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# PROPAGATION OF SINGULARITIES FOR HYPERBOLIC OPERATORS WITH MULTIPLE INVOLUTIVE CHARACTERISTICS

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## 0. Introduction

In this paper we will give a result on propagation of $C^\infty$ singularities generalizing previous results of R.B. Melrose and G.A. Uhlmann [8]; we consider pseudodifferential operators whose principal symbol vanishes at order $m\geqq 2$ on an involutive manifold. Explicity we shall assume:

(i) Let $X$ be a $C^\infty$ manifold of dimension $n$ and let $\Sigma$ be a $C^\infty$ closed conic, non radial, involutive submanifold of codimension $d\geqq 2$ in $T^*(X)\setminus\{0\}$, the cotangent bundle minus the zero section.

We therefore have, denoting by $\omega$ and $\sigma=d\omega$ the canonical 1 and 2 forms in the symplectic manifold $T^*(X)$, $\gamma\in\Sigma\Rightarrow T_\gamma(\Sigma)^\sigma\subset T_\gamma(\Sigma)$ where with $T_\gamma(\Sigma)^\sigma$ we denote the dual with respect to the bilinear form $\sigma$. When $\Sigma$ is given by $\{\gamma\in T^*(X)\backslash 0\,|\, q_1(\gamma)=\cdots=q_d(\gamma)=0\}$ where $q_j\in C^\infty(T^*(X)\setminus\{0\})$, $j=1,\cdots,d$ are positevely homogeneous of degree one and for any $\gamma\in\Sigma$, $dq_j(\gamma)$ and $\omega(\gamma)$ are linearly independent one forms, then we have $\{q_i, q_j\}\ (\gamma)=0$ where $\{q_i, q_j\}$ denotes as usual the Poisson bracket between $q_i$ and $q_j$. Frobenius Theorem then gives that $\Sigma$ is locally foliated of dimension $d$ by the flow out of the Hamiltonian fields of the $q_j$. The leaf through $\gamma^0\in\Sigma$, whose tangent space in $\gamma^0$ is $T_{\gamma^0}(\Sigma)^\sigma$ will be denoted by $F_{\gamma^0}$. Moreover for any $\gamma\in\Sigma$ the bilinear form $\sigma$ induces an isomorphism

$$J_\sigma\colon\ T_\gamma(T^*(X)\backslash 0)/T_\gamma(\Sigma)\to T^*_\gamma(F_\gamma)\,.$$

(ii) Let $\varphi\in C^\infty(X)$ real valued and $\tilde{\varphi}=\varphi\circ\pi$ where $\pi$ from $T^*(X)$ to $X$ is the canonical projection.

Let $P(x, D_x)$ be a classical properly supported pseudodifferential operator of order $m+k$ in $X$, $m\in\boldsymbol{N}$, $k\in\boldsymbol{R}$. Let $P_{m+k}$ be its principal symbol. We assume: $P$ is hyperbolic with respect to the level surfaces of $\varphi$([5]), the Hamiltonian field of $\tilde{\varphi}$, $H_{\tilde{\varphi}}$ is transversal to $\Sigma$ and $P_{m+k}$ vanishes exactly of order $m$ on $\Sigma$.

(iii) (Microlocal Levi Condition) ([9])

Microlocally near every point $\gamma^0\in\Sigma$ in a neighborhood of which $\Sigma$ is given as in (i):

$$P(x, D_x) \equiv \sum_{|\alpha| \leqq m} A_\alpha(x, D_x) Q_1^{\alpha_1}(x, D_x) \cdots Q_d^{\alpha_d}(x, D_x) \tag{0.1}$$

where $Q_1, \cdots, Q_d$ are first order pseudodifferential operators with principal symbol $q_1, \cdots, q_d$, and $A_\alpha$ are pseudodifferential operators of order $k$. Here $A \equiv B$ if there exists $\Gamma \ni \gamma^0$ such that for any $v \in \mathcal{E}'(X)$ $WF(v) \subset \Gamma \Rightarrow (A-B)\, v \in C^\infty(X)$.
It is well known that $P$ induces on $F_{\gamma^0}$ a differential operator $P^0$ homogeneous of order $m$ in the fibers of $T^*(F_{\gamma^0})$: for its principal symbol one has:

$$P^0_{\gamma^0;\, m}(v) = \lim_{t \to 0} t^{-m} P_{m+k}(\gamma^0 + tv) \tag{0.2}$$

$P^4$ is hyperbolic with respect to $J_\sigma(H_\varphi(\gamma^0)) = N(\gamma^0)$. Finally we shall assume here that:
(iv) for any $\gamma \in \Sigma$, $P^0_\gamma$ is strictly hyperbolic with respect to $N(\gamma)$.
Now denoting by $\Gamma_\gamma$ the component of $N(\gamma)$ in the complement of $\{v \in T^*(F_\gamma) \mid P^0_\gamma(v) = 0\}$ and by $(\Gamma_\gamma)^0$ the (euclidean) polar of $\Gamma_\gamma$, ([2]) let $E^+(\gamma)$ ($E^-(\gamma)$) be the forward (backward) emission from $\gamma$ along the field of cones $(\Gamma_\gamma)^0$, cfr. (4.11). Then the result of our paper is given in the following:

**Theorem.** *Let $P$ satisfy assumption* (i)-(iv). *Let $v \in \mathcal{D}'(X)$ and $\gamma^0 \in \Sigma \setminus WF(Pv)$. If there exists a conic neiborhood $\Gamma$ of $\gamma^0$, and a choice of sign $+$ or $-$ such that*:

$$\Gamma \cap WF(v) \cap (E^\pm(\gamma^0) \setminus (\gamma^0)) = \phi$$

*Then $\gamma^0$ does not belong to $WF(v)$.*

REMARKS.

(i) R.B. Melrose and G.A. Uhlmann proved the theorem when $m=2$ (and $d \geqq 3$: if $d=2$ see [10] for the construction of a microlocal parametrix). In that case assumption (iii) reduces to the Levi condition that the subprincipal symbol of $P$ vanishes on $\Sigma$ ([3], [6]). Always in the case of double characteristics similar results have been obtained by R. Lascar [7] and Ivrii [6]. (Ivrii's results which are proved by means of microlocal energy estimates are more general for $\Sigma$ may have symplectic components also, see however [1] for a precise formulation of some results). In the case of $m \geqq 2$ and involutive characteristics only, assuming an ellipticity and a Calderon's type uniqueness condition for the operator induced on the leaf J. Sjostrand [9] proved that in general there is propagation in any direction on the leaf. The results in [7] are proved essentialy by means of Carleman estimates on the leaf and techniques formerly developed in [9]. For the construction of a parametrix in case of multiple characteristics (of constant multiplicity) see also the work of Chazarain [12].

(ii) Here we shall prove the theorem by constructing as in [8] a microlocal parametrix for the operator and the diffusion result will be clear from direct inspection. We want to point out that under assumptions (i)-(iv) $P$ behaves like a principal type operator outside $\Sigma$ and moreover null bicharacteristics starting outside $\Sigma$ do not have limit points on $\Sigma$ ([6]). As in [8] simple examples of operators satisfying (i)-(iv) are provided by taking $X=X_1\times X_2$, dim $X_1=d$ and $P$ a strictly hyperbolic operator of order $m$ in $X_1$ extended trivially in $X$. As in [8] however this is not a microlocal model of the general case.

(iii) The proof is given in four steps: 1) we reduce the operator to a standard simpler form using the invariance under canonical transformation and conjugation via Fourier Integral Operator of the assumptions; 2) we solve Hamilton-Jacobi equations in polar coordinates; 3) we construct a microlocal parametrix for the Cauchy problem; 4) we finally compute the $WF$'s and conclude.

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## 1. Some preparations

Let $\gamma^0\in\Sigma=\{\gamma\in T^*(X)\backslash 0\,|\,q_1(\gamma)=\cdots=q_d(\gamma)=0\}$ as in assumption (i). Then $T_\gamma(\Sigma)=[H_{q_1}(\gamma),\cdots,H_{q_d}(\gamma)]^\sigma$. By (ii) there exists $j\in\{1,\cdots,d\}$ such that $\sigma(H_{q_j}(\gamma),H_\varphi(\gamma))\neq 0$. Let us consider $\Sigma'=\{\gamma\in T^*(X)\backslash 0\,|\,q_1(\gamma)=\cdots=q_d(\gamma)=0,\ \varphi(\gamma)=0\}$. We have $\operatorname{rank}(\sigma)_{|\Sigma'}=2n-d-1-\dim\operatorname{Ker}(M)$, where $M$ is the matrix:

$$\begin{pmatrix}\{q_i,q_j\}_{|\Sigma',i,j=1,\cdots,d}; & \{q_i,\varphi\}_{|\Sigma',i=1,\cdots,d}\\ \{q_i,\varphi\}_{|\Sigma',i=1\cdots,d} & 0\end{pmatrix}$$

It is obvious that $M$ has rank 2, therefore rank $(\sigma)_{|\Sigma'}=2(n-d)$. Then ([5] Th. 21.2.4) there is a canonical homogeneous transformation sending $\Sigma'$ in $\{(x,\xi)\in T^*(\boldsymbol{R}^n)\backslash\{0\}\,|\,\xi_{n-d+1}=\cdots=\xi_n=x_n=0\}$. Since $\Sigma$ is involutive, it is sent into $\{(x,\xi)\in T^*(\boldsymbol{R}^n)\backslash\{0\}\,|\,\xi_{n-d+1}=\cdots=\xi_n=0\}$ and $\varphi$ in these new canonical coordinates is sent to $x_n$. Setting $\boldsymbol{R}^n\ni x=(x',x'',x_n)\in\boldsymbol{R}^{n-d}\times\boldsymbol{R}^{d-1}\times\boldsymbol{R}$ and $x'''=(x'',x_n)$, $\Sigma$ is then given by $\xi'''=0$. Let $F$ be a Fourier Integral operator elliptic in $\gamma^0$, of order zero and such that, with $D_j=D_{xj}$ $F^{-1}Q_jF=D_{j+n-d}+r_j(x,D_x)$, $j=1,\cdots,d$ with $r_j$ of order zero. Assumption (iii) and Lemma 0.1 in the first chapter of [7] now give that microlocally near $\gamma^0=(x=0;\xi'=(0,\cdots,0,1),\xi''=0,\xi_n=0)$:

$$P(x,D_x)\equiv\sum_{|\alpha|\le m}A_\alpha(x,D_x)\,D_{n-d+1}^{\alpha_1}\cdots D_n^{\alpha_d}$$

where now $A_\alpha$ are pseudodifferential operators of order zero. After composition with a pseudodifferential operator of order zero in view of the hyperbolicity of $P$ we can assume that the complete symbol of $P$ is given by:

$$p(x,\xi)=\xi_n^m+\sum_0^m{}_j\Big[\sum_{|\alpha''|\le j}{}_{\alpha'',j}A^0_{\alpha'',j}(x,\xi)\,(\xi'')^{\alpha''}\Big]\,\xi_n^{m-j}\tag{1.1}$$

Now we have $\partial^h_{\xi_n} p_m(\delta^0)=\delta_{hm}\, m!$ where $\delta_{hm}$ is the Kronecker symbol and $h\in \{1, \cdots, m\}$. So by using a pseudodifferential version of the Malgrange preparation theorem:

$$(1.2)\qquad P(x, D_x)\cong Q(x, D_x)\,[D_n^m+\sum_1^m{}_j\, E_j(x, D_x, D_{x''})\, D_n^{m-j}]$$

in a conic neighborhood of $\gamma^0$ and $Q$ elliptic at $\gamma^0$. Comparison of (1.1) with (1.2) and composition with a parametrix of $Q$ finally gives that near $\gamma^0$ with different $A_{\alpha,j}$:

$$(1.3)\qquad p(x, \xi) = \xi_n^m+\sum_1^m{}_j\,[\sum_{|\alpha''|\leq j} A_{\alpha'',j}(x, \xi', \xi'')\,(\xi'')^{\alpha''}]\,\xi_n^{m-j}\,.$$

In these coordinates the leaf $F_{\gamma^0}$ through $\gamma^0\in\Sigma$, $\gamma^0=(x^0, \xi'^0, \xi''=0, \xi_n=0)$ is given by:

$$F_{\gamma^0} = \{(x, \xi)\in T^*(\boldsymbol{R}^n)\setminus\{0\}\,|\,x' = x'^0, \xi' = \xi'^0, \xi'' = 0, \xi_n = 0\}\ .$$

The principal symbol of the operator $P^0$ is:

$$(1.4)\qquad p^0_{(x'^0,\xi^0)}(x''', \xi'', \xi_n) = \xi_n^m+\sum_1^m{}_j\,[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x'^0, x''', \xi'^0, 0)\,(\xi'')^{\alpha''}]\,\xi_n^{m-j}\,.$$

Therefore assumption (iv) requires that $p^0_{(x',\xi)}(x''', \xi'', \xi_n)=0$ has $m$ real distinct roots $\xi_n$.
Let us now study the local structure of Char $(P)\setminus\Sigma$. We shall assume $d>2$. We introduce polar coordinates near $\Sigma$: $\xi''=\rho\omega$, $\rho\in[0, +\infty[$, $\omega\in S^{d-2}$. The principal symbols of $P$ and $P^0$ are then given by:

$$(1.5)\qquad p_m = \xi_n^m+\sum_1^m{}_j\,[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x, \xi', \rho\omega)\,(\omega)^{\alpha''}]\,\rho^j\xi_n^{m-j}$$

$$(1.6)\qquad p^0_m = \xi_n^m+\sum_1^m{}_j\,[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x, \xi', 0)\,(\omega)^{\alpha''}]\,\rho^j\xi_n^{m-j}$$

Let us blow up again singularities at $\xi_n=\rho=0$, $u=\xi_n/\rho$:

$$(1.5)'\qquad p_m = u^m+\sum_1^m{}_j\,[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x, \xi', \rho\omega)\,(\omega)^{\alpha''}]\,u^{m-j}$$

$$(1.6)'\qquad p^0_m = u^m+\sum_1^m{}_j\,[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(x, \xi', 0)\,(\omega)^{\alpha''}]\,u^{m-j}$$

By Rouche's Theorem and assumption (iv) we have that $p_m=0$ has for positive and sufficiently small $\rho m$ real zeros $\mathrm{u}_h=\rho u_h$ and:

$$(1.7)\qquad u_h\neq u_k \quad \text{if}\quad h\neq k\,.$$

This shows that in Char $(P)\backslash\Sigma$ near $\Sigma$, $P$ is of principal type and Char $(P)\backslash\Sigma$ has $m$ local components intersecting over $\Sigma$. Moreover $p_m$ is there factorized as:

$$p_m = \prod_1^m{}_h\, q_h \text{ where } q_h = \xi_n - |\xi''|\, u_h(x, \xi', \xi'') . \tag{1.8}$$

By considering the Hamilton systems for one of these factors one gets from Gronwall's Lemma:

$$|\xi''(\gamma_1)| \leqq M(\gamma_1, \gamma_2)\, |\xi''(\gamma_2)| . \tag{1.9}$$

$\gamma_1, \gamma_2$ belonging to the same null bicharacteristic of $p_m$. This proves that the simple Hamiltonian flow in Char $(P)\backslash\Sigma$ has no limit point in $\Sigma$ (see [6], Proposition 0.3, (ii)).
Finally the case $d=2$ is treated in the same way with $\omega=\pm 1$. Moreover in the following we will always deal with $d>2$, leaving the trivial extensions $d=2$ to the reader.

## 2. The eikonal equation

As in [8] and already in (1.5), (1.6) we introduce polar coordinates taking near $\Sigma$: $\xi''=\rho\omega$, $\rho\in[0, +\infty[$, $\omega\in S^{d-2}$. We want to solve:

$$P_m(x, \nabla_x\varphi(x)) = 0\,, \quad \varphi(x', x'', 0; \omega, \rho, \eta') = \rho\langle\omega, x''\rangle + \langle\eta', x'\rangle \tag{2.1}$$

In order to use Hamilton-Jacoby theory let us look for $\varphi$ of the form:

$$\varphi(y', y'', y_n; \omega, \rho, \eta') = \langle\eta', y'\rangle + \rho\psi(y, \omega, \rho, \eta') \tag{2.2}$$

$\psi$ homogeneous of degree zero in $(\rho, \omega)$. Then (2.1) goes into:

$$\begin{aligned} &(\partial_n\psi)^m + \sum_1^m{}_j \Big[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(y, \eta'+\rho\nabla_{y'}\psi, \rho\nabla_{y''}\psi)\,(\nabla_{y''}\psi)^{\alpha''}\Big]\,(\partial_n\psi)^{m-j} = 0 \\ &\psi(y', y'', 0; \omega, \rho, \eta') = \langle\omega, y''\rangle \end{aligned} \tag{2.3}$$

Let us denote by $q_m$ the Hamiltonian function in (2.3):

$$\begin{aligned} &q_m(y', y'', y_n, \xi', \xi'', \xi_n, \rho, \eta') = \\ &\quad (\xi_n)^m + \sum_1^m{}_j \Big[\sum_{|\alpha''|=j} A^0_{\alpha'',j}(y, \eta'+\rho\xi', \rho\xi'')\,(\xi'')^{\alpha''}\Big]\,(\xi_n)^{m-j} \end{aligned}$$

Therefore:

$$q_m(y', y'', y_n, \xi', \xi'', \xi_n, 0, \eta') = p^0_{m,(y',\eta')}(y'', y_n, \xi'', \xi_n)$$

Now the equation:

$$0 = q_m(0, 0, 0, \xi', \xi'' = \omega, \xi_n, 0, \eta') =$$
$$(\xi_n)^m + \sum_1^m {}_j [\sum_{|\alpha''|=j} A^0_{\alpha'',j}(0, \eta', 0)(\omega)^{\alpha''}](\xi_n)^{m-j} = p^0_{m,(0,\eta')}(0, \omega, \xi_n)$$

has $m$ real distinct roots $\xi_n$ as $\omega \neq 0$. If $0<\rho$ is sufficiently small then $q_m(y', y'', y_n, \xi', \xi'', \xi_n, \rho, \eta')=0$ has $m$ real distinct roots $(\xi_n)^h$, $h=1, \cdots, m$ by Rouchè's Theorem and the hyperbolicity assumption. By theorem 6.4.5 in [4] there exists $m$ functions $\psi_h(y, \omega, \rho, \eta')$ $C^\infty$ in a conic neighborhood of $(y'=0, y''=0, y_n=0, \rho=0, \omega, \eta=(0, \cdots, 1))$ such that:

$$(2.3)' \qquad \begin{aligned} & q_m(y', y'', y_n, \nabla_{y'} \psi_h, \nabla_{y''} \psi_h, \partial_n \psi_h, \rho, \eta') = 0 \\ & \psi_h(y', y'', 0; \omega, \rho, \eta') = \langle \omega, y'' \rangle \\ & \partial_n \psi_h(0, 0, 0; \omega, \rho, \eta') = (\xi_n)^h(\omega, \rho, \eta'), h = 1, \cdots, m \end{aligned}$$

From $(2.3)'$ if $|y|<\delta$, $\rho<\delta|\eta'|$, $|\eta'-(0, \cdots, 1)|\eta'||<\delta|\eta'|$, $\omega \in S^{d-2}$ and $0<\delta$ sufficiently small we have $m$ $C^\infty$ functions $\varphi_h(y, \omega, \rho, \eta')=\langle \eta', y' \rangle + \rho\psi_h(y, \omega, \rho, \eta')$ solutions of equation (2.1).

### 3. Microlocal Cauchy problem

In this section we want to solve the following microlocal Cauchy problem:

$$(3.1) \qquad \begin{aligned} & Pv = D_n^m v + \sum_1^m {}_j [\sum_{|\alpha''| \leq j} A_{\alpha'',j}(x, D_{x'}, D_{x''}) D_{x''}^{\alpha''}] D_n^{m-j}(v) \equiv 0 \\ & D_n^h v(x', x'', x_n = 0) \equiv \delta_{h,m-1} \delta(x', x'') \quad h \in \{0, \cdots, m-1\} \end{aligned}$$

microlocally near $\gamma^0=(x=0; \xi'=(0, \cdots, 0, 1), \xi''=0, \xi_n=0)$, where $\delta_{h,m-1}$ denotes the Kronecker symbol.
Let us look for $v$ as a sum of oscillotary integrals:

$$(3.2) \qquad v = \sum_1^m {}_j I_{\varphi_j}(a_j)$$

where $I_{\varphi_j}(a_j)(x)=\int_{S^{d-2}} \int_0^{+\infty} \int_{R^{n-d}} \exp(i\varphi_j(x, \omega, \rho, \eta'))\, a_j(x, \omega, \rho, \eta')\, d\eta'\, d\rho d\omega$ the $\varphi_j$'s are the phase functions found in section 2 and $a_j$ are classical symbols to be determined.
Let us recall that:

$$(3.4) \qquad e^{-i\varphi} P(e^{+i\varphi} a_j) =$$
$$\sum_{\alpha \geq 0} 1/\alpha! \partial_\xi^\alpha P(x, \nabla_x \varphi) D_z^\alpha \{\exp(i\varphi_2(x, \omega, \rho, \eta', z))\, a_j(z, \omega, \rho, \eta')\}_{|x=z}$$

where

$$\varphi_2(x, \omega, \rho, \eta', z) = \varphi(z, \omega, \rho, \eta') - \varphi(x, \omega, \rho, \eta') + \langle x-z, \nabla_x \varphi(x, \omega, \rho, \eta') \rangle.$$

Now:

$$Pv = \sum_{1}^{m}{}_j I_{\varphi_j}(b_j) \tag{3.5}$$

with

$$\begin{aligned} b_j = {} & P_m(x, \nabla_x\varphi_j)\, a_j + P_{m-1}(x, \nabla_x\varphi_j)\, a_j + \cdots + \\ & \sum_{1}^{n}{}_h\, \partial_{\xi_n} P_m(x, \nabla_x\varphi_j)\, D_{x_h} a_j + \sum_{1}^{n}{}_h\, \partial_{\xi_h} P_{m-1}(x, \nabla_x\varphi_j)\, D_{x_h} a_j + \cdots + \\ & (-i/2) \sum_{1}^{n}{}_{hk}\, \partial^2_{\xi_h\xi_k} P_m(x, \nabla_x\varphi_j)\, (\partial^2_{x_h x_k}\varphi_j)\, a_j + \\ & (1/2) \sum_{1}^{n}{}_{hk}\, \partial^2_{\xi_k\xi_k} P_m(x, \nabla_x\varphi_j)\, D^2_{x_h x_k} a_j + \\ & (-i/2) \sum_{1}^{\infty}{}_t \sum_{1}^{n}{}_{hk}\, \partial^2_{\xi_h\xi_k} P_{m-t}(x, \nabla_x\varphi_j)\, (\partial_{x_h x_k}\varphi_j)\, a_j + \\ & \quad + (1/2) \sum_{1}^{\infty}{}_t \sum_{1}^{n}{}_{hk}\, \partial^2_{\xi_h\xi_k} P_{m-t}(x, \nabla_x\varphi_j)\, D^2_{x_h x_k} a_j + \\ & \quad + \sum_{|\alpha|\geqq 3}{}_{\alpha} (1/\alpha!)\, \partial^{\alpha}_{\xi} P(x, \nabla_x\varphi_j)\, D^{\alpha}_z \{\exp(i\varphi_{2,j}(x, \omega, \rho, \eta', z)) \\ & \quad a_j(z, \omega, \rho, \eta')\}_{|x=z}\,. \end{aligned} \tag{3.6}$$

Since $\varphi_j$ solves equation (2.1) we have:

$$\begin{aligned} b_j = {} & \sum_{1}^{n}{}_h\, \partial_{\xi_k} P_m(x, \nabla_x\varphi_j)\, D_{x_h} a_j + \\ & (P_{m-1}(x, \nabla_x\varphi_j) - (+i/2) \sum_{1}^{n}{}_{hk}\, \partial^2_{\xi_h\xi_k} P_m(x, \nabla_x\varphi_j)\, \partial^2_{x_h x_k}\varphi_j)\, a_j + R_j(a_j) \end{aligned} \tag{3.7}$$

with $R_j(a_j)$ easily determined from (3.6). Now from (1.3) and the form of $\varphi_j$, setting:

$$\alpha_{hj} = \partial_{\xi_h} P_m(x, \nabla_x\varphi_j(x, \omega, \rho, \eta'))\, j = 1, \cdots, m;\; h = 1, \cdots, n \tag{3.8}$$

we have:

$$\alpha_{nj} = \rho^{m-1}\, \partial_{\xi_n} q_m(y', y'', y_n, \nabla_{y'}\psi_j, \nabla_{y''}\psi_j, \partial_n\psi_j, \rho, \eta') \tag{3.9}$$

From the discussion in section 2 $\alpha_{nj} = \rho^{m-1}\tilde{\alpha}_{nj}$ with $\tilde{\alpha}_{nj}(x, \omega, \rho, \eta') \neq 0$ in a conic neighborhood of $\gamma^0 \in \Sigma$ for every $j = 1, \cdots, m$.
If $1 \leqq h \leqq n-d$:

$$\alpha_{hj} = \sum_{1}^{m}{}_t \Big[\sum_{|\alpha''|=t} \partial_{\xi_h} A^0_{\alpha'',t}(y, \eta' + \rho\nabla_{y'}\psi_j, \rho\nabla_{y''}\psi_j)\, (\nabla_{y''}\psi_j)^{\alpha''}\Big] (\partial_n\psi_j)^{m-t} \rho^m \tag{3.10}$$

If $n-d+1 \leqq h \leqq n-1$ then $\alpha_{hj} = \rho^{m-1}\tilde{\alpha}_{hj}$ where $\tilde{\alpha}_{hj}$ is a similar although slightly more involved expression as (3.10). Now:

$$\begin{aligned} & P_{m-1}(x, \nabla_x\varphi_j) - (-i/2) \sum_{1}^{n}{}_{hk}\, \partial^2_{\xi_h\xi_k} P_m(x, \nabla_x\varphi_j)\, \partial^2_{x_h x_k}\varphi_j = \\ & \quad = \rho^{m-1}\, \tilde{b}_j(x, \omega, \rho, \eta') \end{aligned} \tag{3.11}$$

which follows from an easy but tedious calculation. Therefore if $Pv\equiv 0$ then $b_j$ has to be $\sim 0$. From (3.6) we get:

$$R_j=\sum_{2}^{\infty}{}_t\,\rho^{m-t}R_{j,t}(x, D_x, \rho) \tag{3.12}$$

Let $L_j$ be the first order differential operator:

$$L_j=\tilde{\alpha}_{nj}D_{x_n}+\sum_{1}^{n-1}{}_h\,\tilde{\alpha}_{hj}D_{x_h}+\tilde{b}_j \tag{3.13}$$

The transport equations $b_j\sim 0$ then become:

$$L_j(a_j)+\sum_{2}^{\infty}{}_t\,\rho^{1-t}R_{j,t}(x, D_x, \rho)\,a_j\cong 0 \tag{3.14}$$

Let us consider the initial conditions in (3.1). First recall that all the $\varphi_j$ coincide at $x_n=0$ with $\varphi_0(y', y'', \omega, \rho, \eta')=\langle\eta', y'\rangle+\rho\langle\omega, y''\rangle$. Moreover as in [8] microlocally near $\gamma^0=(x=0;\ \xi'=(0, \cdots, 0, 1),\ \xi''=0,\ \xi_n=0)$ the Dirac delta is represented by:

$$\begin{aligned}(3.15)\quad &\delta(x', x'')\equiv v_0(x', x'')=\\ &=1/(2\pi)^{n-1}\int_{S^{d-2}}\int_0^{+\infty}\int_{R^{n-d}}\exp(i\varphi_0(x', x'', \omega, \rho, \eta')\,\rho^{d-2}\sigma_1(\rho/\delta|\eta'|)\\ &\quad d\eta'\,d\rho d\omega\end{aligned}$$

with $\sigma_1\in C^\infty(\boldsymbol{R})$, $\sigma_1(t)=1$ if $|t|\leqq 1/2$, $\sigma_1(t)=0$ if $|t|\geqq 1$.
Then we obtain:

$$\begin{aligned}(3.16)\qquad &\exp(i\varphi_0)\,(a_1+\cdots+a_m)|_{x_n=0}=0\\ &\exp(i\varphi_0)\,((\partial_{x_n}a_1+\cdots+\partial_{x_n}a_m)|_{x_n=0}+(\rho D_{x_n}\psi_1)|_{x_n=0}\\ &\quad a_1|_{x_n=0}+\cdots+(\rho D_{x_n}\psi_m)|_{x_n=0}\,a_m|_{x_n=0}=0\end{aligned}$$

$\exp(i\varphi_0)\ [(\rho D_{x_n}\psi_1)^{m-1}|_{x_n=0}\,a_1|_{x_n=0}+\cdots+(\rho D_{x_n}\psi_m)^{m-1}|_{x_n=0}\,a_m|_{x_n=0}+$terms with powers of $\rho$ strictly lower than $m-1]=\exp(i\varphi_0)\ 1/(2\pi)^{n-1}\rho^{d-2}\sigma_1(\rho/\delta|\eta'|)$
The last equation suggests that $a_j$ should be of the form:

$$a_j(x, \omega, \rho, \eta')=\sum_{0}^{d-m-1}{}_k\,\rho^k a_{jk}(x, \omega, \rho, \eta')+\sum_{1}^{\infty}{}_k\,\rho^{-k}b_{jk}(x, \omega, \rho, \eta') \tag{3.17}$$

if $d-m-1>0$ which will be the case treated first $a_{jk}$ and $b_{jk}$ here are homogeneous of degree zero in $(\rho, \eta')$.
From (3.15) we have:

$$\begin{bmatrix}1\cdots & & \cdots 1\\ (D_{x_n}\psi_1)|_{x_n=0} & \cdots & (D_{x_n}\psi_m)|_{x_n=0}\\ \cdots & & \cdots\\ (D_{x_n}\psi_1)^{m-1}|_{x_n=0} & \cdots & (D_{x_n}\psi_m)^{m-1}|_{x_n=0}\end{bmatrix}\cdot\begin{bmatrix}a_{1d-m-1}|_{x_n=0}\\ \\ \\ a_{md-m-1}|_{x_n=0}\end{bmatrix}=\begin{bmatrix}0\\ \\ \\ 1/(2\pi)^{n-1}\sigma_1\end{bmatrix} \tag{3.18}$$

Since $\partial_n\psi|_{x_n=0}$ is the $j$-th root of $q_m(0, 0, 0, \xi', \xi''=\omega, \xi_n, \rho, \eta')=0$ and for $j\in\{1, \cdots, m\}$ all these roots are distinct, the linear system has a unique solution giving initial data at $x_n=0$ for $a_{jd-m-1}$. On the other hand by ordering (3.13) according to descending powers of $\rho$ we have:

$$\begin{aligned}(3.19)\qquad & L_j(a_{jd-m-1}) = 0\\ & a_{jd-m-1}|_{x_n=0} = \text{data}\end{aligned}$$

As $\tilde{\alpha}_{nj}\neq 0$ from (3.9), (3.19) has a (local) unique $C^\infty$ solution $a_{jd-m-1}$, $j\in\{1, \cdots, m\}$.

Let us solve for $a_{jp}$ with $p<d-m-1$. (3.13) then yields:

$$\begin{aligned}(3.20)\qquad 0\cong{}&\rho^{d-m-1} L_j(a_{jd-m-1})+\rho^{d-m-2}(L_j(a_{jd-m-2})+R_{j,2}(a_{jd-m-1}))+\\ &\rho^{d-m-3}(L_j(a_{jd-m-3})+R_{j,3}(a_{jd-m-1})+R_{j,2}(a_{jd-m-2})+\cdots+\\ &\rho(L_j(a_{j1})+\sum_{k=2}^{d-m-1} R_{j,k}(a_{jk}))+L_j(a_{j0})+\sum_{k=1}^{d-m-1} R_{j,k+1}(a_{jk})+\\ &\rho^{-1}(L_j(b_{j1})+\sum_{k=0}^{d-m-1} R_{j,k+2}(a_{jk}))+\\ &\rho^{-2}(L_j(b_{j2})+\sum_{k=0}^{d-m-1} R_{j,k+3}(a_{jk})+R_{j,2}(b_{j1}))+\cdots\end{aligned}$$

From (3.16) we have, with $V_{hj}$ the element of place $(h, j)$ in the matrix (3.18):

$$(3.21)\qquad \rho^h \sum_{j=1}^{m} V_{hj}\, a_j+\sum_{t=1}^{h} \rho^{h-t} S_{t,h}(a_j)\cong\delta_{hm-1}\, 1/(2\pi)^{n-1} \rho^{d-2} \sigma_1(\rho/\delta|\eta'|)$$

where $S_{t,h}$ are differential operators with coefficients homogeneous of degree zero in $(\rho, \eta')$. Inserting (3.17) in (3.21):

$$\begin{aligned}(3.22)\qquad &\sum_{k=0}^{d-m-1} \rho^{h+k} (\sum_{j=1}^{m} V_{hj}\, a_{jk})+\sum_{k=1}^{\infty} \rho^{h-k}(\sum_{j=1}^{m} V_{hj}\, b_{jk})+\sum_{k=0}^{d-m-1}\sum_{t=1}^{h} \rho^{h+k-t} S_{t,h}(a_{jk})+\\ &\sum_{k=1}^{\infty}\sum_{t=1}^{h} \rho^{h-k-t} S_{t,h}(b_{jk})\cong\delta_{hm-1}\, 1/(2\pi)^{n-1} \rho^{d-2} \sigma_1(\rho/\delta|\eta'|)\end{aligned}$$

For instance let us find the initial conditions for $a_{jd-m-2}$:

$$(3.23)\qquad \rho^{h+d-m-2}(\sum_{j=1}^{m} V_{hj}\, a_{jd-m-2}+S_{1,h}(a_{jd-m-1})) = 0 \text{ at } x_r = 0,\ h\in\{0, \cdots, m-1\}.$$

(3.23) together with:

$$(3.24)\qquad L_j(a_{jd-m-2})+R_{j,2}(a_{jd-m-1}) = 0$$

enables us to find a (local) $C^\infty$ solution $a_{jd-m-2}$, $j\in\{1, \cdots, m\}$. And so on for all $a_{jp}$ $p\in\{1, \cdots, d-m-1\}$.

As far as $b_{j1}$ is concerned:

$$(3.24)\qquad \rho^{h-1}(\sum_{j=1}^{m} V_{hj}\, b_{j1})+\rho^{h-1}\sum_{k=0}^{d-m-1}\sum_{t=1}^{h} \delta_{tk+1} S_{t,h}(a_{jk})=0,\ h\in\{0, \cdots, m-1\} \text{ at } x_n=0.$$

$$(3.25)\qquad L_j(b_{j1})+\sum_0^{d-m-1}{}_k R_{j,k+2}(a_{jk})=0$$

Therefore it is possible to find all $a_{jk}$ and $b_{jk}$.
By Taylor's formula:

$$(3.26)\qquad b_{jk}(y,\omega,\rho,\eta')=\sum_0^{k-1}{}_h \rho^h b_{jhk}(y,\omega,\eta')+\rho^k b_{jk}(y,\omega,\rho,\eta')$$

$b_{jhk}$, $(b_{jk})$ homogeneous in $\eta'((\rho,\eta'))$ of degree $-h$ $(-k)$. Hence:

$$(3.27)\qquad a_j=\sum_0^{d-m-1}{}_k \rho^k a_{jk}(x,\omega,\rho,\eta')+\sum_{k<0}{}_k \{\sum_1^{-k}{}_t \rho^{-t} a_{jkt}(x,\omega,\eta')+ \\ +\mu_{jk}(x,\omega,\rho,\eta')\}\ .$$

Now to define the classical part $v^{(1)}$ of the solution as in [8], let us put:

$$a_j^{(1)}(x,\omega,\rho,\eta')=\sum_0^{d-m-1}{}_k \rho^k a_{jk}(x,\omega,\rho,\eta')+ \\ +\sum_{k<0}{}_k \mu_{jk}(x,\omega,\rho,\eta')\,\sigma_1(\rho/\delta|\eta'|)$$

and the $a_j^{(1)}$'s are supported where the phase functions are defined.

To construct the "singular" part of the solution $v^{(2)}$, for any $j\in\{1,\cdots,m\}$ and for any $h\geqq 1$ choose symbols using the standard asymptotics $a_{jh}^{(2)}\in S^0(\boldsymbol{R}^n\times S^{d-2}\times\boldsymbol{R}^{n-d})$ such that $a_{jh}^{(2)}\sim\sum_k a_{jkh}\sigma_1$. In the same way select $a_j^{(2)}\in S^{-1}(\boldsymbol{R}^+;S^0)$, $j\in\{1,\cdots,m\}$ (for a definition of $S^m(\boldsymbol{R}^+;S^{m'}(\boldsymbol{R}^n\times S^{d-2}\times\boldsymbol{R}^{n-d}))$ see [8] page 577) such that

$$(3.28)\qquad a_j^{(2)}-\sum_1^N{}_k \rho^{-k} a_{jk}^{(2)}\in S^{-N-1}(\boldsymbol{R}^+;S^0)\quad\text{for any}\quad N\geqq 1$$

If $v=v^{(1)}+v^{(2)}=\sum_1^m{}_j I_\varphi(a_j^{(1)})+\sum_1^m{}_j I_\varphi(a_j^{(2)})$ let us calculate $Pv$: (3.5) and the preceeding constuction show that

$$(3.29)\qquad Pv=I_\psi(c)=\int_{\boldsymbol{R}^{n-d}} e^{i\langle x',\xi'\rangle}c(x,\xi')\,d\xi'$$

where $c\in S^0(\boldsymbol{R}^n\times\boldsymbol{R}^{n-d})$. It is easily verified that $\partial_n^h v|_{x_n=0}=I_\psi(b_h)+\delta_{hm-1}v_0$ with $b_h\in S^0(\boldsymbol{R}^n\times\boldsymbol{R}^{n-d})$, $h\in\{0,\cdots,m-1\}$
Now let us look for a solution of:

$$(3.30)\qquad Pv=I_\psi(c)\quad\text{and}\quad \partial_n^h v|_{x_n=0}=I_\psi(b_h)\,,\quad h\in\{0,\cdots,m-1\}\ .$$

$v$ will be searched of the form $I_\psi(a^{(3)})$. Now:

$$P(I_\psi(a^{(3)}))=I_\psi(D_n^m a^{(3)}+\sum_1^m{}_j[\sum_{|\alpha''|\leq j} A_{\alpha,j}(x',x''',\xi',0)(D_{x''})^{\alpha''}]D_n^{m-j}a^{(3)}+Ma^{(3)})$$

Because of the strict hyperbolicity assumption (iv) and the fact $M$: $S^k\to S^{k-1}$ we begin by solving:

$$(3.31)_0 \qquad P_\gamma(a_0^{(3)}) = c \text{ and } \partial_n^h a_0^{(3)}|_{x_n=0} = b_h\,, \quad h\in\{0,\cdots,m-1\}$$

with $a_0^{(3)}\in S^0(\boldsymbol{R}^n\times\boldsymbol{R}^{n-d})$, $\gamma=(x',x''',\xi',0)\in\Sigma$.
Then by recursively solving:

$$(3.31)_j \qquad P_\gamma(a_j^{(3)}) = -Ma_{j+1}^{(3)}\in S^j \quad\text{and}\quad \partial_n^h a^{(3)}|_{x_n=0} = 0\,, \qquad j=-1,-2,-3,\cdots h\in\{0,\cdots,m-1\}$$

we can select $a^{(3)}\sim\sum_{j<0} a^{(3)}$ and it is obvious that
$(P_\gamma+M)\,(a^{(3)})\in S^{-\infty}(\boldsymbol{R}^n\times\boldsymbol{R}^{n-d})$ and the choice can be made in order to have $\delta_n^h a^{(3)}|_{x_n=0}=b_h$, $h\in\{0,\cdots,m-1\}$
Finally setting $u=v^{(1)}+v^{(2)}-I_\psi(a^{(3)})$ we have $Pu=0$ and $\partial_n^h u|_{x_n=0}=\delta_{hm-1}v_0\equiv \delta_{hm-1}\,\delta(x',x'')$, $h\in\{0,\cdots,m-1\}$, which is the solution of (3.1) if $d-m-1>0$.
If $d-m-1\leqq 0$ choosing at once

$$(3.32) \qquad a_j(x,\omega,\rho,\eta') = \sum_{1+m-d}^{\infty}{}_k\,\rho^{-k}\,a_{jk}(x,\omega,\rho,\eta')$$

the proof goes as before and this gives in any case the solution of (3.1).

## 4. Propagation of singularities

First of all the construction of section 3 can be repeated uniformly when $s\in]-\delta,\delta[$ and we find:

$$(4.1) \qquad E^{(s)}:\mathcal{D}'(\boldsymbol{R}^{n-1})\to\mathcal{E}'(\boldsymbol{R}^n),\ P\,E^{(s)}\equiv 0 \text{ near } (\gamma^0,\gamma^{0'})\,,$$
$$\partial_n^h E^{(s)}|_{x_n=s}\equiv\delta_{hm-1}\,Id \text{ near } (\gamma^0,\gamma^{0'})\ h\in\{0,\cdots,m-1\}$$

where $\gamma^{0'}=(x=0;\ \xi'=(0,\cdots,0,1),\ \xi''=0)$.
We shall now reason when $d-m-1>0$ remarking that the case $d-m-1\leqq 0$ is dealt with exactly in the same way.

$$(4.2) \qquad (E^{(s)}f)\,(x) = (E_{(1)}^{(s)}+E_{(2)}^{(s)}+E_{(3)}^{(s)})\,(f)\,(x)$$

where we have set:

$$(4.3) \qquad (E_{(1)}^{(s)})\,(f)\,(x) = \sum_1^m{}_j \int_{S^{d-2}}\int_0^{+\infty}\int_{\boldsymbol{R}^{n-d}} \exp(i\varphi_j(x,s,\omega,\rho,\eta') -\langle z',\eta'\rangle-\rho\langle\omega,z''\rangle) \times a_j^{(1)}(x,s,\omega,\rho,\eta')f(z',z'')\,dz'\,dz''\,d\eta'\,d\rho d\omega$$

$$(4.4) \qquad (E_{(2)}^{(s)})\,(f)\,(x) = \sum_1^m{}_j \int_{S^{d-2}}\int_0^{+\infty}\int_{\boldsymbol{R}^{n-d}} \exp(i\varphi_j(x,s,\omega,\rho,\eta') -\langle z',\eta'\rangle-\rho\langle\omega,z''\rangle) \times a_j^{(2)}(x,s,\omega,\rho,\eta')f(z',z'')\,dz'\,dz''\,d\eta'\,d\rho d\omega$$

$$(4.5)\quad (E^{(s)}_{(3)})(f)(x)=\int_{\mathbf{R}^{n-d}}\int_{\mathbf{R}^{n-1}} e^{j\langle x'-z',\eta'\rangle}a^{(3)}(x,s,\eta')f(z',z'')\,dz'\,dz''\,d\eta'$$

where in (4.3) $a_j^{(1)}\in S^{d-m-1}$, in (4.4) $a_j^{(2)}\in S^{-1}(\mathbf{R}^+;S^0)$ and in (4.5) $a^{(3)}\in S^0$. By Duhamel's principle let us put:

$$(4.6)\qquad E_+(f)(x)=-\int_{-\delta}^{x_n}((E^{(s)}\circ\gamma_{(s)}\circ A)(f)(x)\,ds,\ \delta>0$$

with conesupp(A) compactly supported near $\gamma^{0'}$ and $\gamma_{(s)}u=u$ restricted to $x_n=s$. Since $\gamma^{0'}\notin\{(x,\xi)\,|\,x_n=s,\xi'=0,\xi''=0,\xi_n=0\}$ $\gamma_{(s)}\circ A$ is well defined and it is clear that:

$$(4.7)\qquad P\,E_+(f)\equiv f \text{ near } \gamma^0$$

Now the rules to compute wave fronts sets given ([4]):

$$(4.8)\qquad WF(E^{(s)}f)|_{x=s}\subset[\bigcup_1^m{}_j\,K_j^+\cup\tilde{K}_F^+]\circ(i_s^*)^{-1}(WF(f))$$

where $f\in\mathcal{D}'(\mathbf{R}^{n-1})$, $(i_s^*)^{-1}(WF(f))=\{(z',z'',s,\zeta',\zeta'',\zeta_n)\,|\,\zeta_n\in\mathbf{R},(z',z'',\zeta',\zeta'')\in WF(f)\}$, $K_j^+$ denotes $\exp(tH_{q_j})$ (cfr. (1.8)), $t\geqq0$ and $\tilde{K}_F^+$ is the relation defined by:

(4.9) $(\rho,\bar\rho)\in\tilde{K}_F^+$ if and only if $\rho$ and $\bar\rho$ belong to same leaf $F_\rho$ through $\rho\in\Sigma$ and $x_n(\rho)\geqq x_n(\bar\rho)$. Then we have

$$(4.10)\qquad WF(E^+)\subset\bigcup_1^m{}_j\,K_j^+\cup\tilde{K}_F^+$$

It is clear that we will also have an other microlocal parametrix $E^-$ satisfying (4.7) and (4.10) with the reversed time orientation. We now want to be more precise on $\Sigma$. Let us introduce as in [8] the following relations:

(4.11) $K_F^+\ni(\rho,\bar\rho)$ if and only if $\rho$ and $\bar\rho\in F_\rho\subset\Sigma$ and $\exists\gamma:[0,1]\to F_\rho$ Lipschitz continous curve $\gamma(0)=\rho$, $\gamma(1)=\bar\rho$ and $\dot\gamma(t)\in(\Gamma_\gamma)^0$ a.e.

$\partial K_F^+\ni(\rho,\bar\rho)$ if and only if there exists a null bicharacteristics of the operator $P^0$ induced on the leaf $F_\rho$ that joints $\rho$ with $\bar\rho$.
$\partial K_F^+$ is the boundary of $K_F^+$, see e.g. Duistermaat [2]. Let now $\gamma^0\in K\subset\Sigma\times\Sigma\backslash\partial K_F^+$, $K$ compact, $U\Supset K$ open neighborhood of $K$. Denoting by $k(x,z',z'')$ the kernel of $E^{(0)}$ we have in $U'$ (proj. of $U$ at $s=0$).

$$(4.12)\qquad k(x,z',z'')=\int e^{i\langle x'-z',\xi\rangle}a(x,z'',\xi')\,d\xi'$$

This is clear since the term in (4.5) is already of this type and in (4.3), (4.4) at $\rho=0$ the phase is stationary exactly on null bicharacteristics of $P^0$. Therefore integrating by parts in $d\rho\,d\omega$ (4.12) follows. Now $k(x,z',z'')$ solves $Pk=0$ and $\partial_n^h k|_{x_n=0}=\delta_{hm-1}\,\delta(x'-z',x''-z'')$ $h\in\{0,\cdots,m-1\}$. Therefore a has to be a solution of $P_\gamma(a)=-Ma$ and $\partial_n^h a|_{x_n=0}\in S^{-\infty}$, if $x'\neq z'$ and $x''\neq z''$ $h\in\{0,\cdots,m-1.\}$

Since $P_\gamma$ is strictly hyperbolic we obtain that a is still in $S^{-\infty}$ outside the set obtained by emanating from $(z', z'', 0; \eta', 0, 0)$ along curves defined in $K_F^+$. Finally:

$$WF(E^{\pm}f)\subset K^{\pm}\circ WF(f) \tag{4.13}$$

Where $K^{\pm}$ is the generalized flow as defined in [11]. Passing to a parametrix for ${}^tP$, the transpose of $P$, and microlocalizing near $\gamma^0$ now ends in a standard way the proof of the theorem.

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