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ON FOX'S CONGRUENCE CLASSES OF KNOTS, II

Dedicated to Professor Akio Hattori on his sixtieth birthday

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R.H. Fox introduced the notion of congruence class of knots in [3], and he gave a necessary condition for congruence in terms of Alexander matrices and polynomials. S. Suzuki and the author [9] improved his condition and showed that there exist infinitely many congruence classes of knots modulo n, q if $n \neq 1$ and $(n, q) \neq (2, 1), (2, 2)$. Further, they conjectured that all knots are congruent modulo 2, 1 and 2, 2. In this note we will generalize the notion of congruence class for links and give an alternate condition. And we will prove that two links are congruent modulo 2, 1 if and only if the two links are \mathbb{Z}_2 -link-homologous. (The number of congruence classes of μ -component links modulo 2, 1 is just $2^{\mu(\mu-1)/2}$.) As a corollary, we have that all knots are congruent modulo 2, 1.

1. Definitions and Theorems

In this note we only consider a μ -component link $L = K_1 \cup \cdots \cup K_\mu$, that is an ordered collection of μ disjoint simple closed oriented curves K_i 's in a three dimensional oriented sphere S^3 . Two links are said to be *equivalent*, if there is an orientation preserving homeomorphism of S^3 onto itself, which maps one link onto the other preserving the orientation and order of the components. And such an equivalence class of links is called a *link type*. A μ -component link $L = K_1 \cup \cdots \cup K_\mu$ is called *trivial* if there exist μ disjoint disks $D_1 \cup \cdots \cup D_\mu$ in S^3 with $\partial D_i = K_i (i=1, \dots, \mu)$. Especially, we call a 1-component link a *knot*, a 1-component link type a *knot type*, and a 1-component trivial link a *trivial knot*.

R.H. Fox introduced the notion of congruence classes of knots in [3], which can be generalized for links as follows.

DEFINITION. Let n and q be non-negative integers. Two μ -component link types κ and λ are said to be *congruent modulo n, q* , written $\kappa \equiv \lambda \pmod{n, q}$,

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iff there are μ -component links $L_0, L_1, L_2, \dots, L_l$, integers c_1, c_2, \dots, c_l , and trivial knots m_1, m_2, \dots, m_l such that

- (1) L_{i-1} and m_i are disjoint,
- (2) L_i is obtained from L_{i-1} by $(1/c_i n)$ -surgery along m_i (see [10, 11] for a/b -surgery),
- (3) the sum of the linking numbers $\sum_{j=1}^{\mu} lk(K_{i-1,j}, m_i) \equiv 0 \pmod{q}$ where $L_{i-1} = K_{i-1,1} \cup \dots \cup K_{i-1,\mu}$, and
- (4) L_0 represents κ , and L_l represents λ .

REMARK. These relations are equivalence relations. Congruence modulo 0, q is just the link equivalence. Any two μ -component link types are congruent modulo 1, q . If the number of components are distinct, then the two link types are incongruent modulo n, q .

S. Suzuki and the author [9] have studied a necessary condition for congruence modulo n, q in terms of Alexander matrices, Alexander polynomials, and elementary ideals in the sense of Fox [2]. They gave the condition only for knot types, but their condition is clearly generalized for link types as in Theorem 1. From an Alexander matrix $A_{\kappa}(t_1, t_2, \dots, t_{\mu})$ of a μ -component link type κ , we obtain a reduced Alexander matrix $\bar{A}_{\kappa}(t)$ by rewriting t_i 's ($i=1, 2, \dots, \mu$) in entries of the matrix to the same t . Similarly, we obtain the reduced Alexander polynomial $\bar{\Delta}_{\kappa}(t)$ and the reduced elementary ideals. In the following, $\sigma_n(t)$ means $(1-t^n)/(1-t) = 1+t+t^2+\dots+t^{n-1}$.

Theorem 1. *If $\kappa \equiv \lambda \pmod{n, q}$, then, for properly chosen $\bar{A}_{\kappa}(t)$ and $\bar{A}_{\lambda}(t)$, we have*

$$\bar{A}_{\kappa}(t) \equiv \bar{A}_{\lambda}(t) \pmod{\left\{ \begin{array}{l} n(1-t) = (1-t) \sigma_n(t^{0 \times q}), (1-t) \sigma_n(t^{1 \times q}), \\ (1-t) \sigma_n(t^{i_1 \times q}), \dots, (1-t) \sigma_n(t^{i_* \times q}) \end{array} \right\}},$$

and hence

$$\bar{\Delta}_{\kappa}(t) \equiv \pm t^r \bar{\Delta}_{\lambda}(t) \pmod{\left\{ \begin{array}{l} n(1-t) = (1-t) \sigma_n(t^{0 \times q}), (1-t) \sigma_n(t^{1 \times q}), \\ (1-t) \sigma_n(t^{i_1 \times q}), \dots, (1-t) \sigma_n(t^{i_* \times q}) \end{array} \right\}},$$

where i_1, \dots, i_* are all divisors of n and $1 < i_1 < \dots < i_* < n$. Furthermore, we have similar statements for the reduced elementary ideals of deficiency greater than 1.

In the above, $f(t) \equiv g(t) \pmod{\{h_1(t), h_2(t), \dots, h_j(t)\}}$ means that $f(t)$ and $g(t)$ are in the same class of the quotient $\mathbb{Z}\langle t \rangle / (h_1(t), h_2(t), \dots, h_j(t))$, where $(h_1(t), h_2(t), \dots, h_j(t))$ is the ideal generated by $h_1(t), h_2(t), \dots, h_j(t)$ in $\mathbb{Z}\langle t \rangle$. The proof of Theorem 1 is parallel to the proof of [9, Theorem 2], so we omit it.

Applying Theorem 1, we can find infinitely many link types that are incongruent modulo n, q .

Theorem 2. *Let n be an integer greater than 1 and q a non-negative integer such that $(n, q) \neq (2, 1), (2, 2)$. For congruence modulo n, q , there exist infinitely many distinct classes of μ -component link types for each μ (cf. [9, Theorem 3]).*

The proof of Theorem 2 will be given in the next section.

To consider the case $(n, q) = (2, 1)$, we use the following notion.

DEFINITION. Two links $L = K_1 \cup \cdots \cup K_\mu$ and $L' = K'_1 \cup \cdots \cup K'_\mu$ are said to be \mathbb{Z}_2 -link-homologous if and only if $\mu = \nu$ and $lk(K_i, K_j) \equiv lk(K'_i, K'_j) \pmod{2}$ for every $1 \leq i < j \leq \mu$.

Theorem 3. *Two given link types are congruent modulo 2, 1 if and only if two links representing the link types are \mathbb{Z}_2 -link-homologous.*

The proof of Theorem 3 will be given in the section 3. Since all knots are \mathbb{Z}_2 -link-homologous, we have the following.

Corollary. *All knot types are congruent modulo 2, 1.*

This is an answer in the affirmative to one of conjectures in [9]. By his experiment, the author have not found the difference between congruence modulo 2, 1 and modulo 2, 2 for knot types. But we can see the difference for link types as follows.

Proposition 4. *The Borromean rings and a 3-component trivial link are congruent modulo 2, 1, but incongruent modulo 2, 2.*

Proof. The linking number of each pair of components of the Borromean rings is 0 and that of a 3-component trivial link is also 0. Since the two links are \mathbb{Z}_2 -link-homologous, the two links are congruent modulo 2, 1 from Theorem 3. On the other hand, a reduced Alexander matrix of the Borromean rings is

$$\begin{bmatrix} (1-t)^2 & 0 & 0 \\ 0 & (1-t)^2 & 0 \end{bmatrix},$$

and that of a 3-component trivial link is (0). It can be seen that

$$\begin{aligned} \mathbb{Z}\langle t \rangle / (2(1-t), (1-t)(1+t^2)) &\cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \\ \mathbb{Z}\langle t \rangle / (2(1-t), (1-t)(1+t^2), (1-t)^2) &\cong \mathbb{Z} \oplus \mathbb{Z}_2. \end{aligned}$$

Therefore, $(1-t)^2 \not\equiv 0 \pmod{\{2(1-t), (1-t)(1+t^2)\}}$. From Theorem 1, the Borromean rings and a 3-component link are incongruent modulo 2, 1. We complete the proof.

Here, we raise the following conjectures which are extensions of the original conjectures in [9].

CONJECTURE B: If two links are link-homotopic, then the links are con-

gruent modulo 2,2.

CONJECTURE A: If two links are link-homotopic, then the links are deformable to each other by a finite sequence of operations τ^2 's, which are shown in Fig. 1.

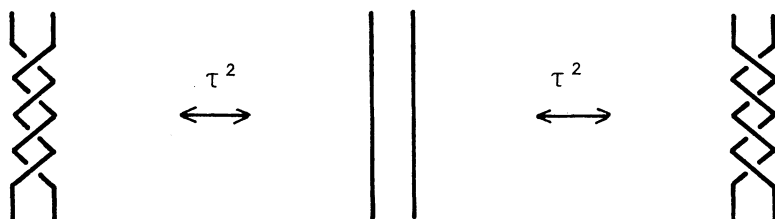


Fig. 1.

NOTE. Conjecture A is proposed by A. Kawauchi. If Conjecture A is true, then Conjecture B is true.

For knot types, all Alexander matrices and polynomials are equivalent up to our condition for congruence modulo 2, 2 as follows.

Theorem 5. *For all knot types, their Alexander matrices, which are chosen properly, are congruent modulo $\{2(1-t), (1-t)\sigma_2(t^2)\}$. Moreover, their Alexander polynomials and elementary ideals are also congruent modulo $\{2(1-t), (1-t)\sigma_2(t^2)\}$ (cf. [9, Theorem 4]).*

The proof of Theorem 5 will be given in the section 4.

2. Proof of Theorem 2

For a non-negative integer $j \in \mathbb{N}_0$, let κ_j be the connected sum of j copies of a trefoil knot. We get a μ -component link λ_j as the split sum of κ_j and a $(\mu-1)$ -component trivial link. Then, the i th reduced elementary ideal $\tilde{E}_i(t)$ of λ_j is (0) for $1 \leq i \leq \mu-1$, $((t^2-t+1)^{\mu+j-i})$ for $\mu \leq i \leq \mu+j-1$, and (1) for $i \geq \mu+j$. As in the proof in [9, §3], their reduced elementary ideals of λ_j 's are mutually distinct mod $\{2(1-t), (1-t)\sigma_n(t^q), (1-t)\sigma_n(t^{i_1 \times q}), \dots, (1-t)\sigma_n(t^{i_* \times q})\}$. Hence, by Theorem 1, there exist infinitely many distinct classes for congruence modulo n, q , completing the proof.

3. Proof of Theorem 3

To prove Theorem 3, we use the notion of Δ -unknotting operation [6] as follows.

DEFINITION. A Δ -unknotting operation is a local move on a link diagram as in Fig. 2. If a diagram of a link L' is a result of a Δ -unknotting operation

on a diagram of L , then we say that L' is obtained from L by a Δ -unknotting operation.

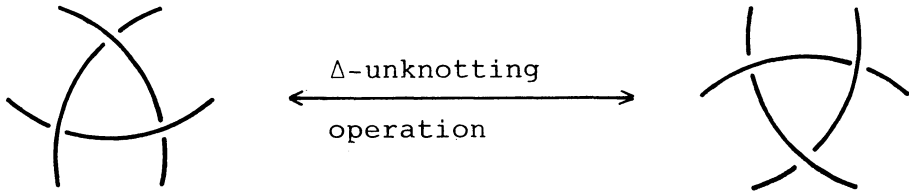


Fig. 2.

DEFINITION. Two links $L=K_1 \cup \dots \cup K_\mu$ and $L'=K'_1 \cup \dots \cup K'_\nu$ are said to be *link-homologous* if and only if $\mu=\nu$ and $lk(K_i, K_j)=lk(K'_i, K'_j)$ for every $1 \leq i < j \leq \mu$.

Proposition 6 ([6]). *Two given links are link-homologous if and only if the two links can be deformed to each other by a finite sequence of Δ -unknotting operations.*

From the above Proposition 6 and the following Lemma, it can be seen that if two links are link-homologous then two link types represented by the links are congruent modulo 2,1.

Lemma. *A Δ -unknotting operation can be realized by $(\pm 1/2)$ -surgeries along trivial knots.*

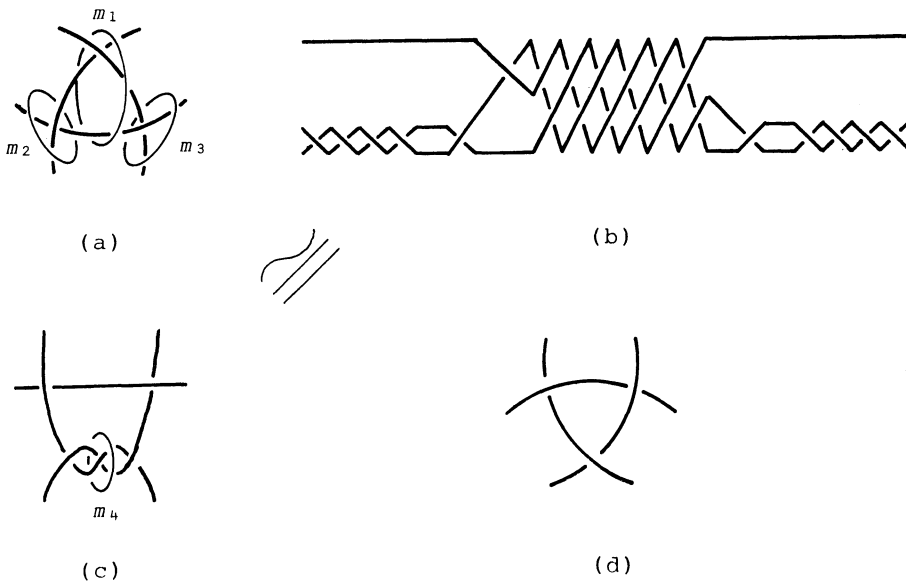


Fig. 3.

Proof. Watch Fig. 3. Take three trivial knots m_1, m_2 , and m_3 as in Fig. 3 (a). By $-1/2$ -surgery along m_1 , $1/2$ -surgery along m_2 , and $1/2$ -surgery along m_3 , we obtain (b) from (a). Take a trivial knot m_4 as in Fig. 3 (c). By $1/2$ -surgery along m_4 , we obtain (d) from (c). Deformation from (a) to (d) is just a Δ -unknotting operation, completing the proof.

Proof of Theorem 3. Let $L = K_1 \cup \cdots \cup K_\mu$ and $L' = K'_1 \cup \cdots \cup K'_\mu$ be \mathbb{Z}_2 -link-homologous, i.e. $lk(K_i, K_j) \equiv lk(K'_i, K'_j) \pmod{2}$ for every pair i, j . If there is a pair i, j such that $lk(K_i, K_j) \not\equiv lk(K'_i, K'_j) \pmod{2}$, then take a trivial knot m such that a disk bounded by m intersects K'_i (and K'_j) in a single point respectively. By $(1/2c)$ -surgery along m for some integer c , the result link $L^* = K_1^* \cup \cdots \cup K_\mu^*$ has $lk(K_i^*, K_j^*) = lk(K_i, K_j)$. Therefore, there exists a link L^* such that L^* and L are link-homologous, and that two link types represented by L^* and L' are congruent modulo 2, 1. As mentioned in the above, two link types represented by L^* and L are congruent modulo 2, 1. Since congruence is an equivalence relation, two link types represented by L and L' are congruent modulo 2, 1. Conversely, a $(1/2c)$ -surgery along a trivial knot does not change the linking number of components modulo 2 for every integer c . Therefore, if two link types represented by two links L and L' are congruent modulo 2, 1, then L and L' are \mathbb{Z}_2 -link-homologous. Hence, we complete the proof.

4. Proof of Theorem 5

In order to show Theorem 5, we use the following.

For a μ -component link $L = K_1 \cup \cdots \cup K_\mu$, let $E = E(L) = S^3 - L$ and take the universal abelian covering $p: \tilde{E}_a \rightarrow E$, associated with the epimorphism $\pi_1(E) \rightarrow \langle t_1, \dots, t_\mu \rangle$ sending each meridian of K_i to $t_i (i=1, \dots, \mu)$, where $\langle t_1, \dots, t_\mu \rangle$ is the free abelian group with a basis t_1, \dots, t_μ . The first integral homology group $H_1(\tilde{E}_a; \mathbb{Z})$ is a finitely generated $\mathbb{Z}\langle t_1, \dots, t_\mu \rangle$ -module and has a presentation matrix as a $\mathbb{Z}\langle t_1, \dots, t_\mu \rangle$ -module, written $P_L(t_1, \dots, t_\mu)$.

Concerning presentation matrices, we note the following well-known fact (cf. [11, pp. 204–205]). For a coefficient ring Λ , the Λ -module presented by a given matrix P is unchanged, up to Λ -isomorphism, by any one of the following operation on P :

- (1) Interchange two rows or two columns.
- (2) Add to any row a Λ -linear combination of other rows.
- (3) Add to any column a Λ -linear combination of other columns.
- (4) Multiply a row or column by a unit of Λ .

- (5) Replace P with the matrix
$$\begin{pmatrix} 1 & * & \cdots & * \\ 0 & & & \\ \vdots & P & & \\ 0 & & & \end{pmatrix}.$$

- (6) The reverse of (5).
- (7) Adjoin a new row which is a Λ -linear combination of rows of P .
- (8) Delete a row which is a Λ -linear combination of other rows.

Proposition 7. *Two matrices, with entries in Λ , present isomorphic Λ -module if and only if one can be deformed into the other by a finite sequence of applications of the above operations (1)–(8).*

For a knot type κ , an Alexander matrix $A_\kappa(t)$ is very similar to a presentation matrix $P_\kappa(t)$ of $H_1(\tilde{E}_\kappa; \mathbf{Z})$ as a $\mathbf{Z}\langle t \rangle$ -module. To say more strictly, we know the following (cf. [2]).

Proposition 8. *Two matrices $A_\kappa(t)$ and $(P_\kappa(t) \ O)$, where O has only one column with all entries 0, are equivalent up to fundamental deformations of presentation matrices. Furthermore, there exists a knot group presentation of κ whose Alexander matrix is $(P_\kappa(t) \ O)$.*

D. Rolfsen gave a characterization of $P_\kappa(t)$ in [11].

Proposition 9. *Up to fundamental deformations of presentation matrices, $P_\kappa(t)$ is equivalent to a matrix of the following form; $(a_{ij}(t))$ where $a_{ij}(t) = a_{ji}(t^{-1})$ and $a_{ij}(1)$ is 0 when $i \neq j$, and 1 when $i = j$.*

The entry $a_{ii}(t)$ has the same properties as Alexander polynomials of knots, so we have $a_{ii}(t) \equiv \pm t^r \cdot 1 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$ like as in [9, Theorem 4]. And we can regard that all $a_{ii}(t)$ are $1 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$ up to fundamental deformations (4). When $i \neq j$, the property $a_{ij}(1) = 0$ implies that $a_{ij}(t) = (1-t) \cdot h(t)$ for a certain Laurent polynomial $h(t)$. It can be seen that $h(t) \equiv 0, \pm t^r \cdot 1$, or $\pm t^r \cdot (1-t) \pmod{\{2, 1+t^2\}}$. So, we have $a_{ij}(t) \equiv 0, \pm t^r \cdot (1-t)$, or $\pm t^r \cdot (1-t)^2 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$ for $i \neq j$.

First, we take the smallest integer i such that $a_{ij}(t) \equiv \pm t^r \cdot (1-t) \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$, and fix i . And we take the smallest integer j such that $a_{ij}(t) \equiv \pm t^r \cdot (1-t) \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$. Adding to the i th row the product of the j th row by $\mp t^r \cdot (1-t)$ (which is one of fundamental deformation), $a_{ij}(t)$ becomes $0 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$. Then, $a_{ii}(t)$ becomes $1, 1 \mp t^s \cdot (1-t)^2$, or $1 \mp t^s \cdot (1-t)^3 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$. Since $(1-t)^3 \equiv 0 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$, $1 \mp t^s \cdot (1-t)^3 \equiv 1 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$. Since $(1-t)^2 \equiv -t(1-t)^2 \equiv t(1-t)^2 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$, $1 \mp t^s \cdot (1-t)^2 \equiv 1 + (1-t)^2 \equiv t^2 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$. Therefore, new $a_{ii}(t)$ can be regarded to be $1 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$ up to fundamental deformation (4). We remark that the other entries $a_{ik}(t) \equiv 0$ or $\pm t^s \cdot (1-t)^2 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$ if $k < j$. Perform these operations inductively, and we have $a_{ij}(t) \equiv 0$ or $\pm t^r \cdot (1-t)^2 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$ for every pair $i \neq j$.

Secondly, for $a_{ij}(t) \equiv \pm t^r \cdot (1-t)^2 \pmod{\{2(1-t), (1-t)\sigma_2(t^2)\}}$, we add to the

i th row the product of the j th row by $\mp t' \cdot (1-t)^2$, and $a_{ij}(t)$ becomes 0 mod $\{2(1-t), (1-t) \sigma_2(t^2)\}$. Since $(1-t)^4 \equiv 0 \pmod{\{2(1-t), (1-t) \sigma_2(t^2)\}}$, we can regard the other entries are unchanged mod $\{2(1-t), (1-t) \sigma_2(t^2)\}$.

Hence, we have the fact that $(a_{ij}(t))$ is congruent to the unit matrix mod $\{2(1-t), (1-t) \sigma_2(t^2)\}$ up to fundamental deformations. Furthermore, $(a_{ij}(t))$ is equivalent to (1) by fundamental deformations (6). Since the fundamental deformations using in the above can be realized by exchanges of knot group presentations of κ , we have the required Theorem 5.

5. Concluding Remarks

We consider an operation τ^n , which is a local move cancelling n full-twists on a link diagram, as shown in Fig. 4. About τ^n operations, S. Kinoshita gave results in [4], which is for a special case of congruence modulo 2,2. Here, we note a μ -variable version as follows. Since the proof is parallel to that in [4], so we omit it.

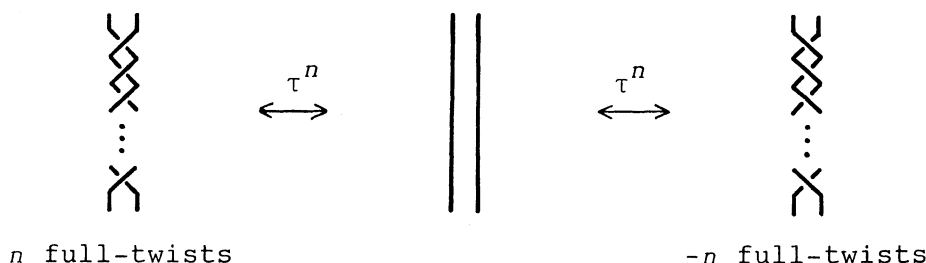


Fig. 4.

Theorem 10. *Let two links L and L^* be deformable to each other by a finite sequence of operations τ^n s. Then, for properly chosen $A_L(t_1, \dots, t_\mu)$ and $A_{L^*}(t_1, \dots, t_\mu)$, we have*

$$A_L(t_1, \dots, t_\mu) \equiv A_{L^*}(t_1, \dots, t_\mu) \pmod{\left\{ \begin{array}{l} (1-t_i) \sigma_n(t_i t_j^{-1}), \\ (1-t_i) \sigma_n(t_i t_j) \\ (1 \leq i, j \leq \mu) \end{array} \right\}}.$$

Furthermore, the Alexander polynomials and elementary ideals of A_L and A_{L^*} are congruent modulo $\{(1-t_i) \sigma_n(t_i t_j^{-1}), (1-t_i) \sigma_n(t_i t_j) \mid (1 \leq i, j \leq \mu)\}$.

In the previous note [8], the author tried to give the P_L -version of the above, but the proof has gaps. Therefore, we give it here as conjecture. If this conjecture is valid, then we can show that the Borromean rings and a 3-component trivial link are never deformed to each other by a finite sequence of operations τ^2 s and link-homotopies.

CONJECTURE. Let two links L and L^* be deformable to each other by a finite sequence of operations τ^n 's. Then, for properly chosen $P_L(t_1, \dots, t_\mu)$ and $P_{L^*}(t_1, \dots, t_\mu)$, we have

$$P_L(t_1, \dots, t_\mu) \equiv P_{L^*}(t_1, \dots, t_\mu) \pmod{\left\{ \begin{array}{l} \sigma_n(t_i t_j), \sigma_n(t_i t_j^{-1}), \\ (1-t_i)(1-t_j)\sigma_{n-1}(t_i t_j), \\ (1-t_i)(1-t_j)\sigma_{n-1}(t_i t_j^{-1}) \\ (1 \leq i, j \leq \mu) \end{array} \right\}}.$$

Furthermore, the Alexander polynomials and elementary ideals of P_L and P_{L^*} are congruent modulo $\{\sigma_n(t_i t_j), \sigma_n(t_i t_j^{-1}), (1-t_i)(1-t_j)\sigma_{n-1}(t_i t_j), (1-t_i)(1-t_j)\sigma_{n-1}(t_i t_j^{-1}) \mid 1 \leq i, j \leq \mu\}$.

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