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| **Note** | }
An Example of a Null-Boundary Riemann Surface

By Zenjiro Kuramochi

We have proved that the Green's function is not uniquely determined, when its pole is at an ideal boundary point of a null-boundary Riemann surface. M. Heins introduced the notion of the minimal function due to R. S. Martin and constructed a boundary point of dimension of preassigned number and conjectured that there would exist a boundary point of dimension infinity. We show by an example that his conjecture holds good.

1) Example. We denote by $G$ the domain bounded by straight lines $L_1$, $L_2$ and the semi-circle $C$ such that

$\begin{align*}
L_1 : & \quad 1 \leq |z| < \infty, \quad \arg z = 0, \\
L_2 : & \quad 1 \leq |z| < \infty, \quad \arg z = \pi \\
C : & \quad |z| = 1, \quad 0 \leq \arg z \leq \pi.
\end{align*}$

On $G$ we define a sequence of slits such that

$\begin{align*}
I_1^i : & \quad 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{4i}}, \quad \arg z = \frac{\pi}{2} : i = 2, 3, 4, \ldots \\
I_2^i : & \quad 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{4i}}, \quad \arg z = \frac{\pi}{4} : i = 3, 4, 5, \ldots \\
& \quad \vdots \\
I_n^i : & \quad 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{4i}}, \quad \arg z = \frac{\pi}{2^n} : i = n+1, n+2, \ldots
\end{align*}$

$n = 1, 2, 3, \ldots$

Let $G^1$ and $G^2$ be the same examplars with the same boundary and connect $G^1$ with $G^2$ by identifying $L_1$, $L_2$ and $\{I_j^i\}$ of them, to con-
struct the symmetric surface with respect to \( L_1 + L_2 + \{I_i^j\} \). We denote such a Riemann surface by \( F \); then \( F \) has only one compact relative boundary lying on \( C \) and is of infinite genus and further has it one ideal boundary point at \( z = \infty \), and it is clear that \( F \) has a null ideal boundary.

2) Let \( B_n \) be the subsurface of \( F \) with projection on the part \( \frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n-1}} \). Then \( B_n \) has boundary on \(|z| = 1\), \( \frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n-1}} \) and

\[
\overline{J}_n^t : 1 \leq |z| \leq 2^{n-1}, \quad \arg z = \frac{\pi}{2^n}, \quad \overline{J}_n^t : 1 \leq |z| \leq 2^n, \quad \arg z = \frac{\pi}{2^n}
\]

and

\[
\overline{J}_{n+1}^t : 2^t \leq |z| \leq 2^t - \delta_{n+1}^t, \quad \arg z = \frac{\pi}{2^n}, \quad \overline{J}_{n+1}^t : 2^t \leq |z| \leq 2 - \delta_{n+1}^t, \quad \arg z = \frac{\pi}{2^n} \cdot \delta_{n+1}^t = \frac{2}{2^{n+1}} \cdot i = n+2, n+3, ...
\]

We transform \( B_n \) by the mapping \( w = \pm (z - \frac{\pi}{2^n})^{2^n} \), where \( \pm \) corresponds to the mapping of upper or lower exemplars respectively, then \( B_n \) is mapped onto the \( w \)-plane slits \( \overline{J}_w, \overline{J}_w^t \) lying on \( \arg w = 0 \), or \( \arg w = \pi \) and having the boundary on \(|w| = 1\), and \( \overline{J}_n^t, \overline{J}_n^t \):

\[
\overline{J}_w^t : (2^t - \delta_n^t)^a \leq |w| \leq 2^ta^t, \quad \alpha = 2^n,
\]

\[
\overline{J}_w^t : (2^t - \delta_{n+1}^t)^a \leq |w| \leq 2^ta^t.
\]
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Then we have

\[ +J'_{w}: \quad 2^{t^*} - \delta^n \leq |w| \leq 2^{t^*}, \quad \frac{2^n}{2^{t^*+1}} \delta^n \leq \delta^n \leq \frac{2^n}{2^{t^*}} \delta^n \]

\[ -J'_{w}: \quad 2^{t^*} - \delta^n+1 \leq |w| \leq 2^{t^*}, \quad \frac{2^n}{2^{t^*+1}} \delta^{n+1} \leq \delta^n+1 \leq \frac{2^n}{2^{t^*}} \delta^{n+1}. \]

Denote by \( \omega^{+t}(w) \) the harmonic measure of \( +J'_{w} \) with respect to the domain \( |w| > 1 \). Then we have by elementary calculation the following inequality

\[
\omega_{n}^{+}(w) \leq \frac{\log \left| 1 - \frac{aw}{a - w} \right|}{\log \left| \frac{a^2 - 1}{\delta_{n}^2} - a \right|} : \quad a = (2^t)^n.
\]

On the other hand, denote by \( U_{n}^{+}(p) \) the harmonic function on \( F \) in the part \( |z| < \gamma_{n} : \gamma_{n} = \left( \frac{2^{t^*} + 2^{t^*+1}/2}{2} \right) \), such that \( U_{n}^{+}(p) = \log |z| \), when \( |z| = \gamma_{n} \), \( \pi \delta^{n} \leq \arg z \leq \pi /2^{t^*+1} \) and \( U_{n}^{+}(p) = 0 \), when \( |z| = \gamma_{n} \), \( \pi \leq \arg z \geq \pi /2^{t^*+1} \) or \( \pi /2^{t^*+1} \leq \arg z \geq 0 \) and further \( U_{n}^{+}(p) = 0 \), when \( |z| = 1 \). Since \( U_{n}^{+}(p) \geq 0 \), we define \( U_{n}^{-}(p) \) by a uniformly convergent subsequence \( \{ U_{n}^{+}(p) \} \); then it is clear \( U_{n}^{+}(p) \leq \log |z| \). On the other hand, let \( V_{n}^{+}(p) \) be a harmonic function such that \( V_{n}^{+}(p) \) is harmonic in \( B_{n} \cap \{ |z| < \gamma_{n} \} \), \( V_{n}^{+}(p) = \log |z| \) on \( \gamma_{n} B_{n} \), \( V_{n}^{+}(p) = 0 \) on \( 2^{t^*} - \delta_{n} \leq |z| \leq 2^{t^*} \), \( \arg z = \pi /2^{t^*} \) or and \( 2^{t^*} - \delta_{n+1} \leq |z| \leq 2^{t^*} \), \( \arg z = \pi /2^{t^*+1} \) and on \( f^{1}_{n}, +J'_{n} \) and consider \( V^{+}(z) = \log |z| - \sum_{t=n}^{\infty} \log 2^{t} \omega^{+t}(z) - \sum_{t=n}^{\infty} \log 2^{t} \omega^{-t}(z) \), where \( \omega^{+t}(z) (\omega^{-t}(z)) \) is the harmonic measure of the boundary of \( B_{n} \) lying on \( \{ f^{1}_{n}, +J'_{n+1} \} \).

Then we have \( V^{+}(z) \leq 0 \), \( z \in \sum_{t=n}^{\infty} \{ I_{n}, I'_{n+1} \} \). Consider \( \omega^{+t}(z) : |z^{|t}} = \gamma_{n}^{t} \), \arg \( \omega^{+t}(z) \) \( r = 2^{t} + 2^{t+1}/2 \), \arg \( \omega^{+t}(z) \) \( r = \frac{3\pi}{2^{t+1}} \), i.e. the value of \( \omega^{+t}(z) \) at \( w = e^{\pm \pi/2} \). Then we have

\[
\omega^{+t}(z_{j}) \leq \frac{\log \left| 1 + a^2 r^2 \right|}{\log \left| \frac{a^2 - 1}{\delta_{n}^2} - a \right|} \]

\[
\leq \frac{1}{2} \log 2^{a^{n}} \leq \frac{1}{2} \left( \log 2 \right) \left( 1 + i \alpha \right) \leq \frac{1}{i \alpha} \log 2^{2} + \log 2 - i \left( n \log 2 - \frac{1 + i \alpha}{2^{t} n} \right) : \quad a = (2^t)^n.
\]

Thus

\[
\sum_{t=n}^{\infty} \log 2^{t} \omega_{n}^{+t}(z) + \sum_{t=n}^{\infty} \log 2^{t} \omega_{n}^{-t}(z) \quad \text{at} \quad z_{j} (j = n, n+1, \ldots)
\]
where \( k_i \) is a finite constant, from which follows the unboundedness of \( V^*(z) \) at \( z_j (j = 1, 2, \ldots) \), and hence \( U_n(p) \geq V_n^*(p) \geq V^*(p) \) yields the non-constancy of \( U_n(p) \).

3) Next we consider the Dirichlet integral of \( U_n(p) \) on \( F \). In \( B_{n-1} \) and \( B_{n+1} \) and we denote by \( R_j^{-1} \) and \( R_j^{n+1} \), the ring-domains contained in \( F - B_n \) with projection such that

\[
R_j^{-1} : \frac{\delta_j^{n-1}}{2} \leq |z - \rho_j^{n-1}| \leq 2^j \sin \frac{\pi}{2^{n+2}} : 0 \leq \arg z - \rho_j^{n-1} \leq \pi
\]

\[
R_j^{n+1} : \frac{\delta_j^n}{2} \leq |z - \rho_j^{n+1}| \leq 2^j \sin \frac{\pi}{2^{n+3}} : 0 \leq \arg z - \rho_j^{n+1} \leq \pi
\]

respectively where

\[
\rho_j^{n-1} : \arg z = \frac{\pi}{2^{n-1}}, \quad |z| = \left(2^j - \frac{\delta_j^{n-1}}{2}\right)
\]

\[
\rho_j^{n+1} : \arg z = \frac{\pi}{2^{n}}, \quad |z| = \left(2 - \frac{\delta_j^{n+1}}{2}\right).
\]

Then we have

\[
\mathfrak{M}_j^{-1} = \text{module of } R_j^{-1} = \log \frac{2^j \sin \frac{\pi}{2^{n+2}}}{2} \geq (j^*n + j - n - 2) \log 2 + \log \pi \geq
\]

\[
(j^*n + j + n - 2) \log 2 \quad \text{and}
\]

\[
\mathfrak{M}_j^{n+1} = \text{module of } R_j^{n+1} \geq (j^*n + 1 + j - n - 3) \log 2.
\]

We denote by \( w_n^t(p) \) the harmonic function such that

\[
w_n^t(p) \text{ is harmonic in } (F - B_n) \cap \{|z| < \frac{1}{2} (2^t + 2^{t+1})\}
\]

\[
w_n^t(p) = 0 : |z| = \frac{1}{2} (2^t + 2^{t+1}), \quad z \in (C_r \cap (F - B_n)) : C_r = |z| = \frac{2^t + 2^{t+1}}{2}
\]

\[
w_n^t(p) = \log 2^t : 2^t - \delta_j^n \leq |z| \leq 2^t, \quad \arg z = \frac{\pi}{2^{n-1}} : j = n, n + 1, \ldots,
\]

\[
w_n^t(p) = \log 2 : 2^t - \delta_j^{n+1} \leq |z| \leq 2^j, \quad \arg z = \frac{\pi}{2^n} : j = n + 1; n + 2, \ldots.
\]

Then

\[
D_{F - B_n}(U_n^t(p)) \leq D_{F - B_n}(w_n^t(p)),
\]

and further \( \bar{w}_n^t(p) \) is a continuous function such that
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\[ \hat{w}_n^t(p) = \log 2^t : \quad |z - p_n^t| \leq \frac{\delta_n^t}{2}, \quad z \in B_{n-1} : j = n, \ldots \]

\[ \hat{w}_n^t(p) = \log 2^t : \quad |z - p_{n+1}^t| \leq \frac{\delta_{n+1}^t}{2}, \quad z \in B_{n+1} : j = n + 1, \ldots \]

In \( R_{n-1}^t \) and \( R_{n+1}^t \) and \( \hat{w}_n^t(p) \) is harmonic and

\[ \hat{w}_n^t(p) = \log 2^t : \quad |z - p| = \frac{\delta_n^t}{2}, \]

\[ \hat{w}_n^t(p) = 0 : \quad |z - p| = \sin \frac{\pi}{2^n}, \]

\[ \hat{w}_n^t(p) = \log 2^t : \quad |z - p| = \frac{\delta_{n+1}^t}{2}, \]

\[ \hat{w}_n^t(p) = 0 : \quad |z - p| = 2^t \sin \frac{\pi}{2^{n+1}}, \]

\[ \hat{w}_n^t(p) = 0 : \quad p \in F - B_n - (\sum_j R_{j-1}^t + \sum_j R_{j+1}^t). \]

Then by Dirichlet principle

\[ D_{F - B_n}(w_n^t(p)) \leq D_{F - B_n}(\hat{w}_n^t(p)) \leq \sum_j \left( \frac{\log 2^t}{2^{j-1}} + \sum_j \frac{(\log 2^t)^2}{2^{n+1}} + A, \right. \]

for every \( t \), where \( A < \infty \).

Thus

\[ D_{F - B_n}(U_n(p)) \leq D_{F - B_n}(\hat{w}_n^t(p)) \leq D_{F - B_n}(\hat{w}_n^t(p)) \leq + \infty. \]

4) Since \( F \) has a null-boundary, \( D_F(U_n(p)) = \infty \), because if \( D_F(U_n(p)) < \infty \), it follows \( U_n(p) = 0 \), whence

\[ D_{B_n}(U_n(p)) = \infty. \]

Since

\[ D_{B_n}(U_m(p)) \leq \infty, \quad \text{if} \quad m \neq n, \]

all \( U_n(p) \) are linearly independent.

We show in reality that 1°) \( U_n(p) \) are all minimal functions, and 2°) each \( B_n \) has only one minimal function.

We denote by \( B_n^j \) the ring-domain \( 2^j \leq |z| \leq 2^{j+1} - \delta_n^j, \quad \frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n+1}} \) contained in \( B_n \), and denote by \( \max U_{p \in \gamma}(p) \) the maximum when \( p \) is on \( B_n \cap C_j : C_j = \{ |z| = \gamma_n \} \). If there exists at least a Jordan curve \( j \) in \( B_n^j \) starting from \( p_0 \), which is on \( C_j \), and reaching at least one boundary component of the ring \( 2^j \leq |z| \leq 2^{j+1} - \delta_n^j, \quad \frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n+1}}, \)
and if we denote by \( \omega(p) \) the harmonic measure of \( J \) with respect to this ring, then there exists a constant \( K \) depending only on the module of this ring such that

\[
\operatorname{Min}_{p \in C_j} U(p) \geq K \omega(p) \operatorname{Max}_{p \in C_j} U(p) \geq K \operatorname{Max}_{p \in C_j} U(p).
\]

5) According to R. S. Martin's theorem any positive harmonic function can be expressed uniquely by a linear form of minimal functions, thus

\[
U = \int V d\mu,
\]

therefore there exists at least a function \( V \) such that \( KU \leq V : K < \infty \).

a) \( V \) is unbounded on \( |z| = \gamma' \) in \( B_n \).

Proof. If \( V(p) \leq M < \infty \), we define \( V_j(p) \) such that \( V_j(p) \) is harmonic in \( |z| < \gamma' \), \( V_j = 0 \) (\( V_j = M \) : \( p \in C_j \cap B_n \) and \( V_j = V \) : \( p \in C_j \cap (F - B_n) \)), and take \( V^1(p) \) from the uniformly convergent sequences \( \{ V_j \} \). Then

\[
0 \leq V_j - V^1 \leq M \omega_j(p),
\]

where \( \omega_j(p) \) is the harmonic measure of \( C_j \cap B_n \) with respect to domain of \( F \) contained in \( |z| < \gamma' \). Since \( F \) has a null-boundary, we have \( V^1(p) = V^2(p) \) in \( \gamma < \gamma' \). On the other hand, let \( U_j(p) \) be the harmonic function in \( |z| < \gamma' \) such that \( U_j(p) = 0 \) : \( p \in C_j \cap B_n \), \( U_j(p) = U(p) : p \in C_j \cap (F - B_n) \), and let \( U^*(p) \) be the harmonic function obtained by taking a uniformly convergent subsequence from \( \{ U_j(p) \} \). We construct ring-domains contained in \( B_n \) such that

\[
R_t^n : \delta_t^n \leq |z - p_t^n| \leq 2^t \sin \frac{\pi}{2^{n+2}}, \quad 0 \leq (\arg z - \arg p_t^n) \leq \pi \quad \text{Fig. 1.}
\]

\[
R_t^{n+1} : \delta_t^{n+1} \leq |z - p_t^{n+1}| \leq 2^{n+1} \sin \frac{\pi}{2^{n+2}}, \quad 0 \leq (\arg z - \arg p_t^{n+1}) \leq \pi
\]

where

\[
|p_t^n| = \frac{1}{2} \left( 2^t - \frac{1}{2^{(n+1)k+4}} \right), \quad \arg p_t^n = \frac{\pi}{2^{n-1}},
\]

\[
|p_t^{n+1}| = \frac{1}{2} \left( 2^t - \frac{2}{2^{(n+1)k+4}} \right), \quad \arg p_t^{n+1} = \frac{\pi}{2^n}
\]

and define a continuous function as in the case (3). Then we have

\[
D_B(U^*(p)) < \infty, \quad \text{and} \quad D_F(U^*_n(p)) < \infty.
\]

This implies that \( U^*_n(p) = 0 \). By assumption \( V \leq U, V \leq M \) in \( B_n \),
we have
\( V = V^1 = V^2, V \leq U_j(p) \), and it follow that \( V^1 \equiv U^*(p) \equiv V \equiv 0 \), therefore \( V \) is not bounded on \( C^1 \cap B_n \) and by (4) \( V(z) \) is not bounded on the sequence \( \{z_t\} : |z_t| = \gamma, \arg z_t = \frac{3\pi}{2n} \).

b) \( V(p) \) is invariant by generalized extremisation.\(^{4)}\)

Let \( V_j(p) \) be harmonic in \( F \cap \{ |z| < \gamma \} \), \( V_j(p) = V(p) : p \in C_j \cap B_n \) \( V_j(p) = 0 \) : \( p \in C_j \cap (F - B_n) \). From the unboundedness of \( V(p) \) on \( \{z_t\} \) and from \( V(p) \leq U(p) \), we can prove as (2) and (3), that there exists a harmonic function \( V^*(p) \) from \( \{V_t(p)\} \), such that

\[ D_{x-B_n}(V^*(p)) < \infty, \quad V^*(p) \equiv \text{const}. \]

Since \( |V_j(p) - V(p)| \leq 2U(p) \) on \( \arg z = \frac{\pi}{2n} \) or \( \arg z = \frac{\pi}{2n-1}, \quad V(p) = V(p) : p \in C_j \cap B_n \), and hence we have by the same manner used in (2) (3), \( D_{x}(V^*(p) - V(p)) < \infty \), therefore \( V^*(p) = V(p) \) \( V^*(p) \) is obtained by generalized extremisation from \( V(p) \) and thus, since \( U(p) \geq V(p) \geq 0 \), it follows that \( V(p) \) is invariant by generalized extremisation with respect to \( B_n \).

c) There is only one minimal function smaller than \( U(p) \).

Since \( V^*(p) \equiv 0 \), if there are two functions \( V_1(p) \geq V_2(p) \) such that \( V_1(p) \leq U(p) \), then there are two constants \( K_1, K_2 \) such that

\[ \lim_{j} \max_{p \in c_j(p)} V_t = K_1 \log |z| : i = 1, 2, \]

but from (4) there exist constants \( K_1, K_2 \), such that

\[ V_1(p) \leq K_3 V_2(p), \quad V_1(p) \geq K_4 V_2(p), \quad p \in C_j. \]

Put \( \min V_1(z) = K_1, \quad z \in C_j \cap B_n. \)

Then

\[ \lim_{j} K_1 = K, \quad K < \infty, \]

because

\[ \max V_1(z) = K'_1 \log |z| : z \in B_n, \]
\[ \min V_2(z) = K'_2 \log |z| : z \in B_n, \]

therefore there exists a subsequence

\( 4) \) We such operation generalized extremisation for convenience. This is certainly different from the extremisation.
\[
\lim_{n \to \infty} K_n = K
\]
and
\[
0 \leq V_1(z) - K_1 V_2(z) = \varepsilon_1 V_2(z) : \lim_{n \to \infty} \varepsilon_n = 0 : z \in C_1 \cap B_n,
\]
thus
\[
V_{1,2}(z) - K_1 V_{2,2}(z) \leq \varepsilon_1 V_{2}(z) \leq \varepsilon_s S \log |z| : S < \infty
\]
\[
\lim_{t \to \infty} V_{1,t}(z) - K_n V_{2,t}(z) = 0,
\]
which implies that
\[
V_1(z) = K V_2(z),
\]
thus \(V_{1,t}(p)\) is a minimal function, and if we compare \(U(p)\) with \(V_{1,t}(p)\) by \(\{U_t(p), V_{1,t}(p)\}\) we have \(U(p) \preceq K V_{1,t}(p)\), \(K^r\) being constant, thus \(U(p)\) is a minimal function and we see that on our example there exist exactly enumerable infinity of minimal functions.

6) **Positive harmonic function in the neighbourhood of an ideal boundary point.**

Let \(F\) be a null-boundary Riemann surface with a compact relative boundary \(\Gamma_0\) and \(p^\infty\) be an ideal boundary point.

We denote by \(G^i(p, p^\infty)\) \((i = 1, 2, \ldots)\), the positive minimal function with a pole at \(p^\infty\) and denote by \(E[G^i > N]\) the domain \(E[G^i \geq N]\) and by \(C_1^{p,n}\) the niveau curve \(E[G^i = N]\). Then \(\sum_i G^i\) is an open set with a compact boundary.

Proof. If \(G^i \in \mathcal{P}_j\); \(\lim p_j = p^\infty\) and if \(G(p, p_j)\) is the Green’s function with its pole at \(p_j\), then since \(\frac{\partial G^i(p, p^\infty)}{\partial n} \geq 0, p \in C_1^{p,n}\), we have by Green’s formula

\[
N \supseteq G^i(p, p^\infty) = \frac{1}{2\pi} \int_{C_1^{p,n}} G(p, p_j) \frac{\partial G^i(p, p^\infty)}{\partial n} ds.
\]

\[5) \quad 2\pi = \int_{r_0} \frac{\partial G(z, p_1)}{\partial n} ds = \int_{r_0} \frac{\partial G(z, p^\infty)}{\partial n} ds
\]
and
\[
D_{r_n \cap (F - \Delta X)}(G(z, p^\infty)) \leq D_{F \cap (F - \Delta X)} (G(z, p_m)) \leq 2\pi,
\]
for sufficiently large \(m(n)\). Let \(m\) and \(n \to \infty\). We have
\[
D_{F \cup (F - \Delta X)}(G(z, p^\infty)) \leq 2\pi N.
\]
It follows that
\[
\lim_{n \to \infty} \int_{r_n \cap (F - \Delta X)} \frac{\partial G(z, p^\infty)}{\partial n} ds = 0, \quad \text{and} \quad \int_{r_n} \frac{\partial G(z, p^\infty)}{\partial n} ds = 2\pi.
\]
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Let $V_M(p)$ be a neighbourhood with compact boundaries such that
\[
\int_{C_M^* \cap V_M} \frac{\partial G}{\partial n} \, ds = \pi.
\]
Then since $N \geq \frac{2}{2\pi} \int_{C_M^* \cap V_M} G(p, p_j) \frac{\partial G(p, p^\infty)}{\partial n} \, ds$,
there exists at least one point $q_j$ on $C_M^* - (C_M^* \cap V_M(p))$ for every
$p_j \in C_M^*$ such that $\lim G(q_j, p^\infty) \leq 2N$ for every $M$, whence $\lim G(p, p_j)$
($\lim G'(p, p^\infty)$) is free from the minimal functions $G_i(p, p^\infty)$. If $G^* = \sum_i G_i^*$ is not compact, there exists a sequence $r_1, r_2, \ldots, r_k \in G^*$, and
there exists at least one general Green’s function $G(p, r^\infty)$, but $G(p, r^\infty)$
must be expressed by a linear form of $G_i(p, p^\infty), G_i^2(p, p^\infty) \ldots$, which
contradicts the preceding assertion.

Since at any point $p$ there exists a constant $k(p)$ such that $U(p) \leq k(p)$ for any positive harmonic function $U(p)$ satisfying
\[
\frac{1}{2\pi} \int_{I_0} \frac{\partial U}{\partial n} \, ds = 1, \quad U(p) = 0, \text{ we have } G_N \supset G_{2N} \ldots, \int G_N = p^\infty.
\]

If $(\omega_n(p)$ denotes the harmonic measure of the boundary $G_{2N}$ with
respect to the domain $F - G_{2N}$, then $Nn\omega_n(p)$ is a monotonically
increasing function. Hence if $\lim \int_{I_0} \frac{\partial \omega_n(p)}{\partial n} \, ds < \infty$, then $\lim \omega_n(p) = \omega^*(p)$ is harmonic and $\lim \omega^*(p) = \infty$. Then as a special case we have

**Corollary.** If $p^\infty$ is finite dimensional, then the solution of Evans’s problem exists.

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6) See 2).
7) See 1) and 2).