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CONVERGENCE TO EQUILIBRIA FOR A CLASS OF REACTION-DIFFUSION SYSTEMS

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1. Introduction

An important question in the study of reaction-diffusion systems is the identification of all steady states and the classification of their stability. In this note, we give a contribution to this question for a class of systems with two diffusing and reacting components.

\[(1.1) \begin{aligned}
\partial_t u_i - d_i \Delta u_i &= f_i(u_1, u_2) \quad (i = 1, 2)
\end{aligned}\]

in some cylindrical time-space domain \(\Omega \times (0, \infty) \subset \mathbb{R}^{N+1}\), together with zero Neumann boundary data on the lateral boundary \(\partial \Omega \times (0, \infty)\) and suitable initial data on \(\Omega \times \{0\}\). Then all equilibria of the associated system of ordinary differential equations

\[(1.2) \begin{aligned}
\dot{y}_i &= f_i(y_1, y_2) \quad (i = 1, 2)
\end{aligned}\]

are also spatially constant steady states of (1.1). Consider now the special case

\[(1.3) f_i(u_1, u_2) = u_i(a_{10} - a_{11}u_1 - a_{12}u_2)\]

with \(a_{ij} > 0\) for all \(i, j\). In this case, it turns out that the unique positive equilibrium of (1.2) (if it exists) is globally asymptotically stable for (1.1) if and only if it is so for (1.2). This will be the case if and only if

\[(1.4) \frac{a_{11}}{a_{21}} > \frac{a_{10}}{a_{20}} > \frac{a_{12}}{a_{22}} ;\]

see [12] for a detailed study and for results on other possible combinations of coefficients.

Note that in this case \(\frac{\partial f_i}{\partial u_j} \leq 0\) for \(i \neq j\); such a vector field is called competitive. On the other hand, it was shown in [11] that for competitive vector
field with two zeroes that are asymptotically stable for (1.2), non-constant stable steady states of (1.1) can exist, if the domain $\Omega$ and the diffusion coefficients $d_1$, $d_2$ are chosen appropriately. This "pattern formation" phenomenon cannot occur if $\Omega$ is convex ([8]) or if the diffusion coefficients are large ([2]).

The purpose of this note is a study of a certain borderline case in which the zero set $Z$ of the vector field $\vec{f}$ in the first quadrant

\[(1.5) \quad Z = \{(u_1, u_2) | u_1, u_2 \geq 0, f_i(u_1, u_2) = 0 \quad \text{for} \quad i = 1, 2\}
\]

is a continuum. An example is the vector field (1.3) with all inequalities in (1.4) replaced by equality. In this case, the zero set $Z$ consists of the coordinate axes and a straight line that divides the positive quadrant $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$ into two components, one of which is bounded, and all these zeroes are stable equilibria of (1.2).

We shall discuss two possible generalizations of this example and give sufficient conditions that imply that every solution of (1.1) converges to a unique spatially constant solution with values in $Z$. Our first result uses the maximum principle and is given in section 2; our second result uses Lyapunov functions and is given in section 3. It will turn out that the vector field $\vec{f}$ does not have to be competitive for either generalization, but the competitive case is included in both results. In section 4, we discuss the stability of these steady states and give comments on extensions to systems with more than two components and on relations to recent work on order preserving semiflows. For other results on the asymptotic behavior of solutions of reaction-diffusion systems, the reader is referred to the bibliographies in [4], [10], [18], [19]. For some related work on predator-prey systems of the form (1.1) with vector fields of the form (1.3), see [15]. For other uses of Lyapunov techniques in the study of parabolic systems, see e.g. [1] and [14].

To conclude this introduction, we list the main assumptions and give a basic existence and uniqueness result that serves as the framework of what follows. The set $\Omega \subset \mathbb{R}^n$ is an open and bounded domain with $C^{2+\alpha}$-boundary $\partial \Omega$ and unit exterior normal vector field $n$. The constants $d_i$ are positive; after rescaling, we may assume that $d_1 = 1$ and $d_2 = d > 0$. Problem (1.1) is considered together with zero Neumann boundary conditions

\[(1.6) \quad \partial_n u_i = 0 \quad (i = 1, 2)
\]
on the lateral boundary $\partial \Omega \times (0, \infty)$ and initial conditions

\[(1.7) \quad u_i(\cdot, 0) = u_i^0 \quad (i = 1, 2)
\]
on $\Omega$. The initial data are always assumed to be non-negative and not identi-
cally zero. The vector field \( f: \mathbb{R}^2 \to \mathbb{R}^2 \) is of class \( C^1 \). The zero set \( Z \) is assumed to be of the form \( Z = \{0\} \times (0, \infty) \cup ((0, \infty) \times \{0\}) \cup Z_1 \) with

\[
Z_1 = \{(u_1, u_2) | u_i > 0, f_i(u_1, u_2) = 0 \quad (i = 1, 2)\}
\]

We also assume that there exists a number \( M > 0 \) such that \( f_i(u_1, u_2) < 0 \) if \( u_i \geq M \) or \( u_2 \geq M \).

**Lemma 1.** For \((u_0^1, u_0^2) \in L^\infty(\Omega, \mathbb{R}^2)\) there exists a unique solution \( u = u(\cdot; u^0): \Omega \times [0, \infty) \to \mathbb{R}^2 \) of (1.1), (1.6), (1.7) for which \( \partial_x \partial_x u \) and \( \partial_y u \) are Hölder continuous on \( \Omega \times (0, \infty) \). Let \( u(\cdot, t; u^0) \) denote the restriction of \( u(\cdot; u^0) \) to \( \Omega \times \{t\} \), then \([0, \infty) \subseteq t \to u(\cdot, t; u^0)\) is a continuous curve in \( L^p(\Omega, \mathbb{R}^2) \) for any \( p < \infty \). The components \( u_i \) are strictly positive on \( \Omega \times (0, \infty) \) and essentially uniformly bounded on this set.

Sketch of proof. Choose \( p > n \) and consider (1.1), (1.6), (1.7) as an abstract semilinear evolution equation in the Banach space \( X = L^p(\Omega, \mathbb{R}^2) \), after continuing the vector field \( f \) on all \( \mathbb{R}^2 \) in a \( C^1 \)-fashion. By standard arguments (see [6]), the problem has a unique local mild solution which is Hölder continuous for positive times. Applying the regularity theory for parabolic equations again, it follows that also the time derivative and all second spatial derivatives of the solution are Hölder continuous up to the boundary of \( \Omega \) for all positive times (see [9]). The positivity of both components follows from the strong maximum principle (see [3]). Using a large invariant rectangle, a uniform bound for the solution components can be given which implies global existence of the solution (see [19]). Q.E.D.

## 2. Convergence to Constant Equilibria I

In this section, we discuss (1.1), (1.6), (1.7) under the following assumption:

There exist \( \beta > 0 \) and \( \gamma: [0, \beta] \to [0, \infty) \), continuous, strictly decreasing, with \( \gamma(\beta) = 0 \), such that the set \( Z_1 \) is given by

\[
Z_1 = \{(u_1, u_2) | u_1, u_2 > 0, 0 < u_1 < \beta, u_2 = \gamma(u_1)\}
\]

Both \( f_i \) are negative on \( \{(u_1, u_2) | u_i > 0, u_2 > \gamma(u_1)\} \) and positive on the complement of this set in the first quadrant.

We still assume throughout that both components of the initial data \( u^0 \) are non-negative and not identically zero.

**Theorem 2.1.** If (2.1) holds, then every solution \( u(\cdot; u^0) \) of (1.1), (1.6), (1.7) converges to some spatially constant vector \( (\rho_1, \rho_2) \) as \( t \to \infty \), uniformly on \( \bar{\Omega} \), together with its first and second spatial derivatives. Here either \( \rho_1 = 0 \) and \( \rho_2 \geq \gamma(0) \),
or  $0<\rho_1<\beta$ and $\rho_2=\beta(\rho_1)$, or $\rho_1\geq\beta$ and $\rho_2=0$.

Proof. Define the usual $\omega$-limit set

$$\omega(u^0) = \{ \phi \in L^p(\Omega, R^2) \mid u(\cdot, t_k; u^0) \rightharpoonup \phi \text{ in } L^p \text{ as } t_k \to \infty \text{ for some sequence } t_k \}.$$

Then $\omega(u^0)$ is a compact non-empty subset of $C^2(\Omega)$ and independent of $p$, and the convergence is actually in $C^2(\Omega)$ (see [16]). We must show that $\omega(u^0)$ is a singleton with two constant components. Thus let $v=(v_1, v_2)\in\omega(u^0)$ and set $\alpha=\max v_2, \delta=\max v_1$.

Case 1: $\alpha>\gamma(0)$. We claim that in this case $\omega(u^0) = \{v^*\} = \{(0, \alpha)\}$ on $\Omega$. To show this claim, suppose first that $v_2$ is not identically equal to $\alpha$. By the strong maximum principle, the second component of $u(\cdot; v^*)$ is strictly less than $\alpha$ for all positive $t$. Since $u(\cdot; v^*)\in\omega(u^0)$ for all $t$, the solution $u(\cdot, t; u^0)$ must therefore have its range in some rectangle $[0, M_1] \times [0, \alpha-\varepsilon]$ for some positive $\varepsilon$ and for some positive $t$. Such a rectangle is invariant, if $M_1$ is large, and therefore $v^*\in\omega(u^0)$, contradicting the assumption. Thus $v_2=\alpha$. The same argument shows that $v_1=0$. It remains to show that $v^*$ is the only element of $\omega(u^0)$. To see this, let $w^*=(w_1, w_2)\in\omega(u^0)$, and set $\bar{\alpha}=\max w_2$. Then $\bar{\alpha}=\alpha$, since otherwise an invariant rectangle could be found that contains one of the two solutions but not the other. By the previous argument, $w^*=\max(0, \bar{\alpha})=v^*$, which proves everything in this case.

Case 2: $\delta>\beta$. In this case, $v^*=(\delta, 0)$ and $\omega(u^0) = \{v^*\}$. The proof is as in case 1.

Case 3: $0\leq\alpha\leq\gamma(0), 0\leq\delta\leq\beta$. This is the remaining case. Let $[A, B] \times [C, D]$ be the smallest rectangle with sides parallel to the coordinate axes and corners $(A, D)$ and $(B, C)$ on the graph of $\gamma$ that still contains the range of $v^*$. Then $\delta\leq B \leq \beta$ and $\alpha \leq D \leq \gamma(0)$. All such rectangles are invariant. If this rectangle were not a single point, then by the strong maximum principle the function $u(\cdot, t; v^*)$ would have its range in a strictly smaller rectangle with the same properties, as soon as $t>0$. Again, this implies that $v^*\in\omega(u^0)$. Thus all elements of $\omega(u^0)$ must be constants on the graph of $\gamma$. Let $v^*$ and $w^*$ be two such elements of $\omega(u^0)$. If they were not the same, then one could find a small invariant rectangle about $v^*$ that does not contain $w^*$. Thus $\omega(u^0)$ must reduce to a single such point. This proves the theorem completely. Q.E.D.

Remark 1. In the above result, it cannot be excluded that the limit $(\rho_1, \rho_2)$ lies on one of the coordinate axes. An example is given by a vector field that has the form

$$f_i(u_1, u_2) = -u_i(u_1+u_2-1)$$

(2.2)
for \(i=1, 2\), near the point \(P=(0, 1)\). For solutions of (1.2) with initial data \((u^0_1, u^0_2)\) in a neighborhood of \(P\), \(u_1-u_2\) remains constant along solutions, and thus the limit will be \((\rho_1, \rho_2)\) if \(u^0_2 > u^0_1 + 1\) and \((\rho_1, \rho_2) = \left(\frac{1-c}{2}, \frac{1+c}{2}\right)\), \(c = u^0_2-u^0_1\), if \(u^0_2 \leq u^0_1 + 1\). Obviously, solutions of (1.2) can be viewed as solutions of (1.1), (1.6), (1.7).

**Remark 2.** A crucial assumption in the arguments above is that \(\gamma\) is strictly decreasing. More precisely, it was used to establish that in case 3, \(\omega(u^0)\) contains only one element. All other arguments are still valid if \(\gamma\) is only non-increasing; in particular, all elements of \(\omega(u^0)\) must still be componentwise constants with their range on the graph of \(\gamma\). We suspect that the general result is still true in this case, but we can only prove it under additional assumptions such as

\[
(2.3) \quad f \in C^2, \quad \frac{\partial f_z}{\partial u_2} = 0 \quad (u_1, u_2) \in Z.
\]

By the implicit function theorem, \(\gamma\) is a \(C^2\)-curve in this situation, and writing \(f_{ij} = \frac{\partial f_i}{\partial u_j}\), we must have \(f_{11} + \gamma' f_{12} = 0\) on \(Z\). Note that necessarily \(f_{22} < 0\) on \(Z_1\) due to (2.1) and (2.3). Suppose now that only \(\gamma' \leq 0\). To show that \(\omega(u^0)\) is a singleton, we first can exclude the possibility that \(\omega(u^0)\) contains an element \(v^* = (v_1, v_2) = (v_1, \gamma(v_1))\) with \(\gamma'(v_1) < 0\), since in that case arbitrarily small invariant rectangles about \(v^*\) can again be found. Since \(\omega(u^0)\) is a continuum, it must therefore have the form \(\omega(u^0) = [v_1, w_1] \times \{\gamma(v_1)\}\), with \(\gamma'\) vanishing on \([v_1, w_1]\). We now refer to the results in [5], in particular to the proof of Theorem 3.4 in that paper, which implies that \(\omega(u^0)\) will be a singleton if the spectrum of the linearization of (1.1), (1.6), (1.7) at each equilibrium point contains 0 as an algebraic simple eigenvalue and if the remainder of the spectrum is bounded away from the imaginary axis. By the above arguments, we only have to show this at points \((v_1, \gamma(v_1)) \in Z_1\) for which \(\gamma'(v_1) = 0\). Let \(\mu_0 = 0 < \mu_2 \leq \mu_3 \leq \ldots\) be the sequence of eigenvalues of \(-\Delta\) with zero Neumann boundary values, then the linearization of (1.1), (1.6), (1.7) about any such point on \(Z_1\) has the spectrum

\[
(2.4) \quad \sigma = \{\varepsilon \in \mathbb{C} | \varepsilon = -\mu_i \text{ or } \varepsilon = -d\mu_1 + f_{22}(v_1, \gamma(v_1)) \text{ for some } i > 0\}
\]
as a computation shows. The required condition on the spectrum therefore holds in this situation, and \(\omega(u^0)\) must be a singleton.

3. **Convergence to Constant Equilibria. II**

In this section, we present a convergence result for systems of the form (1.1) that have a zero set \(Z_1\) with a more complicated structure. We assume in
this section that

\[(3.1) \quad f_i(u_1, u_2) = h_i(u_i)g(u_1, u_2) \quad (i = 1, 2, u_i \geq 0).\]

The functions \(h_i\) and \(g\) are assumed to be continuously differentiable, and

\[(3.2) \quad h_i(0) = 0, \quad h_i(v) > 0 \quad (v > 0).\]

We assume further that there exist continuous functions \(k_1, k_2: [0, \infty) \rightarrow \mathbb{R}\) such that

\[(3.3) \quad Z_i = \{(u_1, u_2) | u_i > 0, k_1(u_i) + k_2(u_2) = 0\}.

The following additional assumptions have to hold:

\[(3.4) \quad g(u_1, u_2)(k_1(u_1) + k_2(u_2)) \leq 0 \quad (u_i \geq 0), \quad \text{with equality only if} \quad g(u_1, u_2) = 0.

\[(3.5) \quad k_i(0) < 0 \quad (i = 1, 2), \quad k_i(v_1) + k_2(v_2) > 0 \quad \text{for large} \quad v_1, v_2.

\[(3.6) \quad \text{The functions} \quad z \rightarrow \frac{k_1(z)}{h_1(z)} \quad \text{are non-decreasing on} \quad (0, \infty).

\[(3.7) \quad \text{Each} \quad k_i \quad \text{has exactly one zero.}

Let \(I \subset (0, \infty)\) be any open interval and \(x: I \rightarrow (0, \infty)\) be a solution of the ordinary differential equation \(h_i(x(s)) = h_2(x(s))\) on \(I\), then

\[(3.8) \quad \{s \in I | k_1(s) + k_2(x(s)) = 0\} \quad \text{has empty interior.}

A few remarks are in order.

1. The functions \(k_i\) need not be differentiable.

2. In general, the functions \(k_i\) are not uniquely defined. Clearly, both \(k_i\) can be modified for large arguments. Also, one can replace, e.g., \(k_i(\cdot)\) by \(k_i(\cdot) + \delta\) and \(k_2(\cdot)\) by \(k_2(\cdot) - \delta\), as long as (3.6) still holds. Thus condition (3.7) is not too restrictive.

3. Conditions (3.4) and (3.5) imply that \(g\) is positive near the origin and negative for large arguments, in agreement with the general assumptions in section 1.

4. Suppose now that the \(h_i\) are linear functions, \(h_i(x) = x\) and \(h_2(x) = \alpha x\) and that \(Z_i\) is the graph of a function \(\gamma\) as in the previous section which we assume to be continuously differentiable. We want to choose

\[k_i(v) = \begin{cases} \frac{\varepsilon \gamma(v)}{\beta} & 0 \leq v \leq \beta \\ \frac{\varepsilon}{\beta} v & v < \beta \end{cases}\]
and \( k_2(v) = v - \varepsilon \) for some suitable \( \varepsilon \). Then (3.6) and (3.7) will hold for all small \( \varepsilon \) if \( \gamma'(\beta) < 0 \) and \( \gamma(x) - x\gamma'(x) > 0 \) on \([0, \beta]\). In this case, (3.8) holds if every parabola of the form \( u_2 - C\varepsilon^2 \) with \( C > 0 \) intersects \( Z_1 \) only at isolated points. However, for linear \( h_i, Z_1 \) need not be graph; for instance it can be the intersection of the boundary of any disk containing the origin with the first quadrant.

5. Using nonlinear functions \( h_i \), it is not hard to construct examples in which \( Z_1 \) has more than one connected component.

6. The transversality condition (3.8) is a weak form of requiring that every equilibrium of the system (1.2) with the right hand side (3.1) has a one-dimensional center manifold.

As before we discuss (1.1), (1.6), (1.7) for non-negative initial data \( u^0 \) for which neither component vanishes identically.

**Theorem 3.1.** If (3.1)-(3.8) hold, then for any \( u^0 \in L^\infty(\Omega, R^2) \) the solution of (1.1) (1.6), (1.7) satisfies

\[
(3.9) \quad u(\cdot, t; u^0) \to (\rho_1, \rho_2)
\]
as \( t \to \infty \), uniformly in \( \Omega \) together with its first and second derivatives. Here \( \rho_1 \) and \( \rho_2 \) are positive constants and \( g(\rho_1, \rho_2) = 0 \).

Proof. We define the \( \omega \)-limit set \( \omega(u^0) \) as in the proof of Theorem 2.1 and must show that it is a singleton with two constant components \((\rho_1, \rho_2) \in Z_1\). Set

\[
V_0(u_1, u_2) = \int_1^u k_1(r) \frac{dr}{h_1(r)} + \int_1^u k_2(r) \frac{dr}{h_2(r)}
\]

for \( u_1, u_2 > 0 \). Then \( V_0 \) is continuously differentiable, \( V_0 \to +\infty \) if \( u_1 \downarrow 0 \) or \( u_2 \downarrow 0 \) (due to (3.2), (3.5)), and

\[
\nabla V_0(u_1, u_2) \cdot \mathbf{j}(u_1, u_2) \leq 0
\]

for all \( u_i > 0 \), with equality only if \( g(u_1, u_2) = 0 \). We then define

\[
V(\phi) = \int_\Omega V_0(\phi(x)) \, dx
\]

for any continuous function \( \phi: \Omega \to (0, \infty)^2 \). We want to use \( V \) as a Lyapunov functional for solutions of (1.1), (1.6), (1.7). To do this, we compute for \( t > 0 \), writing \( u(\cdot, t; u^0) = (u_1, u_2) \):
If \( \frac{k_1}{h_1} \) and \( \frac{k_2}{h_2} \) are smooth, then the first two integrals are non-positive, as an integration by parts shows. An approximation argument implies that this is also the case if these functions are only continuous and non-decreasing. The last integral is negative, unless \( (u_1, u_2) \) has all its values in the zero set \( Z_1 \). Thus \( V \) never increases along orbits with initial data \( u^0 \). In particular, \( V(v^*) < \infty \) for any \( v^* \in \omega(u^0) \), and thus \( v^* \) cannot have an identically vanishing component. By the results in [17], \( V \) is constant on \( \omega(u^0) \), and this set is invariant under the solution flow of (1.1), (1.6), (1.7). This implies that

\[
\nabla V_\delta(v_1, v_2) \cdot \tilde{f}(v_1, v_2) = 0 \quad \text{on} \quad \Omega
\]

for any \( v^* = (v_1, v_2) \in \omega(u^0) \). Thus any function in \( \omega(u^0) \) must have values on \( Z_1 \). The backwards invariance of \( \omega(u^0) \) now implies that such functions must be constants which cannot be zero.

It remains to show that \( \omega(u^0) \) is a singleton. Thus let \( v^* = (v_1, v_2) \) and \( \omega^* = (w_1, w_2) \) be two different elements of \( \omega(u^0) \). Without loss of generality, \( v_1 < w_2 \). Then there exists a continuum of points \( (\xi, \zeta) \in \omega(u^0) \) that connects \( v^* \) to \( \omega^* \). Since \( V \) is constant on \( \omega(u^0) \), this implies that

\[
\int_1^\xi \frac{k_i(r)}{h_i(r)} \, dr + \int_1^\zeta \frac{k_i(r)}{h_i(r)} \, dr = \text{const} .
\]

and

\[
k_i(\xi) + k_2(\xi) = 0
\]

for \( v_2 \leq \zeta \leq w_2 \) and the corresponding \( \xi \). Without loss of generality, \( k_2(\zeta) = 0 \) for \( v_2 \leq \xi \leq w_2 \). Then the solution set of (3.10) is locally the graph of a \( C^1 \)-function \( p \), and we obtain the differential equation

\[
\frac{k_i(\xi)}{h_i(\xi)} + p'(\xi) \frac{k_2(\xi)}{h_2(\xi)} = 0
\]

on some \( \xi \)-interval. Using (3.11), it follows that

\[
h_1(\xi) h_2(p(\xi)) = h_2(\xi)
\]

on some \( \xi \)-interval and that the solution \( p \) of this differential equation lies entirely in \( Z_1 \). This contradicts assumption (3.8). Therefore \( \omega(u^0) \) must be a singleton, and the theorem is completely proved. Q.E.D.
There is a systematic way to get the function $V_0$: Choose a $C^1$-function $\chi$ that has the same sign behavior as $-g$ and solve the first order partial differential equation

$$
\frac{\partial V_0}{\partial u_1} h_1 + \frac{\partial V_0}{\partial u_2} h_2 = \chi.
$$

If for such a solution the matrix

$$
(3.15) \quad \text{diag}(1, d) \cdot \nabla^2 V_2
$$

is positive semidefinite, then $V_0$ can serve as a Lyapunov functional. In the argument above, we took both $\chi$ and $V_0$ as sums of functions of $u_1$ and $u_2$ alone. In the special linear case $h_1(u_1) = u_1$, $h_2(u_2) = \alpha u_2$ with $\alpha > 0$, the general solution of (3.14) is given by

$$
(3.16) \quad V_0(u_1, u_2) = \phi \left( \frac{u_2}{u_1^\alpha} \right) + \int_{u_1}^{u_2} \frac{\chi(ru_1, r^\alpha u_2)}{r} \, dr,
$$

where $\phi$ is an arbitrary $C^1$-function. It would be interesting to have more general conditions on $\phi$ and $\chi$ that imply that the matrix in (3.15) is positive semidefinite.

4. Additional Remarks

1. The results in section 2 remain true if the operator $\text{diag}(-\Delta, -d\Delta)$ is replaced by any diagonal second order elliptic operator that allows the use of the maximum principle. The results in section 3 remain true if this operator is replaced by a diagonal elliptic operator of divergence form.

2. The techniques of section 3 permit an obvious extension to systems with $M > 2$ components and coupling terms

$$
f_i(\bar{u}) = h_i(u_i) g(\bar{u}), \quad (i = 1, \ldots, M).
$$

The zero set of $g$ should be given in the form

$$
Z_i = \{ \bar{u} > 0 | g(\bar{u}) = 0 \} = \{ \bar{u} | \sum_{i=1}^{M} k_i(u_i) = 0 \},
$$

and conditions (3.4)-(3.8) have to be generalized accordingly. It is not clear how the results of section 2 can be generalized to the case of $M > 2$ components. The principal difficulty is that such a system will have very few invariant sets, if all diffusion coefficients are different and the coupling terms have a common zero set $Z_i$ that is, say, an $(M-1)$-dimensional hypersurface. We note that indeed few general results are known for competitive systems with more than two species, even in the absence of diffusion.
3. Although we showed that $\omega(u^0)$ is a subset of $Z_1$ and a singleton, not all points on $Z_1$ will be stable. Unstable points on $Z_1$ can occur in particular in the situation of section 3, if $Z_1$ is locally a graph and $g$ is negative below and positive above $Z_1$. Such a behavior is possible, if $Z_1$ is globally not a graph. On the other hand, if $Z_1$ is locally a decreasing graph and $g$ is negative above and positive below $Z_1$, then all points on this portion of $Z_1$ are stable for (1.1), (1.6), (1.7), as was shown implicitly in section 2. A more subtle problem occurs if $Z_1$ is locally an increasing graph and $g$ is locally positive below and negative above this segment of $Z_1$. Such a segment will again be unstable if the slope of $Z_1$ is larger than the slope of the characteristics of the vector field $f$ (given in (3.8)). In the opposite case, it can be shown to be stable under suitable restrictions on the diffusion coefficient $d$, using the arguments in [5].

4. As indicated in section 1, there is some overlap between the class of competitive vector fields and the classes that are considered in section 2 and 3. In particular, if a competitive vector field $\tilde{f}$ has the property that $f_1$ and $f_2$ vanish on the same set $Z$ which is in addition the graph of a function $\gamma$, then $\gamma$ must be non-increasing, and the results of section 2 (possibly modified by the remarks after the proof of the main result) imply that $\omega(u^0)$ is a singleton for all initial data $u^0$. On the other hand, it is well-known that competitive vector fields together with diffusion generate strongly order-preserving semiflows which will converge for "almost all" $u^0$ to some equilibrium; see [7], [13], [20], [21], [22]. Thus the question arises whether there are general properties for semiflows (not necessarily the property of preserving order) that could extend the results of section 2 and 3.

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**References**


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