $i=0, 2, 4$ or 6.

iii) When $n=4s+3$, $\psi_R^k=A_{k,2s+2}$, $k^2A_{k,2s+2}$ or 1 according as $i=0$, 4 or otherwise.

By the aid of (4.3) we next observe the behavior of the Adams operation ψ_R^k for RP^n.

Lemma 4.4. *The Adams operation ψ_R^k acts normally in $KO_i\Sigma^1 RP^n$ $(0\leqq i\leqq 7)$ for each $k\neq 0$ as follows:*

i) *When n is even or infinite, $\psi_R^k=k^2$ or 1 according as $i=4$ or otherwise.*

ii) *When $n=4s+1$, $\psi_R^k=1/k^{2s}$, k^2, $1/k^{2s-2}$ or 1 according as $i=2$, 4, 6 or otherwise.*

iii) *When $n=4s+3$, $\psi_R^k=A_{k,2s+2}$, $k^2A_{k,2s+2}$ or 1 according as $i=0$, 4 or otherwise.*

Proof. i) In the $n=\infty$ case our result follows from Lemma 4.1 and (1.6). Use the cofiber sequence $\Sigma^0\to X_{2t+1}\to\Sigma^1RP^{2t}\to\Sigma^1$ in the $n=2t$ case. Evidently (4.3) i) implies our result except $\psi_R^k=1$ in $KO_1RP^{8r+6}\cong KO_1RP^{8r+8}\cong Z/2\oplus Z/2$. As is observed in ii) and iii) below, $\psi_R^k=1/k^{4r+2}$ in $KO_1RP^{8r+5}\cong Z\oplus Z/2$ and $\psi_R^k=1$ in $KO_1RP^{8r+7}\cong Z/2\oplus Z/2\oplus Z/2$. By means of these results we can easily show the rest of our result.

ii) By Lemma 4.1 and Theorem 4.3 ii) we note that Σ^1RP^{4s+1} satisfies the property $(\mathrm{II}_{2^{2s}})_{2s+1}$ and it is quasi KO_*-equivalent to $M_{2^{2s}}$. Our result is immediate from (2.1).

iii) Use the cofiber sequence $\Sigma^0\to X_{4s+4}\to\Sigma^1RP^{4s+3}\to\Sigma^1$. Then (4.3) iii) implies immediately our result except $\psi_R^k=1$ in $KO_1 RP^{8s+7}\cong Z/2\oplus Z/2\oplus Z/2$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & KO_2X_n & \to & KO_1RP^{n-1} & \to & KO_1\Sigma^0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & KO_2X_{n+1} & \to & KO_1RP^n & \to & KO_1\Sigma^0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & KO_1\Sigma^n & = & KO_1\Sigma^n & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

with $n=8r+7$. Since $\psi_R^k=1$ in $KO_1RP^{n-1}\cong KO_2X_{n+1}\cong Z/2\oplus Z/2$, a routine computation shows that $\psi_R^k=1$ in $KO_1RP^n\cong Z/2\oplus Z/2\oplus Z/2$ as in the proof of Lemma 2.2.

Under the assumption that CW-spectra X and Y are finite, X is quasi KO_*-equivalent to Y if and only if the Spanier-Whitehead dual DY is quasi KO_*-equivalent to DX (see [15, Corollary I.1.6]). Therefore Theorem 4.3 ii) implies that

(4.4) the Spanier-Whitehead dual DRP^n is quasi KO_*-equivalent to the following elementary spectrum: $SZ/2^{4r}$, $\Sigma^{-1}M'_{2^{4r}}$, $\Sigma^4 V_{2^{4r+1}}$, $\Sigma^5 \vee \Sigma^4 V_{2^{4r+1}}$, $\Sigma^4 V_{2^{4r+2}}$, $\Sigma^{-1}M'_{2^{4r+2}}$, $SZ/2^{4r+3}$, $\Sigma^1 \vee SZ/2^{4r+3}$ according as $n=8r, 8r+1, \cdots, 8r+7$ (cf. [9, Theorem 1]).

As a dual of Lemma 4.4 we can easily show

Lemma 4.5. *The Adams operation ψ_R^k acts normally in $KO_i DRP^n \cong KO^{-i}RP^n$ $(0 \leqq i \leqq 7)$ for each $k \neq 0$ as follows*:

i) *When n is even, $\psi_R^k = k^2$ or 1 according as $i=4$ or otherwise.*

ii) *When $n=4s+1$, $\psi_R^k = k^{2s+2}, k^2, k^{2s+4}$ or 1 according as $i=3, 4, 7$ or otherwise.*

ii) *When $n=4s+3$, $\psi_R^k = k^{2s+2}, k^2, k^{2s+4}$ or 1 according as $i=1, 4, 5$ or otherwise.*

For the Moore spectrum $SZ/2^t$ of type $Z/2^t$ the bottom cell inclusion i: $\Sigma^0 \to SZ/2^t$ and the top cell projection j: $SZ/2^t \to \Sigma^1$ are here written as i_t and j_t with emphasis. Similarly the maps i_V: $C(\bar{\eta}) \to V_{2^t}$, j_V: $V_{2^t} \to \Sigma^1$, i'_V: $\Sigma^2 \to V'_{2^t}$ and j'_V: $V'_{2^t} \to C(\tilde{\eta})$ are written as $i_{V,t}, j_{V,t}, i'_{V,t}$ and $j'_{V,t}$. By virtue of Lemmas 4.1 and 4.4 we may now apply Theorems 1.2, 1.7 and 2.6 with (2.8) to determine the K_*-local types of X_{n+1} and RP^n.

Theorem 4.6. i) *The symmetric square $X_{n+1} = \Sigma^{-n-1}SP^2S^{n+1}$ of the $n+1$-sphere has the same K_*-local type as the following elementary spectrum*: Σ^0, $C(\mu_{4r+1})$, $C(\bar{\eta})$, $C(a_{4r+2})$, $C(\tilde{\eta})$, $C(m_{4r+3})$, Σ^0, $C(\alpha_{4r+4})$ *according as $n=8r, 8r+1, \cdots, 8r+7$.*

ii) *The real projective n-space $\Sigma^1 RP^n$ has the same K_*-local type as the following elementary spectrum*: $SZ/2^{4r}$, $C(i_{4r}\mu_{4r+1})$, $V_{2^{4r+1}}$, $C(i_{V,4r+1}a_{4r+2})$, $V_{2^{4r+2}}$, $C(i_{V,4r+2}\, m_{4r+3})$, $SZ/2^{4r+3}$, $C(i_{4r+3}\,\alpha_{4r+4})$ *according as $n=8r, 8r+1, \cdots, 8r+7$. In addition, $\Sigma^1 RP^\infty$ has the same K_*-local type as $SZ/2^\infty$.*

In order to determine the K_*-local type of the Spanier-Whitehead dual DRP^n the following result is useful (cf. [15, Corollary I.1.6]).

Lemma 4.7. *Assume that CW-spectra X and Y are finite. Then X is quasi S_{K*}-equivalent to Y if and only if the Spanier-Whitehead dual DY is quasi S_{K*}-equivalent to DX.*

Proof. It is sufficient to show the "only if" part. If X is quasi S_K*-equivalent to Y, then we get a K_*-equivalence f: $Y \to S_K \wedge X$. Choose an adjoint map Df: $DX \to DY \wedge S_K$ such that $(1 \wedge e_X)(f \wedge 1) = (e_Y \wedge 1)(1 \wedge Df)$: $Y \wedge DX \to S_K$ where e_W: $W \wedge DW \to \Sigma^0$ denotes the evaluation map for $W=X$ or Y. Consider the diagram

$$\begin{array}{ccccc} K_iDX & \xrightarrow{Df_*} & K_iDY_\wedge S_K & \xleftarrow{\simeq} & K_iDY \\ \downarrow\simeq & & & & \downarrow\simeq \\ K^{-i}X & \xleftarrow{\simeq} & K^{-i}S_{K\wedge}X & \xrightarrow[f^*]{} & K^{-i}Y \end{array}$$

where vertical arrows are the duality isomorphisms. As is easily checked, the above diagram is commutative. Therefore the adjoint map Df: $DX \to DY_\wedge S_K$ becomes a K_*-equivalence because f: $Y \to S_{K\wedge}X$ is a K^*-equivalence, too. Thus DY is quasi S_{K^*}-equivalent to DX.

Theorem 4.6 combined with Lemma 4.7, (1.4) and (3.7) implies

Theorem 4.8. *The Spanier-Whitehead dual DRP^n of the real projective n-space has the same K_*-local type as the following elementary spectrum*: $SZ/2^{4r}$, $\Sigma^{-8r-1}C(\mu'_{4r+1}j_{4r})$, $U_{2^{4r+1}}$, $\Sigma^{-8r-3}C(a'_{4r+2}j'_{V,4r+1})$, $U_{2^{4r+2}}$, $\Sigma^{-8r-5}C(m'_{4r+3}j'_{V,4r+2})$, $SZ/2^{4r+3}$, $\Sigma^{-8r-7}C(\alpha_{4r+4}j_{4r+3})$ *according as* $n=8r, 8r+1, \cdots, 8r+7$.

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