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ON THE ACTION OF Θ^{2n-1}

Dedicated to Professor Atuo Komatu for his 60th birthday

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Introduction

The group Θ^m of homotopy m-spheres acts on the diffeomorphism classes of manifolds by connected sum. What properties of the manifolds determine this action? In the former part of this paper we give sufficient conditions for manifolds to vary or not to vary with the action of Θ^m. The inertia group $I(M)$ of a closed manifold M^m is the subgroup of Θ^m consisting of the elements Σ such that $M\#\Sigma=M$. ([4], [5])

Theorem 1. *Let M^{2n-1} be a closed π-manifold, $n\geqslant 3$. If there exists a manifold W^{2n}, $bW=M$, such that $H_k(W, M)=0$ for each $k\leqslant n$ and the natural map: $\pi_1(M)\to\pi_1(W)$ is onto, then $M\#\Sigma$ can be imbedded in S^{2n} if and only if $\Sigma=S^{2n-1}$. Hence particularly $I(M)=0$.*

Theorem 2. *Let Σ^{2n-1} be a homotopy sphere which bounds an $(n-1)$-connected manifold W^{2n}, $n\geqslant 3$. If a closed simply connected manifold M^{2n-1} is "resonant" with W^{2n}, then we have*

$$\Sigma\in I(M)\,.$$

The definition of the word "*resonant*" is given in §2.

In the latter part of the present paper we define the signature for some kind of bounded manifolds (§3) and improve a theorem of Novikov [6, Theorem 5.2].

Theorem 4. *Let M^{2n-1} be a closed simply connected manifold, n even>3. Let f, g: $S^{2n-1+k}\to T^k(\nu)$ be two admissible maps in the sense of Novikov ([6]), where $T^k(\nu)$ is the Thom complex of the normal bundle ν^k of M, k large. If there is a homotopy F between f and g such that the signature of F is zero, then the manifold $f^{-1}(M)$ is diffeomorphic to $g^{-1}(M)$ with degree$+1$.*

Theorem 4 is used to define a homomorphism β: $\pi_{4n+k}(T^k(\nu), S^k)\to I(M^{4n-1})$. In the forthcoming paper we shall discuss the relation between the image of β and the extensibility of "almost diffeomorphisms".

Theorem 1 and Theorem 2 are proved by the technique of *surgery*. Lemma

3.2 is fundamental for the proof of Theorem 4.

See W. Browder [2], A. Kosinski [5] for another investigations on the action of Θ^m.

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0. Notations

In this paper all manifolds, with or without boundary, are to be compact, oriented, connected, and differentiable of class C^∞. The boundary of M will be denoted by bM. $M_1 = M_2$ will mean M_1 is diffeomorphic to M_2 by an orientation preserving diffeomorphism. $M_1 \# M_2$ will mean the connected sum of two manifolds M_1 and M_2. Let Θ^m be the m-th homotopy sphere goup. (See [4].) Define bP^{2n} to be the subgroup of Θ^{2n-1} consisting of homotopy spheres which bound π-manifolds (i.e., stably parallelizable manifolds). For a closed manifold M^m the inertia group $I(M^m)$ is defined to be a subgroup of Θ^m consisting of the elements Σ^m such that $M^m \# \Sigma^m = M^m$. Define $I(M; bP)$ to be the subgroup of Θ^{2n-1}/bP^{2n} consisting of classes $\{\Sigma\}$ such that $M \# \Sigma = M \# \Sigma'$ for some $\Sigma' \in bP$. As usual S^m, D^m denote the ordinary m-sphere and the m-disk respectively.

1. Proof of Theorem 1

Let W^{2n}, M^{2n-1} be manifolds, $bW = M$, and let ϕ: $S^p \times D^{2n-1-p} \to M$ be an imbedding. Then a new manifold W' is formed from the disjoint sum $W \cup (D^{p+1} \times D^{2n-1-p})$ by identifying $\phi(x, y) \in M = bW$ with $(x, y) \in S^p \times D^{2n-1-p} = bD^{p+1} \times D^{2n-1-p}$ and "rounding the corners". By definition $M' = bW'$ is the manifold obtained from M by the spherical modification $\chi(\phi)$ (See [4]). We shall say that this modification is p-dimensional, $\dim \chi(\phi) = p$. Let i: $M \subset W$, i': $M' \subset W'$ be the inclusion maps. Put $M_0 = M - \operatorname{Int} \phi(S^p \times D^{2n-1-p})$, and assume $n \geqslant 3$.

Lemma 1.

(1) *If* $1 \leqslant p \leqslant 2n-3$ *and if* i_*: $\pi_1(M) \to \pi_1(W)$ *is onto, then* i'_*: $\pi_1(M') \to \pi_1(W')$ *is also onto.*

(2) *If* $1 \leqslant p \leqslant n-2$ *and if* $H_k(W, M) = 0$ *for* $k \leqslant n$, *then* $H_k(W', M') = 0$ *for* $k \leqslant n$.

(3) *Let* $p = n-1$. *Let* $\phi | S^{n-1} \times 0$ *represent an element* $\lambda \in H_{n-1}(M)$. *If* $i_* \lambda \in H_{n-1}(W)$ *is free, then* $H_k(W) = H_k(W')$ *for* $k \neq n-1$. *If* λ *is primitive in the sense that* $\mu \cdot \lambda = 1$ *for some* $\mu \in H_n(M)$ *and if* $H_{n-1}(M) \xrightarrow[\approx]{} H_{n-1}(W)$, *then* $H_{n-1}(M') \xrightarrow[\approx]{} H_{n-1}(W')$.

Proof. (1) The general position arguments show that $\pi_1(W) \to \pi_1(W')$ and $\pi_1(M') \to \pi_1(M \cup_\phi D^{p+1} \times D^{2n-p-1})$ are onto. There is a commutative diagram:

$$\begin{array}{ccccc} & & \pi_1(M) & \longrightarrow & \pi_1(W) \\ & & \downarrow & & \downarrow \\ \pi_1(M') & \longrightarrow & \pi_1(M \cup_\phi D^{p+1}\times D^{2n-p-1}) & \longrightarrow & \pi_1(W')\,. \end{array}$$

Hence $i'_*\colon \pi_1(M')\to\pi_1(W')$ is onto. (2) From the homology sequence of triple (W, M, M_0), we see that $H_k(W, M_0)=H_k(W, M)=0$ for $k\leqslant n$ since $H_k(M, M_0)=H_k(S^p\times(S^{2n-1-p}, *))=0$ for $k\leqslant n$ if $1\leqslant p\leqslant n-2$. Hence $H_k(W', M_0)\to H_k(W', W)$ is one to one for $k\leqslant n$. Since there is a commutative diagram:

$$\begin{array}{ccc} H_k(D^{p+1}\times S^{2n-2-p}, S^p\times S^{2n-2-p}) & \xrightarrow[\approx]{} & H_k(M', M_0) \\ \downarrow & & \downarrow \\ H_k(D^{p+1}\times D^{2n-1-p}, S^p\times D^{2n-1-p}) & \xrightarrow[\approx]{} & H_k(W', W)\,, \end{array}$$

it follows that $H_k(M', M_0)\xrightarrow[\approx]{}H_k(W', W)$ for $k\leqslant 2n-2$. From a commutative diagram:

$$\begin{array}{ccc} H_k(M', M_0) & \longrightarrow & H_k(W', M_0) \\ & \searrow & \downarrow \\ & & H_k(W', W) \end{array}$$

we see that $H_k(M', M_0)\xrightarrow[\approx]{}H_k(W', M_0)$ for $k\leqslant n$. It follows that $H_k(W', M')=0$ for $k\leqslant n$. (3) Let $p=n-1$. If $i_*\lambda$ is free, then $\partial\colon H_n(W', W)\to H_{n-1}(W)$ is one to one. Hence $H_n(W)\to H_n(W')$ is onto. Since $H_i(W', W)=0$ for $i\neq n$ we see that $H_k(W)=H_k(W')$ for $k\neq n-1$. This proves the former part of (3). There is a commutative diagram:

$$\text{(d)}\qquad \begin{array}{ccccccc} H_n(M', M_0) & \longrightarrow & H_{n-1}(M_0) & \longrightarrow & H_{n-1}(M') & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H_n(W', W) & \longrightarrow & H_{n-1}(W) & \longrightarrow & H_{n-1}(W') & \longrightarrow & 0\,. \end{array}$$

Since λ is primitive, it follows that $H_{n-1}(M_0)\xrightarrow[\approx]{}H_{n-1}(M)$. (See [4, p. 516]). Hence $H_{n-1}(M_0)\xrightarrow[\approx]{}H_{n-1}(W)$ by our assumption. We see also that $H_n(M', M_0)\xrightarrow[\approx]{}H_n(W', W)$ as in the proof of (2). Applying five lemma to (d) we see that $H_{n-1}(M')\xrightarrow[\approx]{}H_{n-1}(W')$. This completes the proof of Lemma 1.

DEFINITION. A sequence of spherical modifications,

$$M=M_0\xrightarrow{\chi(\phi_1)}M_1\xrightarrow{\chi(\phi_2)}M_2\longrightarrow\cdots\xrightarrow{\chi(\phi_r)}M_r\,,$$

is *admissible* if the following conditions are satisfied:

(i) $M_r \in \Theta^{2n-1}$.

(ii) $\dim \chi(\phi_i) \leqslant \dim \chi(\phi_j)$ for $i \leqslant j$.

(iii) $\dim \chi(\phi_i) \leqslant n-1$ for each i.

(iv) if $\dim \chi(\phi_i)=n-1$, then $\{\phi_i | S^{n-1}\times 0\}$ is primitive in $H_{n-1}(M_{i-1})$.

(v) if $\dim \chi(\phi_i)=n-1$, then M_{i-1} is $(n-2)$-conected.

Lemma 2.

(a) *If M satisfies the assumption of Theorem* 1, *then there is at least one admissible sequence of modifications.*

(b) *If M satisfies the assumption of Theorem* 1, *then any admissible sequence of modifications always reduces M to S^{2n-1}.*

Proof of Lemma 2. (a) Let M satisfy the assumption of Theorem 1. Since M is a π-manifold, M can be reduced to a $(n-2)$-connected π-manifold M' by a sequence of framed modifications. ([4]) By (1), (2) of Lemma 1 there is a simply connected $2n$-manifold W', $bW'=M'$, such that $H_k(W', M')=0$ for $k\leqslant n$. Then $H_k(M') \xrightarrow[\approx]{} H_k(W')$ for $k\leqslant n-1$, and so W' is $(n-2)$-connected. Applying the Poincaré duality theorem to (W', M') one sees that $H_{n-1}(M')$ is free. Therefore we can reduce M' to a homotopy sphere M'' by killing primitive elements of $H_{n-1}(M')$. ([4, p. 516]) Since the sequence of modifications which is given in the proof of [4, Theorem 6.6] satisfies the conditions (ii), (iii), the former part (a) is proved. (b) Let M satisfy the assumption of Theorem 1. Let an admissible sequence of modifications,

$$M = M_0 \xrightarrow{\chi(\phi_1)} M_1 \longrightarrow \cdots \xrightarrow{\chi(\phi_r)} M_r\,,$$

be given. If $\dim \chi(\phi_r)\leqslant n-2$, then we see that $H_k(W_r, M_r)=0$ for $k\leqslant n$ by (2) of Lemma 1 and that $\pi_1(M_r)\to\pi_1(W_r)$ is onto by (1) of Lemma 2. Since $M_r\in\Theta^{2n-1}$ by (i), we see that W_r is n-connected. Hence W_r is contractible by the Poincaré duality. By the Smale theorem $M_r=bW_r=S^{2n-1}$. Thus we may assume that $\dim \chi(\phi_i)\leqslant n-2$ and $\dim \chi(\phi_{i+1})=n-1$ for some i. Then M_i is $(n-2)$-connected by (v) and $H_k(W_i, M_i)=0$ for $k\leqslant n$ by (2) of Lemma 1. Hence $H_k(M_i)\xrightarrow[\approx]{} H_k(W_i)$ for $k\leqslant n-1$ and W_i is $(n-2)$-connected. Applying the Poincaré duality theorem to (W_i, M_i) one sees that $H_n(W_i)=0$. By (3) of Lemma 1 and (iv), $H_k(W_i)=H_k(W_{i+1})$ for $k\neq n-1$ and $H_{n-1}(M_{i+1})\xrightarrow[\approx]{} H_{n-1}(W_{i+1})$. Thus we see that W_{i+1} is $(n-2)$-connected, $H_n(W_{i+1})=0$ and $H_{n-1}(M_{i+1})\xrightarrow[\approx]{} H_{n-1}(W_{i+1})$. Since $\dim \chi(\phi_{i+1})=\dim \chi(\phi_{i+2})=\cdots=\dim \chi(\phi_r)=n-1$ by (ii), (iii), we see that W_r is $(n-2)$-connected, $H_n(W_r)=0$ and $H_{n-1}(M_r)\xrightarrow[\approx]{} H_{n-1}(W_r)$ by (3) of Lemma 1 and (iv). Hence W_r is n-connected by (i). By the Poincaré duality and the Smale theorem we see that $M_r = bW_r = S^{2n-1}$. This completes the proof of Lemma 2.

Lemma 3.
(c) *If M satisfies the assumption of Theorem 1 and $M \# \Sigma$ is a submanifold of S^{2n}, then there is at least one admissible sequence of modifications by which $M \# \Sigma$ is reduced to S^{2n-1}.*
(d) *If M is reduced to S^{2n-1} by an admissible sequence of modifications, then $M \# \Sigma$ can be reduced to Σ by some admissible sequence of modifications.*

Proof of Lemma 3. (c) Note that there is a topological manifold $\tilde{W}$, $b\tilde{W} = M \# \Sigma$, such that $H_i(\tilde{W}, M \# \Sigma) = 0$ for each $i \leqslant n$ and the natural map: $\pi_1(M \# \Sigma) \to \pi_1(\tilde{W})$ is onto. Lemma 1 is valid for this topological pair $(\tilde{W}, M \# \Sigma)$. By "exchanging halndles" M is reduced to M' which is an $(n-2)$-connected submanifold of S^{2n} ([1]). As in the proof of Lemma 2 we see that $H_{n-1}(M')$ is free. Hence M' is reduced to a homotopy sphere M'' by killing primitive elements of $H_{n-1}(M')$. Since $M'' \subset S^{2n}$, it follows that $M'' = S^{2n-1}$ by the Smale theorem. This completes the proof of (c). The latter part (d) is trivial.

Theorem 1. *Let M^{2n-1} be a closed π-manifold, $n \geqslant 3$. Assume there exists a manifold W^{2n}, $bW = M$, such that $H_k(W, M) = 0$ for each $k \leqslant n$ and the natural map: $\pi_1(M) \to \pi_1(W)$ is onto. Then $M \# \Sigma$ can be imbedded in S^{2n} if and only if $\Sigma = S^{2n-1}$.*

Proof. From (a), (b) if follows that M, W can be imbedded in S^{2n}. If $M \# \Sigma \subset S^{2n}$, then it follows from (c), (d) and (b) that $\Sigma^{2n-1} = S^{2n-1}$.

Corollary. *Let M^{2n-1} satisfy the assumption of Theorem 1. Let D^{2n-1} be an imbedded disk in M^{2n-1}, Any orientation preserving diffeomorphism of M-Int D can be extended to a diffeomorphism of M.*

Proof. It is an easy consequence of the well-known isomorphism $\Theta^m = \mathrm{Diff}^+(S^{m-1})/i^* \mathrm{Diff}^+(D^m)$.

REMARK. There exists a manifold M^{2n-1} with $I(M) \neq 0$ such that M satisfies all the conditions of Theorem 1 except that $H_n(W, M) \neq 0$. See Remark 1 in §2.

EXAMPLE. Let N^{n-1} be a closed manifold imbedded in $\boldsymbol{R}^{2n}$, $n \geqslant 3$. Then the closed tubular neighbourhood of N is a manifold W^{2n} with boundary. It is easily verified that the closed manifold $bW = M^{2n-1}$ satisfies the conditions of Theorem 1.

2. Proof of Theorem 2

Let M^{2n-1} be a closed manifold and let W^{2n} be an $(n-1)$-connected manifold whose boundary is a homotopy sphere.

DEFINITION. M is *resonant* with W if there is a basis $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_r\}$ of the free group $H_n(W)$ such that the following conditions are satisfied:

(1) $\lambda_i \cdot \lambda_j = 0, \quad \lambda_i \cdot \mu_j = \delta_{ij}$

for all i, j (where δ_{ij} denotes a Kronecker delta, and a dot $\cdot$ denotes the intersection number).

(2) there are imbeddings $\phi_i: S^n \to W$, and $\psi_i: S^n \to M \times I$ such that ϕ_i represents the homology class $\lambda_i \in H_n(W)$ and $c(\phi_i) + c(\psi_i) = 0 \in \pi_{n-1}(\mathrm{SO}(n))$

for all i, where $c(\phi_i)$, $c(\psi_i)$ denote the characteristic classes of the normal bundles of ϕ_i, ψ_i.

DEFINITION. An $(n-1)$-connected $2n$-manifold W with a homotopy sphere boundary is *elementary* if $H_n(W) = \boldsymbol{Z} \oplus \boldsymbol{Z}$ and in the case n even the signature of W^{2n} is zero. Then it is known that there is a basis $\{\lambda, \mu\}$ of $H_n(W)$ satisfying the condition $\lambda \cdot \mu = 1$, $\lambda \cdot \lambda = 0$. [4, p. 529]

Proposition 2.1. *Let W^{2n} be elementary with a basis $\{\lambda, \mu\}$ such that $\lambda \cdot \mu = 1$, $\lambda \cdot \lambda = 0$. Let $\phi: S^n \to W^{2n}$ be an imbedding which is a representative of λ. Let M^{2n-1} be a simply connected closed manifold, $n \geqslant 3$. If there is an imbedding $\psi: S^n \to M^{2n-1} \times I$ such that $c(\phi) + c(\psi) = 0 \in \pi_{n-1}(\mathrm{SO}(n))$, then $bW^{2n} \in I(M)$.*

Proof. Let N^{2n} be a manifold obtained by the connected sum along the boundary of W and $M \times I$ (using $M \times 0$). Then the boundary of N is $M_1 \cup -M_2$, where $M_1 = M \# bW$, $M_2 = M$. Using ϕ and ψ we can construct a new imbedding $\Phi: S^n \to N$ with trivial normal bundle, since $c(\phi) + c(\psi) = 0$. Then Φ represents an element $x \in H_n(N, M_1)$ such that $x = i_* \lambda$, where $i: W \subset (N, M_1)$. Let $\tilde{\Phi}: S^n \times D^n \to N$ be an imbedding which is an extension of Φ. Let N' be the new manifold obtained by $\chi(\tilde{\Phi})$.

Assertion 1. $H_i(N', M_\alpha) = 0$ for $i \leqslant n-2$,

2. $H_{n-1}(N', M_\alpha) = 0$ and $H_n(N', M_\alpha) = 0$, $\alpha = 1, 2$.

Proof of 1. Let $N_0 = N\text{-Int}\, \tilde{\Phi}(S^n \times D^n) = N'\text{-Int}\, \tilde{\Phi}'(D^{n+1} \times S^{n-1})$. Consider the diagram:

$$\begin{array}{l} \qquad\quad\downarrow \\ \to H_i(N_0, M_1) \to H_i(N, M_1) \to H_i(N, N_0) \to H_{i-1}(N_0, M_1) \to \\ \qquad\quad\downarrow \\ \quad H_i(N', M_1) \\ \qquad\quad\downarrow \\ \quad H_i(N', N_0) \\ \qquad\quad\downarrow \\ \quad H_{i-1}(N_0, M_1) \\ \qquad\quad\downarrow \end{array}$$

where the horizontal line is the homology exact sequence of the triple $\{N, N_0, M_1\}$ and the vertical line is the one of the triple $\{N', N_0, M_1\}$. Since $H_i(N', N_0)=H_i(D^{n+1}\times S^{n-1}, S^n\times S^{n-1})$ and $H_i(N, N_0)=H_i(S^n\times D^n, S^n\times S^{n-1})$, we see that $H_i(N_0, M_1)\underset{\approx}{\longrightarrow}H_i(N', M_1)$ for $i\leqslant n-1$ and that $H_i(N_0, M_1)\underset{\approx}{\longrightarrow}H_i(N, M_1)$ for $i\leqslant n-2$. Since $H_i(N, M_1)=H(W, *)=0$ for $i\leqslant n-1$, it is proved that $H_i(N', M_1)=0$ for $i\leqslant n-2$. Similarly $H_i(N', M_2)=0$ for $i\leqslant n-2$.

Proof of 2. Consider the diagram:

$$H_n(W)\underset{\approx}{\xrightarrow{i_*}}H_n(N, M_1)\xrightarrow{j_*}H_n(N, N_0)\underset{\approx}{\xleftarrow{e_*}}H_n(S^n\times D^n, S^n\times S^{n-1})=\boldsymbol{Z}$$

where j is the inclusion $(N, M_1)\subset(N, N_0)$ and e is the excision map. We see that $e_*^{-1}\circ j_*\circ i_*(\lambda)=0$ and $e_*^{-1}\circ j_*\circ i_*(\mu)=1$ since $\lambda\cdot\lambda=0$ and $\lambda\cdot\mu=1$. Hence $H_n(N, M_1)\to H_n(N, N_0)$ is onto. Since $H_{n-1}(N, M_1)=0$, we have $H_{n-1}(N_0, M_1)=0$. From the above vertical sequence it follows that $H_{n-1}(N', M_1)=0$. In the case $i=n$, the above diagram is

$$\begin{array}{ccccccc}
 & H_{n+1}(N', N_0) & \underset{\approx}{\xleftarrow{e'_*}} & H_{n+1}(D^{n+1}\times S^{n-1}, S^n\times S^{n-1})=\boldsymbol{Z} & & & \\
 & \downarrow\partial & & & & & \\
0\longrightarrow & H_n(N_0, M_1) & \xrightarrow{k_*} & H_n(N, M_1) & \xrightarrow{e_*^{-1}\circ j_*} & \boldsymbol{Z} & \longrightarrow 0 \\
 & \downarrow & & & & & \\
 & H_n(N', M_1) & & & & & \\
 & \downarrow & & & & & \\
 & 0\,. & & & & &
\end{array}$$

We see that $k_*\circ\partial\circ e'_*(1)=x=i_*\lambda$. Since $H_n(N, M_1)=\boldsymbol{Z}\oplus\boldsymbol{Z}$ has the basis $\{i_*\lambda, i_*\mu\}$, the map ∂ is onto. Hence $H_n(N', M_1)=0$. Similarly $H_{n-1}(N', M_2)=0$, $H_n(N', M_2)=0$. This completes the proof of Assertion. By the Poincaré duality theorem we see that $H_i(N', M_\alpha)=0$ for all i, $\alpha=1, 2$. The general position arguments show that N' is simply connected. By the h-cobordism theorem we see that $M_1=M_2$, i.e., $M\# bW=M$. This completes the proof of Proposition 2.1.

Lemma 2. *Let H be a free abelian group of rank $2r$. Let Φ be a symmetric or skew-symmetric bilinear form over H. Assume that there is a basis $\{\lambda_1, \cdots, \lambda_r, \mu_1, \cdots, \mu_r\}$ such that $\Phi(\lambda_i, \lambda_j)=0$, $\Phi(\lambda_i, \mu_j)=\delta_{ij}$, $i, j=1, \cdots, r$. Then we can choose a new basis $\{x_1, y_1, x_2, y_2, \cdots, x_r, y_r\}$ which satisfies the conditions;*

$$\Phi(x_i, x_j)=0,\quad \Phi(x_i, y_j)=\delta_{ij}\,,$$
$$\Phi(y_i, y_j)=0\quad \textit{if } i\neq j, \textit{ and } x_i=\lambda_i \textit{ for } i, j=1, \cdots, r\,.$$

Proof. Define x_i, y_i inductively by $x_1=\lambda_1$, $y_1=\mu_1$, and

$$x_i = \lambda_i,\ y_i = \mu_i \mp \sum_{k=1}^{i-1} \Phi(y_k, \mu_i)x_k\,,$$

where $-(+)$ is used in the symmetric (skew-symmetric) case. Then the condition is satisfied.

Proposition 2.2. *Let W be an $(n-1)$-connected $2n$-manifold with a homotopy sphere boundary, $n\geqslant 3$. Assume that the signature of W is zero in the case n even. Then there are elementary manifolds $W_1, \cdots, W_r$ such that $W=W_1\#\cdots\# W_r$, where $2r$ is the rank of $H_n(W)$ and $\#$ denotes the connected sum along the boundary.*

Proof. Since $H_n(W)=H$ is a free abelian group of rank $2r$, and the intersection number matrix has determinant ± 1 by the Poincaré duality theorem, we see that the matrix of intersection numbers, with respect to a suitable basis, is a direct sum of matrices of the form

$$\begin{pmatrix} 0 & 1 \\ \pm 1 & * \end{pmatrix}$$

by Lemma 2 ([4, p. 529].). Hence Corollary 2 is a direct consequence of Wall [7, Lemma 4]. This completes the proof of Proposition 2.2.

From Proposition 2.1 and Proposition 2.2 we obtain the following result which is slightly sharper than the statement in the introduction.

Theorem 2. *Let Σ^{2n-1} be a homotopy sphere which bounds an $(n-1)$-connected $2n$-manifold W, $n\geqslant 3$. If a closed simply connected manifold M^{2n-1} is resonant with W, then $bW_1, bW_2, \cdots, bW_r \in I(M)$, and so especially $bW\in I(M)$, where $\{W_i\}$ is a suitable elementary decomposition.*

Remark 1. (E. H. Brown and B. Steer [3]) Let Σ be the $(4k+1)$-dimensional Kervaire sphere. Here we assume that $bP^{4k+2}=\boldsymbol{Z}_2$. It is known that there is $2k$-connected manifold W^{4k+2}, $bW=\Sigma$, which has following properties:

(i) $H_{2k+1}(W)=\boldsymbol{Z}\oplus\boldsymbol{Z}$ with a basis $\{\lambda, \mu\}$.

(ii) there is an imbedding ϕ: $S^{2k+1}\to W$ which represents λ.

(iii) $c(\phi)$ is equal to the characteristic class $c(\tau(S^{2k+1}))$ of the tangent bundle of S^{2k+1}.

On the other hand $V_{2k+2,2}$, the tangent sphere bundle to S^{2k+1}, satisfies the condition that there is an imbedding ψ: $S^{2k+1}\to V_{2k+2,2}\times I$ with $c(\psi)=-c(\tau(S^{2k+1}))$, since the tangent disk bundle to S^{2k+1} has a nowhere zero cross-section($2k+1$; odd!). Hence the hypothesis of Proposition 2.1 is satisfied. Thus $\Sigma\in I(V)$.

Remark 2. For each elementary $(n-1)$-connected manifold W^{2n} we have always the manifolds M^{2n-1} such that $bW\in I(M)$. For example, if W^{2n} satisfies the hypothesis of Proposition 2.1, there is $x \in \pi_{n-1}(\mathrm{SO}(n-1))$ such that $i_* x = -c(\phi)$, because $\lambda \cdot \lambda = 0$ ([7, Lemma 2]). Let B be the $(n-1)$-disk bundle over S^n with the characteristic class x. Let B_1, B_2 be two copies of the manifold B^{2n-1}. Form M^{2n-1} by identifying bB_1 with bB_2 in the disjoint sum $B_1 \cup -B_2$. Then clearly M is resonant with W, hence $bW \in I(M)$.

3. The relative signature

Let n be even. Let N^{2n} be a $2n$-manifold with $bN = M_1 \cup -M_2$. Suppose there exist continuous maps $r_\alpha: N \to M_\alpha$, $\alpha = 1, 2$, such that

a) the map $r_\alpha \circ i_\alpha$ is homotopic to the identity map of M_α, where i_α is the inclusion map: $M_\alpha \subset N$, $\alpha = 1, 2$,

b) there exists an orientation preserving homotopy equivalence $h: M_1 \to M_2$ such that the map $h \circ r_1$ is homotopic to the map r_2.

Lemma 3.1. *Let j_α: $N \subset (N, M_\alpha)$, k_α: $(N, M_\alpha) \subset (N, M_1 \cup M_2)$ be the inclusion maps. Let $G = \boldsymbol{Z}$ or $G = \boldsymbol{R}$. Then*

(1) $i_\alpha^*: H^*(N; G) \to H^*(M_\alpha; G)$ *is onto,*

(2) $j_\alpha^*: H^*(N, M_\alpha; G) \to H^*(N; G)$ *is injective,*

(3) $k_\alpha^*: H^*(N, M_1 \cup M_2; G) \to H^*(N, M_\alpha; G)$ *is onto,*

(4) $\mathrm{Im}\, j_1^* = \mathrm{Im}\, j_2^* \subset H^*(N; G)$, $\mathrm{Ker}\, i_1^* = \mathrm{Ker}\, i_2^* \subset H^*(N; G)$ *and*

(5) *the following diagram is commutative:*

$$\begin{array}{ccc} H^*(N; G) & \xrightarrow{i_1^*} & H^*(M_1; G) \\ & \searrow^{i_2^*} & \uparrow h^* \\ & & H^*(M_2; G) \end{array}.$$

Proof. By a) we have the split exact sequences:

$$(3.1) \quad \begin{array}{l} 0 \to H_*(M_\alpha; G) \longrightarrow H_*(N; G) \longrightarrow H_*(N, M_\alpha; G) \to 0\,, \\ 0 \to H^*(N, M_\alpha; G) \xrightarrow{j_\alpha^*} H^*(N; G) \xrightarrow{i_\alpha^*} H^*(M_\alpha; G) \to 0\,. \end{array}$$

Applying the Poincaré duality theorem to $(3.1)_*$, we see that k_α^* is onto. From the commutative diagram:

$$\begin{array}{ccccc} & & H^*(N, M_1 \cup M_2; G) & & \\ & \swarrow^{k_1^*} & & \searrow^{k_2^*} & \\ H^*(N, M_1; G) & & & & H^*(N, M_2; G) \\ & \searrow_{j_1^*} & & \swarrow_{j_2^*} & \\ & & H^*(N; G)\,, & & \end{array}$$

it follows that $\operatorname{Im} j_1^* = \operatorname{Im} j_2^*$. By (3.1)* we see $\operatorname{Ker} i_1^* = \operatorname{Ker} i_2^*$. Thus (1)~(4) are proved. Since $i_2^* \circ r_2^* = \mathrm{id}$, we see $r_2^* \circ i_2^* x - x \in \operatorname{Ker} i_2^* = \operatorname{Ker} i_1^*$ for $x \in H^*(N; G)$. Hence $i_1^* \circ r_2^* \circ i_2^* x - i_1^* x = 0$. Since $r_2^* = r_1^* \circ h^*$, it follows $h^* \circ i_2^* x = i_1^* x$. This completes the proof of Lemma 3.1.

Now we may define a homomorphism ν: $H^*(N, M_1; G) \to H^*(N, M_2; G)$ by the relation $\nu \circ k_1^* = k_2^*$. By (2) and (3), we see that the map ν is well-defined and an isomorphism. Define a symmetric bilinear form

$$\phi: H^n(N, M_1; \boldsymbol{R}) \otimes H^n(N, M_1; \boldsymbol{R}) \to \boldsymbol{R}$$

by the formula

$$\phi(x, y) = \langle x \cup \nu(y),\ [N, M_1 \cup M_2] \rangle ,$$

where $[N, M_1 \cup M_2] \in H_{2n}(N, M_1 \cup M_2)$ is the fundamental class of the manifold N.

DEFINITION. The *relative signature* of N is the signature of the bilinear form ϕ.

Let ρ be an orientation preserving homeomorphism: $M_1 \to M_2$. Then an oriented topological manifold W^{2n} is formed from the disjoint sum $N \cup M_1 \times I$ by identifying $(x, 0) \in M_1 \times I$ with $x \in M_1 \subset N$ and $(x, 1)$ with $\rho(x) \in M_2 \subset N$.

Now we can state a fundamental lemma to prove Theorem 4.

Lemma 3.2. *The signature of the manifold W is equal to the signature of the bilinear form ϕ.*

Proof of Lemma 3.2. Consider the Mayer-Vietoris sequence in cohomology for the couple $\{N, M_1 \times I\}$, where N, $M_1 \times I$ are considered the subspaces of W:

$$\to H^{j-1}(N) \oplus H^{j-1}(M_1 \times I) \to H^{j-1}(M_1 \cup M_2) \to H^j(W) \to H^j(N) \oplus H^j(M_1 \times I) \to$$

Identifying $H^*(M_1 \times I)$ with $H^*(M_1)$ by the inclusion: $M_1 \to M_1 \times 0$, and $H^*(M_1 \cup M_2)$ with $H^*(M_1) \oplus H^*(M_2)$, the above sequence becomes:

$$\text{(3.2)} \qquad \to H^{j-1}(N) \oplus H^{j-1}(M_1) \xrightarrow{\ell} H^{j-1}(M_1) \oplus H^{j-1}(M_2) \xrightarrow{\Delta} H^j(W) \\ \to H^j(N) \oplus H^j(M_1) \to \cdots ,$$

where

$$\ell(x, y) = (i_1^* x - y,\ i_2^* x - (\rho^{-1})^* y) \in H^*(M_1) \oplus H^*(M_2)$$

for

$$(x, y) \in H^*(N) \oplus H^*(M_2) .$$

We denote by X the kernel of the restriction:

$$H^n(W; \boldsymbol{R}) \to H^n(M_1; \boldsymbol{R}) .$$

Considering (3.2) with real coefficients, we see that $(\mathrm{Im}\ \Delta)^n \subset X$. We put $B=(\mathrm{Im}\ \Delta)^n$. Let $i: W \subset (W, M_1\times I)$, $j: N \subset W$ be the inclusion maps. Since $\mathrm{Im}\ i^* = X$, we may define the linear map

$$\mu: X \to H^n(N, M_1; \boldsymbol{R})$$

so that the following diagram is commutative:

$$(3.3)\qquad \begin{array}{ccccc} & & H^n(W, M_1\times I; \boldsymbol{R}) & \xrightarrow{i^*} & X \\ & \swarrow{\scriptstyle e^*\ \approx} & & \swarrow{\scriptstyle \mu} & \downarrow{\scriptstyle j^*|X} \\ H^n(N, M_1\cup M_2; \boldsymbol{R}) & & & & \\ & \searrow{\scriptstyle k_1^*} & & & \\ & & H^n(N, M_1; \boldsymbol{R}) & \xrightarrow{j_1^*} & H^n(N; \boldsymbol{R}), \end{array}$$

where e is the excision map. From (2) of Lemma 3.1, we see that the map μ is well-defined and $\mathrm{Ker}\ \mu = \mathrm{Ker}\ (j^*|X)$. From the exactness of (3.2) it follows $\mathrm{Ker}\ \mu = B$. Since μ is onto by (3) of Lemma 3.1, there is a subspace A of $H^n(W; \boldsymbol{R})$ such that

$$\mu|A: A \to H^n(N, M_1; \boldsymbol{R})$$

is the isomorphism and $X = A \oplus B$. Let C be a subspace of $H^n(W; \boldsymbol{R})$ such that $H^n(W; \boldsymbol{R}) = X \oplus C$.

Assertion. (1) $\langle x \cup y, [W] \rangle = \phi(\mu(x), \mu(y))$ *for* $x, y \in A$,
(2) $\langle x \cup y, [W] \rangle = 0$ *for* $x \in B, y \in X$,
(3) $\dim B = \dim C$,

where $[W] \in H_{2n}(W)$ is the fundamental class of W.

Proof. Using (3.3) we see that for $x, y \in X$

$$\begin{aligned} \langle x \cup y, [W] \rangle &= \langle i^*x' \cup i^*y', [W] \rangle && (i^*;\ \text{onto}) \\ &= \langle e^*x' \cup e^*y', [N, M_1 \cup M_2] \rangle \\ &= \langle k_1^*e^*x' \cup k_2^*e^*y', [N, M_1 \cup M_2] \rangle && (\text{naturality}) \\ &= \langle \mu(x) \cup \nu\circ\mu(y), [N, M_1 \cup M_2] \rangle && (\text{definitions of } \mu, \nu) \\ &= \phi(\mu(x), \mu(y)) \,. \end{aligned}$$

Hence (1) is proved. Since $B = \mathrm{Ker}\ \mu$, we have (2).

(3) By (3.2) we see that

$$\dim H^n(W; \boldsymbol{R}) = \dim B + \dim (\mathrm{Ker}\ \ell)^n,$$

$$\dim (\mathrm{Ker}\, \ell)^n = \dim H^n(N; \boldsymbol{R}) + \dim H^n(M_1; \boldsymbol{R}) - \dim (\mathrm{Im}\, \ell)^n \quad \text{and}$$
$$\dim B = 2 \dim H^{n-1}(M_1; \boldsymbol{R}) - \dim (\mathrm{Im}\, \ell)^{n-1}.$$

The Poincaré duality theorem implies that

$$\dim H^{n-1}(M_1; \boldsymbol{R}) = \dim H^n(M_1; \boldsymbol{R}).$$

From (3.1)* it follows

$$\dim A = \dim H^n(N, M_1; \boldsymbol{R}) = \dim H^n(N; \boldsymbol{R}) - \dim H^n(M_1; \boldsymbol{R}).$$

Using these formulas we have

$$\dim B - \dim C = \dim (\mathrm{Im}\, \ell)^n - \dim (\mathrm{Im}\, \ell)^{n-1}. \tag{3.4}$$

Define

$$\ell' : H^*(M_1; \boldsymbol{R}) \oplus H^*(M_1; \boldsymbol{R}) \to H^*(M_1; \boldsymbol{R}) \oplus H^*(M_2; \boldsymbol{R})$$

by

$$\ell'(x, y) = (x - y, (h^{-1})^* x - (\rho^{-1})^* y), \ (x, y) \in H^*(M_1; \boldsymbol{R}) \oplus H^*(M_1; \boldsymbol{R}),$$

where h^{-1} denotes the homotopy inverse of h. From (5) of Lemma 3.1 it follows that the diagram:

$$\begin{array}{ccc} H^*(M_1; \boldsymbol{R}) \oplus H^*(M_1; \boldsymbol{R}) & \xrightarrow{\ \ell'\ } & H^*(M_1; \boldsymbol{R}) \oplus H^*(M_2; \boldsymbol{R}) \\ \uparrow {\scriptstyle i_1^* \oplus \mathrm{id}} & \nearrow {\scriptstyle \ell} & \\ H^*(N; \boldsymbol{R}) \oplus H^*(M_1; \boldsymbol{R}) & & \end{array}$$

is commutative. By (1) of Lemma 3.1 we see that $\mathrm{Im}\, \ell' = \mathrm{Im}\, \ell$. Hence $\dim (\mathrm{Im}\, \ell)^* = \dim (\mathrm{Im}\, \ell')^*$. By (3.4) we have

$$\dim B - \dim C = \dim (\mathrm{Im}\, \ell')^n - \dim (\mathrm{Im}\, \ell')^{n-1}.$$

In order to prove $\dim (\mathrm{Im}\, \ell')^n = \dim (\mathrm{Im}\, \ell')^{n-1}$, we need the following diagrams:

$$\begin{array}{ccc} H^{n-1}(M_1; \boldsymbol{R}) & \xleftarrow{\ (h^{-1} \circ \rho)^*\ } & H^{n-1}(M_1; \boldsymbol{R}) \\ \downarrow {\scriptstyle \kappa} & & \downarrow {\scriptstyle \kappa} \\ \mathrm{Hom}\,(H_{n-1}(M_1; \boldsymbol{R}), \boldsymbol{R}) & \xleftarrow[\ \mathrm{Hom}\,((h^{-1} \circ \rho)_*, 1)\ ]{} & \mathrm{Hom}\,(H_{n-1}(M_1; \boldsymbol{R}), \boldsymbol{R}), \end{array}$$

(3.5)

$$\begin{array}{ccc} H_{n-1}(M_1; \boldsymbol{R}) & \xrightarrow{\ (h^{-1} \circ \rho)_*\ } & H_{n-1}(M_1; \boldsymbol{R}) \\ \uparrow {\scriptstyle \pi} & & \uparrow {\scriptstyle \pi} \\ H^n(M_1; \boldsymbol{R}) & \xleftarrow[\ (h^{-1} \circ \rho)^*\ ]{} & H^n(M_1; \boldsymbol{R}), \end{array}$$

where κ is the isomorphism defined by Kronecker product, π is the isomorphism of Poincaré duality. The upper square is commutative by the naturality and the lower square is commutative since $h^{-1}\circ\rho$ is degree $+1$. Let E^{n-1}, E^n be the subspaces of $H^{n-1}(M_1; \boldsymbol{R})$, $H^n(M_1; \boldsymbol{R})$ corresponding to the eigenvalue 1 of $(h^{-1}\circ\rho)^*$: $H^*(M_1; \boldsymbol{R})\to H^*(M_1; \boldsymbol{R})$. From (3.5) it follows that $\dim E^{n-1}=\dim E^n$. Since $(\operatorname{Ker}\ell')^{n-1}=E^{n-1}$ and $(\operatorname{Ker}\ell')^n=E^n$, we see that $\dim(\operatorname{Ker}\ell')^{n-1}=\dim(\operatorname{Ker}\ell')^n$. Hence $\dim(\operatorname{Im}\ell')^{n-1}$ is equal to $\dim(\operatorname{Im}\ell')^n$. This completes the proof of Assertion. Lemma 3.2 is a direct consequence of Assertion and the following lemma.

Lemma 3.3. *Let Φ be a real symmetric non-degenerate bilinear form over V a real finite dimensional vector space. Let A, B, C be the subspaces of V such that V is isomorphic to the direct sum $A\oplus B\oplus C$. If $\Phi(x, y)=0$ for $x\in B$, $y\in A\oplus B$ and* $\dim B=\dim C$, *then the signature of the form Φ is equal to the one of the form $\Phi|A$ the restriction of Φ to A.*

Proof. We can prove the lemma if we note the following facts.

1) Let $\{b_i\}$, $\{c_i\}$ be the basis of B, C. Then the matrix $(\Phi(b_i, c_j))$ is nonsingular.

2) If a symmetric non-degenerate bilinear form Ψ over $D\oplus E$, $\dim D=\dim E<\infty$, satisfies the condition $\Psi(x, y)=0$ for $x, y\in D$, then the signature of Ψ is zero.

4. A homomorphism β

In this section it is assumed that M^{2n-1} is a closed simply connected manifold with a base point b, $n\geqslant 3$. Denote by ν^k the normal bundle of M, k large. Denote by $T^k(\nu)$ and S^k the Thom complexes of ν^k and the fibre over b respectively. Hence $S^k\subset T^k(\nu)$.

Lemma 4.1. *For a map h: $(D^{2n+k}, S^{2n-1+k})\to(T^k(\nu), S^k)$ we can find a new map*

$$h'\colon (D^{2n+k}, S^{2n-1+k}) \to (T^k(\nu), S^k)$$

such that

a) *the map h' is homotopic to the map h,*
b) *the map $h'|D^{2n+k}$: $D^{2n+k}\to T^k(\nu)$ is t-regular over M,*
c) *the map $h'|S^{2n-1+k}$: $S^{2n-1+k}\to S^k$ is t-regular over b,*
d) *the manifold*

$$(h'|S^{2n-1+k})^{-1}(b)$$

is a homotopy $(2n-1)$-sphere.

Proof. We can easily find a map h'' satisfying the conditions a), b) c).

From [4, Theorem 6.6] it follows that h'' can be deformed to a map h' satisfying all the conditions. The following lemma is clear.

Lemma 4.2. *Suppose two maps* $h, h'\colon (D^{2n+k}, S^{2n-1+k}) \to (T^k(\nu), S^k)$ *satisfy the conditions* a)$\sim$d). *Then*

$$(h\,|\,S^{2n-1+k})^{-1}(b) = (h'\,|\,S^{2n-1+k})^{-1}(b) \# \Sigma$$

for some $\Sigma \in bP^{2n}$.

Define a map

$$\beta'\colon \pi_{2n+k}(T^k(\nu), S^k) \to \Theta^{2n-1}/bP^{2n}$$

by

$$\beta'(x) = \{(h\,|\,S^{2n-1+k})^{-1}(b)\} \in \Theta^{2n-1}/bP^{2n}$$

for x and the representative h satisfying a)$\sim$d). Then it follows from Lemma 4.2 that β' is well-defined and a homomorphism.

Proposition 4.1. Im $\beta' \subset I(M; bP)$.

Proof. For $x \in \pi_{2n+k}(T^k(\nu), S^k)$, let h be the representative of x satisfying the conditions a)$\sim$d). Put

$$(N, bN) = (M \times I, M \times 1) \# (h^{-1}(M), (h\,|\,S^{2n-1+k})^{-1}(b)),$$

where $\#$ denotes the connected sum along the boundary. Then there is an imbedding

$$\tilde{F}\colon N \to S^{2n-1+k} \times I$$

such that

1) $\tilde{F}(M \times 0) = \tilde{F}(N) \cap S^{2n-1+k} \times 0$,
2) $\tilde{F}(M \times 1 \# (h\,|\,S^{2n-1+k})^{-1}(b)) = \tilde{F}(N) \cap S^{2n-1+k} \times 1$,
3) the manifold $\tilde{F}(N)$ orthogonally approaches the boundaries $S^{2n-1+k} \times \dot{I}$.

By the Pontrjagin-Thom construction we have a map

$$F\colon S^{2n-1+k} \times I \to T^k(\nu)$$

such that

1) F, $F\,|\,S^{2n-1+k} \times \dot{I}$ are t-regular over M,
2) $(F_0)^{-1}(M) = M \times 0$ and the map F_0 induces the homotopy equivalence:

$$(F_0)^{-1}(M) \to M,$$

3) $(F_1)^{-1}(M) = M \times 1 \# (h\,|\,S^{2n-1+k})^{-1}(b)$ and the map F_1 induces the homotopy equivalence:

$$(F_1)^{-1}(M) \to M,$$

where $F_t(x)=F(x, t)$ for $(x, t)\in S^{2n-1+k}\times I$. From the following theorem of Novikov we see that

$$M\times 0 = M\times 1 \# (h\,|\,S^{2n-1+k})^{-1}(b) \# \Sigma$$

for some $\Sigma\in bP^{2n}$. This proves that $\beta'(x)\in I(M; bP)$.

Theorem (Novikov [6, Theorem 5.2]). *Let two maps f and g: $S^{2n-1+k}\to T^k(\nu)$ be t-regular over M. Suppose the maps*

$$f\,|\,f^{-1}(M)\colon f^{-1}(M)\to M \text{ and}$$
$$g\,|\,g^{-1}(M)\colon g^{-1}(M)\to M$$

are the homotopy equivalences. If f and g are homotopic, then we have

$$f^{-1}(M)=g^{-1}(M)\#\Sigma$$

for some $\Sigma\in bP^{2n}$.

Let n be even. Suppose a map F: $S^{2n-1+k}\times I\to T^k(\nu)$ satisfies the following conditions:

1) F and $F\,|\,S^{2n-1+k}\times \dot{I}$ are t-regular over M,
2) $F\,|\,S^{2n-1+k}\times i$ induces the homotopy equivalence:

$$(F\,|\,S^{2n-1+k}\times i)^{-1}(M)\to M$$

for $i=0, 1$. Then it is easily verified that the manifold $F^{-1}(M)$ satisfies the conditions a), b) in §3.

DEFINITION. The *signature* of F is the relative signature of the manifold $F^{-1}(M)$.

Theorem 4. *Let n be even. Let f, g be as in the theorem of Novikov. If there is a homotopy F between f and g such that the signature of F is zero, then we have*

$$f^{-1}(M)=g^{-1}(M)\,.$$

Proof. Recall the proof of the original theorem of Novikov [6, p. 304]. When we reconstruct the map F by the means of spherical modifications to kill the groups $H_*(F^{-1}(M), f^{-1}(M))$, $H_*(F^{-1}(M), g^{-1}(M))$, the relative signature of $F^{-1}(M)$ is invariant by Lemma 3.2, since the signature of closed topological manifolds is invariant under the spherical modifications. Therefore the last obstruction for h-cobordism is the relative signature of $F^{-1}(M)$, which is zero by the assumption. Hence we see that $f^{-1}(M)=g^{-1}(M)$. This completes the proof of Theorem 4.

We also see that the relative signature of $F^{-1}(M)$ is divisible by 8 when we do not assume the triviality of the signature of the homotopy F. This proves

Proposition 4.2. *Let n be even. The signature of any homotopy F between f and g in the theorem of Novikov is always divisible by* 8.

Lemma 4.3. *Let n be even. For $x\in\pi_{2n+k}(T^k(\nu), S^k)$ there exists a map*

$$h: (D^{2n+k}, S^{2n-1+k}) \to (T^k(\nu), S^k)$$

such that

(1) *h represents the element x,*
(2) *h satisfies the conditions* b), c), d) *in Lemma* 4.1,
(3) *the signature of the manifold $h^{-1}(M)$ is zero.*

Proof. By Lemma 4.1 we can find a map h' satisfying the conditions (1), (2). Construct the map F as in the proof of Proposition 4.1. From Proposition 4.2 it follows that the signature of the homotopy F is divisible by 8. Clearly the signature of F is equal to the signature of the manifold $(h')^{-1}(M)$. Hence we see that the signature of $(h')^{-1}(M)$ is divisible by 8. It is known that for any number $\sigma\equiv 0 \pmod 8$ there exists a homotopy sphere Σ which bounds a π-manifold W with the signature σ ([4]). Using this manifold W, we can deform the map h' to the map h which satisfies the conditions (1), (2), (3). This completes the proof of Lemma 4.3.

Lemma 4.4. *If h and h' are two maps satisfying the conditions* (1), (2), (3) *in Lemma* 4.3. *Then we have*

$$(h\,|\,S^{2n-1+k})^{-1}(b) = (h'\,|\,S^{2n-1+k})^{-1}(b)\,.$$

Proof. Let H be the homotopy between h and h'. We may assume that the maps H and $H|S^{2n-1+k}\times I$ are t-regular over M and b respectively. Then the manifold $H^{-1}(M)$ has the boundary

$$(h')^{-1}(M)\cup(H\,|\,S^{2n-1+k}\times I)^{-1}(b)\cup-(h)^{-1}(M)\,.$$

Since the signature of boundaries is zero, we see that the signature of the manifold $(H\,|\,S^{2n-1+k}\times I)^{-1}(b)$ is zero by the condition (3). Since $(H\,|\,S^{2n-1+k}\times I)^{-1}(b)$ is a π-manifold which boundaries are the disjoint sum

$$(h'\,|\,S^{2n-1+k})^{-1}(b)\cup-(h\,|\,S^{2n-1+k})^{-1}(b)\,,$$

it follows from [4, Theorem 7.5] that

$$(h'\,|\,S^{2n-1+k})^{-1}(b) = (h\,|\,S^{2n-1+k})^{-1}(b)\,.$$

This completes the proof of Lemma 4.4.

Now we can define a homomorphism

$$\beta: \pi_{2n+k}(T^k(\nu), S^k) \to \Theta^{2n-1}$$

by

$$\beta(x) = (h \mid S^{2n-1+k})^{-1}(b),$$

where x, h are as in Lemma 4.3 and n is even. From the proof of Proposition 4.1 and Theorem 4 it follows

Proposition 4.3. Im $\beta \subset I(M)$.

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