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An Example of a Null-Boundary Riemann Surface

By Zenjiro KURAMOCHI

We have proved that the Green's function is not[1] uniquely determined, when its pole is at an ideal boundary point of a null-boundary Riemann surface. M. Heins introduced[2] the notion of the minimal function due to R. S. Martin[3] and constructed a boundary point of dimension of preassigned number and conjectured that there would exist a boundary point of dimension infinity. We show by an example that his conjecture holds good.

1) Example. We denote by G the domain bounded by straight lines L_1, L_2 and the semi-circle C such that

$$L_1: 1 \leq |z| < \infty, \quad \arg z = 0, \quad L_2: 1 \leq |z| < \infty, \quad \arg z = \pi$$
$$C: |z| = 1, \quad 0 \leq \arg z \leq \pi.$$

On G we define a sequence of slits such that

$$I_1^i: 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{i4}}, \quad \arg z = \frac{\pi}{2} : i = 2, 3, 4, \ldots$$

$$I_2^i: 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{2i4}}, \quad \arg = \frac{\pi}{4} : i = 3, 4, 5, \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$I_n^i: 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{ni4}}, \quad \arg z = \frac{\pi}{2^n} : i = n+1, n+2, \ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$n = 1, 2, 3, \ldots$$

Let G^1 and G^2 be the same examplars with the same boundary and connect G^1 with G^2 by identifying L_1, L_2 and $\{I_j^i\}$ of them, to con-

1) Z. Kuramochi: Potential theory and its applications, I, Osaka Math. J. **3** (1951), 123-174.

2) M. Heins: Riemann surfaces of infinite genus, Annals of Math. **55** (1952), 296-317.

3) R. S. Martin: Minimal positive harmonic functions, Trans. Amer. Math. Soc. **19** (1941), 137-172.

struct the symmetric surface with respect to $L_1 + L_2 + \{I_j^i\}$. We denote
such a Riemann surface by F; then F has only one compact relative
bonudary lying on C and is of infinite genus and further has it one ideal
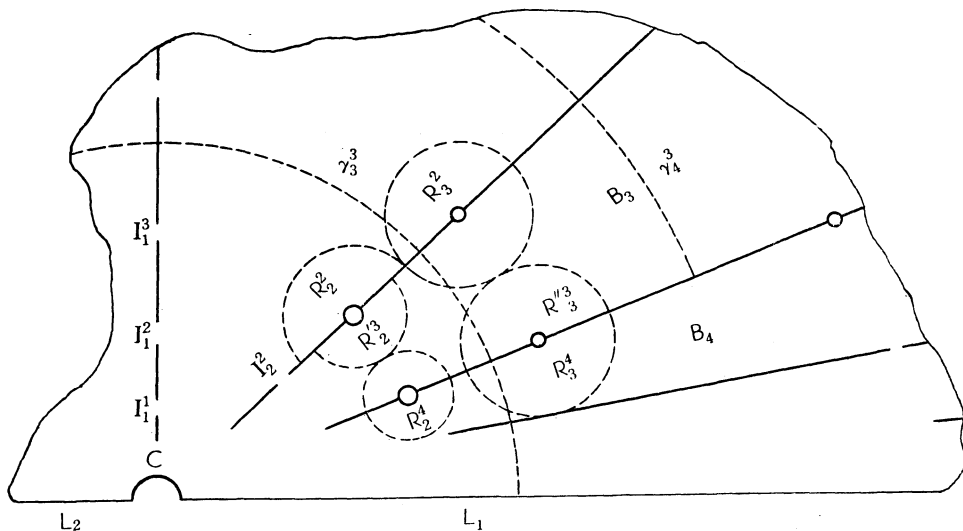boundary point at $z = \infty$, and it is clear that F has a null ideal boundary.

Fig. 1.

2) Let B_n be the subsurface of F with projection on the part
$\frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n-1}}$. Then B_n has boundary on $|z| = 1$, $\frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n-1}}$
and

$$\bar{J}_n^1: 1 \leq |z| \leq 2^{n-1}, \arg z = \frac{\pi}{2^{n-1}}, \overset{+}{J}_1^n: 1 \leq |z| \leq 2^n, \arg z = \frac{\pi}{2^n}$$

and

$$\bar{J}_n^i: 2^i \geq |z| \geq 2^i - \delta_n^i, \arg z = \frac{\pi}{2^{n-1}}: \delta_n^i = \frac{1}{2^{ni4}}: i = n+1, n+2, \ldots$$

$$\overset{+}{J}_{n+1}^i: 2^i \geq |z| \geq 2 - \delta_{n+1}^i, \arg z = \frac{\pi}{2^n}: \delta_{n+1}^i = \frac{2}{2^{(n+1)i4}}: i = n+2, n+3, \ldots$$

We transform B_n by the mapping $w = \pm (z e^{-\frac{\pi}{2^n}})^{2^n}$, where $\pm$ corresponds
to the mapping of upper or lower exemplars respectively, then B_n is
mapped onto the w-plane slits $^+J_w^i$, $^-J_w^i$ lying on $\arg w = 0$, or $\arg w = \pi$
and having the boundary on $|w| = 1$, and $^-J_n^1$, $^+J_n^1$,

$$^+J_w^i: \quad (2^i - \delta_n^i)^\alpha \leq |w| \leq 2^{i\alpha}: \quad \alpha = 2^n,$$

$$^-J_w^i: \quad (2^i - \delta_{n+1}^i)^\alpha \leq |w| \leq 2^{i\alpha}.$$

Then we have

$$^+J_w^i: \quad 2^{i\alpha} - \overset{w}{\delta_n^i} \leq |w| \leq 2^{i\alpha}, \quad \frac{2^n}{2^{i+1}}\delta_i^n \leq \overset{w}{\delta_i^n} \leq \frac{2^n}{2^i}\delta_i^n$$

$$^-J_w^i: \quad 2^{i\alpha} - \overset{w}{\delta_{n+1}^i} \leq |w| \leq 2^{i\alpha}, \quad \frac{2^n}{2^{i+1}}\delta_i^{n+1} \leq \overset{w}{\delta_{n+1}^i} \leq \frac{2^n}{2^i}\delta_i^{n+1}.$$

Denote by $\omega^{+i}(w)$ the harmonic measure of $^+J_w^i$ with respect to the domain $|w| > 1$. Then we have by elementary calculation the following inequality

$$\omega_i^+(w) \leq \frac{\log\left|\dfrac{1-\bar{a}w}{a-w}\right|}{\log\left|\dfrac{a^2-1}{\delta_i^w}-a\right|} : \quad a = (2^i)^\alpha.$$

On the other hand, denote by $U_n^j(p)$ the harmonic function on F in the part $|z| < \gamma_j^n$: $\gamma_j^n = \left(\dfrac{2^{j\alpha}+2^{(j+1)\alpha}}{2}\right)^{\frac{1}{\alpha}}$, such that $U_n^j(p) = \log|z|$, when $|z| = \gamma_j^n$, $\dfrac{\pi}{2^n} \leq \arg z \leq \dfrac{\pi}{2^{n+1}}$ and $U_n^j(p) = 0$, when $|z| = \gamma_j^n$, $\pi \geq \arg z \geq \dfrac{\pi}{2^{n-1}}$ or $\dfrac{\pi}{2^n} \geq \arg z \geq 0$ and further $U_n^j(p) = 0$, when $|z| = 1$. Since $U_n^j(p) \geq 0$, we define $U_n(p)$ by a uniformly convergent subsequence $\{U_n^j(p)\}$; then it is clear $U_n(p) \leq \log|z|$. On the other hand, let $V_n^j(p)$ be a harmonic function such that $V_n^j(p)$ is harmonic in $B_n \cap \{|z| < \gamma_j^n\}$, $V_j^n(p) = \log|z|$ on $\gamma_j^n \cap B_m$, $V_n^j(p) = 0$ on $2^i - \delta_n^i \leq |z| \leq 2^i$, $\arg z = \dfrac{\pi}{2^{n-1}}$ or and $2^i - \delta_{n+1}^i \leq |z| \leq 2^i$, $\arg z = \dfrac{\pi}{2^n}$ and on $\bar{J}_n^i$, $^+J_n^1$ and consider $V^*(z) = \log|z| - \sum\limits_{i=n}^{\infty}\log 2^i \omega^{+i}(z) - \sum\limits_{i=n}^{\infty}\log 2^i \omega^{-i}(z)$, where $\omega^{+i}(z)$ $(\omega^{-i}(z))$ is the harmonic measure of the boundary of B_n lying on $\{J_n^i\}$, $\{J_{n+1}^i\}$. Then we have $V^*(z) \leq 0$, $z \in \sum\limits_{i=n}^{\infty}\{I_n^i, I_{n+1}^i\}$. Consider $\omega^{+i}(z)$: $|z^j| = \gamma_j^n$, $\arg z_j = \dfrac{3\pi}{2^{n+1}}$, i.e. the value of $\omega^{+i}(w)$ at $w = e^{\pm\frac{\pi}{2}}r$: $r = \dfrac{2^{j\alpha}+2^{(j+1)\alpha}}{2}$. Then we have

$$\omega_i^+(z_j) \leq \frac{\log\sqrt{\dfrac{1+a^2r^2}{a^2+r^2}}}{\log\left|\left(\dfrac{a^2-1}{\overset{w}{\delta_i^n}}-a\right)\right|}$$

$$\leq \frac{\dfrac{1}{2}\log 2a^2}{\log\left(\dfrac{a^2}{2\delta_i}\right)} \leq \frac{\dfrac{1}{2}(\log 2)(1+i\alpha)}{i\alpha\log 2 - \log 2 + i^4 n\log 2} \leq \frac{1+i\alpha}{2i^4 n}, \quad a = (2^i)^\alpha.$$

Thus

$$\sum\limits_{i=n}^{\infty}\log 2^i \,\omega_n^{+i}(z) + \sum\limits_{i=n}\log 2^i \,\omega_n^{-i}(z) \quad \text{at} \quad z_j \,(j = n, \, n+1, \, \ldots)$$

$$\leq \log 2 \Big(\sum_{i=n}^{\infty} \frac{i(1+i\alpha)}{2i^4 n} + \sum_{i=n}^{\infty} \frac{i(1+i\alpha)}{2i^4(n+1)} \Big) + k_1 < k_2 < +\infty,$$

where k_i is a finite constant, from which follows the unboundedness of $V^*(z)$ at z_j $(j=1, 2, \ldots)$, and hence $U_n(p) \geq V_n^j(p) \geq V^*(p)$ yields the non-constancy of $U_n(p)$.

3) Next we consider the Dirichlet integral of $U_n(p)$ on F. In B_{n-1} and B_{n+1} and we denote by R_j^{n-1} and R_j^{n+1}, the ring-domains contained in $F - B_n$ with projection such that

$$R_j^{n-1} : \frac{\delta_j^{n-1}}{2} \leq |z - p_j^{n-1}| \leq 2^j \sin \frac{\pi}{2^{n+2}} : 0 \leq \arg z - p_j^{n-1} \leq \pi$$

$$R_j^{n+1} : \frac{\delta_j^n}{2} \leq |z - 'p_j^{n+1}| \leq 2^j \sin \frac{\pi}{2^{n+3}} : 0 \leq \arg z - p_j^{n+1} \leq \pi$$

respectively where

$$p_j^{n-1} : \arg z = \frac{\pi}{2^{n-1}}, \quad |z| = \Big(2^j - \frac{\delta_j^n}{2} \Big)$$

$$p_j^{n+1} : \arg z = \frac{\pi}{2^n}, \quad |z| = \Big(2 - \frac{\delta_j^{n+1}}{2} \Big).$$

Then we have

$$\mathfrak{M}_j^{n-1} = \text{module of } R_j^{n-1} = \log \frac{2^j \sin \frac{\pi}{2^{n+2}}}{\frac{2}{2\ 2^{nj4}}} \geq (j^4 n + j - n - 2) \log 2 + \log \pi \geq$$

$$(j^4 n + j + n - 2) \log 2 \text{ and}$$

$$\mathfrak{M}_j^{n+1} = \text{module of } R_j^{n+1} \geq (j^4(n+1) + j - n - 3) \log 2.$$

We denote by $w_n^i(p)$ the harmonic function such that

$w_n^i(p)$ is harmonic in $(F - B_n) \cap \{ |z| < \frac{1}{2} (2^i + 2^{i+1}) \}$

$w_n^i(p) = 0 : |z| = \frac{1}{2} (2^i + 2^{i+1})$, $z \in \{ C_i \cap (F - B_n) \} : C_r = |z| = \frac{2^i + 2^{i+1}}{2}$

$w_n^i(p) = \log 2^j : 2^j - \delta_n^j \leq |z| \leq 2^j$, $\arg z = \frac{\pi}{2^{n-1}} : j = n, n+1, \ldots,$

$w_n^i(p) = \log 2 : 2^j - \delta_{n+1}^j |z| \leq 2^j$, $\arg z = \frac{\pi}{2^n} : j = n+1; n+2, \ldots.$

Then

$$D_{F-B_n}(U_n^i(p)) \leq D_{F-B_n}(w_n^i(p)),$$

and further $\tilde{w}_n^i(p)$ is a continuous function such that

$$\tilde{w}_n^t(p) = \log 2^j : \quad |z - p_n^j| \leqq \frac{\delta_n^t}{2}, \qquad z \in B_{n-1} : j = n, \ldots$$

$$\tilde{w}_n^t(p) = \log 2^j : \quad |z - p_{n+1}^j| \leqq \frac{\delta_{n+1}^t}{2}, \quad z \in B_{n+1} : j = n+1, \ldots .$$

In R_j^{n-1} and R_j^{n+1} and $\tilde{w}_n^t(p)$ is harmonic and

$$\tilde{w}_n^t(p) = \log 2^j : \quad |z - p| = \frac{\delta_j^n}{2},$$

$$\tilde{w}_n^t(p) = 0 \qquad : \quad |z - p| = \sin \frac{\pi}{2^n},$$

$$\tilde{w}_n^t(p) = \log 2^j : \quad |z - p_j^{n+1}| = \frac{\delta_j^{n+1}}{2},$$

$$\tilde{w}_n^t(p) = 0 \qquad : \quad |z - p_j^{n+1}| = 2^j \sin \frac{\pi}{2^{n+1}},$$

$$\tilde{w}_n^t(p) = 0 \qquad : \quad p \in F - B_n - (\sum_j R_j^{n-1} + \sum_j R_j^{n+1}).$$

Then by Dirichlet principle

$$D_{F-B_n}(w_n^t(p)) \leqq \underset{F-B_n}{D(\tilde{w}_n^t(p))} \leqq \sum_j \frac{(\log 2^j)^2}{\mathfrak{M}_j^{n-1}} + \sum_j \frac{(\log 2^j)^2}{\mathfrak{M}_j^{n+1}} + A,$$

for every i, where $A < \infty$.

Thus

$$D_{F-B_n}(U_n(p)) \leqq \underset{\substack{i=\infty \\ F-B_n}}{D} (U_n^t(p)) \leqq \underset{\substack{i=\infty \\ F-B_n}}{D} (\tilde{w}_n^t(p))) < +\infty .$$

4) Since F has a null-boundary, $D_F(U_n(p)) = \infty$, because if $D_F(U_n(p)) < \infty$, it follows $U_n(p) = 0$, whence

$$D_{B_n}(U_n(p)) = \infty .$$

Since

$$D_{B_n}(U_m(p)) < \infty, \quad \text{if} \quad m \neq n ,$$

all $U_n(p)$ are linearly independent.

We show in reality that $1°)$ $U_n(p)$ are all minimal functions, and $2°)$ each B_n has only one minimal function.

We denote by B_j^n the ring-domain $2^j \leqq |z| \leqq 2^{j+1} - \delta_n^{j+1}$, $\frac{\pi}{2^n} \leqq \arg z \leqq \frac{\pi}{2^{n-1}}$ contained in B_n, and denote by $\max U_{p\in\gamma}(p)$ the maximum when p is on $B_n \cap C_j : C_j = \{|z| = \gamma_n^j\}$. If there exists at least a Jordan curve J in B_j^n starting from p_0, which is on C_j, and reaching at least one boundary component of the ring $2^j \leqq |z| \leqq 2^{j+1} - \delta_n^{j+1}$, $\frac{\pi}{2^n} \leqq \arg z \leqq \frac{\pi}{2^{n-1}}$,

and if we denote by $\omega(p)$ the harmonic measure of J with respect to this ring, then there exists a constant K depending only on the module of this ring such that

$$\underset{p \in c_j}{\text{Min}}\, U\,(p) \geqq K\omega(p) \quad \underset{p \in c_j}{\text{Max}}\, U\,(p) \geqq K\, \underset{p \in c_j}{\text{Max}}\, U(p)\,.$$

5) According to R. S. Martin's theorem any positive harmonic function can be expressed uniquely by a linear form of minimal functions, thus

$$U = \int V d\mu\,,$$

therefore there exists at least a function V such that $KU_n \geqq V : K < \infty$.

a) *V is unbounded on* $|z| = \gamma^j$ *in* B_n.

Proof. If $V(p) \leqq M < \infty$, we define $V_j^1(V_j^2(p))$ such that $V_j^1(p)(V_j^2(p))$ is harmonic in $|z| < \gamma^j$, $V_j^1 = 0$ ($V_j^2 = M$) : $p \in C_j \cap B_n$ and $V_j^1 = V_j^2 = V$: $p \in C_j \cap (F - B_n)$, and take $V^1(p)$, $V^2(p)$ from the uniformly convergent sequences $\{V_j^1\}$, $\{V_j^2\}$. Then $0 \leqq V_j^2 - V_j^1 \leqq M \omega_j(p)$, where $\omega_j(p)$ is the harmonic measure of $C_j \cap B_n$ with respect to domain of F contained in $|z| \leqq \gamma^j$. Since F has a null-boundary, we have $V^1(p) = V^2(p)$. On the other hand, let $U_j(p)$ be the harmonic function in $|z| \leqq \gamma^j$ such that $U_j(p) = 0 : p \in C_j \cap B_n$, $U_j(p) = U(p) : p \in C_j \cap (F - B_n)$, and let $U^*(p)$ be a harmonic function obtained by taking a uniformly convergent subsequence from $\{U_j(p)\}$. We construct ring-domains contained in B_n such that

$$R_i'^n \quad : \ \delta_i^n \leqq |z - p_i^n| \leqq 2^i \sin \frac{\pi}{2^{n+2}}\,, \qquad 0 \leqq (\arg z - p_i^n) \leqq \pi$$

Fig. 1.

$$R_i''^{n+1} : \ \delta_i^{n+1} \leqq |z - p_i^{n+1}| \leqq 2^i \sin \frac{\pi}{2^{n+2}}\,, \quad 0 \leqq (\arg z - p_i^{n+1}) \leqq \pi$$

where

$$|p^n| = \frac{1}{2}\left(2^i - \frac{1}{2^{(n+1)i4}}\right),\ \arg p_i^n = \frac{\pi}{2^{n-1}}\,,$$

$$|p_i^{n+1}| = \frac{1}{2}\left(2^i - \frac{2}{2^{(n+1)i4}}\right),\ \arg p_i^{n+1} = \frac{\pi}{2^n}\,,$$

and define a continuous function as in the case (3). Then we have

$$D_{B_n}(U_n^*(p)) < \infty\,, \quad \text{and} \quad D_F(U_n^*(p)) < \infty\,.$$

This implies that $U_n^*(p) = 0$. By assumption $V \leqq U$, $V \leqq M$ in B_n,

we have
$V = V^1 = V^2$, $V_j^1 \leq U_j(p)$, and it follow that $V^1 \equiv U^*(p) \equiv V \equiv 0$, therefore V is not bounded on $C^J \cap B_n$ and by (4) $V(z)$ is not bounded on the sequence $\{z_i\} : |z_i| = \gamma^i$, $\arg z_i = \dfrac{3\pi}{2^n}$.

b) $V(p)$ *is invariant by generalized extremisation.*[4]

Let $V_j(p)$ be harmonic in $F \cap \{|z| < \gamma^J\}$, $V_j(p) = V(p) : p \in C_j \cap B_n$ $V_j(p) = 0 : p \in C_j \cap (F - B_n)$: From the unboundedness of $V(p)$ on $\{z_i\}$ and from $V(p) \leq U(p)$, we can prove as (2) and (3), that there exists a harmonic function $V^*(p)$ from $\{V_i(p)\}$, such that

$$D_{F-B_n}(V^*(p)) < \infty, \quad V^*(p) \not\equiv \text{const}.$$

Since $|V_j(p) - V(p)| \leq 2U(p)$ on $\arg z = \dfrac{\pi}{2^n}$ or $\arg z = \dfrac{\pi}{2^{n-1}}$, $V(p) = V(p) : p \in C_j \cap B_n$, and hence we have by the same manner used in (2) (3), $D_F(V^*(p) - V(p)) < \infty$, therefore $V^*(p) = V(p)\{V^*(p)$ is obtained by generalized extremisation from $V(p)\}$ and thus, since $U(p) \geq V(p) \geq 0$., it follows that $V(p)$ is invariant by generalized exteremisation with respect to B_n.

c) *There is only one minimal function smaller than* $U(p)$.

Since $V^*(p) \neq 0$, if there are two functions $V_1(p) \geq V_2(p)$ such that $V_i(p) \leq U(p)$, then there are two constants K_1, K_2 such that

$$\varlimsup_j \operatorname{Max} V_{i \ p \in c_j}(p) = K_i \log |z| : \quad i = 1, 2,$$

but from (4) there exist constants K_3, K_4, such that

$$V_1(p) \leq K_3 V_2(p) \qquad V_1(p) > K_4 V_2(p) , \ p \in C_j .$$

Put $\operatorname{Min} \dfrac{V_1(z)}{V_2(z)} = K_i , \quad z \in C_i \cap B_n .$

Then

$$\varlimsup_j K_i = K, \quad K < \infty ,$$

because

$$\max V_1(z) = K_1' \log |z| : \quad z \in B_n ,$$
$$\min V_2(z) = K' \log |z| : \quad z \in B_n ,$$

therefore there exists a subsequence

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4) We such operation generalized extremisation for convenience. This is certainly different from the extremisation.

$$\lim_i K'_n = K$$

and

$$0 \leq V_1(z) - K'_i V_2(z) = \varepsilon'_i V_2(z) : \quad \lim_i \varepsilon'_i = 0 : \quad z \in C_i \cap B_n ,$$

thus

$$V_{1,2}(z) - K_i V_{2,i}(z) \leq \varepsilon_i V_{i2}(z) \leq \varepsilon_i S \log |z| : \quad S < \infty$$

$$\lim_{i=\infty} V_{1,i}(z) - K'_n V_{2,i}(z) = 0 ,$$

which implies that

$$V_1(z) = K V_2(z) ,$$

thus $V_{1,i}(p)$ is a minimal function, and if we compare $U(p)$ with $V_1(p)$ by $\{U_i(p), V_{1,i}(p)\}$ we have $U(p) \leq K'' V_1(p)$, K'' being constant, thus $U(p)$ is a minimal function and we see that on our example there exist exactly enumrable infinity of minimal functions.

6) **Positive harmonic function in the neighbourhood of an ideal boundary point.**

Let F be a null-boundary Riemann surface with a compact relative boundary Γ_0 and p^∞ be an ideal boundary point.

We denote by $G^i(p, p^\infty)$ $(i = 1, 2, \ldots)$, the positive minimal function with a pole at p^∞ and denote by G_i^N the domain $E[G^i \geq N]$ and by C_i^{*N} the niveau curve $E[G^i = N]$. Then $\sum_i G_i^N$ is an open set with a compact boundary.

Proof.   If $G_i^N \in p_j$: $\lim_j p_j = p^\infty$ and if $G(p, p_j)$ is the Green's function with its pole at p_j, then since $\dfrac{\partial G^i(p, p^\infty)}{\partial n} \geq 0$, $p \in C_i^{*N}$, we have by Green's formula

$$N \geq G^i(p_j, p^\infty) = \frac{1}{2\pi} \int_{C_i^{*M}} G(p, p_j) \frac{\partial G^i(p. p^\infty)}{\partial n} \, ds. \quad ^{5)}$$

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5) $$2\pi = \int_{\Gamma_0} \frac{\partial G(z, p_i)}{\partial n} \, ds = \int_{\Gamma_0} \frac{\partial G(z, p^\infty)}{\partial n} \, ds$$

and

$$D_{F_n \cap (F_n - D_N)}(G(z, p^\infty)) \leqq \underset{F \cap (F - D_N)}{D} (G(z, p_m)) \leqq 2\pi ,$$

for sufficiently larrge $m(n)$. Let m and $n \to \infty$. We have

$$D_{F \cup (F - D_N)}(G(z, p^\infty)) \leqq 2\pi N.$$

It follows that

$$\lim_{n=\infty} \int_{\Gamma_n \cap (F - D_N)} \frac{\partial G(z, p^\infty)}{\partial n} \, ds = 0, \quad \text{and} \quad \int_{\Gamma_n} \frac{\partial G(z, p^\infty)}{\partial n} \, ds = 2\pi .$$

Let $V_M(p^\infty)$ be a neighbourhood with compact boundaries such that $\int_{\overset{*}{C}_i^M \cap V_M} \dfrac{\partial G_1^i}{\partial n}\, ds = \pi$. Then since $N \geq \dfrac{2}{2\pi} \int_{\overset{*}{C}_i^M \cap V_M} G(p, p_j) \dfrac{\partial G^i(p,\ p^\infty)}{\partial n}\, ds$, there exists at least one point q_j on $\overset{*}{C}_i^M - (C_i^M \cap V_M(p))$ for every $p_j \overline{\in} G_i^N$ such that $\lim\limits_{q_j \in C_i^M} G(q_j, p^\infty) \leq 2N$ for every M, whence $\lim\limits_{j=\infty} G(p, p_j)$ ($\lim G'(p,\ p^\infty)$) is free from the minimal functions $G^i(p,\ p^\infty)$. If $G^N = \sum\limits_i G_i^N$ is not compact, there exists a sequence $r_1,\ r_2,\ \ldots,\ r_i \overline{\in} G^N$, and there exists at least one general Green's function $G(p,\ r^\infty)$, but $G(p,\ r^\infty)$ must be expressed by a linear form of $G^1(p,\ p^\infty),\ G^2(p,\ p^\infty) \cdots$, which contradicts the preceding assertion.

Since at any point p there exists a constant $k(p)$ such that $U(p) \leq k(p)$ for any positive harmonic function $U(p)$ satisfying $\dfrac{1}{2\pi} \int_{\varGamma_0} \dfrac{\partial U}{\partial n}\, ds = 1$, $\underset{P \in \varGamma_0}{U(p)} = 0$, we have $G_N > G_{2N} \cdots$, $\prod\limits_{i=1}^{\infty} \overline{G}_{Ni} = p^\infty$.

If $\omega_{Nn}(p)$ denotes the harmonic measure of the boundary G_{Nn} with respect to the domain $F - G_{Nn}$, then $Nn\omega_{nN}(p)$ is a monotonously increasing function. Hence if $Nn \int_{\varGamma_0} \dfrac{\partial \omega_{nN}(p)}{\partial n}\, ds < \infty$, then $\lim\limits_{n=\infty} \omega_{nN}(p) = \omega^*(p)$ is harmonic and $\lim\limits_{p \to p_\infty} \omega^*(p) = \infty$. Then as a special case we have

Corollary. *If $p_\infty^{6)}$ is finite dimensional, then the solution of Evans's[7] problem exists.*

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6) See 2).

7) See 1) and 2).