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THE K_* -LOCALIZATIONS OF WOOD AND ANDERSON SPECTRA AND THE REAL PROJECTIVE SPACES

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0. Introduction

Let E be an associative ring spectrum with unit. For any CW -spectra X and Y we say that X is *quasi E_* -equivalent* to Y (see [15] or [16]) if there exists a map $f: Y \rightarrow E \wedge X$ such that the composite map $(\mu \wedge 1)(1 \wedge f): E \wedge Y \rightarrow E \wedge X$ is an equivalence where $\mu: E \wedge E \rightarrow E$ denotes the multiplication of E . Let KO , KU and KT be the real, the complex and the self-conjugate K -spectrum respectively (see [3] or [7]). It is known that there is no difference among the KO_* -, KU_* - and KT_* -localizations ([11], [5] or [13]). So we denote by S_K the K_* -localization of the sphere spectrum $S = \Sigma^0$. These spectra KO , KU , KT and S_K are all associative ring spectra with unit.

In [15] we studied the quasi K_* -equivalences, especially the quasi KO_* -equivalence, and in [16] and [17] we determined the quasi KO_* -types of the real projective spaces RP^n and the stunted real projective spaces RP^n/RP^m . In this note we will be interested in the quasi S_{K_*} -equivalence in advance of the quasi KO_* -equivalence. According to the smashing theorem [6, Corollary 4.7] (or [13]), for any CW -spectrum X the smash product $S_{K \wedge} X$ is actually the K_* -localization of X . Hence we notice that two CW -spectra X and Y have the same K_* -local type if and only if X is quasi S_{K_*} -equivalent to Y .

For any map $f: X \rightarrow Y$ its cofiber is usually denoted by $C(f)$. Let $\eta: \Sigma^1 \rightarrow \Sigma^0$ be the stable Hopf map of order 2. The KO -homologies of the cofibers $C(\eta)$ and $C(\eta^2)$ are well known as follows: $KO_i C(\eta) \cong \pi_i KU \cong Z$ or 0 according as i is even or odd, and $KO_i C(\eta^2) \cong \pi_i KT \cong Z, Z/2, 0$ or Z according as $i \equiv 0, 1, 2$ or $3 \pmod{4}$. A CW -spectrum X is said to be a *Wood spectrum* if it is quasi KO_* -equivalent to the cofiber $C(\eta)$, and an *Anderson spectrum* if it is quasi KO_* -equivalent to the cofiber $C(\eta^2)$ (see [12], [15] or [18]).

Let $\bar{\eta}: \Sigma^1 SZ/2 \rightarrow \Sigma^0$ and $\bar{\eta}: \Sigma^2 \rightarrow SZ/2$ be an extension and a coextension of η with $\bar{\eta}i = \eta$ and $j\bar{\eta} = \eta$, where $SZ/2$ denotes the Moore spectrum of type $Z/2$ constructed by the cofiber sequence $\Sigma^0 \xrightarrow{2} \Sigma^0 \xrightarrow{i} SZ/2 \xrightarrow{j} \Sigma^1$. Choose two maps $\bar{h}: \Sigma^3 SZ/2 \rightarrow C(\bar{\eta})$ and $\bar{k}: \Sigma^5 SZ/2 \rightarrow C(\bar{\eta})$ with $j\bar{h} = \bar{\eta}j$ and $j\bar{k} = \bar{\eta}j$ where $\bar{j}: C(\bar{\eta}) \rightarrow \Sigma^2 SZ/2$ denotes the bottom cell collapsing. Using a fixed Adams' K_* -equiva-

lence $A_2: \Sigma^8SZ/2 \rightarrow SZ/2$ in [2] we can introduce four kinds of maps $f_t (t \geq 1)$ as follows:

$$\begin{aligned} \alpha_{4r} &= jA_2^r i: \Sigma^{8r-1} \rightarrow \Sigma^0, & \mu_{4r+1} &= \bar{\eta}A_2^r i: \Sigma^{8r+1} \rightarrow \Sigma^0, \\ a_{4r+2} &= \bar{h}A_2^r i: \Sigma^{8r+3} \rightarrow C(\bar{\eta}) \quad \text{and} \quad m_{4r+3} &= \bar{k}A_2^r i: \Sigma^{8r+5} \rightarrow C(\bar{\eta}). \end{aligned}$$

Setting $\bar{\alpha}_{4r} = jA_2^r i, \bar{\mu}_{4r+1} = \bar{\eta}A_2^r i, \bar{a}_{4r+2} = \bar{h}A_2^r i$ and $\bar{m}_{4r+3} = \bar{k}A_2^r i$, we can also introduce four kinds of maps $f_{-t} (t \geq 1)$ as follows:

$$\begin{aligned} \alpha_{-4r} &: \Sigma^{-8r-1} C(\bar{\alpha}_{4r}) \rightarrow \Sigma^0, & \mu_{-4r-1} &: \Sigma^{-8r-3} C(\bar{\mu}_{4r+1}) \rightarrow \Sigma^0, \\ a_{-4r-2} &: \Sigma^{-8r-5} C(\bar{a}_{4r+2}) \rightarrow \Sigma^0 \quad \text{and} \quad m_{-4r-3} &: \Sigma^{-8r-7} C(\bar{m}_{4r+3}) \rightarrow \Sigma^0 \end{aligned}$$

of which each cofiber $C(f_{-t})$ coincides with $\Sigma^{-2t}C(f_t)$.

In §1 and §3 we will determine the K_* -local types of Wood and Anderson spectra as our results (Theorems 1.7 iii) and 3.4 ii):

Theorem 1. *Let X be a Wood spectrum whose rationalization $X \wedge SQ$ is $(\Sigma^0 \vee \Sigma^{2t}) \wedge SQ$ for some odd integer $t \geq 1$. Then X has the same K_* -local type as the following cofiber $C(\mu_t)$ or $C(m_t)$ according as $t = 4r + 1$ or $4r + 3$.*

Theorem 2. *Let X be an Anderson spectrum whose rationalization $X \wedge SQ$ is $(\Sigma^0 \vee \Sigma^{2t+1}) \wedge SQ$ for some odd integer t . Assume that $t \neq -1$. Then X has the same K_* -local type as the following cofiber $C(\eta\mu_t)$ or $C(\eta m_t)$ according as $t = \pm(4r + 1)$ or $\pm(4r + 3)$.*

For the Moore spectrum $SZ/2^t$ of type $Z/2^t$ we denote by $i_t: \Sigma^0 \rightarrow SZ/2^t$ and $j_t: SZ/2^t \rightarrow \Sigma^1$ with the subscript “ t ” the bottom cell inclusion and the top cell projection. Abbreviating the cofiber $C(i_{t-1} \bar{\eta})$ to be V_{2^t} we have a cofiber sequence $\Sigma^0 \xrightarrow{2^{t-1}i} C(\bar{\eta}) \xrightarrow{i_{V,t}} V_{2^t} \xrightarrow{j_{V,t}} \Sigma^1$. In §4 the K_* -local types of the real projective spaces $RP^n (2 \leq n \leq \infty)$ will be determined as our main result (Theorem 4.6 ii):

Theorem 3. *The real projective space $\Sigma^1 RP^n$ has the same K_* -local type as the following elementary spectrum: $SZ/2^{4r}, C(i_{4r} \mu_{4r+1}), V_{2^{4r+1}}, C(i_{V,4r+1} a_{4r+2}), V_{2^{4r+2}}, C(i_{V,4r+2} m_{4r+3}), SZ/2^{4r+3}, C(i_{4r+3} \alpha_{4r+4})$ according as $n = 8r, 8r + 1, \dots, 8r + 7$. In addition, $\Sigma^1 RP^\infty$ has the same K_* -local type as $SZ/2^\infty$ (cf. [8, Theorem 4.2] or [13, Theorem 9.1]).*

In order to prove the above theorems we will need the following powerful tool due to Bousfield [7, Theorems 7.11 and 7.12].

Theorem 4. *Let Y be a certain CW-spectrum satisfying either of the following two conditions: i) $KU_* Y$ is either free or divisible and $\text{Hom}(\pi_i Y \otimes Q, \pi_{i+1} Y \otimes Q) = 0$ for each i ; ii) $KU_1 Y = 0$ (or $KU_0 Y = 0$). Assume that a CW-spectrum X is quasi KO_* -equivalent to Y , and the real Adams operations ψ_R^k in*

KO_*X and KO_*Y behave as the same action for each $k \neq 0$ when KO_*X is identified with KO_*Y as a KO_* -module. Then X is quasi S_{K_*} -equivalent to Y , thus X has the same K_* -local type as Y (cf. [7, 9.8]).

In §1 we will mainly deal with CW -spectra X satisfying the following property:

- (I) $KU_0X \cong Z$ with $\psi_c^k = 1$ and $KU_1X = 0$;
- (I_{2m}) $KU_0X \cong Z/2m$ with $\psi_c^k = 1$ and $KU_1X = 0$; or
- (II)_t $KU_0X \cong Z \oplus Z$ with $\psi_c^k = A_{k,t}$ and $KU_1X = 0$.

Here $A_{k,t} = \begin{pmatrix} 1/k^t & 0 \\ 1 - k^t/2k^t & 1 \end{pmatrix}$, which operates on $(Z \oplus Z) \otimes Z[1/k]$ as left action.

After investigating the behavior of the real Adams operation ψ_R^k for CW -spectra X with the above property we will determine their K_* -local types (Theorems 1.2 and 1.7). In §2 and §3 we will next deal with CW -spectra X satisfying the following property:

- (II_{2m})_t $KU_0X \cong Z \oplus Z/2m$ with $\psi_c^k = A_{k,t}$ and $KU_1X = 0$; or
- (III)_t $KU_0X \cong Z$ with $\psi_c^k = 1$ and $KU_1X \cong Z$ with $\psi_c^k = 1/k^t$.

As in §1 we will also determine the K_* -local types of such CW -spectra X (Theorems 2.6 and 3.4). In §4 we will finally deal with the symmetric squares SP^2S^n of the n -spheres and the real projective n -spaces RP^n . After investigating the behavior of the Adams operations ψ_c^k and ψ_R^k for the spaces SP^2S^n and RP^n , we will determine their K_* -local types (Theorem 4.6) by applying Theorems 1.2, 1.7 and 2.6.

In the forthcoming paper [19] we will completely determine the K_* -local types of the stunted real projective spaces RP^n/RP^m ($0 \leq m < n \leq \infty$) along our line.

1. K_* -local types of Wood spectra

1.1. Let X be a CW -spectrum with $KU_0X \cong Z$ and $KU_1X = 0$. For such a CW -spectrum X we may assume that the stable complex Adams operation ψ_c^k acts identically on $KU_0X \otimes Z[1/k]$ for each $k \neq 0$. Thus X satisfies the following property:

- (I) $KU_0X \cong Z$ in which $\psi_c^k = 1$ and $KU_1X = 0$.

Whenever a CW -spectrum X satisfies the property (I), it is quasi KO_* -equivalent to either of Σ^0 and Σ^4 (see [7, Theorem 3.2] or [15, Theorem I.2.4]). In this case it is easily seen that the stable real Adams operation ψ_R^k acts always on $KO_iX \otimes Z[1/k]$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:

$$(1.1) \quad \psi_R^k = k^2 \text{ or } 1 \text{ according as } i=4 \text{ or otherwise.}$$

The Moore spectrum $SZ/2m$ of type $Z/2m$ is constructed as the cofiber of multiplication by $2m$ on Σ^0 . Thus we have a cofiber sequence $\Sigma^0 \xrightarrow{2m} \Sigma^0 \xrightarrow{i} SZ/2m \xrightarrow{j} \Sigma^1$. Let $\bar{\eta}_{2m}: \Sigma^1 SZ/2m \rightarrow \Sigma^0$ and $\tilde{\eta}_{2m}: \Sigma^2 \rightarrow SZ/2m$ be an extension and a coextension of η satisfying $\bar{\eta}_{2m} i = \eta$ and $j \tilde{\eta}_{2m} = \eta$ respectively, where $\eta: \Sigma^1 \rightarrow \Sigma^0$ denotes the stable Hopf map of order 2. The maps $\bar{\eta}_2$ and $\tilde{\eta}_2$ are often abbreviated to be $\bar{\eta}$ and $\tilde{\eta}$. Consider the two cofiber sequences

$$\Sigma^1 SZ/2 \xrightarrow{\bar{\eta}} \Sigma^0 \xrightarrow{i} C(\bar{\eta}) \xrightarrow{\bar{j}} \Sigma^2 SZ/2 \quad \text{and} \quad \Sigma^2 \xrightarrow{\tilde{\eta}} SZ/2 \xrightarrow{\tilde{i}} C(\tilde{\eta}) \xrightarrow{\tilde{j}} \Sigma^3$$

in which the cofibers $C(\bar{\eta})$ and $C(\tilde{\eta})$ are denoted by P'_2 and P_2 respectively in [15, I.4.1]. Between these cofibers there holds a Spanier-Whitehead duality as $C(\tilde{\eta}) = \Sigma^3 DC(\bar{\eta})$. By observing [15, Propositions I.4.1 and I.4.2] we verify that

(1.2) both $C(\bar{\eta})$ and $\Sigma^{-3}C(\tilde{\eta})$ satisfy the property (I), and they are quasi KO_* -equivalent to Σ^4 .

Let X be a CW -spectrum with $KU_0 X \cong Z/2m$ and $KU_1 X = 0$. In this case we assume that the Adams operation ψ^k acts identically in $KU_0 X$ for each $k \neq 0$. Thus we here deal with a CW -spectrum X satisfying the following property:

(I_{2m}) $KU_0 X \cong Z/2m$ in which $\psi^k = 1$ and $KU_1 X = 0$.

Consider the cofibers $C(i\bar{\eta})$ and $C(\bar{\eta}j)$ of the composite maps $i\bar{\eta}: \Sigma^1 SZ/2 \rightarrow SZ/m$ and $\bar{\eta}j: \Sigma^1 SZ/m \rightarrow SZ/2$, which are denoted by V_{2m} and V'_{2m} respectively as in [15, I.4.4]). Between them we have a Spanier-Whitehead duality as $V'_{2m} = \Sigma^3 DV_{2m}$. Since there exist cofiber sequences

$$\Sigma^0 \xrightarrow{mi} C(\bar{\eta}) \xrightarrow{i_V} V_{2m} \xrightarrow{j_V} \Sigma^1 \quad \text{and} \quad \Sigma^2 \xrightarrow{i'_V} V'_{2m} \xrightarrow{j'_V} C(\tilde{\eta}) \xrightarrow{m\tilde{j}} \Sigma^3,$$

it follows from [15, Corollaries I.4.6 and I.5.4] that

(1.3) both V_{2m} and $\Sigma^{-2}V'_{2m}$ satisfy the property (I_{2m}), and $\Sigma^2V'_{2m}$ is quasi KO_* -equivalent to V_{2m} , whose KO -homology $KO_i V_{2m} \cong Z/m, 0, Z/2, Z/2, Z/4m, Z/2, Z/2, 0$ according as $i = 0, 1, \dots, 7$.

Notice that a CW -spectrum X is quasi KO_* -equivalent to one of the four elementary spectra $SZ/2m, \Sigma^4 SZ/2m, V_{2m}$ and $\Sigma^4 V_{2m}$ whenever it satisfies the property (I_{2m}) (see [15, Theorem II.2 or Theorem I.5.2]).

Lemma 1.1. *Let W and Y be CW -spectra satisfying the property (I), and $g: W \rightarrow Y$ be a map whose cofiber $C(g)$ satisfies the property (I_{2m}). Then the cofiber $C(g)$ is quasi KO_* -equivalent to $W \wedge SZ/2m$ or $W \wedge V_{2m}$ according as W is*

quasi KO_* -equivalent to Y or not. In the latter case the Adams operation ψ_R^k acts normally in $KO_i C(g) \cong KO_i W \wedge V_{2m}$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows: $\psi_R^k = k^2$ or 1 according as $i=4$ or otherwise.

Proof. The induced homomorphism $g_*: KO_i W \rightarrow KO_i Y$ is trivial in dimension $i=1, 2, 5$ or 6 because $g_*: KU_0 W \rightarrow KU_0 Y$ is multiplication by $2m$ on Z . Therefore it is immediate that $KO_i C(g) = 0$ if both W and Y are quasi KO_* -equivalent to Σ^0 , and $KO_2 C(g) \cong Z/2$ and $KO_1 C(g) = 0$ if W and Y are quasi KO_* -equivalent to Σ^0 and Σ^4 respectively. Thus $C(g)$ is quasi KO_* -equivalent to $SZ/2m$ in the first case, and it is quasi KO_* -equivalent to V_{2m} in the second case. In the other two cases we can similarly observe the quasi KO_* -type of $C(g)$. When $C(g)$ is quasi KO_* -equivalent to either of V_{2m} and $\Sigma^4 V_{2m}$, it is easily checked that $\psi_R^k = 1$ or k^2 in $KO_i C(g)$ for each $k \neq 0$ according as $i=0$ or 4 .

Since the maps $\bar{\eta}: \Sigma^1 SZ/2 \rightarrow \Sigma^0$ and $\tilde{\eta}: \Sigma^2 \rightarrow SZ/2$ have order 4 [4, (4.2)], we can choose maps

$$\bar{\eta}_{4m/2}: \Sigma^2 SZ/2 \rightarrow SZ/4m \quad \text{and} \quad \tilde{\eta}_{4m/2}: \Sigma^2 SZ/4m \rightarrow SZ/2$$

with $j\bar{\eta}_{4m/2} = \bar{\eta}$ and $\tilde{\eta}_{4m/2} i = \tilde{\eta}$. Denote by U_{2m} and U'_{2m} their cofibers $C(\bar{\eta}_{4m/2})$ and $C(\tilde{\eta}_{4m/2})$ respectively. Between them there holds a Spanier-Whitehead duality as $U'_{2m} = \Sigma^4 D U_{2m}$. Using the cofiber sequences

$$C(\bar{\eta}) \xrightarrow{m\bar{\lambda}} \Sigma^0 \xrightarrow{i_U} U_{2m} \xrightarrow{j_U} \Sigma^1 C(\bar{\eta}) \quad \text{and} \quad \Sigma^3 \xrightarrow{m\tilde{\lambda}} C(\tilde{\eta}) \xrightarrow{i'_U} U'_{2m} \xrightarrow{j'_U} \Sigma^4$$

with $\bar{\lambda}^i = 4$ and $\tilde{\lambda}^j = 4$, we can easily show by the aid of Lemma 1.1 that

(1.4) both U_{2m} and $\Sigma^1 U'_{2m}$ satisfy the property (I_{2m}) , and they are quasi KO_* -equivalent to $\Sigma^4 V_{2m}$.

If a CW -spectrum X satisfies the property (I_{2m}) , then the smash product $X \wedge C(\bar{\eta})$ does the same property, but it is quasi KO_* -equivalent to $\Sigma^4 X$ because of (1.2). Whenever $X = SZ/2m, V_{2m}, \Sigma^{-2} V'_{2m}, U_{2m}$ or $\Sigma^{-3} U'_{2m}$, the Adams operation ψ_R^k behaves normally in $KO_i X$ and $KO_i X \wedge C(\bar{\eta})$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:

(1.5) $\psi_R^k = k^2$ or 1 according as $i=4$ or otherwise.

Because the $X = SZ/2m$ case is well known, and the other four cases are immediately shown by Lemma 1.1.

Let X be a CW -spectrum satisfying the property:

(I_{2^∞}) $KU_0 X \cong Z/2^\infty$ in which $\psi_C^k = 1$ and $KU_1 X = 0$.

Such a CW -spectrum X is quasi KO_* -equivalent to either of $SZ/2^\infty$ and

$\Sigma^4SZ/2^\infty$ (see [7, Theorem 3.3]). In this case it is easily seen that the Adams operation ψ_R^k behaves always in KO_iX ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:

$$(1.6) \quad \psi_R^k = k^2 \text{ or } 1 \text{ according as } i=4 \text{ or otherwise.}$$

Use (1.1), (1.5) or (1.6) to apply Theorem 4 for CW -spectra X with the property (I), (I_{2m}) or (I_{2^∞}) . Then we obtain

Theorem 1.2. i) *Let X be a CW -spectrum satisfying the property (I). Then it has the same K_* -local type as either of Σ^0 and $C(\bar{\eta})$.*

ii) *Let X be a CW -spectrum satisfying the property (I_{2m}) . Assume that the real Adams operation ψ_R^k behaves normally in KO_*X in the sense of (1.5). Then X has the same K_* -local type as one of the following spectra $SZ/2m$, $SZ/2m \wedge C(\bar{\eta})$, V_{2m} and U_{2m} .*

iii) *Let X be a CW -spectrum satisfying the property (I_{2^∞}) . Then X has the same K_* -local type as either of $SZ/2^\infty$ and $SZ/2^\infty \wedge C(\bar{\eta})$.*

1.2. Let X be a CW -spectrum with $KU_0X \cong Z \oplus Z$ and $KU_1X = 0$. For such a CW -spectrum X we may assume that $X \wedge SQ = (\Sigma^{2t} \vee \Sigma^0) \wedge SQ$ for some integer $t \geq 0$. In this case the complex Adams operation ψ_C^k on $KU_0X \otimes Z[1/k]$ is represented as the matrix $C^{-1}A_{k,t,0}C$ for each $k \neq 0$ where the matrix $C \in GL(2, Q)$ associated with the Chern character is independent of k and $A_{k,t,0} = \begin{pmatrix} 1/k^t & 0 \\ 0 & 1 \end{pmatrix}$.

When t is odd, we may regard that the conjugation ψ_C^{-1} on $KU_0X \cong Z \oplus Z$ is expressed by either of the matrices $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ (see [7, Proposition 3.7] or [15, I.2.1]). This observation implies easily that the Adams operation ψ_C^k in KU_0X for each $k \neq 0$ can be expressed by the following matrix

$$A_{k,t,0} = \begin{pmatrix} 1/k^t & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad A_{k,t} = \begin{pmatrix} 1/k^t & 0 \\ 1-k^t/2k^t & 1 \end{pmatrix}$$

according as $\psi_C^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$, whenever t is odd.

Let X be a CW -spectrum satisfying the following property:

$$(II)_{t,0} \quad KU_0X \cong Z \oplus Z \text{ in which } \psi_C^k = A_{k,t,0} \text{ and } KU_1X = 0.$$

Then X is quasi KO_* -equivalent to one of the wedge sums $\Sigma^0 \vee \Sigma^0$, $\Sigma^0 \vee \Sigma^4$ and $\Sigma^4 \vee \Sigma^4$ when t is even, and it is quasi KO_* -equivalent to one of the wedge sums $\Sigma^2 \vee \Sigma^0$, $\Sigma^2 \vee \Sigma^4$, $\Sigma^6 \vee \Sigma^0$ and $\Sigma^6 \vee \Sigma^4$ when t is odd (see [7, Theorem 3.2] or [15, Theorem I.2.4]). By an easy argument using the long exact sequence induced by the Bott cofiber sequence $\Sigma^1KO \rightarrow KO \rightarrow KU \rightarrow \Sigma^2KO$ we can show that in the case when t is even the Adams operation ψ_R^k behaves always in KO_iX

($0 \leq i \leq 7$) for each $k \neq 0$ as follows (cf. [2, Proposition 7.14]):

(1.7) i) If X is quasi KO_* -equivalent to either of $\Sigma^0 \vee \Sigma^0$ and $\Sigma^4 \vee \Sigma^4$, then $\psi_R^k = A_{k,t,0}, k^2 A_{k,t,0}$ or 1 according as $i=0, 4$ or otherwise.

ii) If X is quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^4$, then $\psi_R^k = A_{k,t,\varepsilon}, k^2 A_{k,t,\varepsilon'}$ or 1 according as $i=0, 4$ or otherwise where $(\varepsilon, \varepsilon')=(0, 0), (0, 1)$ or $(1, 0)$ and $A_{k,t,1} = A_{k,t}$.

Let X be a CW -spectrum satisfying the following property:

(II) _{t} $KU_0 X \cong Z \oplus Z$ in which $\psi_C^k = A_{k,t}$ and $KU_1 X = 0$.

Then X is quasi KO_* -equivalent to one of the wedge sums $\Sigma^0 \vee \Sigma^0, \Sigma^0 \vee \Sigma^4$ and $\Sigma^4 \vee \Sigma^4$ when t is even, but it is only quasi KO_* -equivalent to the cofiber $C(\eta)$ when t is odd (see [7, Theorem 3.2] or [15, Theorem I.2.4]). Thus X is always a Wood spectrum in the case when t is odd. By a similar argument to (1.7) we can also show that the Adams operation ψ_R^k behaves always in $KO_i X$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:

(1.8) i) If X is quasi KO_* -equivalent to either of $\Sigma^0 \vee \Sigma^0$ and $\Sigma^4 \vee \Sigma^4$, then $\psi_R^k = A_{k,t}, k^2 A_{k,t}$ or 1 according as $i=0, 4$ or otherwise.

ii) If X is quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^4$, then $\psi_R^k = A_{k,t,\varepsilon}, k^2 A_{k,t,2-\varepsilon}$ or 1 according as $i=0, 4$ or otherwise where $\varepsilon=0$ or 2 and $A_{k,t,j} = \begin{pmatrix} 1/k^t & 0 \\ 1-k^t/2^j k^t & 1 \end{pmatrix}$.

iii) If X is a Wood spectrum, then $\psi_R^k = 1, 1/k^{t-1}, k^2$ or $1/k^{t-3}$ according as $i=0, 2, 4$ or 6.

For any map $\alpha_{2s/j}: \Sigma^{4s-1} \rightarrow \Sigma^0$ whose e_C -invariant $e_C(\alpha_{2s/j}) \equiv 1/2^j \pmod{1}$, we notice that the Adams operation ψ_C^k in $KU_0 C(\alpha_{2s/j}) \cong Z \oplus Z$ is represented by the matrix $A_{k,2s,j}$ as given in (1.8) ii) for each $k \neq 0$ [2, Proposition 7.5]. Consider the maps

$$(1.9) \quad \alpha_{2s}: \Sigma^{4s-1} \rightarrow \Sigma^0, \quad i\alpha_{2s/2}: \Sigma^{4s-1} \rightarrow C(\bar{\eta}) \quad \text{and} \quad \alpha_{2s/2} \tilde{j}: \Sigma^{4s-4} C(\tilde{\eta}) \rightarrow \Sigma^0$$

where $s \geq 1$ and $\alpha_{2s/1}$ is abbreviated as α_{2s} .

Proposition 1.3. *The cofibers $C(\mu_{2s}), C(i\alpha_{2s/2})$ and $C(\alpha_{2s/2} \tilde{j})$ satisfy the property (II) _{$2s$} , and they are quasi KO_* -equivalent to the wedge sum $\Sigma^{4s} \vee \Sigma^0, \Sigma^{4s} \vee \Sigma^4$ and $\Sigma^{4s-4} \vee \Sigma^0$ respectively.*

Proof. The first half is easy, and the latter half is immediate because $\pi_{4s-1} KO = 0$.

1.3. Let us fix an Adams' K_* -equivalence $A_2: \Sigma^8 SZ/2 \rightarrow SZ/2$ [2]. We first consider the composite maps $A_2^r i: \Sigma^{8r} \rightarrow SZ/2$ and $jA_2^r: \Sigma^{8r-1} SZ/2 \rightarrow \Sigma^0$ ($r \geq 0$).

Lemma 1.4. *The cofibers $\Sigma^{-8r-1}C(A'_2 i)$ and $C(jA'_2)$ satisfy the property (I), and they are quasi KO_* -equivalent to Σ^0 .*

Proof. Since the Adams' K_* -equivalence $A_2: \Sigma^8SZ/2 \rightarrow SZ/2$ induces an isomorphism in KU -homology, we obtain that $KU_1C(A'_2 i) \cong KU_1\Sigma^{8r+1} \cong Z$, $KU_0C(jA'_2) \cong KU_0\Sigma^0 \cong Z$ and $KU_0C(A'_2 i) = 0 = KU_1C(jA'_2)$. Moreover it follows that $\Sigma^{-1}C(A'_2 i)$ and $C(jA'_2)$ are both quasi KO_* -equivalent to Σ^0 but not to Σ^4 because $KO_6C(A'_2 i) = 0 = KO_5C(jA'_2)$.

Lemma 1.5. *Let X be a CW-spectrum satisfying the property (I).*

i) *Let $f: \Sigma^{2t-1}SZ/2 \rightarrow X$ be a map whose cofiber $C(f)$ satisfies the property (I). For the composite map $fA'_2 i: \Sigma^{8r+2t-1} \rightarrow X$ its cofiber $C(fA'_2 i)$ satisfies the property (II) $_{4r+t}$, and it is quasi KO_* -equivalent to $\Sigma^{2t} \vee C(f)$ or $C(\eta)$ according as t is even or odd.*

ii) *Let $g: \Sigma^{2t}X \rightarrow SZ/2$ be a map whose cofiber $\Sigma^{2t-1}C(g)$ satisfies the property (I). For the composite map $jA'_2 g: \Sigma^{8r+2t-1}X \rightarrow \Sigma^0$ its cofiber $C(jA'_2 g)$ satisfies the property (II) $_{4r+t}$, and it is quasi KO_* -equivalent to $\Sigma^{-1}C(g) \vee \Sigma^0$ or $C(\eta)$ according as t is even or odd.*

Proof. i) Consider the commutative diagram

$$\begin{array}{ccccccc}
 \Sigma^{8r+2t-1} & \xrightarrow{A'_2 i} & \Sigma^{2t-1}SZ/2 & \rightarrow & \Sigma^{2t-1}C(A'_2 i) & \rightarrow & \Sigma^{8r+2t} \\
 \parallel & & \downarrow f & & \downarrow F & & \parallel \\
 \Sigma^{8r+2t-1} & \xrightarrow{fA'_2 i} & X & \rightarrow & C(fA'_2 i) & \rightarrow & \Sigma^{8r+2t} \\
 & & \downarrow i_f & & \downarrow i_F & & \\
 & & C(f) & = & C(f) & &
 \end{array}$$

involving four cofiber sequences. It is obvious that $KU_0C(fA'_2 i) \cong KU_0\Sigma^{8r+2t} \oplus KU_0X \cong Z \oplus Z$ and $KU_1C(fA'_2 i) = 0$. Observe that the induced homomorphism $F_*: KU_0\Sigma^{2t-1}C(A'_2 i) \rightarrow KU_0C(fA'_2 i)$ is given by $F_*(1) = (2, a)$ for some integer a . Since the integer a must be odd, we may take a to be 1. By an easy argument we can then show that $\psi_c^k = A_{k,4r+t}$ in $KU_0C(fA'_2 i)$ for each $k \neq 0$. Since $\Sigma^{-1}C(A'_2 i)$ is quasi KO_* -equivalent to Σ^0 by Lemma 1.4 and $C(f)$ is quasi KO_* -equivalent to either of Σ^0 and Σ^4 , the cofiber $C(fA'_2 i)$ becomes quasi KO_* -equivalent to the wedge sum $C(f) \vee \Sigma^{2t}$ in the case when t is even. On the other hand, it is exactly a Wood spectrum in the case when t is odd, because $\psi_c^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$ on $KU_0C(fA'_2 i)$.

ii) is similarly shown by a dual argument.

Consider the composite map $\tilde{\eta}\tilde{\eta}: \Sigma^3SZ/2 \rightarrow SZ/2$. Since $KO_7C(\tilde{\eta}\tilde{\eta}) \cong KO_3$

$SZ/2 \cong Z/2$ and $KO_6 C(\bar{\eta}\bar{\eta}) \cong KO_2 SZ/2 \cong Z/4$, a routine argument with (1.2) shows that

(1.10) $\Sigma^{-2}C(\bar{\eta}\bar{\eta})$ satisfies the property (I_4) , and it is quasi KO_* -equivalent to $\Sigma^4SZ/4$.

Since the composite maps $\bar{\eta}\bar{\eta}j: \Sigma^3SZ/2 \rightarrow \Sigma^1, i\bar{\eta}\bar{\eta}: \Sigma^3 \rightarrow SZ/2, \bar{\eta}\bar{\eta}\bar{\eta}: \Sigma^5SZ/2 \rightarrow \Sigma^1$ and $\bar{\eta}\bar{\eta}\bar{\eta}: \Sigma^5 \rightarrow SZ/2$ are all trivial [4, §4], we can choose the following maps

$$(1.11) \quad \begin{aligned} \bar{h}: \Sigma^3SZ/2 &\rightarrow C(\bar{\eta}), & \tilde{h}: \Sigma^1C(\bar{\eta}) &\rightarrow SZ/2, \\ \bar{k}: \Sigma^5SZ/2 &\rightarrow C(\bar{\eta}) & \text{and } \tilde{k}: \Sigma^3C(\bar{\eta}) &\rightarrow SZ/2 \end{aligned}$$

such that $j\bar{h} = \bar{\eta}j, \tilde{h}i = i\tilde{h}, j\bar{k} = \bar{\eta}j$ and $\tilde{k}i = \bar{\eta}i$. Among their cofibers there hold Spanier-Whitehead dualities as $C(\tilde{h}) = \Sigma^5DC(\tilde{h})$ and $C(\tilde{k}) = \Sigma^7DC(\tilde{k})$. Since $KU_0 C(\bar{\eta}j) \cong KU_0 C(i\bar{\eta}) \cong KU_0 C(\bar{\eta}\bar{\eta}) \cong Z/4$ by (1.3) and (1.10), we can easily observe that

(1.12) the cofibers $C(\bar{h}), \Sigma^{-5}C(\tilde{h}), C(\bar{k})$ and $\Sigma^{-7}C(\tilde{k})$ satisfy the property (I), and the first two and the last two are respectively quasi KO_* -equivalent to Σ^4 and Σ^0 ,

because $KO_1C(\bar{h}) = KO_7C(\tilde{h}) = KO_5C(\bar{k}) = 0$ and $KO_1C(\tilde{k}) \cong KO_3SZ/2 \cong Z/2$.

By taking f in Lemma 1.5 i) as the map $j, \bar{\eta}, \bar{h}$ or \bar{k} , and g in Lemma 1.5 ii) as the map $i, \bar{\eta}, \tilde{h}$ or \tilde{k} , we can now introduce the following maps of order 2:

$$(1.13) \quad \begin{aligned} \alpha_{4r} &= jA_2^r i: \Sigma^{8r-1} \rightarrow \Sigma^0, \\ \mu_{4r+1} &= \bar{\eta}A_2^r i: \Sigma^{8r+1} \rightarrow \Sigma^0, & \mu'_{4r+1} &= jA_2^r \bar{\eta}: \Sigma^{8r+1} \rightarrow \Sigma^0 \\ a_{4r+2} &= \bar{h}A_2^r i: \Sigma^{8r+3} \rightarrow C(\bar{\eta}), & a'_{4r+2} &= jA_2^r \bar{h}: \Sigma^{8r}C(\bar{\eta}) \rightarrow \Sigma^0 \\ m_{4r+3} &= \bar{k}A_2^r i: \Sigma^{8r+5} \rightarrow C(\bar{\eta}), & m'_{4r+3} &= jA_2^r \bar{k}: \Sigma^{8r+2}C(\bar{\eta}) \rightarrow \Sigma^0. \end{aligned}$$

Among their cofibers we may regard that there hold Spanier-Whitehead dualities as $C(f'_i) = \Sigma^{2i}DC(f_i)$ for $f_i = \alpha_{4r}, \mu_{4r+1}, a_{4r+2}$ or m_{4r+3} where $r \geq 0$ and $\alpha'_{4r} = \alpha_{4r}$.

Combining Lemma 1.5 with (1.2) and (1.12) we obtain

Proposition 1.6. *Set $f_i = \alpha_{4r}, \mu_{4r+1}, \mu'_{4r+1}, a_{4r+2}, a'_{4r+2}, m_{4r+3}$ or m'_{4r+3} ($r \geq 0$). Then each cofiber $C(f_i)$ satisfies the property $(II)_i$. Moreover $C(\alpha_{4r}), C(a'_{4r+2})$ and $\Sigma^4C(a_{4r+2})$ are quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^0$, and $C(\mu_{4r+1}), C(\mu'_{4r+1}), C(m_{4r+3})$ and $C(m'_{4r+3})$ are all Wood spectra.*

Use Proposition 1.6 combined with (1.8) to apply Theorem 4. Then we obtain the following result, which contains Theorem 1.

Theorem 1.7. *Let X be a CW-spectrum satisfying the property $(II)_t$ with $t \geq 0$.*

i) *If X is quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^0$, then it has the same K_* -local*

type as $C(\alpha_{4r})$ or $C(a'_{4r+2})$ according as $t=4r$ or $4r+2$.

ii) If X is quasi KO_* -equivalent to $\Sigma^4 \vee \Sigma^4$, then it has the same K_* -local type as $C(\alpha_{4r}) \wedge C(\bar{\eta})$ or $C(a_{4r+2})$ according as $t=4r$ or $4r+2$.

iii) If X is a Wood spectrum, then it has the same K_* -local type as $C(\mu_{4r+1})$ or $C(m_{4r+3})$ according as $t=4r+1$ or $4r+3$.

2. K_* -local types of spectra with the property $(II_{2m})_t$

2.1. Consider the cofibers $C(i\eta)$, $C(\bar{\eta}_{2m})$ and $C(\eta^2\bar{\eta}_{2m})$ of the maps $i\eta: \Sigma^1 \rightarrow SZ/2$, $\bar{\eta}_{2m}: \Sigma^1 SZ/2m \rightarrow \Sigma^0$ and $\eta^2\bar{\eta}_{2m}: \Sigma^3 SZ/2m \rightarrow \Sigma^0$, which are denoted by M_{2m} , P'_{2m} and R'_{2m} respectively in [15, I.4.1]. Recall that $KU_0 M_{2m} \cong Z \oplus Z/2m$ on which $\psi_c^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$, $KU_0 P'_{2m} \cong Z \oplus Z/m$ on which $\psi_c^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, $KU_0 R'_{2m} \cong Z \oplus Z/2m$ on which $\psi_c^{-1} = 1$, and $KU_1 M_{2m} = KU_1 P'_{2m} = KU_1 R'_{2m} = 0$ [15, Proposition I.4.1]. Note that $\Sigma^{-2} P'_{2m}$ is quasi KO_* -equivalent to M_{2m} , whose KO -homology $KO_i M_{2m} \cong Z/2m, 0, Z \oplus Z/2, Z/2, Z/4m, 0, Z, 0$ according as $i=0, 1, \dots, 7$ (see [15, Proposition I.4.2 and Corollary I.5.4]).

Let X be a CW -spectrum satisfying the following property:

$(II_{2m})_t$ $KU_0 X \cong Z \oplus Z/2m$ in which $\psi_c^k = A_{k,t}$ and $KU_1 X = 0$.

Then X is quasi KO_* -equivalent to one of the following elementary spectra $\Sigma^{4i} \vee \Sigma^{4j} SZ/2m$, $\Sigma^{4i} \vee \Sigma^{4j} V_{2m}$ and $\Sigma^{4j} R'_{2m}$ for $i, j=0$ or 1 when t is even, and it is quasi KO_* -equivalent to either of M_{2m} and $\Sigma^4 M_{2m}$ when t is odd.

Lemma 2.1. *Let X, Y and W be CW -spectra satisfying the property (I). Let $f: \Sigma^{2t-1} X \rightarrow Y$ and $g: W \rightarrow Y$ be maps whose cofibers $C(f)$ and $C(g)$ satisfy the properties $(II)_t$ and (I_{2m}) respectively. Then the cofiber $C(i_g f)$ of the composite map $i_g f: \Sigma^{2t-1} X \rightarrow Y \rightarrow C(g)$ satisfies the property $(II_{2m})_t$. Moreover it is quasi KO_* -equivalent to $\Sigma^{2t} X \vee C(g)$ when t is even, and it is quasi KO_* -equivalent to M_{2m} or $\Sigma^4 M_{2m}$ according as W is quasi KO_* -equivalent to Σ^0 or Σ^4 when t is odd.*

Proof. Use the commutative diagram

$$\begin{array}{ccccccc}
 & & & W & = & W & \\
 & & & \downarrow g & & \downarrow G & \\
 \Sigma^{2t-1} X & \xrightarrow{f} & Y & \rightarrow & C(f) & \rightarrow & \Sigma^{2t} X \\
 \parallel & & \downarrow i_g & & \downarrow i_G & & \parallel \\
 \Sigma^{2t-1} X & \rightarrow & C(g) & \rightarrow & C(i_g f) & \rightarrow & \Sigma^{2t} X
 \end{array}$$

involving four cofiber sequences. Obviously $KU_0 C(i_g f) \cong KU_0 \Sigma^{2t} X \oplus KU_0 C(g)$ in which $\psi_c^k = A_{k,t}$ and $KU_1 C(i_g f) = 0$. When t is even, $C(i_g f)$ is quasi KO_* -equivalent to the wedge sum $\Sigma^{2t} X \vee C(g)$ since $\Sigma^{2t-1} X$ is quasi KO_* -equivalent to Σ^3 or Σ^7 and Y is quasi KO_* -equivalent to Σ^0 or Σ^4 . On the other

hand, $C(i_g f)$ is quasi KO_* -equivalent to either of M_{2m} and $\Sigma^4 M_{2m}$ when t is odd. However we notice that $KO_3 C(i_g f) \cong KO_2 W$ because $C(f)$ is a Wood spectrum in the case when t is odd.

Let X be a CW -spectrum with $(II_{2m})_{2s+1}$, which is quasi KO_* -equivalent to either M_{2m} or $\Sigma^4 M_{2m}$. Using the long exact sequence induced by the Bott cofiber sequence $\Sigma^1 KO \rightarrow KO \rightarrow KU \rightarrow \Sigma^2 KO$ we can easily show that the Adams operation ψ_R^k behaves always in $KO_i X$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:

$$(2.1) \quad \psi_R^k = 1/k^{2s}, k^2, 1/k^{2s-2} \text{ or } 1 \text{ according as } i=2, 4, 6 \text{ or otherwise.}$$

Lemma 2.2. *Let X, Y and W be CW -spectra satisfying the property (I). Let $f: \Sigma^{4s-1} X \rightarrow Y$ and $g: W \rightarrow Y$ be maps whose cofibers $C(f)$ and $C(g)$ satisfy the properties $(II)_{2s}$ and (I_{2m}) respectively. Assume that the Adams operation ψ_R^k behaves normally in $KO_* C(g)$ in the sense of (1.5). Then the Adams operation ψ_R^k acts normally in $KO_i C(i_g f)$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:*

- i) *If both $\Sigma^{4s} X$ and Y are quasi KO_* -equivalent to either of Σ^0 and Σ^4 , then $\psi_R^k = A_{k,2s}, k^2 A_{k,2s}$ or 1 according as $i=0, 4$ or otherwise.*
- ii) *If $\Sigma^{4s} X$ and Y are respectively quasi KO_* -equivalent to Σ^0 and Σ^4 , then $\psi_R^k = A_{k,2s,2s}, k^2 A_{k,2s,0}$ or 1 according as $i=0, 4$ or otherwise.*
- iii) *If $\Sigma^{4s} X$ and Y are respectively quasi KO_* -equivalent to Σ^4 and Σ^0 , then $\psi_R^k = A_{k,2s,0}, k^2 A_{k,2s,2}$ or 1 according as $i=0, 4$ or otherwise.*

Proof. Use the cofiber sequence $W \xrightarrow{G} C(f) \xrightarrow{i_G} C(i_g f) \xrightarrow{j_G} \Sigma^1 W$ appeared in the proof of Lemma 2.1 where $C(f)$ and $C(i_g f)$ are quasi KO_* -equivalent to $\Sigma^{4s} X \vee Y$ and $\Sigma^{4s} X \vee C(g)$ respectively. Since W is quasi KO_* -equivalent to either of Σ^0 and Σ^4 , the map i_G induces epimorphisms $i_{G*}: KO_i C(f) \rightarrow KO_i C(i_g f)$ in dimensions $i=0, 1, 4$ and 5 . By using (1.8) i) and ii) we can immediately observe the behavior of ψ_R^k in $KO_i C(i_g f)$ for $i=0, 1, 4$ or 5 . We will next show that $\psi_R^k = 1$ in $KO_i C(i_g f)$ for $i=2$ or 6 . It is obvious that $KO_2 C(i_g f)$ is isomorphic to $KO_2 C(g), KO_2 C(f)$ or $KO_2 C(f) \oplus KO_2 C(g)$ according as $\Sigma^{4s} X, W$ or Y is quasi KO_* -equivalent to Σ^4 . Therefore it is easy to see that $\psi_R^k = 1$ in $KO_2 C(i_g f)$ in these three cases. Assume that $\Sigma^{4s} X, W$ and Y are all quasi KO_* -equivalent to Σ^0 . Then we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & KO_2 Y & \rightarrow & KO_2 C(f) & \rightarrow & KO_2 \Sigma^{4s} X \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & KO_2 C(g) & \rightarrow & KO_2 C(i_g f) & \rightarrow & KO_2 \Sigma^{4s} X \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & KO_1 W & = & KO_1 W & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

with exact rows and columns, where $C(g)$ is quasi KO_* -equivalent to $SZ/2m$ by Lemma 1.1. Then a routine computation shows that $\psi_R^k=1$ in $KO_2C(i_g f)$ as desired, because $\psi_R^k=1$ in $KO_2C(f)$ and $KO_2C(g)$. Similarly as to $KO_6 C(i_g f)$.

We remark that the Adams operation ψ_R^k acts normally in $KO_*C(i_g f) \wedge C(\bar{\eta})$ as stated in the above lemma if it behaves normally in $KO_*C(g) \wedge C(\bar{\eta})$ in the sense of (1.5).

Take f in Lemma 2.1 as the map $\alpha_{4r}, \mu_{4r+1}, a_{4r+2}, a'_{4r+2}$ or m_{4r+3} given in (1.13) and g in Lemma 2.1 as the map $2m: \Sigma^0 \rightarrow \Sigma^0, m\bar{\lambda}: C(\bar{\eta}) \rightarrow \Sigma^0, 2m: C(\bar{\eta}) \rightarrow C(\bar{\eta})$ or $m\bar{i}: \Sigma^0 \rightarrow C(\bar{\eta})$ whose cofiber is $SZ/2m, U_{2m}, SZ/2m \wedge C(\bar{\eta})$ or V_{2m} . Then we can introduce the composite maps $i_g f_t (t \geq 0)$ as follows:

$$\begin{aligned}
 (2.2) \quad & i\alpha_{4r}: \Sigma^{8r-1} \rightarrow SZ/2m, & i_U\alpha_{4r}: \Sigma^{8r-1} \rightarrow U_{2m}, \\
 & i\mu_{4r+1}: \Sigma^{8r+1} \rightarrow SZ/2m, & i_U\mu_{4r+1}: \Sigma^{8r+1} \rightarrow U_{2m}, \\
 & ia'_{4r+2}: \Sigma^{8r}C(\bar{\eta}) \rightarrow SZ/2m, & i_Ua'_{4r+2}: \Sigma^{8r}C(\bar{\eta}) \rightarrow U_{2m}, \\
 & (i_{\wedge 1})a_{4r+2}: \Sigma^{8r+3} \rightarrow SZ/2m \wedge C(\bar{\eta}), & i_Va_{4r+2}: \Sigma^{8r+3} \rightarrow V_{2m}, \\
 & (i_{\wedge 1})m_{4r+3}: \Sigma^{8r+5} \rightarrow SZ/2m \wedge C(\bar{\eta}), & i_Vm_{4r+3}: \Sigma^{8r+5} \rightarrow V_{2m}.
 \end{aligned}$$

Applying Lemmas 2.1 and 2.2 and (2.1) with the aid of Proposition 1.6, (1.3), (1.4) and (1.5), we obtain

Proposition 2.3. *For each composite map $i_g f_t (t \geq 0)$ given in (2.2), its cofiber $C(i_g f_t)$ satisfies the property $(II_{2m})_t$, and the Adams operation ψ_R^k behaves normally in $KO_*C(i_g f_t)$ as stated in Lemma 2.2 i) when t is even, or as stated in (2.1) when t is odd. Moreover $C(i\alpha_{4r}), C(ia'_{4r+2})$ and $\Sigma^4C((i_{\wedge 1})a_{4r+2})$ are quasi KO_* -equivalent to $\Sigma^0 \vee SZ/2m$, and $C(i_U\alpha_{4r}), C(i_Ua'_{4r+2})$ and $\Sigma^4C(i_Va_{4r+2})$ are quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^4V_{2m}$. On the other hand, $C(i\mu_{4r+1}), C(i_Vm_{4r+3}), \Sigma^4C(i_U\mu_{4r+1})$ and $\Sigma^4C((i_{\wedge 1})m_{4r+3})$ are all quasi KO_* -equivalent to M_{2m} .*

2.2. Let $f: \Sigma^{2t-1}X \rightarrow Y$ be a map of order 2. Then we have extensions

$$\begin{aligned}
 \bar{f}_{2m}: \Sigma^{2t-1}X \wedge SZ/2m \rightarrow Y, \quad \bar{f}_{U,4m}: \Sigma^{2t-1}X \wedge U_{4m} \rightarrow Y \quad \text{and} \\
 \bar{f}_{V,4m}: \Sigma^{2t-1}X' \wedge V_{4m} \rightarrow Y \quad \text{when} \quad X = X' \wedge C(\bar{\eta})
 \end{aligned}$$

such that $\bar{f}_{2m}(1_{\wedge i})=f, \bar{f}_{U,4m}(1_{\wedge i_U})=f$ and $\bar{f}_{V,4m}(1_{\wedge i_V})=f$ because U_{4m} and V_{4m} are constructed as the cofibers of the maps $2m\bar{\lambda}: C(\bar{\eta}) \rightarrow \Sigma^0$ and $2m\bar{i}: \Sigma^0 \rightarrow C(\bar{\eta})$ respectively.

Lemma 2.4. *Let X and Y be CW-spectra satisfying the property (I), and $f: \Sigma^{2t-1}X \rightarrow Y$ be a map of order 2 whose cofiber $C(f)$ satisfies the property $(II)_t (t \neq 0)$.*

(i) *The cofiber $C(\bar{f}_2)$ satisfies the property (I), and it is quasi KO_* -equivalent*

to Y or $\Sigma^4 Y$ according as t is even or odd.

ii) For $\bar{\varphi}_{4m} = \bar{f}_{4m}, \bar{f}_{U,4m}$ or $\bar{f}_{V,4m}$ each cofiber $\Sigma^{-2t}C(\bar{\varphi}_{4m})$ satisfies the property $(\Pi_{2m})_{-t}$. Whenever t is odd, all of $C(\bar{f}_{4m}), \Sigma^4 C(\bar{f}_{U,4m})$ and $C(\bar{f}_{V,4m})$ are quasi KO_* -equivalent to P'_{4m} or $\Sigma^4 P'_{4m}$ according as $\Sigma^{2t} X$ is quasi KO_* -equivalent to Σ^2 or Σ^6 .

Proof. Consider the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & \Sigma^{2t} X & = & \Sigma^{2t} X \\
 & & & & \downarrow \lambda & & \downarrow 2m \\
 \Sigma^{2t-1} X & \xrightarrow{f} & Y & \rightarrow & C(f) & \rightarrow & \Sigma^{2t} X \\
 \downarrow 1 \wedge i & & \parallel & & \downarrow & & \downarrow \\
 \Sigma^{2t-1} X \wedge SZ/2m & \xrightarrow{\bar{f}_{2m}} & Y & \rightarrow & C(\bar{f}_{2m}) & \rightarrow & \Sigma^{2t} X \wedge SZ/2m
 \end{array}$$

involving four cofiber sequences. The induced homomorphism $\lambda_*: KU_0 \Sigma^{2t} X \rightarrow KU_0 C(f)$ is given by $\lambda_*(1) = (2m, m) \in KU_0 C(f) \cong KU_0 \Sigma^{2t} X \oplus KU_0 Y \cong Z \oplus Z$, since $\psi_c^k = A_{k,t}$ in $KU_0 C(f)$. Hence it is immediate that $KU_0 C(\bar{f}_{2m}) \cong Z \oplus Z/m$ and $KU_1 C(\bar{f}_{2m}) = 0$. Moreover the Adams operation ψ_c^k in $KU_0 C(\bar{f}_{2m})$ is represented as the matrix $\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} A_{k,t} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} = k^{-t} A_{k,-t}$. In other words, $\psi_c^k = 1$ in $KU_0 C(\bar{f}_2) \cong Z$ and $\psi_c^k = A_{k,-t}$ in $KU_0 \Sigma^{-2t} C(\bar{f}_{2m}) \cong Z \oplus Z/m$ unless $m=1$.

Assume that Y is quasi KO_* -equivalent to Σ^0 . Then it is obvious that $KO_7 \Sigma^{2t} X \wedge SZ/2 = 0$ because $KO_7 C(\bar{f}_2) = 0 = KO_6 Y$. Therefore $\Sigma^{2t} X \wedge SZ/2$ becomes quasi KO_* -equivalent to $SZ/2$ or $\Sigma^2 SZ/2$ according as t is even or odd. This implies easily that $C(\bar{f}_2)$ is quasi KO_* -equivalent to Σ^0 or Σ^4 according as t is even or odd. When Y is quasi KO_* -equivalent to Σ^4 , a similar result can be shown. Since $C(f)$ is a Wood spectrum when t is odd, it is immediate that $KO_1 C(\bar{f}_{2m}) \cong KO_0 \Sigma^{2t} X$. Hence $C(\bar{f}_{2m})$ is quasi KO_* -equivalent to P'_{2m} or $\Sigma^4 P'_{2m}$ according as $\Sigma^{2t} X$ is quasi KO_* -equivalent to Σ^2 or Σ^6 .

We can similarly prove as for $C(\bar{f}_{U,4m})$ and $C(\bar{f}_{V,4m})$.

Since $[\Sigma^3 SZ/2, C(\bar{\eta})] \cong [\Sigma^1 C(\bar{\eta}), SZ/2] \cong Z/2$ (use [4, §4]), the maps $\bar{h}: \Sigma^3 SZ/2 \rightarrow C(\bar{\eta})$ and $\bar{h}: \Sigma^1 C(\bar{\eta}) \rightarrow SZ/2$ have order 2. So there exist maps

$$\bar{h}_{2m/2}: \Sigma^4 SZ/2 \rightarrow SZ/2m \wedge C(\bar{\eta}) \quad \text{and} \quad \bar{h}_{2m/2}: \Sigma^1 SZ/2m \wedge C(\bar{\eta}) \rightarrow SZ/2$$

satisfying $(j \wedge 1) \bar{h}_{2m/2} = \bar{h}$ and $\bar{h}_{2m/2}(i \wedge 1) = \bar{h}$. We now set

$$\begin{aligned}
 (2.3) \quad & \bar{\alpha}_{4r} = jA_2^r: \Sigma^{8r-1} SZ/2 \rightarrow \Sigma^0, & \bar{\mu}_{4r+1} &= \bar{\eta}A_2^r: \Sigma^{8r+1} SZ/2 \rightarrow \Sigma^0, \\
 & \bar{a}_{4r+2} = \bar{h}A_2^r: \Sigma^{8r+3} SZ/2 \rightarrow C(\bar{\eta}), & \bar{m}_{4r+3} &= \bar{k}A_2^r: \Sigma^{8r+5} SZ/2 \rightarrow C(\bar{\eta}), \\
 & \bar{a}'_{4r+2} = jA_2^r \bar{h}_{2/2}: \Sigma^{8r} SZ/2 \wedge C(\bar{\eta}) \rightarrow \Sigma^0.
 \end{aligned}$$

Then Lemma 2.4 i) combined with Proposition 1.6 shows that

(2.4) the cofibers $C(\bar{\alpha}_{4r}), C(\bar{\mu}_{4r+1}), C(\bar{a}_{4r+2}), C(\bar{m}_{4r+3})$ and $C(\bar{a}'_{4r+2})$ satisfy the property (I), and the first, the fourth and the last are quasi KO_* -equivalent to Σ^0 and the other two are quasi KO_* -equivalent to Σ^4 .

Let $f: \Sigma^{2t-1}X \rightarrow Y$ be a map of order 2 and $\bar{f}: \Sigma^{2t-1}X \wedge SZ/2 \rightarrow Y$ be its extension with $\bar{f}(1 \wedge i) = f$. Then there exists a map $\varphi: \Sigma^{-2t-1}C(\bar{f}) \rightarrow X$ of order 2 whose cofiber $C(\varphi)$ coincides with $\Sigma^{-2t}C(f)$. Hence we can choose the following maps of order 2:

$$(2.5) \quad \begin{aligned} \alpha_{-4r}: \Sigma^{-8r-1}C(\bar{\alpha}_{4r}) &\rightarrow \Sigma^0, & \mu_{-4r-1}: \Sigma^{-8r-3}C(\bar{\mu}_{4r+1}) &\rightarrow \Sigma^0, \\ a_{-4r-2}: \Sigma^{-8r-5}C(\bar{a}_{4r+2}) &\rightarrow \Sigma^0, & m_{-4r-3}: \Sigma^{-8r-7}C(\bar{m}_{4r+3}) &\rightarrow \Sigma^0, \\ b_{-4r-2}: \Sigma^{-8r-5}C(\bar{a}'_{4r+2}) &\rightarrow \Sigma^{-3}C(\bar{\eta}) \end{aligned}$$

of which each cofiber $C(f_{-t})$ coincides with $\Sigma^{-2t}C(f_t)$ where $f_t = \alpha_{4r}, \mu_{4r+1}, a_{4r+2}, m_{4r+3}$ or b_{4r+2} ($r \geq 0$) with $b_{4r+2} = a'_{4r+2}$.

Take f in Lemma 2.1 as the above map $\alpha_{-4r}, \mu_{-4r-1}, a_{-4r-2}, m_{-4r-3}$ or b_{-4r-2} , and g in Lemma 2.1 as the map $2m: \Sigma^0 \rightarrow \Sigma^0, m\bar{\lambda}: C(\bar{\eta}) \rightarrow \Sigma^0, 2m: C(\bar{\eta}) \rightarrow C(\bar{\eta})$ or $m\bar{\lambda}: \Sigma^3 \rightarrow C(\bar{\eta})$. Then we obtain the following composite maps $i_g f_{-t}$ ($t \geq 0$):

$$(2.6) \quad \begin{aligned} i\alpha_{-4r}: \Sigma^{-8r-1}C(\bar{\alpha}_{4r}) &\rightarrow SZ/2m, & i_U\alpha_{-4r}: \Sigma^{-8r-1}C(\bar{\alpha}_{4r}) &\rightarrow U_{2m}, \\ i\mu_{-4r-1}: \Sigma^{-8r-3}C(\bar{\mu}_{4r+1}) &\rightarrow SZ/2m, & i_U\mu_{-4r-1}: \Sigma^{-8r-3}C(\bar{\mu}_{4r+1}) &\rightarrow U_{2m}, \\ ia_{-4r-2}: \Sigma^{-8r-5}C(\bar{a}_{4r+2}) &\rightarrow SZ/2m, & i_Ua_{-4r-2}: \Sigma^{-8r-5}C(\bar{a}_{4r+2}) &\rightarrow U_{2m}, \\ im_{-4r-3}: \Sigma^{-8r-7}C(\bar{m}_{4r+3}) &\rightarrow SZ/2m, & i_Um_{-4r-3}: \Sigma^{-8r-7}C(\bar{m}_{4r+3}) &\rightarrow U_{2m}, \\ (i \wedge 1)b_{-4r-2}: \Sigma^{-8r-5}C(\bar{a}'_{4r+2}) &\rightarrow \Sigma^{-3}SZ/2m \wedge C(\bar{\eta}) \quad \text{and} \\ i'_U b_{-4r-2}: \Sigma^{-8r-5}C(\bar{a}'_{4r+2}) &\rightarrow \Sigma^{-3}U'_{2m}. \end{aligned}$$

By making use of Lemmas 2.1 and 2.2 and (2.1) we obtain

Proposition 2.5. *For each composite map $i_g f_{-t}$ ($t \geq 0$) given in (2.6), its cofiber $C(i_g f_{-t})$ satisfies the property $(II_{2m})_{-t}$, and the Adams operation ψ_R^k behaves normally in $KO_*C(i_g f_{-t})$ as stated in Lemma 2.2 i) when t is even, or as stated in (2.1) when t is odd. Moreover $C(i\alpha_{-4r}), C(ia_{-4r-2})$ and $\Sigma^4C((i \wedge 1)b_{-4r-2})$ are quasi KO_* -equivalent to $\Sigma^0 \vee SZ/2m$, and $C(i_U\alpha_{-4r}), C(i_Ua_{-4r-2})$ and $\Sigma^4C(i'_U b_{-4r-2})$ are quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^4V_{2m}$. On the other hand, $C(i\mu_{-4r-1}), C(im_{-4r-3}), \Sigma^4C(i_U\mu_{-4r-1})$ and $\Sigma^4C(i_Um_{-4r-3})$ are all quasi KO_* -equivalent to M_{2m} .*

By virtue of Propositions 2.3 and 2.5 we can apply Theorem 4 to show the following result.

Theorem 2.6. *Let X be a CW-spectrum satisfying the property $(II_{2m})_t$.*

i) *Assume that X is quasi KO_* -equivalent to $\Sigma^0 \vee SZ/2m$. If the Adams operation ψ_R^k behaves normally in KO_*X for each $k \neq 0$ as stated in Lemma 2.2*

i), then X has the same K_* -local type as $C(i\alpha_{4r})$, $C(ia'_{4r+2})$, $C(i\alpha_{-4r})$ or $C(ia_{-4r-2})$ according as $t=4r, 4r+2, -4r$ or $-4r-2$ ($r \geq 0$).

ii) Assume that X is quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^4 V_{2m}$. If the Adams operation ψ_R^k behaves normally in KO_*X for each $k \neq 0$ as stated in Lemma 2.2 i), then X has the same K_* -local type as $C(i_U \alpha_{4r})$, $C(i_U a'_{4r+2})$, $C(i_U \alpha_{-4r})$ or $C(i_U a_{-4r-2})$ according as $t=4r, 4r+2, -4r$ or $-4r-2$ ($r \geq 0$).

iii) Assume that X is quasi KO_* -equivalent to M_{2m} . Then X has the same K_* -local type as $C(i_{\mu_{4r+1}})$, $C(i_{\nu m_{4r+3}})$, $C(i_{\mu_{-4r-1}})$ or $C(im_{-4r-3})$ according as $t=4r+1, 4r+3, -4r-1, -4r-3$ ($r \geq 0$).

iv) Assume that X is quasi KO_* -equivalent to $\Sigma^4 M_{2m}$. Then X has the same K_* -local type as $C(i_U \mu_{4r+1})$, $C((i_{\wedge 1}) m_{4r+3})$, $C(i_U \mu_{-4r-1})$ or $C(i_U m_{-4r-3})$ according as $t=4r+1, 4r+3, -4r-1$ or $-4r-3$ ($r \geq 0$).

In the above theorem we may replace the map $i_U: \Sigma^0 \rightarrow U_{2m}$ by the map $i'_U: \Sigma^0 \rightarrow \Sigma^{-2} V'_{2m}$, and also the maps $\mu_{4r+1}: \Sigma^{8r+1} \rightarrow \Sigma^0$, $i_{\nu} m_{4r+3}: \Sigma^{8r+5} \rightarrow V_{2m}$ and $(i_{\wedge 1}) m_{4r+3}: \Sigma^{8r+5} \rightarrow SZ/2m \wedge C(\bar{\eta})$ by $\mu'_{4r+1}: \Sigma^{8r+1} \rightarrow \Sigma^0$, $im'_{4r+3}: \Sigma^{8r+2} C(\bar{\eta}) \rightarrow SZ/2m$ and $i'_U m'_{4r+3}: \Sigma^{8r+2} C(\bar{\eta}) \rightarrow \Sigma^{-2} V'_{2m}$ respectively. Thus

(2.7) i) $C(i'_U f_i)$ has the same K_* -local type as $C(i_U f_i)$ for $f_i = \alpha_{\pm 4r}, \mu_{\pm(4r+1)}, a'_{4r+2}, a_{-4r-2}, m'_{4r+3}$ or m_{-4r-3} .

ii) $C(i_{\mu'_{4r+1}})$ and $C(i_U \mu'_{4r+1})$ have the same K_* -local types as $C(i_{\mu_{4r+1}})$ and $C(i_U \mu_{4r+1})$ respectively.

iii) $C(im'_{4r+3})$ and $C(i'_{\nu} m'_{4r+3})$ have the same K_* -local types as $C(i_{\nu} m_{4r+3})$ and $C((i_{\wedge 1}) m_{4r+3})$ respectively.

When X is quasi KO_* -equivalent to $\Sigma^4 \vee \Sigma^4 SZ/2m$ or $\Sigma^4 \vee V_{2m}$, we can obtain a similar result corresponding to the above theorem i) or ii). In fact, if the Adams operation ψ_R^k behaves normally in KO_*X for each $k \neq 0$ as stated in Lemma 2.2 i), then X has the same K_* -local type as the cofiber appeared in Theorem 2.6 i) or ii) smashed with $C(\bar{\eta})$ (see the remark following Lemma 2.2). In particular, by means of Propositions 2.3 and 2.5 again we obtain that

(2.8) $C((i_{\wedge 1}) a_{4r+2})$, $C(i_{\nu} a_{4r+2})$, $C((i_{\wedge 1}) b_{-4r-2})$ and $C(i'_U b_{-4r-2})$ have the same K_* -local types as $C(ia'_{4r+2}) \wedge C(\bar{\eta})$, $C(i_U a'_{4r+2}) \wedge C(\bar{\eta})$, $C(ia_{-4r-2}) \wedge C(\bar{\eta})$ and $C(i_U a_{-4r-2}) \wedge C(\bar{\eta})$ respectively.

3. K_* -local types of Anderson spectra

3.1. Let X be a CW -spectrum with $KU_0 X \cong KU_1 X \cong Z$. For such a CW -spectrum X we may assume that $X_{\wedge} SQ = (\Sigma^0 \vee \Sigma^{2t+1})_{\wedge} SQ$ for some integer t . In this case X satisfies the following property:

(III)_t: $KU_0 X \cong Z$ with $\psi_C^k = 1$ and $KU_1 X \cong Z$ with $\psi_C^k = 1/k^t$.

If X satisfies the property (III)_{2s+1}, then it is quasi KO_* -equivalent to one of the

following spectra $\Sigma^0 \vee \Sigma^3, \Sigma^0 \vee \Sigma^7, \Sigma^4 \vee \Sigma^3, \Sigma^4 \vee \Sigma^7$ or $C(\eta^2)$ (see [7, Theorem 3.2] or [15, Theorem I.3.4]).

Lemma 3.1. *Let X and Y be CW-spectra satisfying the property (I) and $f: \Sigma^{2t-1}X \rightarrow Y$ be a map whose cofiber $C(f)$ satisfies the property (II)_{*i*}. Then the cofiber $C(\eta f)$ of the composite map $\eta f: \Sigma^{2t}X \rightarrow Y$ satisfies the property (III)_{*i*}, and it is quasi KO_* -equivalent to $Y \vee \Sigma^{2t+1}X$ or $C(\eta^2)$ according as t is even or odd.*

Proof. Obviously $KU_0 C(\eta f) \cong KU_0 Y \cong Z$ and $KU_1 C(\eta f) \cong KU_1 \Sigma^{2t+1}X \cong Z$. In the case when t is even, $C(\eta f)$ is quasi KO_* -equivalent to the wedge sum $Y \vee \Sigma^{2t+1}X$ since $C(f)$ is quasi KO_* -equivalent to $Y \vee \Sigma^{2t}X$. On the other hand, $C(\eta f)$ is just an Anderson spectrum in the case when t is odd, because $KO_2 C(\eta f) = 0 = KO_6 C(\eta f)$.

Let X be an Anderson spectrum satisfying the property (III) _{$2s+1$} . Then we can easily observe that the Adams operation ψ_R^k behaves always in $KO_i X$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:

$$(3.1) \quad \psi_R^k = 1/k^{2s}, k^2, 1/k^{2s-2} \text{ or } 1 \text{ according as } i=3, 4, 7 \text{ or otherwise.}$$

Lemma 3.2. *Let X and Y be CW-spectra satisfying the property (I) and $f: \Sigma^{4s-1}X \rightarrow Y$ be a map whose cofiber $C(f)$ satisfies the property (II) _{$2s$} . Then the Adams operation ψ_R^k acts normally in $KO_i C(\eta f)$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows: $\psi_R^k = 1/k^{2s}, k^2, 1/k^{2s-2}$ or 1 according as $i=1, 4, 5$ or otherwise.*

Proof. Use the cofiber sequence $\Sigma^1 C(f) \rightarrow C(\eta f) \rightarrow C(\eta) \wedge Y \rightarrow \Sigma^2 C(f)$ where $C(f), C(\eta f)$ and $C(\eta) \wedge Y$ are quasi KO_* -equivalent to $Y \vee \Sigma^{4s}X, Y \vee \Sigma^{4s+1}X$ and $C(\eta)$ respectively. Then the result follows immediately from (1.8) i) and ii).

Take f in Lemma 3.1 as the map $\alpha_{\pm 4r}, \mu_{\pm(4r+1)}, a_{\pm(4r+2)}, m_{\pm(4r+3)}, a'_{4r+2}$ or b_{-4r-2} given in (1.13) or (2.5). Using Lemmas 3.1 and 3.2 and (3.1) by virtue of Proposition 1.6 we obtain

Proposition 3.3. *Set $f_i = \alpha_{\pm 4r}, \mu_{\pm(4r+1)}, a_{\pm(4r+2)}, m_{\pm(4r+3)}, a'_{4r+2}$ or b_{-4r-2} ($r \geq 0$). Then each cofiber $C(\eta f_i)$ satisfies the property (III)_{*i*}, and the Adams operation ψ_R^k behaves normally in $KO_* C(\eta f_i)$ as stated in Lemma 3.2 when t is even, or as stated in (3.1) when t is odd. Moreover the cofibers $C(\eta f_i)$ for $f_i = \alpha_{\pm 4r}, a'_{4r+2}$ and a_{-4r-2} are quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^1$, but $C(\eta a_{4r+2})$ and $C(\eta b_{-4r-2})$ are quasi KO_* -equivalent to $\Sigma^4 \vee \Sigma^5$. On the other hand, the cofibers $C(\eta f_i)$ for $f_i = \mu_{\pm(4r+1)}$ and $m_{\pm(4r+3)}$ are Anderson spectra.*

By applying Theorem 4 combined with Proposition 3.3 we can show the following result, which contains Theorem 2.

Theorem 3.4. *Let X be a CW-spectrum satisfying the property (III)_{*i*} with*

$t \neq -1$.

i) Assume that X is quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^1$. If the Adams operation ψ_R^k behaves normally in KO_*X for each $k \neq 0$ as stated in Lemma 3.2, then X has the same K_* -local type as $C(\eta\alpha_{4r})$, $C(\eta a'_{4r+2})$, $C(\eta\alpha_{-4r})$ or $C(\eta a_{-4r-2})$ according as $t=4r$, $4r+2$, $-4r$ or $-4r-2$ ($r \geq 0$).

ii) When X is an Anderson spectrum, then it has the same K_* -local type as $C(\eta\mu_{4r+1})$, $C(\eta m_{4r+3})$, $C(\eta\mu_{-4r-1})$ or $C(\eta m_{-4r-3})$ according as $t=4r+1$, $4r+3$, $-4r-1$ or $-4r-3$ ($r \geq 0$) where $t \neq -1$.

3.3. As duals of M_{2m} , P'_{2m} and R'_{2m} appeared in §2 we next consider the cofibers $C(\eta j)$, $C(\tilde{\eta}_{2m})$ and $C(\tilde{\eta}_{2m}\eta^2)$ of the maps $\eta j:SZ/2m \rightarrow \Sigma^0$, $\tilde{\eta}_{2m}: \Sigma^2 \rightarrow SZ/2m$ and $\tilde{\eta}_{2m}\eta^2: \Sigma^4 \rightarrow SZ/2m$, which are denoted by M'_{2m} , P_{2m} and R_{2m} respectively in [15, I.4.1]. Then there hold Spanier-Whitehead dualities as $M'_{2m} = \Sigma^2 DM_{2m}$, $P'_{2m} = \Sigma^3 DP_{2m}$ and $R'_{2m} = \Sigma^5 DR_{2m}$. Hence $KU^0 M'_{2m} \cong Z \oplus Z/2m$ on which $\psi_c^{-1} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, $KU^1 P_{2m} \cong Z \oplus Z/m$ on which $\psi_c^{-1} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$, $KU^1 R_{2m} \cong Z \oplus Z/2m$ on which $\psi_c^{-1} = 1$, and $KU^1 M'_{2m} = KU^0 P_{2m} = KU^0 R_{2m} = 0$ (cf. [15, Proposition I.4.1]). Note that $\Sigma^1 P_{4m}$ is quasi KO_* -equivalent to M'_{2m} , whose KO -homology $KO_i M'_{2m} \cong Z, Z/4m, Z/2, Z/2, Z, Z/2m, 0, 0$ according as $i=0, 1, \dots, 7$ (see [15, Proposition I.4.2 and Corollary I.5.4]).

Let X be a CW -spectrum satisfying the following property:

$$(II_{2m})_t^* \quad KU^0 X \cong Z \oplus Z/2m \text{ in which } \psi_c^k = A_{k,t} \text{ and } KU^1 X = 0.$$

If $KU_i X$ is finitely generated for each i , then the property $(II_{2m})_t^*$ implies that $KU_0 X \cong Z$ with $\psi_c^k = k^t$ and $KU_{-1} X \cong Z/2m$ with $\psi_c^k = 1$. Under the assumption that X is finite, we note that X satisfies the property $(II_{2m})_t^*$ if and only if its Spanier-Whitehead dual DX does the property $(II_{2m})_t$. As a dual of Lemma 2.1 we have

Lemma 3.5. Let X, Y and W be CW -spectra satisfying the property (I). Let $f: \Sigma^{2t-1} X \rightarrow Y$ and $g: X \rightarrow W$ be maps whose cofibers $C(f)$ and $C(g)$ satisfy the properties $(II)_t$ and (I_{2m}) respectively. Then for the composite map $fj_g: \Sigma^{2t-2} C(g) \rightarrow \Sigma^{2t-1} X \rightarrow Y$ the cofiber $\Sigma^{-2t} C(fj_g)$ satisfies the property $(II_{2m})_t^*$. Moreover $C(fj_g)$ is quasi KO_* -equivalent to the wedge sum $Y \vee \Sigma^{2t-1} C(g)$ when t is even. On the other hand, under the assumption that $C(fj_g)$ is finite, it is quasi KO_* -equivalent to M'_{2m} or $\Sigma^4 M'_{2m}$ according as $\Sigma^{2t} W$ is quasi KO_* -equivalent to Σ^2 or Σ^6 when t is odd.

Let $f: \Sigma^{2t-1} X \rightarrow Y$ be a map of order 2. Then we have coextensions

$$\begin{aligned} \tilde{f}_{2m}: \Sigma^{2t} X \rightarrow Y \wedge SZ/2m, \quad \tilde{f}_{V,4m}: \Sigma^{2t} X \rightarrow Y \wedge V_{4m} \quad \text{and} \\ \tilde{f}_{U,4m}: \Sigma^{2t} X \rightarrow Y' \wedge U_{4m} \quad \text{when } Y = Y' \wedge C(\bar{\eta}) \end{aligned}$$

such that $(1 \wedge j) \tilde{f}_{2m} = f$, $(1 \wedge j_V) \tilde{f}_{V,4m} = f$ and $(1 \wedge j_U) \tilde{f}_{U,4m} = f$. As a dual of Lemma 2.4 we have

Lemma 3.6. *Let X and Y be CW-spectra satisfying the property (I), and $f: \Sigma^{2t-1}X \rightarrow Y$ be a map of order 2 whose cofiber $C(f)$ satisfies the property (II), ($t \neq 0$).*

i) *The cofiber $\Sigma^{-2t-1}C(\tilde{f}_2)$ satisfies the property (I), and it is quasi KO_* -equivalent to X or Σ^4X according as t is even or odd.*

ii) *For $\tilde{\varphi}_{4m} = \tilde{f}_{4m}, \tilde{f}_{V,4m}$ or $\tilde{f}_{U,4m}$ each cofiber $\Sigma^{-1}C(\tilde{\varphi}_{4m})$ satisfies the property $(II_{2m})_{*,t}^*$. Under the assumption that these cofibers are finite, all of $C(\tilde{f}_{4m}), \Sigma^4C(\tilde{f}_{V,4m})$ and $C(\tilde{f}_{U,4m})$ are quasi KO_* -equivalent to P_{4m} or Σ^4P_{4m} according as Y is quasi KO_* -equivalent to Σ^0 or Σ^4 whenever t is odd.*

As a dual of (2.3) we set

$$(3.2) \quad \begin{aligned} \tilde{\alpha}_{4r} &= A_2^r i: \Sigma^{8r} \rightarrow SZ/2, & \tilde{\mu}'_{4r+1} &= A_2^r \tilde{\eta}: \Sigma^{8r+2} \rightarrow SZ/2, \\ \tilde{a}'_{4r+2} &= A_2^r \tilde{h}: \Sigma^{8r+1} C(\tilde{\eta}) \rightarrow SZ/2, & \tilde{m}'_{4r+3} &= A_2^r \tilde{k}: \Sigma^{8r+3} C(\tilde{\eta}) \rightarrow SZ/2, \\ \tilde{a}_{4r+2} &= \tilde{h}_{2/2} A_2^r i: \Sigma^{8r+4} \rightarrow SZ/2 \wedge C(\tilde{\eta}). \end{aligned}$$

Since $\Sigma^{-2t-1}C(\tilde{f}'_t) = DC(\tilde{f}'_t)$ for $f_t = \alpha_{4r}, \mu_{4r+1}, a_{4r+2}, m_{4r+3}$ or a'_{4r+2} ($r \geq 0$) with $\alpha'_{4r} = \alpha_{4r}$ and $a'_{4r+2} = a_{4r+2}$, (2.5) implies that

(3.3) each cofiber $\Sigma^{-2t-1}C(\tilde{f}'_t)$ satisfies the property (I) for \tilde{f}'_t given in (3.2), and $C(\tilde{\alpha}_{4r}), \Sigma^2C(\tilde{\mu}'_{4r+1}), C(\tilde{a}'_{4r+2}), \Sigma^2C(\tilde{m}'_{4r+3})$ and $\Sigma^4C(\tilde{a}_{4r+2})$ are all quasi KO_* -equivalent to Σ^1 .

Let $f: \Sigma^{2t-1}X \rightarrow Y$ be a map of order 2 and $\tilde{f}: \Sigma^{2t}X \rightarrow Y \wedge SZ/2$ be its coextension with $(1 \wedge j) \tilde{f} = f$. Then there exists a map $\psi: Y \rightarrow C(\tilde{f})$ of order 2 whose cofiber $C(\psi)$ coincides with $\Sigma^1C(f)$. So we can choose the following maps of order 2:

$$(3.4) \quad \begin{aligned} \alpha'_{-4r}: \Sigma^0 &\rightarrow C(\tilde{\alpha}_{4r}), & \mu'_{-4r-1}: \Sigma^0 &\rightarrow C(\tilde{\mu}'_{4r+1}), \\ a'_{-4r-2}: \Sigma^0 &\rightarrow C(\tilde{a}'_{4r+2}), & m'_{-4r-3}: \Sigma^0 &\rightarrow C(\tilde{m}'_{4r+3}) \quad \text{and} \\ b'_{-4r-2}: C(\tilde{\eta}) &\rightarrow C(\tilde{a}_{4r+2}) \end{aligned}$$

of which each cofiber $C(f'_{-t})$ coincides with $\Sigma^1C(f'_t)$ where $f'_t = \alpha_{4r}, \mu'_{4r+1}, a'_{4r+2}, m'_{4r+3}$ and b'_{4r+2} ($r \geq 0$) with $b'_{4r+2} = a_{4r+2}$. Since the maps f'_{-t} given in (3.4) are respectively dual to those f_{-t} given in (2.5), we have Spanier-Whitehead dualities as $C(f'_{-t}) = \Sigma^1DC(f_{-t})$ for $f_{-t} = \alpha_{-4r}, \mu_{-4r-1}, a_{-4r-2}, m_{-4r-3}$ or b_{-4r-2} ($r \geq 0$).

Dually to (2.2) and (2.6) we obtain the composite maps $f_t j_g$ and $f_{-t} j_g$ ($t \geq 0$) as follows:

$$(3.5) \quad \begin{aligned} \alpha_{4r} j: \Sigma^{8r-2}SZ/2m &\rightarrow \Sigma^0, & \alpha_{4r} j_V: \Sigma^{8r-2}V_{2m} &\rightarrow \Sigma^0, \\ \mu'_{4r+1} j: \Sigma^{8r}SZ/2m &\rightarrow \Sigma^0, & \mu'_{4r+1} j_V: \Sigma^{8r}V_{2m} &\rightarrow \Sigma^0, \end{aligned}$$

$$\begin{aligned}
 & a_{4r+2}j: \Sigma^{8r+2}SZ/2m \rightarrow C(\bar{\eta}), & a_{4r+2}j_V: \Sigma^{8r+2}V_{2m} \rightarrow C(\bar{\eta}), \\
 & a'_{4r+2}(j \wedge 1): \Sigma^{8r-1}SZ/2m \wedge C(\bar{\eta}) \rightarrow \Sigma^0, & a'_{4r+2}j'_V: \Sigma^{8r}V'_{2m} \rightarrow \Sigma^0, \\
 & m'_{4r+3}(j \wedge 1): \Sigma^{8r+1}SZ/2m \wedge C(\bar{\eta}) \rightarrow \Sigma^0, & m'_{4r+3}j'_V: \Sigma^{8r+2}V'_{2m} \rightarrow \Sigma^0. \\
 \\
 & \alpha'_{-4r}j: \Sigma^{-1}SZ/2m \rightarrow C(\bar{\alpha}_{4r}), & \alpha'_{-4r}j_V: \Sigma^{-1}V_{2m} \rightarrow C(\bar{\alpha}_{4r}), \\
 & \tilde{\mu}'_{-4r-1}j: \Sigma^{-1}SZ/2m \rightarrow C(\bar{\mu}_{4r+1}), & \mu'_{-4r-1}j_V: \Sigma^{-1}V_{2m} \rightarrow C(\bar{\mu}_{4r+1}), \\
 (3.6) \quad & a'_{-4r-2}j: \Sigma^{-1}SZ/2m \rightarrow C(\bar{a}_{4r+2}), & a'_{-4r-2}j_V: \Sigma^{-1}V_{2m} \rightarrow C(\bar{a}_{4r+2}), \\
 & m'_{-4r-3}j: \Sigma^{-1}SZ/2m \rightarrow C(\bar{m}_{4r+3}), & m'_{-4r-3}j_V: \Sigma^{-1}V_{2m} \rightarrow C(\bar{m}_{4r+3}), \\
 & b'_{-4r-2}(j \wedge 1): \Sigma^{-1}SZ/2m \wedge C(\bar{\eta}) \rightarrow C(\bar{a}_{4r+2}) \text{ and} \\
 & b'_{-4r-2}j_U: \Sigma^{-1}U_{2m} \rightarrow C(\bar{a}_{4r+2}).
 \end{aligned}$$

Then there hold Spanier-Whitehead dualities as

$$\begin{aligned}
 (3.7) \quad & \text{i) } C(f'_t j) = \Sigma^{2t} DC(if_t) \text{ and } C(f'_t j_V) = \Sigma^{2t} DC(i'_V f_t) \text{ for } f_t = \alpha_{4r}, \mu_{4r+1} \text{ or} \\
 & a'_{4r+2} (r \geq 0) \text{ where } \alpha'_{4r} = \alpha_{4r} \text{ and } a'_{4r+2} = a_{4r+2}. \\
 & \text{ii) } C(f'_t j'_V) = \Sigma^{2t} DC(i'_V f_t) \text{ and } C(f'_t(j \wedge 1)) = \Sigma^{2t} DC((i \wedge 1)f_t) \text{ for } f_t = a_{4r+2} \\
 & \text{or } m_{4r+3} (r \geq 0). \\
 & \text{iii) } C(f'_{-t} j) = \Sigma^1 DC(if_{-t}) \text{ and } C(f'_{-t} j_V) = \Sigma^1 DC(i'_V f_{-t}) \text{ for } f_{-t} = \alpha_{-4r}, \\
 & \mu_{-4r-1}, a_{-4r-2} \text{ or } m_{-4r-3} (r \geq 0). \\
 & \text{iv) } C(b'_{-4r-2}(j \wedge 1)) = \Sigma^1 DC((i \wedge 1)b_{-4r-2}) \text{ and } C(b'_{-4r-2} j_U) = \Sigma^1 DC(i'_U b_{-4r-2}) \\
 & (r \geq 0).
 \end{aligned}$$

By making use of Lemma 3.5 we obtain the following result, which is a dual of Propositions 2.3 and 2.5.

Proposition 3.7. i) For each composite map $f_t j_g (t \geq 0)$ given in (3.5) the cofiber $\Sigma^{-2t} C(f_t j_g)$ satisfies the property $(II_{2m})_t^*$.

ii) For each composite map $f_{-t} j_g (t \geq 0)$ given in (3.6) the cofiber $\Sigma^{-1} C(f_{-t} j_g)$ satisfies the property $(II_{2m})_{-t}^*$.

iii) $C(\alpha_{4r} j), \Sigma^4 C(a_{4r+2} j), C(a'_{4r+2}(j \wedge 1)), \Sigma^{-1} C(\alpha'_{-4r} j), \Sigma^{-1} C(a'_{-4r-2} j)$ and $\Sigma^3 C(b'_{-4r-2}(j \wedge 1))$ are quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^{-1} SZ/2m$, and $C(\alpha_{4r} j_V), \Sigma^4 C(a_{4r+2} j'_V), C(a'_{4r+2} j'_V), \Sigma^{-1} C(\alpha'_{-4r} j_V), \Sigma^{-1} C(a'_{-4r-2} j_V)$ and $\Sigma^3 C(b'_{-4r-2} j_U)$ are quasi KO_* -equivalent to $\Sigma^0 \vee \Sigma^{-1} V_{2m}$.

iv) $C(\mu_{4r+1} j), \Sigma^4 C(\mu_{4r+1} j_V), \Sigma^4 C(m'_{4r+3} j'_V), C(m'_{4r+3}(j \wedge 1)), \Sigma^1 C(\mu'_{-4r-1} j), \Sigma^5 C(\mu'_{-4r-1} j_V), \Sigma^1 C(m'_{-4r-3} j)$ and $\Sigma^5 C(m'_{-4r-3} j_V)$ are all quasi KO_* -equivalent to M'_{2m} .

4. K_* -local types of the real projective spaces

4.1. Let RP^n be the real projective n -space and X_n denote the suspension spectrum $\Sigma^{-n} SP^2 S^n$ whose n -th term is the symmetric square $SP^2 S^n$ of the n -sphere as in [16, §2]. The suspension spectra X_n and RP^n are related by the following commutative diagram

$$\begin{array}{ccccccc}
 & & & & \Sigma^n & = & \Sigma^n \\
 & & & & \downarrow & & \downarrow \\
 RP^{n-1} & \rightarrow & \Sigma^0 & \rightarrow & X_n & \rightarrow & \Sigma^1 RP^{n-1} \\
 \downarrow & & \parallel & & \downarrow & & \downarrow \\
 (4.1) \quad RP^n & \rightarrow & \Sigma^0 & \rightarrow & X_{n+1} & \rightarrow & \Sigma^1 RP^n \\
 & & & & \downarrow & & \downarrow \\
 & & & & \Sigma^{n+1} & = & \Sigma^{n+1}
 \end{array}$$

involving four cofiber sequences [10]. Their KU -homologies and KU -cohomologies are well known ([1, Theorem 7.3] and [14, Theorem 3.3]):

- (4.2) i) $KU_0 X_{n+1} \cong Z$ or $Z \oplus Z$ and $KU_{-1} RP^n \cong Z/2^t$ or $Z \oplus Z/2^t$ according as $n=2t$ or $2t+1$, and $KU_1 X_{n+1} = 0 = KU_0 RP^n$.
 ii) $KU^0 X_{n+1} \cong Z$ or $Z \oplus Z$ and $KU^{-1} RP^n \cong 0$ or Z according as n is even or odd, and $KU^1 X_{n+1} = 0$ and $KU^0 RP^n \cong Z/2^t$ when $n=2t$ or $2t+1$.

We here investigate the behavior of the Adams operation ψ_c^k for X_{n+1} and RP^n .

Lemma 4.1. i) $X_{n+1} = \Sigma^{-n-1} SP^2 S^{n+1}$ satisfies the property (I) or (II)_{t+1} according as $n=2t$ or $2t+1$.
 ii) $\Sigma^1 RP^n$ satisfies the property (I_{2t}) or (II_{2t})_{t+1} according as $n=2t$ or $2t+1$. In addition, $\Sigma^1 RP^\infty$ satisfies the property (I_{2^\infty}).

Proof. It is sufficient to show that in both $KU_0 X_{n+1}$ and $KU_0 \Sigma^1 RP^n$, $\psi_c^k = 1$ or $A_{k,t+1}$ according as $n=2t$ or $2t+1$. The $n=0$ case is evident because $X_1 = \Sigma^0$ and $RP^0 = \{pt\}$. Assume that $\psi_c^k = 1$ in $KU_0 X_{2t-1} \cong Z$ and $KU_{-1} RP^{2t-2} \cong Z/2^{t-1}$ ($t \geq 1$). Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
 0 & \rightarrow & KU_0 \Sigma^0 & \rightarrow & KU_0 X_{2t-1} & \rightarrow & KU_{-1} RP^{2t-2} \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & KU_0 \Sigma^0 & \rightarrow & KU_0 X_{2t} & \rightarrow & KU_{-1} RP^{2t-1} \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & KU_0 \Sigma^{2t} & = & KU_0 \Sigma^{2t} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

with exact rows and columns. Then $KU_0 X_{2t} \cong KU_0 \Sigma^{2t} \oplus KU_0 X_{2t-1} \cong Z \oplus Z$ and $KU_{-1} RP^{2t-1} \cong KU_0 \Sigma^{2t} \oplus KU_{-1} RP^{2t-2} \cong Z \oplus Z/2^{t-1}$, in both of which ψ_c^k is expressed by a matrix $\begin{pmatrix} 1/k^t & 0 \\ c_{k,t} & 1 \end{pmatrix}$ for some rational number $c_{k,t}$. We here use another commutative diagram

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & KU_0 \Sigma^{2t} & = & KU_0 \Sigma^{2t} & \\
 & & & h_{2t*} \downarrow & & \downarrow & \\
 0 \rightarrow & KU_0 \Sigma^0 & \rightarrow & KU_0 X_{2t} & \rightarrow & KU_{-1} RP^{2t-1} & \rightarrow 0 \\
 & \parallel & & \downarrow & & \downarrow & \\
 0 \rightarrow & KU_0 \Sigma^0 & \rightarrow & KU_0 X_{2t+1} & \rightarrow & KU_{-1} RP^{2t} & \rightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
 \end{array}$$

with exact rows and columns. Since the right vertical sequence is expressed into the form of $0 \rightarrow Z \rightarrow Z \oplus Z/2^{t-1} \rightarrow Z/2^t \rightarrow 0$, we may regard that the induced homomorphism $h_{2t*}: KU_0 \Sigma^{2t} \rightarrow KU_0 X_{2t}$ is given by $h_{2t*}(1) = (2, 1)$ where $KU_0 X_{2t} \cong KU_0 \Sigma^{2t} \oplus KU_0 X_{2t-1}$. Since the Adams operation ψ_c^k commutes with h_{2t*} , it is easily computed that $c_{k,t} = 1 - k^t/2k^t$. Thus $\psi_c^k = A_{k,t}$ in both $KU_0 X_{2t} \cong Z \oplus Z$ and $KU_{-1} RP^{2t-1} \cong Z \oplus Z/2^{t-1}$. Further it is immediate that $\psi_c^k = 1$ in both $KU_0 X_{2t+1} \cong Z$ and $KU_{-1} RP^{2t} \cong Z/2^t$.

As a dual of Lemma 4.1 we have

Corollary 4.2. i) *The Spanier-Whitehead dual DX_{n+1} satisfies the property (I) or (II) $_{-t-1}$ according as $n = 2t$ or $2t + 1$. Thus $\psi_c^k = 1$ or $A_{k,-t-1}$ in $KU^0 X_{n+1} \cong Z$ or $Z \oplus Z$ according as $n = 2t$ or $2t + 1$.*

ii) *The Spanier-Whitehead dual DRP^{2t} satisfies the property (I $_t$) and $\Sigma^{-1} DRP^{2t+1}$ does the property (II $_t$) $_{i+1}^*$. Thus $\psi_c^k = 1$ in $KU^0 RP^{2t} \cong KU^0 RP^{2t+1} \cong Z/2^t$ and $\psi_c^k = k^{t+1}$ in $KU^{-1} RP^{2t+1} \cong Z$.*

4.2. In [16, Theorem 2.7] we have determined the quasi KO_* -types of the symmetric square $X_n = \Sigma^{-*} SP^2 S^n$ of the n -sphere and the real projective n -space RP^n .

Theorem 4.3. i) *X_{n+1} is quasi KO_* -equivalent to the following elementary spectrum: $\Sigma^0, C(\eta), \Sigma^4, \Sigma^4 \vee \Sigma^4, \Sigma^4, C(\eta), \Sigma^0, \Sigma^0 \vee \Sigma^0$ according as $n \equiv 0, 1, \dots, 7 \pmod 8$.*

ii) *$\Sigma^1 RP^n$ is quasi KO_* -equivalent to the following elementary spectrum: $SZ/2^{4r}, M_{2^{4r}}, V_{2^{4r+1}}, \Sigma^4 \vee V_{2^{4r+1}}, V_{2^{4r+2}}, M_{2^{4r+2}}, SZ/2^{4r+3}, \Sigma^0 \vee SZ/2^{4r+3}$ according as $n = 8r, 8r + 1, \dots, 8r + 7$.*

By virtue of Lemma 4.1 we can easily observe the behavior of the Adams operation ψ_R^k for X_{n+1} . In fact, (1.1) and (1.8) i) and iii) assert that the Adams operation ψ_R^k behaves in $KO_i X_{n+1}$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:

- (4.3) i) When n is even, $\psi_R^k = k^2$ or 1 according as $i = 4$ or otherwise.
 ii) When $n = 4s + 1$, $\psi_R^k = 1, 1/k^{2s}, k^2$ or $1/k^{2s-2}$ according as $i = 0, 2, 4$ or 6.

iii) When $n=4s+3$, $\psi_R^k = A_{k,2s+2}$, $k^2 A_{k,2s+2}$ or 1 according as $i=0, 4$ or otherwise.

By the aid of (4.3) we next observe the behavior of the Adams operation ψ_R^k for RP^n .

Lemma 4.4. *The Adams operation ψ_R^k acts normally in $KO_i \Sigma^1 RP^n$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:*

- i) *When n is even or infinite, $\psi_R^k = k^2$ or 1 according as $i=4$ or otherwise.*
- ii) *When $n=4s+1$, $\psi_R^k = 1/k^{2s}$, k^2 , $1/k^{2s-2}$ or 1 according as $i=2, 4, 6$ or otherwise.*
- iii) *When $n=4s+3$, $\psi_R^k = A_{k,2s+2}$, $k^2 A_{k,2s+2}$ or 1 according as $i=0, 4$ or otherwise.*

Proof. i) In the $n = \infty$ case our result follows from Lemma 4.1 and (1.6). Use the cofiber sequence $\Sigma^0 \rightarrow X_{2t+1} \rightarrow \Sigma^1 RP^{2t} \rightarrow \Sigma^1$ in the $n=2t$ case. Evidently (4.3) i) implies our result except $\psi_R^k = 1$ in $KO_1 RP^{8r+6} \cong KO_1 RP^{8r+8} \cong Z/2 \oplus Z/2$. As is observed in ii) and iii) below, $\psi_R^k = 1/k^{4r+2}$ in $KO_1 RP^{8r+5} \cong Z \oplus Z/2$ and $\psi_R^k = 1$ in $KO_1 RP^{8r+7} \cong Z/2 \oplus Z/2 \oplus Z/2$. By means of these results we can easily show the rest of our result.

ii) By Lemma 4.1 and Theorem 4.3 ii) we note that $\Sigma^1 RP^{4s+1}$ satisfies the property $(II_{2^{2s}})_{2s+1}$ and it is quasi KO_* -equivalent to $M_{2^{2s}}$. Our result is immediate from (2.1).

iii) Use the cofiber sequence $\Sigma^0 \rightarrow X_{4s+4} \rightarrow \Sigma^1 RP^{4s+3} \rightarrow \Sigma^1$. Then (4.3) iii) implies immediately our result except $\psi_R^k = 1$ in $KO_1 RP^{8s+7} \cong Z/2 \oplus Z/2 \oplus Z/2$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & KO_2 X_n & \rightarrow & KO_1 RP^{n-1} & \rightarrow & KO_1 \Sigma^0 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 0 & \rightarrow & KO_2 X_{n+1} & \rightarrow & KO_1 RP^n & \rightarrow & KO_1 \Sigma^0 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & KO_1 \Sigma^n & = & KO_1 \Sigma^n & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

with $n=8r+7$. Since $\psi_R^k = 1$ in $KO_1 RP^{n-1} \cong KO_2 X_{n+1} \cong Z/2 \oplus Z/2$, a routine computation shows that $\psi_R^k = 1$ in $KO_1 RP^n \cong Z/2 \oplus Z/2 \oplus Z/2$ as in the proof of Lemma 2.2.

Under the assumption that CW -spectra X and Y are finite, X is quasi KO_* -equivalent to Y if and only if the Spanier-Whitehead dual DY is quasi KO_* -equivalent to DX (see [15, Corollary I.1.6]). Therefore Theorem 4.3 ii)

implies that

(4.4) the Spanier-Whitehead dual DRP^n is quasi KO_* -equivalent to the following elementary spectrum: $SZ/2^{4r}, \Sigma^{-1}M'_{2^{4r}}, \Sigma^4V_{2^{4r+1}}, \Sigma^5 \vee \Sigma^4V_{2^{4r+1}}, \Sigma^4V_{2^{4r+2}}, \Sigma^{-1}M'_{2^{4r+2}}, SZ/2^{4r+3}, \Sigma^1 \vee SZ/2^{4r+3}$ according as $n=8r, 8r+1, \dots, 8r+7$ (cf. [9, Theorem 1]).

As a dual of Lemma 4.4 we can easily show

Lemma 4.5. *The Adams operation ν_R^k acts normally in $KO; DRP^n \cong KO^{-i}RP^n$ ($0 \leq i \leq 7$) for each $k \neq 0$ as follows:*

- i) *When n is even, $\nu_R^k = k^2$ or 1 according as $i=4$ or otherwise.*
- ii) *When $n=4s+1$, $\nu_R^k = k^{2s+2}, k^2, k^{2s+4}$ or 1 according as $i=3, 4, 7$ or otherwise.*
- ii) *When $n=4s+3$, $\nu_R^k = k^{2s+2}, k^2, k^{2s+4}$ or 1 according as $i=1, 4, 5$ or otherwise.*

For the Moore spectrum $SZ/2^t$ of type $Z/2^t$ the bottom cell inclusion $i: \Sigma^0 \rightarrow SZ/2^t$ and the top cell projection $j: SZ/2^t \rightarrow \Sigma^1$ are here written as i_t and j_t with emphasis. Similarly the maps $i_V: C(\bar{\eta}) \rightarrow V_{2^t}, j_V: V_{2^t} \rightarrow \Sigma^1, i'_V: \Sigma^2 \rightarrow V_{2^t}$ and $j'_V: V_{2^t} \rightarrow C(\bar{\eta})$ are written as $i_{V,t}, j_{V,t}, i'_{V,t}$ and $j'_{V,t}$. By virtue of Lemmas 4.1 and 4.4 we may now apply Theorems 1.2, 1.7 and 2.6 with (2.8) to determine the K_* -local types of X_{n+1} and RP^n .

Theorem 4.6. i) *The symmetric square $X_{n+1} = \Sigma^{-n-1}SP^2S^{n+1}$ of the $n+1$ -sphere has the same K_* -local type as the following elementary spectrum: $\Sigma^0, C(\mu_{4r+1}), C(\bar{\eta}), C(a_{4r+2}), C(\bar{\eta}), C(m_{4r+3}), \Sigma^0, C(\alpha_{4r+4})$ according as $n=8r, 8r+1, \dots, 8r+7$.*

ii) *The real projective n -space Σ^1RP^n has the same K_* -local type as the following elementary spectrum: $SZ/2^{4r}, C(i_{4r}\mu_{4r+1}), V_{2^{4r+1}}, C(i_{V,4r+1}a_{4r+2}), V_{2^{4r+2}}, C(i_{V,4r+2}m_{4r+3}), SZ/2^{4r+3}, C(i_{4r+3}\alpha_{4r+4})$ according as $n=8r, 8r+1, \dots, 8r+7$. In addition, Σ^1RP^∞ has the same K_* -local type as $SZ/2^\infty$.*

In order to determine the K_* -local type of the Spanier-Whitehead dual DRP^n the following result is useful (cf. [15, Corollary I.1.6]).

Lemma 4.7. *Assume that CW-spectra X and Y are finite. Then X is quasi S_{K^*} -equivalent to Y if and only if the Spanier-Whitehead dual DY is quasi S_{K^*} -equivalent to DX .*

Proof. It is sufficient to show the "only if" part. If X is quasi S_{K^*} -equivalent to Y , then we get a K_* -equivalence $f: Y \rightarrow S_{K \wedge} X$. Choose an adjoint map $Df: DX \rightarrow DY \wedge S_K$ such that $(1 \wedge e_X)(f \wedge 1) = (e_Y \wedge 1)(1 \wedge Df): Y \wedge DX \rightarrow S_K$ where $e_W: W \wedge DW \rightarrow \Sigma^0$ denotes the evaluation map for $W=X$ or Y . Consider the diagram

$$\begin{array}{ccccc}
 K_i DX & \xrightarrow{Df_*} & K_i DY \wedge S_K & \xleftarrow{\cong} & K_i DY \\
 \downarrow \cong & & & & \downarrow \cong \\
 K^{-i} X & \xleftarrow{\cong} & K^{-i} S_{K \wedge} X & \xrightarrow{f_*} & K^{-i} Y
 \end{array}$$

where vertical arrows are the duality isomorphisms. As is easily checked, the above diagram is commutative. Therefore the adjoint map $Df: DX \rightarrow DY \wedge S_K$ becomes a K_* -equivalence because $f: Y \rightarrow S_{K \wedge} X$ is a K^* -equivalence, too. Thus DY is quasi S_{K^*} -equivalent to DX .

Theorem 4.6 combined with Lemma 4.7, (1.4) and (3.7) implies

Theorem 4.8. *The Spanier-Whitehead dual DRP^n of the real projective n -space has the same K_* -local type as the following elementary spectrum: $SZ/2^{4r}$, $\Sigma^{-8r-1}C(\mu'_{4r+1} j_{4r})$, $U_{2^{4r+1}}$, $\Sigma^{-8r-3}C(a'_{4r+2} j'_{v,4r+1})$, $U_{2^{4r+2}}$, $\Sigma^{-8r-5}C(m'_{4r+3} j'_{v,4r+2})$, $SZ/2^{4r+3}$, $\Sigma^{-8r-7}C(\alpha_{4r+4} j_{4r+3})$ according as $n=8r, 8r+1, \dots, 8r+7$.*

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