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An Example of a Null-Boundary Riemann Surface

By Zenjiro KURAMOCHI

We have proved that the Green's function is not¹⁾ uniquely determined, when its pole is at an ideal boundary point of a null-boundary Riemann surface. M. Heins introduced²⁾ the notion of the minimal function due to R. S. Martin³⁾ and constructed a boundary point of dimension of preassigned number and conjectured that there would exist a boundary point of dimension infinity. We show by an example that his conjecture holds good.

1) Example. We denote by G the domain bounded by straight lines L_1 , L_2 and the semi-circle C such that

$$\begin{aligned} L_1: 1 \leq |z| < \infty, \quad \arg z = 0, \quad L_2: 1 \leq |z| < \infty, \quad \arg z = \pi \\ C: |z| = 1, \quad 0 \leq \arg z \leq \pi. \end{aligned}$$

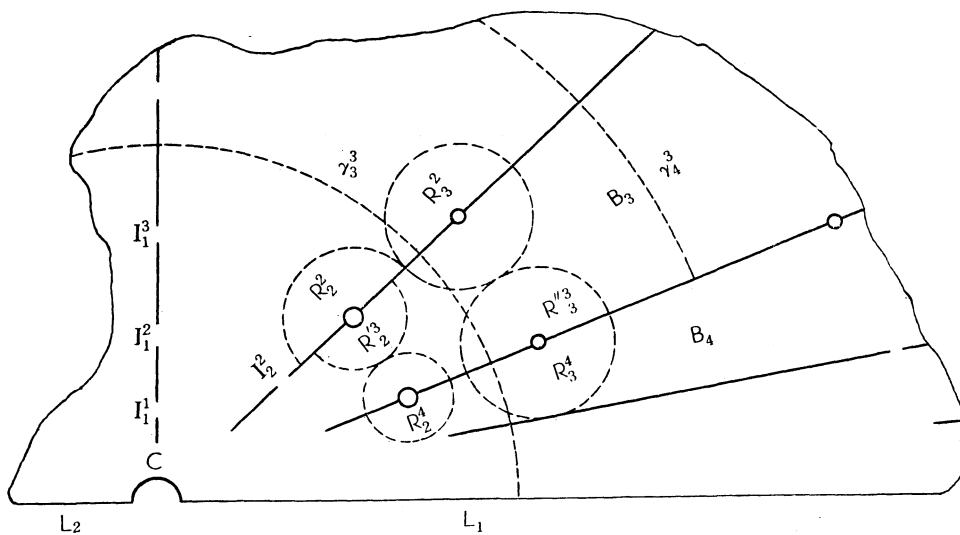
On G we define a sequence of slits such that

$$\begin{aligned} I_1^i: 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{i4}}, \quad \arg z = \frac{\pi}{2}: i = 2, 3, 4, \dots \\ I_2^i: 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{2i4}}, \quad \arg z = \frac{\pi}{4}: i = 3, 4, 5, \dots \\ \dots \dots \dots \\ I_n^i: 2^{i-1} \leq |z| \leq 2^i - \frac{1}{2^{ni4}}, \quad \arg z = \frac{\pi}{2^n}: i = n+1, n+2, \dots \\ \dots \dots \dots \\ n = 1, 2, 3, \dots \end{aligned}$$

Let G^1 and G^2 be the same exemplars with the same boundary and connect G^1 with G^2 by identifying L_1 , L_2 and $\{I_j^i\}$ of them, to con-

1) Z. Kuramochi: Potential theory and its applications, I, Osaka Math. J. **3** (1951), 123-174.
 2) M. Heins: Riemann surfaces of infinite genus, Annals of Math. **55** (1952), 296-317.
 3) R. S. Martin: Minimal positive harmonic functions, Trans. Amer. Math. Soc. **19** (1941), 137-172.

struct the symmetric surface with respect to $L_1 + L_2 + \{I^i\}$. We denote such a Riemann surface by F ; then F has only one compact relative boundary lying on C and is of infinite genus and further has it one ideal boundary point at $z = \infty$, and it is clear that F has a null ideal boundary.



2) Let B_n be the subsurface of F with projection on the part $\frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n-1}}$. Then B_n has boundary on $|z|=1$, $\frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n-1}}$ and

$$\bar{J}_n^1: 1 \leq |z| \leq 2^{n-1}, \arg z = \frac{\pi}{2^{n-1}}, J_n^1: 1 \leq |z| \leq 2^n, \arg z = \frac{\pi}{2^n}$$

and

$$\bar{J}_n^i: 2^i \geq |z| \geq 2^i - \delta_n^i, \arg z = \frac{\pi}{2^{n-1}}: \delta_n^i = \frac{1}{2^{ni4}}: i = n+1, n+2, \dots$$

$$J_{n+1}^+: 2^i \geq |z| \geq 2 - \delta_{n+1}^i, \arg z = \frac{\pi}{2^n}: \delta_{n+1}^i = \frac{2}{2^{(n+1)i4}}: i = n+2, n+3, \dots$$

We transform B_n by the mapping $w = \pm (ze^{-\frac{\pi}{2^n}})^{2^n}$, where \pm corresponds to the mapping of upper or lower exemplars respectively, then B_n is mapped onto the w -plane slits $^+J_w^i, ^-J_w^i$ lying on $\arg w = 0$, or $\arg w = \pi$ and having the boundary on $|w|=1$, and $^-J_n^1, ^+J_n^1$,

$$^+J_w^i: (2^i - \delta_n^i)^\alpha \leq |w| \leq 2^{i\alpha}: \alpha = 2^n,$$

$$^-J_w^i: (2^i - \delta_{n+1}^i)^\alpha \leq |w| \leq 2^{i\alpha}.$$

Then we have

$$\begin{aligned}
 +J_w^i &: 2^{i\alpha} - \delta_n^i \leq |w| \leq 2^{i\alpha}, \quad \frac{2^n}{2^{i+1}} \delta_n^i \leq \delta_n^i \leq \frac{2^n}{2^i} \delta_n^i \\
 -J_w^i &: 2^{i\alpha} - \delta_{n+1}^i \leq |w| \leq 2^{i\alpha}, \quad \frac{2^n}{2^{i+1}} \delta_{n+1}^i \leq \delta_{n+1}^i \leq \frac{2^n}{2^i} \delta_{n+1}^i.
 \end{aligned}$$

Denote by $\omega^{+i}(w)$ the harmonic measure of $+J_w^i$ with respect to the domain $|w| > 1$. Then we have by elementary calculation the following inequality

$$\omega_i^+(w) \leq \frac{\log \left| \frac{1-\bar{a}w}{a-w} \right|}{\log \left| \frac{a^2-1}{\delta_i^w} - a \right|} : \quad a = (2^i)^\alpha.$$

On the other hand, denote by $U_n^j(p)$ the harmonic function on F in the part $|z| < \gamma_j^n$: $\gamma_j^n = \left(\frac{2^{j\alpha} + 2^{(j+1)\alpha}}{2} \right)^{\frac{1}{\alpha}}$, such that $U_n^j(p) = \log |z|$, when $|z| = \gamma_j^n$, $\frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n+1}}$ and $U_n^j(p) = 0$, when $|z| = \gamma_j^n$, $\pi \geq \arg z \geq \frac{\pi}{2^{n-1}}$ or $\frac{\pi}{2^n} \geq \arg z \geq 0$ and further $U_n^j(p) = 0$, when $|z| = 1$. Since $U_n^j(p) \geq 0$, we define $U_n(p)$ by a uniformly convergent subsequence $\{U_n^j(p)\}$; then it is clear $U_n(p) \leq \log |z|$. On the other hand, let $V_n^j(p)$ be a harmonic function such that $V_n^j(p)$ is harmonic in $B_n \cap \{|z| < \gamma_j^n\}$, $V_n^j(p) = \log |z|$ on $\gamma_j^n \cap B_n$, $V_n^j(p) = 0$ on $2^i - \delta_n^i \leq |z| \leq 2^i$, $\arg z = \frac{\pi}{2^{n-1}}$ or and $2^i - \delta_{n+1}^i \leq |z| \leq 2^i$, $\arg z = \frac{\pi}{2^n}$ and on \bar{J}_n^1 , $+J_n^1$ and consider $V^*(z) = \log |z| - \sum_{i=n}^{\infty} \log 2^i \omega^{+i}(z) - \sum_{i=n}^{\infty} \log 2^i \omega^{-i}(z)$, where $\omega^{+i}(z)$ ($\omega^{-i}(z)$) is the harmonic measure of the boundary of B_n lying on $\{J_n^i\}$, $\{J_{n+1}^i\}$. Then we have $V^*(z) \leq 0$, $z \in \sum_{i=n}^{\infty} \{I_n^i, I_{n+1}^i\}$. Consider $\omega^{+i}(z)$: $|z^j| = \gamma_j^n$, $\arg z_j = \frac{3\pi}{2^{n+1}}$, i.e. the value of $\omega^{+i}(w)$ at $w = e^{\pm \frac{\pi}{2}} r$: $r = \frac{2^{j\alpha} + 2^{(j+1)\alpha}}{2}$. Then we have

$$\begin{aligned}
 \omega_i^+(z_j) &\leq \frac{\log \sqrt{\frac{1+a^2r^2}{a^2+r^2}}}{\log \left| \left(\frac{a^2-1}{\delta_i^w} - a \right) \right|} \\
 &\leq \frac{\frac{1}{2} \log 2a^2}{\log \left(\frac{a^2}{2\delta_i} \right)} \leq \frac{\frac{1}{2} (\log 2) (1+i\alpha)}{i\alpha \log 2 - \log 2 + i^n \log 2} \leq \frac{1+i\alpha}{2i^n}, \quad a = (2^i)^\alpha.
 \end{aligned}$$

Thus

$$\sum_{i=n}^{\infty} \log 2^i \omega_n^{+i}(z) + \sum_{i=n}^{\infty} \log 2^i \omega_n^{-i}(z) \quad \text{at } z_j \quad (j = n, n+1, \dots)$$

$$\leq \log 2 \left(\sum_{i=n}^{\infty} \frac{i(1+i\alpha)}{2^{i^4} n} + \sum_{i=n}^{\infty} \frac{i(1+i\alpha)}{2^{i^4} (n+1)} \right) + k_1 < k_2 < +\infty,$$

where k_i is a finite constant, from which follows the unboundedness of $V^*(z)$ at z_j ($j=1, 2, \dots$), and hence $U_n(p) \geq V_n^j(p) \geq V^*(p)$ yields the non-constancy of $U_n(p)$.

3) Next we consider the Dirichlet integral of $U_n(p)$ on F . In B_{n-1} and B_{n+1} and we denote by R_j^{n-1} and R_j^{n+1} , the ring-domains contained in $F - B_n$ with projection such that

$$R_j^{n-1}: \frac{\delta_j^{n-1}}{2} \leq |z - p_j^{n-1}| \leq 2^j \sin \frac{\pi}{2^{n+2}}: 0 \leq \arg z - p_j^{n-1} \leq \pi$$

$$R_j^{n+1}: \frac{\delta_j^n}{2} \leq |z - p_j^{n+1}| \leq 2^j \sin \frac{\pi}{2^{n+3}}: 0 \leq \arg z - p_j^{n+1} \leq \pi$$

respectively where

$$p_j^{n-1}: \arg z = \frac{\pi}{2^{n-1}}, \quad |z| = \left(2^j - \frac{\delta_j^n}{2} \right)$$

$$p_j^{n+1}: \arg z = \frac{\pi}{2^n}, \quad |z| = \left(2 - \frac{\delta_j^{n+1}}{2} \right).$$

Then we have

$$\mathfrak{M}_j^{n-1} = \text{module of } R_j^{n-1} = \log \frac{2^j \sin \frac{\pi}{2^{n+2}}}{2} \geq (j^4 n + j - n - 2) \log 2 + \log \pi \geq \frac{2}{2^{2n^4}}$$

$$(j^4 n + j + n - 2) \log 2 \text{ and}$$

$$\mathfrak{M}_j^{n+1} = \text{module of } R_j^{n+1} \geq (j^4(n+1) + j - n - 3) \log 2.$$

We denote by $w_n^i(p)$ the harmonic function such that

$$w_n^i(p) \text{ is harmonic in } (F - B_n) \cap \left\{ |z| < \frac{1}{2} (2^i + 2^{i+1}) \right\}$$

$$w_n^i(p) = 0: |z| = \frac{1}{2} (2^i + 2^{i+1}), \quad z \in \{C_i \cap (F - B_n)\}: C_r = |z| = \frac{2^i + 2^{i+1}}{2}$$

$$w_n^i(p) = \log 2^j: 2^j - \delta_j^j \leq |z| \leq 2^j, \quad \arg z = \frac{\pi}{2^{n-1}}: j = n, n+1, \dots,$$

$$w_n^i(p) = \log 2: 2^j - \delta_{n+1}^j |z| \leq 2^j, \quad \arg z = \frac{\pi}{2^n}: j = n+1; n+2, \dots$$

Then

$$D_{F-B_n}(U_n^i(p)) \leq D_{F-B_n}(w_n^i(p)),$$

and further $\tilde{w}_n^i(p)$ is a continuous function such that

$$\tilde{w}_n^i(p) = \log 2^j : |z - p_n^j| \leq \frac{\delta_n^j}{2}, \quad z \in B_{n-1} : j = n, \dots$$

$$\tilde{w}_n^i(p) = \log 2^j : |z - p_{n+1}^j| \leq \frac{\delta_{n+1}^j}{2}, \quad z \in B_{n+1} : j = n+1, \dots$$

In R_j^{n-1} and R_j^{n+1} and $\tilde{w}_n^i(p)$ is harmonic and

$$\tilde{w}_n^i(p) = \log 2^j : |z - p| = \frac{\delta_j^n}{2},$$

$$\tilde{w}_n^i(p) = 0 : |z - p| = \sin \frac{\pi}{2^n},$$

$$\tilde{w}_n^i(p) = \log 2^j : |z - p_j^{n+1}| = \frac{\delta_j^{n+1}}{2},$$

$$\tilde{w}_n^i(p) = 0 : |z - p_j^{n+1}| = 2^j \sin \frac{\pi}{2^{n+1}},$$

$$\tilde{w}_n^i(p) = 0 : p \in F - B_n - (\sum_j R_j^{n-1} + \sum_j R_j^{n+1}).$$

Then by Dirichlet principle

$$D_{F-B_n}(w_n^i(p)) \leq D_{F-B_n}(\tilde{w}_n^i(p)) \leq \sum_j \frac{(\log 2^j)^2}{\mathfrak{M}_j^{n-1}} + \sum_j \frac{(\log 2^j)^2}{\mathfrak{M}_j^{n+1}} + A,$$

for every i , where $A < \infty$.

Thus

$$D_{F-B_n}(U_n(p)) \leq D_{\substack{i=\infty \\ F-B_n}}(U_n^i(p)) \leq D_{\substack{i=\infty \\ F-B_n}}(\tilde{w}_n^i(p)) < +\infty.$$

4) Since F has a null-boundary, $D_F(U_n(p)) = \infty$, because if $D_F(U_n(p)) < \infty$, it follows $U_n(p) = 0$, whence

$$D_{B_n}(U_n(p)) = \infty.$$

Since $D_{B_n}(U_m(p)) < \infty$, if $m \neq n$,

all $U_n(p)$ are linearly independent.

We show in reality that 1°) $U_n(p)$ are all minimal functions, and 2°) each B_n has only one minimal function.

We denote by B_j^n the ring-domain $2^j \leq |z| \leq 2^{j+1} - \delta_n^{j+1}$, $\frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n-1}}$ contained in B_n , and denote by $\max_{p \in \gamma} U_{p \in \gamma}(p)$ the maximum when p is on $B_n \cap C_j$: $C_j = \{|z| = \gamma_n^j\}$. If there exists at least a Jordan curve J in B_j^n starting from p_0 , which is on C_j , and reaching at least one boundary component of the ring $2^j \leq |z| \leq 2^{j+1} - \delta_n^{j+1}$, $\frac{\pi}{2^n} \leq \arg z \leq \frac{\pi}{2^{n-1}}$,

and if we denote by $\omega(p)$ the harmonic measure of J with respect to this ring, then there exists a constant K depending only on the module of this ring such that

$$\text{Min}_{p \in c_j} U(p) \geq K \omega(p) \text{Max}_{p \in c_j} U(p) \geq K \text{Max}_{p \in c_j} U_{p \in c_j}(p).$$

5) According to R. S. Martin's theorem any positive harmonic function can be expressed uniquely by a linear form of minimal functions, thus

$$U = \int V d\mu,$$

therefore there exists at least a function V such that $KU_n \geq V: K < \infty$.

a) V is unbounded on $|z| = \gamma^j$ in B_n .

Proof. If $V(p) \leq M < \infty$, we define $V_1^j(V_2^j(p))$ such that $V_1^j(p)(V_2^j(p))$ is harmonic in $|z| < \gamma^j$, $V_1^j = 0$ ($V_2^j = M$): $p \in C_j \cap B_n$ and $V_1^j = V_2^j = V$: $p \in C_j \cap (F - B_n)$, and take $V^1(p), V^2(p)$ from the uniformly convergent sequences $\{V_1^j\}, \{V_2^j\}$. Then $0 \leq V_2^j - V_1^j \leq M \omega_j(p)$, where $\omega_j(p)$ is the harmonic measure of $C_j \cap B_n$ with respect to domain of F contained in $|z| \leq \gamma^j$. Since F has a null-boundary, we have $V^1(p) = V^2(p)$. On the other hand, let $U_j(p)$ be the harmonic function in $|z| \leq \gamma^j$ such that $U_j(p) = 0: p \in C_j \cap B_n, U_j(p) = U(p): p \in C_j \cap (F - B_n)$, and let $U^*(p)$ be a harmonic function obtained by taking a uniformly convergent subsequence from $\{U_j(p)\}$. We construct ring-domains contained in B_n such that

$$R_i^{n'} : \delta_i^n \leq |z - p_i^n| \leq 2^i \sin \frac{\pi}{2^{n+2}}, \quad 0 \leq (\arg z - p_i^n) \leq \pi$$

Fig. 1.

$$R_i'^{n+1}: \delta_i^{n+1} \leq |z - p_i^{n+1}| \leq 2^i \sin \frac{\pi}{2^{n+2}}, \quad 0 \leq (\arg z - p_i^{n+1}) \leq \pi$$

where

$$|p_i^n| = \frac{1}{2} \left(2^i - \frac{1}{2^{(n+1)i^4}} \right), \quad \arg p_i^n = \frac{\pi}{2^{n-1}},$$

$$|p_i^{n+1}| = \frac{1}{2} \left(2^i - \frac{2}{2^{(n+1)i^4}} \right), \quad \arg p_i^{n+1} = \frac{\pi}{2^n},$$

and define a continuous function as in the case (3). Then we have

$$D_{B_n}(U_n^*(p)) < \infty, \quad \text{and} \quad D_F(U_n^*(p)) < \infty.$$

This implies that $U_n^*(p) = 0$. By assumption $V \leq U, V \leq M$ in B_n ,

we have

$V = V^1 = V^2$, $V_j^1 \leq U_j(p)$, and it follows that $V^1 \equiv U^*(p) \equiv V \equiv 0$, therefore V is not bounded on $C^j \cap B_n$ and by (4) $V(z)$ is not bounded on the sequence $\{z_i\} : |z_i| = \gamma^i, \arg z_i = \frac{3\pi}{2^n}$.

b) $V(p)$ is invariant by generalized extremisation.⁴⁾

Let $V_j(p)$ be harmonic in $F \cap \{|z| < \gamma^j\}$, $V_j(p) = V(p) : p \in C_j \cap B_n$, $V_j(p) = 0 : p \in C_j \cap (F - B_n)$: From the unboundedness of $V(p)$ on $\{z_i\}$ and from $V(p) \leq U(p)$, we can prove as (2) and (3), that there exists a harmonic function $V^*(p)$ from $\{V_i(p)\}$, such that

$$D_{F-B_n}(V^*(p)) < \infty, \quad V^*(p) \equiv \text{const.}$$

Since $|V_j(p) - V(p)| \leq 2U(p)$ on $\arg z = \frac{\pi}{2^n}$ or $\arg z = \frac{\pi}{2^{n-1}}$, $V(p) = V_j(p) : p \in C_j \cap B_n$, and hence we have by the same manner used in (2) (3), $D_F(V^*(p) - V(p)) < \infty$, therefore $V^*(p) = V(p)$ { $V^*(p)$ is obtained by generalized extremisation from $V(p)$ } and thus, since $U(p) \geq V(p) \geq 0$, it follows that $V(p)$ is invariant by generalized extremisation with respect to B_n .

c) There is only one minimal function smaller than $U(p)$.

Since $V^*(p) \neq 0$, if there are two functions $V_1(p) \geq V_2(p)$ such that $V_i(p) \leq U(p)$, then there are two constants K_1, K_2 such that

$$\overline{\lim}_j \text{Max}_{p \in C_j} V_i(p) = K_i \log |z| : i = 1, 2,$$

but from (4) there exist constants K_3, K_4 , such that

$$V_1(p) \leq K_3 V_2(p) \quad V_1(p) > K_4 V_2(p), \quad p \in C_j.$$

$$\text{Put } \text{Min} \frac{V_1(z)}{V_2(z)} = K_i, \quad z \in C_i \cap B_n.$$

Then

$$\overline{\lim}_j K_i = K, \quad K < \infty,$$

because

$$\begin{aligned} \max V_1(z) &= K'_1 \log |z| : z \in B_n, \\ \min V_2(z) &= K' \log |z| : z \in B_n, \end{aligned}$$

therefore there exists a subsequence

4) We such operation generalized extremisation for convenience. This is certainly different from the extremisation.

$$\lim_{\varepsilon} K'_n = K$$

and

$$0 \leq V_1(z) - K'_\varepsilon V_2(z) = \varepsilon'_\varepsilon V_2(z) : \lim_{\varepsilon} \varepsilon'_\varepsilon = 0 : z \in C_\varepsilon \cap B_n,$$

thus

$$V_{1,2}(z) - K_\varepsilon V_{2,\varepsilon}(z) \leq \varepsilon_\varepsilon V_{2,\varepsilon}(z) \leq \varepsilon_\varepsilon S \log |z| : S < \infty$$

$$\lim_{\varepsilon \rightarrow \infty} V_{1,\varepsilon}(z) - K'_n V_{2,\varepsilon}(z) = 0,$$

which implies that

$$V_1(z) = K V_2(z),$$

thus $V_{1,\varepsilon}(p)$ is a minimal function, and if we compare $U(p)$ with $V_1(p)$ by $\{U_\varepsilon(p), V_{1,\varepsilon}(p)\}$ we have $U(p) \leq K'' V_1(p)$, K'' being constant, thus $U(p)$ is a minimal function and we see that on our example there exist exactly enumerable infinity of minimal functions.

6) Positive harmonic function in the neighbourhood of an ideal boundary point.

Let F be a null-boundary Riemann surface with a compact relative boundary Γ_0 and p^∞ be an ideal boundary point.

We denote by $G^i(p, p^\infty)$ ($i = 1, 2, \dots$), the positive minimal function with a pole at p^∞ and denote by G_i^N the domain $E[G^i \geq N]$ and by C_i^{*N} the niveau curve $E[G^i = N]$. Then $\sum_i G_i^N$ is an open set with a compact boundary.

Proof. If $G_i^N \bar{\cap} p_j : \lim_j p_j = p^\infty$ and if $G(p, p_j)$ is the Green's function with its pole at p_j , then since $\frac{\partial G^i(p, p^\infty)}{\partial n} \geq 0$, $p \in C_i^{*N}$, we have by Green's formula

$$N \geq G^i(p_j, p^\infty) = \frac{1}{2\pi} \int_{C_i^{*N}} G(p, p_j) \frac{\partial G^i(p, p^\infty)}{\partial n} ds. \quad 5)$$

$$5) \quad 2\pi = \int_{\Gamma_0} \frac{\partial G(z, p_i)}{\partial n} ds = \int_{\Gamma_0} \frac{\partial G(z, p^\infty)}{\partial n} ds$$

and

$$D_{F_n \cap (F - D_N)}(G(z, p^\infty)) \leq D_{F \cap (F - D_N)}(G(z, p_m)) \leq 2\pi,$$

for sufficiently large $m(n)$. Let m and $n \rightarrow \infty$. We have

$$D_{F \cup (F - D_N)}(G(z, p^\infty)) \leq 2\pi N.$$

It follows that

$$\lim_{n \rightarrow \infty} \int_{\Gamma_n \cap (F - D_N)} \frac{\partial G(z, p^\infty)}{\partial n} ds = 0, \quad \text{and} \quad \int_{\Gamma_n} \frac{\partial G(z, p^\infty)}{\partial n} ds = 2\pi.$$

Let $V_M(p^\infty)$ be a neighbourhood with compact boundaries such that $\int_{C_i^M \cap V_M} \frac{\partial G_i^M}{\partial n} ds = \pi$. Then since $N \geq \frac{2}{2\pi} \int_{C_i^M \cap V_M} G(p, p_j) \frac{\partial G^i(p, p^\infty)}{\partial n} ds$, there exists at least one point q_j on $C_i^M - (C_i^M \cap V_M(p))$ for every $p_j \in G_i^N$ such that $\lim_{q_j \in C_i^M} G(q_j, p^\infty) \leq 2N$ for every M , whence $\lim_{j \rightarrow \infty} G(p, p_j)$ ($\lim G'(p, p^\infty)$) is free from the minimal functions $G^i(p, p^\infty)$. If $G^N = \sum_i G_i^N$ is not compact, there exists a sequence $r_1, r_2, \dots, r_i \in G^N$, and there exists at least one general Green's function $G(p, r^\infty)$, but $G(p, r^\infty)$ must be expressed by a linear form of $G^1(p, p^\infty), G^2(p, p^\infty) \dots$, which contradicts the preceding assertion.

Since at any point p there exists a constant $k(p)$ such that $U(p) \leq k(p)$ for any positive harmonic function $U(p)$ satisfying

$$\frac{1}{2\pi} \int_{\Gamma_0} \frac{\partial U}{\partial n} ds = 1, \quad U(p) = 0, \quad \text{we have } G_N > G_{2N} \dots, \quad \prod_{i=1}^{\infty} \bar{G}_{Ni} = p^\infty.$$

If $\omega_{Nn}(p)$ denotes the harmonic measure of the boundary G_{Nn} with respect to the domain $F - G_{Nn}$, then $Nn\omega_{nN}(p)$ is a monotonously increasing function. Hence if $Nn \int_{\Gamma_0} \frac{\partial \omega_{nN}(p)}{\partial n} ds < \infty$, then $\lim_{n \rightarrow \infty} \omega_{nN}(p) = \omega^*(p)$ is harmonic and $\lim_{p \rightarrow p^\infty} \omega^*(p) = \infty$. Then as a special case we have

Corollary. *If p^∞ is finite dimensional, then the solution of Evans's⁷⁾ problem exists.*

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6) See 2).

7) See 1) and 2).

