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ON GALOIS ALGEBRA OVER A COMMUTATIVE RING

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In [1], M. Auslander and O. Goldman introduced the notion of a Galois extension of a commutative ring. Galois theory for separable extension of a commutative ring has been developed by S. U. Chase, D. K. Harrison and A. Rosenberg in [3]. The author, in [6] and [7], generalized the notion of a Galois extension of a commutative ring to the case of non commutative ring, and developed the Galois theory for separable algebra over a commutative ring. We call here an algebra Λ over a commutative ring R a *Galois algebra* if Λ is a Galois extension of R . The study of Galois algebra over a commutative ring has been done by F. R. DeMeyer in [4] and [5], and Y. Takeuchi in [11]. In this note we investigate the structure of such Galois algebra over a commutative ring.

In §2 we prove that if Λ is a Galois algebra over a commutative ring R with group G and if C is the center of Λ then Λ is a direct sum of C -submodule J_σ of Λ with $\sigma \in G$ where $J_\sigma = \{a \in \Lambda \mid \sigma(x)a = ax \text{ for every } x \text{ in } \Lambda\}$. Using this fact, we give shorter proofs of the results of F. DeMeyer in [4] and [5] and Y. Takeuchi in [11]. In §3 we prove that if Λ is a Galois algebra over R with group G then, for each σ in G , $c_\sigma = J_\sigma J_{\sigma^{-1}}$ is an idempotent ideal of the center of Λ which is generated by an idempotent element. As corollary to this theorem, we reduce the following Harrison-DeMeyer's theorems. If Λ is a Galois algebra over R with group G and if the center C of Λ is indecomposable then Λ is a Galois algebra over C and C is a Galois algebra over R . If Λ is a Galois algebra over R with cyclic group G , then Λ is commutative.

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Throughout this note we assume that every ring has an identity element.

1. Definitions and Preliminary results.

Let Λ be a ring, G a finite group of ring automorphisms of Λ , and let $\Delta = \Delta(\Lambda, G) = \sum_{\sigma \in G} \oplus \Lambda U_\sigma$ be the crossed product of Λ and G with trivial

factor set, i.e. $\{U_\sigma\}$ is a Λ -free basis of Δ and $U_\sigma U_\tau = U_{\sigma\tau}$, $U_\sigma \lambda = \sigma(\lambda)U_\sigma$ for $\lambda \in \Lambda$. We let Λ^G denote the totality of elements of Λ which are left invariant by G . For λ in Λ , we let λ_r (or λ_l) denote the right (or left) multiplication by λ on Λ and Γ_r (or Γ_l) denote the totality of λ_r (or λ_l) with $\lambda \in \Gamma$. In [6] we generalized the notion of Galois extension defined first by M. Auslander and O. Goldman [1] to the non commutative case. Our definition of Galois extension is as follows. A ring Λ is called a Galois extension of a ring Γ relative to G , if the following conditions are satisfied;

- I. $\Gamma = \Lambda^G$,
- II. Δ is finitely generated projective Γ -module.
- III. $\delta: \Delta(\Lambda, G) \rightarrow \text{Hom}_{\Gamma_r}(\Lambda, \Lambda)$ is an isomorphism where δ is defined by $\delta(\lambda U_\sigma) = \lambda_r \sigma$ for $\lambda \in \Lambda$.

If Λ is an algebra over a commutative ring R , and if Λ is a Galois extension of R relative to G , then we call Λ a *Galois algebra* over R with group G . If Λ is a Galois algebra over R with Group G , and if R is the center of Λ , then we call Λ *central Galois algebra* over R with group G . In [3], Chase, Harrison and Rosenberg gave another definition of Galois extension for the case of commutative ring which is equivalent to the definition by Auslander and Goldman [1]. We consider the following Definition the case of non commutative ring; Λ is called a Galois extension of Γ with group G , if the following conditions are satisfied;

- I'. $\Gamma = \text{Tr}(\Lambda)$, where $\text{Tr}(x) = \sum_{\sigma \in G} \sigma(x)$ for $x \in \Lambda$,
- II'. there exist x_1, x_2, \dots, x_s and y_1, y_2, \dots, y_s in Λ such that for $\sigma \in G$

$$\sum_{i=1}^s x_i \sigma(y_i) = \begin{cases} 1, & \text{if } \sigma = 1 \\ 0, & \text{if } \sigma \neq 1. \end{cases}$$

In [7], we have seen that if Λ is an algebra over R then “Galois extension Λ of R ” in our sense and that in their sense are equivalent.

Now, let Λ be an arbitrary ring, and C the center of Λ . We generalize the argument for J_σ in [10], §3. For any ring automorphism σ of Λ , let $J_\sigma = \{a \in \Lambda \mid \sigma(x)a = ax \text{ for every } x \in \Lambda\}$. Then J_σ is a C -submodule of Λ and we may show easily the following properties. If σ and τ are ring automorphisms of Λ , then

- 1) $J_\sigma J_\tau \subset J_{\sigma\tau}$,
- 2) $\tau(J_\sigma) = J_{\tau\sigma\tau^{-1}}$,
- 3) $J_\sigma \Lambda = \Lambda J_\sigma$ is a two sided ideal of Λ ,
- 4) for the identity mapping 1 of Λ , $J_1 = C$.

For a central separable algebra Λ over C , using the result in Rosenberg and Zelinsky [10], we have

Lemma 1. *Let Λ be a central separable algebra over C and σ a ring automorphism of Λ which leaves C element wise fixed. Then we have*

- 1) $J_\sigma\Lambda = \Lambda J_\sigma = \Lambda$,
- 2) $J_\sigma J_{\sigma^{-1}} = J_{\sigma^{-1}} J_\sigma = C$,
- 3) σ is an inner automorphism of Λ if and only if there is an element x in J_σ such that $xC = J_\sigma$ (cf. Lemma in 5 in [10]),
- 4) if C is a semi-local ring then σ is an inner automorphism of Λ .

Proof. 1). By Theorem 3.1 in [1], the homomorphism $g: \Lambda \otimes_C J_\sigma \rightarrow \Lambda$, defined by $g(\lambda \otimes a) = \lambda a$ for $\lambda \in \Lambda$ and $a \in J_\sigma$, is an isomorphism as C -module. Therefore, we have $\Lambda J_\sigma = \Lambda$. 2). $\mathfrak{c}_\sigma = J_\sigma J_{\sigma^{-1}}$ is an ideal of C , and $\mathfrak{c}_\sigma \Lambda = J_\sigma J_{\sigma^{-1}} \Lambda = J_\sigma \Lambda = \Lambda$. Since Λ is central separable, $\mathfrak{c}_\sigma = \mathfrak{c}_\sigma \Lambda \cap C = C$. 3) is clear by 1). 4). We suppose that C is semi-local. Let $\mathfrak{p}_1, \mathfrak{p}_2, \dots, \mathfrak{p}_r$ be the maximal ideals of C . We first show that there is an element x in J_σ such that $x \notin \mathfrak{p}_i J_\sigma$ for every maximal ideal \mathfrak{p}_i of C . Since $J_\sigma J_{\sigma^{-1}} = C$, we have $\mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_{i-1} \mathfrak{p}_{i+1} \dots \mathfrak{p}_r J_\sigma \subsetneq \mathfrak{p}_i J_\sigma$ for $i = 1, 2, \dots, r$. For each i , there is an element x_i in J_σ such that

$$x_i \in \mathfrak{p}_1 \mathfrak{p}_2 \dots \mathfrak{p}_{i-1} \mathfrak{p}_{i+1} \dots \mathfrak{p}_r J_\sigma \text{ and } x_i \notin \mathfrak{p}_i J_\sigma.$$

Put $x = \sum_{i=1}^r x_i$. Then x is contained in J_σ , but is not contained in $\mathfrak{p}_i J_\sigma$ for every \mathfrak{p}_i . Now, we shall show $xC = J_\sigma$. Since, by Proposition 4 in [10], J_σ is a finitely generated projective and rank one C -module, we have $[J_\sigma \otimes_C \mathfrak{p}_i : C/\mathfrak{p}_i] = 1$ for every \mathfrak{p}_i . Since $xC \subsetneq \mathfrak{p}_i J_\sigma$, $J_\sigma = xC + \mathfrak{p}_i J_\sigma$ for $i = 1, 2, \dots, r$. By Nakayama's Lemma, we have $J_\sigma = xC$. By 3), this completes the proof.

2. Structure theorem.

Proposition 1. *If Λ is a Galois extension of Γ relative to G , then*

$$V_\Lambda(\Gamma) = \sum_{\sigma \in G} \bigoplus J_\sigma$$

where $V_\Lambda(\Gamma)$ is the commutor ring of Γ in Λ .

Proof. From our definition of Galois extension, we may identify $\Delta = \Delta(\Lambda, G) = \sum_{\sigma \in G} \bigoplus \Lambda U_\sigma$ and $\text{Hom}_{\Gamma, r}(\Lambda, \Lambda)$ by the isomorphism δ . Then we may denote $\Delta(\Lambda, G) = \sum_{\sigma \in G} \bigoplus \Lambda_\sigma \sigma$. It follows that $V_\Delta(\Lambda) = V_{\text{Hom}_{\Gamma, r}(\Lambda, \Lambda)}(\Lambda) = \text{Hom}_{\Lambda, \Gamma, r}(\Lambda, \Lambda) = (V_\Lambda(\Gamma))_r$. On the other hand, an easy computation shows $V_\Delta(\Lambda) = \sum_{\sigma \in G} \bigoplus J_{\sigma^{-1}} U_\sigma = (\sum_{\sigma \in G} \bigoplus J_{\sigma^{-1}})_r$. Therefore, we have $V_\Lambda(\Gamma) = \sum_{\sigma \in G} \bigoplus J_\sigma$. From this proposition we have immediately

Theorem 1. *Let Λ be a Galois algebra over a commutative ring R with group G . Then we have $\Lambda = \sum_{\sigma \in G} \oplus J_{\sigma}$.*

Proposition 2. *Let Λ be a Galois algebra over R with group G , G the center of Λ , and let $c_{\sigma} = J_{\sigma} \Lambda \cap C$ for each σ in G . Then c_{σ} is an ideal of C and $c_{\sigma} \Lambda = J_{\sigma} \Lambda$. For σ, τ in G , we have the following properties;*

- 1) $c_{\sigma} = 0$ if and only if $J_{\sigma} = 0$,
- 2) $J_{\sigma} J_{\tau} = c_{\sigma} J_{\sigma\tau} = c_{\tau} J_{\sigma\tau}$,
- 3) $J_{\sigma} J_{\sigma^{-1}} = J_{\sigma^{-1}} J_{\sigma} = c_{\sigma}$, therefore $c_{\sigma} = c_{\sigma^{-1}}$,
- 4) $c_{\sigma} J_{\sigma} = J_{\sigma}$,
- 5) $c_{\sigma}^2 = c_{\sigma}$,
- 6) $c_{\sigma} = C$ if and only if σ leaves each element of the center C invariant, i.e. $\sigma|C = 1$,
- 7) if $\sigma|C = 1$ or $\tau|C = 1$ then $J_{\sigma} J_{\tau} = J_{\sigma\tau}$.

Proof. Let Λ be a Galois algebra over R with group G . Then Λ is separable over R (cf. Proposition 4 in [6]), therefore Λ is central separable over C and C is separable over R . From the central separability of Λ , we obtain $c_{\sigma} \Lambda = J_{\sigma} \Lambda$ for $c_{\sigma} = C \cap J_{\sigma} \Lambda$. Since $\Lambda = \sum_{\sigma \in G} \oplus J_{\sigma}$, we have $c_{\sigma} \Lambda = \sum_{\tau \in G} \oplus c_{\sigma} J_{\tau}$ and $J_{\sigma} \Lambda = \sum_{\tau \in G} J_{\sigma} J_{\tau}$. Since $J_{\sigma} J_{\tau} \subset J_{\sigma\tau}$ and $c_{\sigma} \Lambda = J_{\sigma} \Lambda$, we have $J_{\sigma} \Lambda = \sum_{\tau \in G} \oplus J_{\sigma} J_{\tau}$ and $c_{\sigma} J_{\sigma\tau} = J_{\sigma} J_{\tau}$. Similarly, $c_{\tau} J_{\sigma\tau} = J_{\sigma} J_{\tau}$. In particular, taking $\tau = \sigma^{-1}$ or $\tau = 1$, we have $J_{\sigma} J_{\sigma^{-1}} = c_{\sigma} = c_{\sigma^{-1}}$ or $c_{\sigma} J_{\sigma} = J_{\sigma}$, and $c_{\sigma}^2 = c_{\sigma} J_{\sigma} J_{\sigma^{-1}} = J_{\sigma} J_{\sigma^{-1}} = c_{\sigma}$. If σ is an automorphism of the central separable algebra Λ over C which leaves C element wise fixed, then by Lemma 1 we have $\Lambda J_{\sigma} = \Lambda$, therefore $c_{\sigma} = C \cap \Lambda J_{\sigma} = C$. Conversely, if $c_{\sigma} = C$, then by definition of J_{σ} we have $(\sigma(x) - x)a = 0$ for every x in C and a in J_{σ} . Since $c_{\sigma} = J_{\sigma} J_{\sigma^{-1}}$, $(\sigma(x) - x)C = (\sigma(x) - x)c_{\sigma} = 0$ for every x in C . Therefore $\sigma(x) = x$ for every x in C . If $\sigma|C = 1$ then by 6) and 2) $J_{\sigma} J_{\tau} = c_{\sigma} J_{\sigma\tau} = J_{\sigma\tau}$.

From Theorem 1 and Proposition 2, we obtain easily the following

Corollary 1. *If Λ is a central Galois algebra over C with group G , then $\Lambda = \sum_{\sigma \in G} \oplus J_{\sigma}$, $J_{\sigma} J_{\tau} = J_{\sigma\tau}$ and $c_{\sigma} = J_{\sigma} J_{\sigma^{-1}} = C$ for every σ in G .*

Corollary 2. (De Meyer and Takeuchi) *If Λ is a central Galois algebra over C with group G , and if every element σ of G is an inner automorphism of Λ associated with a unit u_{σ} in Λ , then $J_{\sigma} = Cu_{\sigma}$ and $\Lambda = \sum_{\sigma \in G} \oplus Cu_{\sigma}$.*

Proposition 3. *Let Λ be a Galois algebra over R with group G , C the center of Λ , and $H = \{\sigma \in G \mid \sigma(x) = x \text{ for every } x \text{ in } C\}$. Then Λ is a central Galois algebra over C with group H if and only if $J_{\tau} = 0$ for every τ in G such that $\tau \notin H$, and then C is a Galois algebra over R with group G/H .*

Proof. Let Λ be Galois algebra over R with group G . Then $\Lambda = \sum_{\sigma \in G} \bigoplus J_\sigma$.

If Λ is a central Galois algebra over C with group H , then $\Lambda^H = C$ and C is a Galois extension of R with group G/H (cf. proof of Theorem 3.1 in [3], or Theorem 1 in [11]). If Λ is a central Galois algebra over C with group H , then, by Theorem 1, $\Lambda = \sum_{\sigma \in H} \bigoplus J_\sigma$, therefore $J_\tau = 0$ for $\tau \notin H$. Conversely, if $J_\tau = 0$ for every $\tau \notin H$, then $\Lambda = \sum_{\sigma \in H} \bigoplus J_\sigma$. Since by Theorem 3 in [6] Λ is a Galois extension of Λ^H relative to H , by Proposition 1 we have $V_\Lambda(\Lambda^H) = \sum_{\sigma \in H} \bigoplus J_\sigma$. therefore $\Lambda = V_\Lambda(\Lambda^H)$, and $\Lambda^H \subset C$. Since $C \subset \Lambda^H$, we have $\Lambda^H = C$, thus Λ is a central Galois algebra over C . This completes the proof.

Proposition 4. *Let Λ be a Galois algebra over R with group G , and let $N(\sigma) = \{\tau \in G \mid \tau\sigma = \sigma\tau\}$ for each $\sigma \in G$, then we have the following statements;*

- 1) *for $J_\sigma \neq 0$ and $J_\tau, J_\sigma = J_\tau$ if and only if $\sigma = \tau$,*
- 2) *each element of $N(\sigma)$ induces an automorphism of C -module J_σ , and if $J_\sigma \neq 0$ then τ is contained in $N(\sigma)$ if and only if $\tau(J_\sigma) = J_\sigma$,*
- 3) *for $\sigma \neq 1$ in G and x in J_σ , if $\tau(x) = x$ for every τ in $N(\sigma)$, then $x = 0$.*
- 4) *for $\sigma \neq 1$ and for every x in J_σ , $\sum_{\tau \in N(\sigma)} \tau(x) = 0$,*
- 5) *for $\sigma \neq 1$ in G and for every x in J_σ , $\text{Tr}(x) = 0$.*

Proof. 1) and 2) are clear. To prove 3), let $\tau_1, \tau_2, \dots, \tau_r$ be the right coset representatives of G modulo $N(\sigma)$. If x in J_σ satisfies $\tau(x) = x$ for every τ in $N(\sigma)$, then we put $y = \sum_{i=1}^r \tau_i(x)$. Since $\nu(y) = \sum_i \nu \tau_i(x) = \sum_i \tau_i(x) = y$ for every ν in G , y is contained in $\Lambda^G = R$, and therefore $y \in J_1 = C$. On the other hand, $\tau_i(x) \in \tau_i(J_\sigma) = J_{\tau_i \sigma \tau_i^{-1}} \neq J_1$, and by 2) $\tau_i(J_\sigma) \neq \tau_i(J_\sigma)$ if $i \neq j$. Since Λ is a direct sum of J_τ for $\sigma \in G$, we have $\tau_i(x) = 0$ $i = 1, 2, \dots, r$, and therefore $x = 0$. 4) is easily proved by 3). Now, for every element x in J_σ , $\text{Tr}(x) = \sum_{\sigma \in G} \sigma(x) = \sum_{i=1}^r \tau_i \left(\sum_{\nu \in N(\sigma)} \nu(x) \right) = 0$, therefore we have 5).

Using this proposition we have

Proposition 5. *Let Λ be a Galois algebra over R with group G , C the center of Λ , and let $H = \{\sigma \in G \mid \sigma|C=1\}$. Then the order $|H|$ of H is a unit in R .*

Proof. By 5) in Proposition 4, $\text{Tr}(J_\sigma) = 0$ for $\sigma \neq 1$ in G . Therefore $\text{Tr}(\Lambda) = \text{Tr}(\sum_{\sigma \in G} J_\sigma) = \sum_{\sigma \in G} \text{Tr}(J_\sigma) = \text{Tr}(J_1) = \text{Tr}(C)$, and $R = \text{Tr}(C)$. Then there

is an element a in C such that $\text{Tr}(a)=1$. Let $G=\sigma_1H+\cdots+\sigma_rH$ be the right decomposition of G modulo H . We have $\text{Tr}(a)=\sum_{\sigma \in G} \sigma(a)=|H|(\sum_{i=1}^r \sigma_i(a))=1$. However, $\sum_{i=1}^r \sigma_i(a)$ is contained in $\Lambda^G=R$. Therefore $|H|$ is a unit in R .

Corollary 3. (De Meyer and Takeuchi) *Let Λ be a central Galois algebra over C with group G . Then the order $|G|$ of G is a unit in C .*

Corollary 4. *Let Λ be a central Galois algebra over C with group R . Then Λ is a strongly separable algebra over C in the sense of [9].*

Proof. By Theorem 1 in [9], Λ is a strongly separable algebra over C if and only if $\Lambda/\mathfrak{p}\Lambda$ is a strongly separable algebra over C/\mathfrak{p} for every maximal ideal \mathfrak{p} of C ¹⁾. For a maximal ideal \mathfrak{p} of C , $\Lambda/\mathfrak{p}\Lambda$ is a central simple algebra with minimum condition over C/\mathfrak{p} , and $[\Lambda/\mathfrak{p}\Lambda : C/\mathfrak{p}] = [\Lambda \otimes_C C_{\mathfrak{p}} : C_{\mathfrak{p}}] = |G|$. Therefore the degree of the central simple algebra $\Lambda/\mathfrak{p}\Lambda$ is a unit in C/\mathfrak{p} . Thus the degree of $\Lambda/\mathfrak{p}\Lambda$ is prime to the characteristic of C/\mathfrak{p} . By definition of strongly separability in [8], $\Lambda/\mathfrak{p}\Lambda$ is a strongly separable algebra over C/\mathfrak{p} for every maximal ideal \mathfrak{p} of C , which completes the proof.

3. Main theorem.

Proposition 6. *Let Λ be a Galois algebra over R with group G , C the center of Λ , and \mathfrak{c}_{σ} the ideal defined in Proposition 2 for each $\sigma \in G$. Then we have the following statements;*

- 1) $\mathfrak{c}_{\sigma}\mathfrak{c}_{\tau} = \mathfrak{c}_{\sigma}\mathfrak{c}_{\sigma\tau} = \mathfrak{c}_{\tau}\mathfrak{c}_{\sigma\tau}$,
- 2) $\mathfrak{c}_{\sigma} \subset \mathfrak{c}_{\sigma i}$ for any integer i , therefore $\mathfrak{c}_{\sigma} \neq 0$ implies $\mathfrak{c}_{\sigma i} \neq 0$,
- 3) for $\tau \in G$, $\tau(\mathfrak{c}_{\sigma}) = \mathfrak{c}_{-\sigma\tau^{-1}}$,
- 4) for $H = \{\sigma \in G \mid \sigma|C=1\}$, if $\sigma \equiv \tau \pmod{H}$ then $\mathfrak{c}_{\sigma} = \mathfrak{c}_{\tau}$.

Proof. 1) and 3) are clear by Proposition 2, and 2) and 4) are easily proved by 1).

Lemma 2. *Let C be a commutative algebra over R , and \mathfrak{c} an ideal of C such that \mathfrak{c} is idempotent and finitely generated over R . Then \mathfrak{c} is generated by an idempotent element in C .²⁾*

Proof. Let $\mathfrak{c} = \sum_{i=1}^r Rx_i$. Since \mathfrak{c} is idempotent, $\mathfrak{c}^2 = \mathfrak{c} = \sum_{i=1}^r cx_i$. Then, we

1) Let Λ be a central separable algebra over C . Then Λ is strongly separable over C if and only if $\Lambda/\mathfrak{p}\Lambda$ is strongly separable over C/\mathfrak{p} for every maximal ideal \mathfrak{p} of C . (Cf. proof of Theorem 1 in [9].)

2) This lemma suggested to me by M. Harada, I express here my thanks to him,

have $x_i = \sum_j a_{ij}x_j$ with some a_{ij} in \mathfrak{c} . Let Δ be the determinant of the matrix $E - (a_{ij})$, where E is the unit matrix. Then, we can easily see that $\Delta = 1 - e$ with some e in \mathfrak{c} and $xe = 0$ for every x in \mathfrak{c} . Therefore, we have $e^2 = e$ and $ex = x$ for every x , thus $\mathfrak{c} = eC$.

From this lemma, we have the following main theorem ;

Theorem 2. *Let Λ be a Galois algebra over R with group G, C , the center of Λ . Then $\mathfrak{c}_\sigma = J_\sigma J_{\sigma^{-1}}$ is generated by an idempotent element e_σ in C for each σ in G .*

As a corollary of Theorem 2, we have

Theorem 3. (Harrison, De Meyer) *Let Λ be a Galois algebra over R with group G , and let C be the center of Λ . If C is indecomposable, then Λ is a central Galois algebra over C with group H , and C is a Galois algebra over R with group G/H , where $H = \{\sigma \in G \mid \sigma|C = 1\}$.*

Proof. Since the idempotent elements in C are only 0 and 1, for each $\sigma \in G$, by Theorem 2, \mathfrak{c}_σ is either 0 or C . Therefore, if $\tau \notin H$ then $J_\tau = 0$. By Proposition 3, the proof is completed.

Proposition 7. *Let Λ be a Galois algebra over R with group G , and let $\mathfrak{a}_\sigma = \{x \in C \mid xc_\sigma = 0\}$. Then we have the following statements ;*

- 1) $\mathfrak{a}_\sigma = \mathfrak{a}_{\sigma^{-1}} \supset \mathfrak{a}_{\sigma^i}$ for any integer i ,
- 2) $\mathfrak{a}_\sigma \Lambda = \{x \in \Lambda \mid xJ_\sigma = 0\}$,
- 3) $\mathfrak{a}_\sigma \Lambda \cap J_\tau = \mathfrak{a}_\sigma J_\tau$,
- 4) for $x \in J_\sigma$, $x = 0$ if and only if $xJ_\sigma = 0$ (or $xc_\sigma = 0$).
- 5) if $x \in J_\sigma$ and $xJ_{\sigma^i} = 0$ for some integer i , then $x = 0$.

Proof. 1) and 2) are clear by 4) in Proposition 6. Since $\Lambda = \sum_{\sigma \in G} \oplus J_\sigma$, we have $\mathfrak{a}_\sigma \Lambda = \sum_{\tau \in G} \oplus \mathfrak{a}_\sigma J_\tau$, therefore $\mathfrak{a}_\sigma \Lambda \cap J_\tau = \mathfrak{a}_\sigma J_\tau$. In particular, taking $\sigma = \tau$, we have $\mathfrak{a}_\sigma \Lambda \cap J_\sigma = \mathfrak{a}_\sigma J_\sigma = \mathfrak{a}_\sigma \mathfrak{c}_\sigma J_\sigma = 0$, which proves 4). 5) is clear by 1).

For a Galois algebra with abelian group, we have the following proposition with a weaker assumption than Theorem 3.

Proposition 8. *Let Λ be a Galois algebra over R with abelian group G . Then Λ is a strongly separable algebra over R . If R is indecomposable, then Λ is a central Galois algebra over the center C and the center C is a Galois algebra over R .*

Proof. We prove first the second part. Since G is abelian, for every τ in G , $\tau(\mathfrak{c}_\sigma) = \mathfrak{c}_{\tau\sigma\tau^{-1}} = \mathfrak{c}_\sigma$. If $\mathfrak{c}_\sigma \neq 0$, then there is a non zero idempotent element e_σ in C such that $\mathfrak{c}_\sigma = e_\sigma C$, and for every τ in G , $\tau(e_\sigma) = e_\sigma$.

Therefore, e_σ is contained in $\Lambda^G=R$. It must be $e_\sigma=1$. Therefore $\mathfrak{c}_\sigma=C$. By Proposition 3, this completes the proof of the second part. By Theorem 1 in [9] and the second part of this proposition, we can prove the first part; for every maximal ideal \mathfrak{p} of R , $\Lambda \otimes_R R_{\mathfrak{p}}$, is strongly separable over $R_{\mathfrak{p}}$, therefore Λ is strongly separable over R^3 .

Proposition 9. *Let Λ be a Galois algebra over R with group G . If Λ is a strongly separable algebra over R , then we have the following statements;*

- 1) *for each $\sigma \in G$, $\sigma|J_\sigma=1$, i.e. $\sigma(x)=x$ for all x in J_σ ,*
- 2) *for each integer i , if $a \in J_\sigma$ and $b \in J_{\sigma^i}$ then $ab=ba$.*

Proof. If Λ is a strongly separable algebra over R , then by Proposition 1 in [9], $\Lambda=C \oplus [\Lambda, \Lambda]$ where C is the center of Λ and $[\Lambda, \Lambda]$ is a C -submodule of Λ generated by $xy-yx$ for every x, y in Λ . For any x, y in J_σ and z in $J_{\sigma^{-1}}$, it follows that $\sigma(x)yz=yxz=xzy$. Since zy and yz are in $J_\sigma J_{\sigma^{-1}}=J_{\sigma^{-1}}J_\sigma=\mathfrak{c}_\sigma \subset C$, we have $zy-yz \in [\Lambda, \Lambda] \cap C=0$, and therefore $zy=yz$. Thus $\sigma(x)yz=xzy$, and $(\sigma(x)-x)yz=0$. Therefore, $(\sigma(x)-x)J_\sigma J_{\sigma^{-1}}=(\sigma(x)-x)\mathfrak{c}_\sigma=0$, and hence $\sigma(x)=x$. Thus we have 1). By 1), we obtain the statement 2); for every $a \in J_\sigma$ and $b \in J_{\sigma^i}$, $ab=\sigma^i(a)b=ba$.

We now obtain the following Harrison- De Meyers, Theorem.

Theorem 4. (Harrison-De Meyer) *Let Λ be a Galois algebra over R with cyclic group G . Then Λ is commutative.*

Proof. Since G is abelian, by Proposition 8, Λ is strongly separable over R . Now, suppose Λ is non commutative. Let $G=(\sigma)$. Since $\Lambda=\sum_i \oplus J_{\sigma^i}$, there is $J_{\sigma^i} \neq 0$. Let $k=\min(i>0|J_{\sigma^i} \neq 0)$. If $k \nmid i$ then, by 1) in Proposition 6, $\mathfrak{c}_{\sigma^k} \mathfrak{c}_{\sigma^i}=\mathfrak{c}_{\sigma^k} \mathfrak{c}_{\sigma^{i-nk}}=0$ where n is an integer such that $0 < i-nk < k$. Therefore, if $k \nmid i$ then $J_{\sigma^i} J_{\sigma^k}=J_{\sigma^k} J_{\sigma^i}=0$. If $k|i$, i.e. $i=kr$, then by 2) in Proposition 9, $ab=ba$ for every $a \in J_{\sigma^k}$ and $b \in J_{\sigma^{kr}}=J_{\sigma^i}$. Thus $J_{\sigma^k} \neq 0$ is contained in the center $C=J_1$, this is a contradiction. Therefore Λ is commutative.

Now, let Λ be a Galois algebra over R with group G , and C the center of Λ . Then for each $\sigma \in G$, there is an idempotent element e_σ such that $e_\sigma C=\mathfrak{c}_\sigma$. Let $e_\sigma=\sum_{i=1}^r a_i b_i$, $a_i \in J_\sigma$, $b_i \in J_{\sigma^{-1}}$. Then we have

Proposition 10. *Under the above assumption, $e'_\sigma=\sum_{i=1}^r b_i a_i$ is an element in \mathfrak{c}_σ , and satisfies the following conditions;*

3) By Theorem 1 in [9], if Λ is a separable algebra over R , then Λ is strongly separable over R if and only if $\Lambda \otimes_R R_{\mathfrak{p}}$ is strongly separable over $R_{\mathfrak{p}}$ for every maximal ideal \mathfrak{p} of R .

1) $\sigma(x) = e'_\sigma x$ for every $x \in J_\sigma$,
 2) $e'^2 = e_\sigma$ and $e'_\sigma C = c_\sigma$, therefore $\sigma^2|J_\sigma = 1$.

Proof. Since $\sigma(x) \in J_\sigma$ for every $x \in J_\sigma$, we have $\sigma(x) = e_\sigma \sigma(x) = \sum_{i=1}^r a_i b_i \sigma(x) = \sum_{i=1}^r b_i \sigma(x) a_i = \sum_{i=1}^r b_i a_i x = e'_\sigma x$ for $x \in J_\sigma$. Now, $e'^2 = \sum_{ij} b_i (a_i b_j) a_j = \sum_{ij} (a_i b_j) b_i a_j = \sum_{ij} a_i b_j (b_i a_j) = \sum_{ij} a_i (b_i a_j) b_j = e_\sigma^2 = e_\sigma$. It follows that $e'_\sigma C = c_\sigma$ and $\sigma^2(x) = \sigma(\sigma(x)) = e'_\sigma \sigma(x) = e'^2 x = e_\sigma x = x$ for all x in J_σ .

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