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EQUIVARIANT ALGEBRAIC VECTOR BUNDLES OVER A PRODUCT OF AFFINE VARIETIES

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0. Introduction

Let G be a reductive complex affine algebraic group and Z a complex affine G-variety with a G-fixed base point $z_0 \in Z$. Throughout this paper, the base field is the field $\boldsymbol{C}$ of complex numbers. Let Q be a G-module. We denote by $\mathrm{Vec}_G(Z,Q)$ the set of algebraic G-vector bundles over Z whose fiber at z_0 is Q and by $\mathrm{VEC}_G(Z,Q)$ the set of G-isomorphism classes in $\mathrm{Vec}_G(Z,Q)$. We denote by $[E]$ the isomorphism class of $E \in \mathrm{Vec}_G(Z,Q)$.

There are many interesting problems concerning $\mathrm{VEC}_G(Z,Q)$, especially when the base space Z is a G-module P. One of them is the Equivariant Serre Problem, which asks whether $\mathrm{VEC}_G(P,Q)$ is the trivial set consisting of the isomorphism class of the product bundle $P \times Q$. When G is trivial, the Quillen-Suslin Theorem says that $\mathrm{VEC}_G(P,Q)$ is the trivial set. More generally, Masuda-Moser-Petrie [9] recently have shown that $\mathrm{VEC}_G(P,Q)$ is trivial for any abelian group G. However, when G is not abelian, $\mathrm{VEC}_G(P,Q)$ is non-trivial in general. Schwarz [13] (see Kraft-Schwarz [5] for details) first presented counter examples to the Equivariant Serre Problem by proving that $\mathrm{VEC}_G(P,Q) \cong \boldsymbol{C}^p$ when the algebraic quotient space $P/\!/G$ is one dimensional i.e. isomorphic to affine line $\boldsymbol{A}$. When $\dim P/\!/G \geq 2$, there are many non-trivial examples of $\mathrm{VEC}_G(P,Q)$ ([11], [4]) but it remains open to classify elements in $\mathrm{VEC}_G(P,Q)$ in general.

The results of [13] extend to the case where the base space is a weighted G-cone with smooth one dimensional quotient (for a precise definition, see §1; a G-module with one dimensional quotient is an example of such a cone):

Theorem A ([8]). *Let X be a weighted G-cone with smooth one dimensional quotient and Q be a G-module. Then $\mathrm{VEC}_G(X,Q) \cong \boldsymbol{C}^p$ for a non-negative integer p. Moreover, there is a G-vector bundle $\mathfrak{B}$ over $X \times \boldsymbol{C}^p$ such that the map $\boldsymbol{C}^p \ni z \mapsto [\mathfrak{B}|_{X \times \{z\}}] \in \mathrm{VEC}_G(X,Q)$ gives a bijection.*

Masuda-Petrie have made the following observation. Let X and p be as above and Y an irreducible affine variety with trivial G-action. We denote by $\mathrm{Mor}(Y, \boldsymbol{C}^p)$ the set of morphisms from Y to $\boldsymbol{C}^p$. Then there is a map

$$\Phi: \mathrm{Mor}(Y, C^p) \to \mathrm{VEC}_G(X \times Y, Q)$$

defined by $\Phi(f) = [(id_X \times f)^* \mathfrak{B}]$ for $f \in \mathrm{Mor}(Y, C^p)$. It is bijective when Y is a point by Theorem A. Moreover, Theorem A implies that Φ is injective. Masuda-Petrie have shown that Φ is bijective in some examples. We prove

Main Theorem. *Let X be a weighted G-cone with smooth one dimensional quotient and Y an irreducible affine variety such that every vector bundle over Y and $(A - \{0\}) \times Y$ is trivial. If a G-module Q is multiplicity free with respect to a principal isotropy group of X, then*

$$\Phi: \mathrm{Mor}(Y, C^p) \to \mathrm{VEC}_G(X \times Y, Q)$$
$$f \mapsto [(id_X \times f)^* \mathfrak{B}]$$

is bijective and hence $\mathrm{VEC}_G(X \times Y, Q) \cong \mathrm{Mor}(Y, C^p)$ *where p and $\mathfrak{B}$ are given in Theorem A.*

Here a G-module Q is called *multiplicity free* with respect to a reductive subgroup H if in the decomposition of Q as a direct sum of irreducible H-modules, each irreducible H-module occurs with multiplicity at most 1. G-modules which satisfy the multiplicity free condition with respect to some reductive subgroup are abundant. Moreover, the integer p in Theorem A is computed or estimated mainly in the case where Q is multiplicity free with respect to a principal isotropy group of X ([5], [10]).

When Y is m-dimensional affine space A^m, the assumptions on Y in the Main Theorem are satisfied by Swan's Theorem ([15]). So we have

Corollary. *Let X, Q and p be the same as in the Main Theorem. Then*

$$\mathrm{VEC}_G(X \times A^m, Q) \cong \mathrm{Mor}(A^m, C^p).$$

We show the Main Theorem by calculating $\mathrm{VEC}_G(X \times Y, Q)$. For the calculation of $\mathrm{VEC}_G(X \times Y, Q)$, we apply the techniques of Kraft-Schwarz [5] (or [8]). In order to extend the glueing argument of Kraft-Schwarz we need the hypotheses on Y (cf. remark after Theorem 3.4). But it is still difficult to apply their method directly to $\mathrm{VEC}_G(X \times Y, Q)$ for any G-module Q since the dimension of the algebraic quotient space of the base space is greater than 1 (unless Y is a point). However, when Q is multiplicity free with respect to a principal isotropy group of X, the argument in [5] and [8] becomes drastically simplified and even in the case where the base space is $X \times Y$ the argument does not become difficult so much. For example, thanks to the multiplicity free condition, the approximation property established in [5] (or [8]) becomes obvious. It is not hard to check that a similar argument to that in [5] and [8] works in our case.

The organization of this paper is as follows. In §1 we recall the definition of a weighted G-cone with smooth one dimensional quotient and discuss its properties. In §2, under the multiplicity free condition, we investigate the action of a cyclic group Γ and prove the vanishing of a group cohomology of Γ (Lemma 2.2) which is needed to show the key fact that every G-vector bundle over $X\times Y$ is trivial when restricted to $(X-\pi_X^{-1}(0))\times Y$ where $\pi_X: X\to X//G\cong A$ denotes the algebraic quotient map (Theorem 3.3 (1)). Its proof is elementary by virtue of the multiplicity free condition. In §3 we show that every G-vector bundle over $X\times Y$ has a trivialization over $(X-\pi_X^{-1}(0))\times Y$ which reduces $\mathrm{VEC}_G(X\times Y, Q)$ to a double coset of transition functions. Furthermore, from the multiplicity free condition, the double coset turns out to be a quotient group of some abelian group. In order to analyze the quotient group, we prove the decomposition property established in [5] (or [8]) in §4. Thanks to the multiplicity free condition, its proof also becomes elementary. In §5 we give a proof of the Main Theorem.

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1. Weighted G-cone with one dimensional quotient

Let G be a reductive algebraic group and Z an affine G-variety (reduced but not necessarily irreducible). We denote by $\mathcal{O}(Z)$ the ring of regular functions on Z and by $\mathcal{O}(Z)^G$ the ring of G-invariants. The quotient space $Z//G$ is the affine variety corresponding to $\mathcal{O}(Z)^G$ and the quotient map $\pi_Z: Z\to Z//G$ is the morphism corresponding to the inclusion $\mathcal{O}(Z)^G \subsetneq \mathcal{O}(Z)$.

We recall the definition of a weighted G-cone with smooth one dimensional quotient ([10]). Let X be a $G\times \boldsymbol{C}^*$-affine variety. The $\boldsymbol{C}^*$-action defines an integer-valued grading on $\mathcal{O}(X)$.

DEFINITION. An affine $G\times\boldsymbol{C}^*$-variety X is called *a weighted G-cone with smooth one dimensional quotient* if it satisfies the following conditions:

(1) $\mathcal{O}(X)^{\boldsymbol{C}^*}=\boldsymbol{C}$ and $\mathcal{O}(X)$ is positively graded with respect to the $\boldsymbol{C}^*$-action.

(2) $\mathcal{O}(X)^G=\boldsymbol{C}[t]$ where $t\in\mathcal{O}(X)^G$ is homogeneous.

REMARK. A G-module P with $\dim P//G=1$ is a weighted G-cone with smooth one dimensional quotient. In fact, the $\boldsymbol{C}^*$-action corresponds to the scalar multiplication, so that condition (1) is clearly satisfied. It is known that $P//G\cong A$

when $\dim P//G=1$ ([5, p.13]), this implies that condition (2) is also satisfied.

From now on, X will denote a weighted G-cone with smooth one dimensional quotient. It follows from condition (1) that X has a unique closed $\boldsymbol{C}^*$-orbit, in fact a $G\times\boldsymbol{C}^*$-fixed point, which we denote by x_0. Condition (2) means that the quotient space $X//G$ is isomorphic to the affine line $A=\mathrm{Spec}\,\boldsymbol{C}[t]$. We identify $X//G$ with A. Then the quotient map $\pi_X: X\to X//G\cong A$ is given by the function $t\in\mathcal{O}(X)^G\subset\mathcal{O}(X)$. Since t is homogeneous, every fiber of π_X over $\dot{A}:=A-\{0\}$ is isomorphic to each other.

Let H be a principal isotropy group of X, that means it is the minimal one among isotropy groups of points of closed orbits in X up to conjugation (cf. [7]). Since every fiber over $\dot{A}$ is isomorphic to each other, isotropy groups of points of closed orbits in $X-\pi_X^{-1}(0)$ are all conjugate to H. Let $x\in X-\pi_X^{-1}(0)$ be a point whose isotropy group is H. Set $X_{cl}:=\overline{(G\times\boldsymbol{C}^*)x}$. It is a closed $G\times\boldsymbol{C}^*$-subvariety of X. Hence clearly, $\mathcal{O}(X_{cl})^{\boldsymbol{C}^*}=\boldsymbol{C}$. Since π_X maps a G-closed set to a closed set ([3]), $\pi_X(X_{cl})=\overline{\pi_X((G\times\boldsymbol{C}^*)x)}=\bar{\dot{A}}=A$. Thus $X_{cl}//G=X//G=A$, i.e. $\mathcal{O}(X_{cl})^G=\mathcal{O}(X)^G=\boldsymbol{C}[t]$. Hence X_{cl} is also a weighted G-cone with smooth one dimensional quotient. We denote the restriction map of π_X to X_{cl} by $\pi_{cl}: X_{cl}\to X_{cl}//G=X//G$. We set $F:=\pi_{cl}^{-1}(1)$. Then $F\cong G/H$ ([10]).

Let Y be an irreducible affine variety with trivial G-action. Then $(X\times Y)//G=(X_{cl}\times Y)//G=A\times Y$.

Lemma 1.1. *Let Q be a G-module. If every vector bundle over Y is trivial then for every $E\in\mathrm{Vec}_G(X\times Y,Q)$ there exists $f\in\mathcal{O}(X\times Y)^G=\mathcal{O}(A\times Y)$ such that $f(0,y)=1$ and E is trivial over $(X\times Y)_f:=\{(x,y)\in X\times Y\,|\,f(x,y)\neq 0\}$.*

Proof. Let $E\in\mathrm{Vec}_G(X\times Y,Q)$. Since $\{x_0\}\times Y$ is fixed under the G-action and every vector bundle over Y is trivial by assumption, E restricts to a trivial G-vector bundle $\{x_0\}\times Y\times Q$ ([2]). The Equivariant Nakayama Lemma ([1]) implies that the G-isomorphism $E|_{\{x_0\}\times Y}\to\{x_0\}\times Y\times Q$ extends to a G-homomorphism $E\to X\times Y\times Q$ which is an isomorphism over a G-invariant open neighborhood U of $\{x_0\}\times Y$. Note that $U\supset\pi_X^{-1}(0)\times Y$ since the set of G-closed orbits in $\pi_X^{-1}(0)\times Y$ is just $\{x_0\}\times Y$. Let V be the complement of U in $X\times Y$. Since V is a G-invariant closed set, $V//G$ is also closed in $A\times Y$. Let $f_i\in\mathcal{O}(A\times Y)$, $1\leq i\leq r$ be the generators of the defining ideal of $V//G$. Since $V//G\cap(\{0\}\times Y)=\emptyset$, the ideal $(f_1,\cdots,f_r,t)$ is equal to $\mathcal{O}(A\times Y)$. Restricting the functions to $\{0\}\times Y$, we obtain $(f_1(0,y),\cdots,f_r(0,y))=\mathcal{O}(Y)$. Hence there exist $g_i(y)\in\mathcal{O}(Y)$, $1\leq i\leq r$ such that $\Sigma_{i=1}^r g_i(y)f_i(0,y)=1$. Let $f:=\Sigma_{i=1}^r g_if_i\in\mathcal{O}(A\times Y)$. Then f is contained in the defining ideal of $V//G$ and $f(0,y)=1$. This means that the image of U under the quotient map $X\times Y\to(X\times Y)//G\cong A\times Y$ contains $(A\times Y)_f$, hence $U\supset(X\times Y)_f$.

□

Note that $X_{cl}\times Y$ contains all closed G-orbits in $X\times Y$. Hence it follows from the Equivariant Nakayama Lemma that the restriction map $\mathrm{VEC}_G(X\times Y, Q) \to \mathrm{VEC}_G(X_{cl}\times Y, Q)$ is an injection (cf. [1]).

2. The multiplicity free condition and the action of Γ

Let Q be a G-module and H be a principal isotropy group of X. Note that H is a reductive subgroup of G by the Theorem of Matsushima. Decompose Q as a direct sum of irreducible H-modules

$$Q\cong \oplus_{i=1}^{q} n_i W_i$$

where W_i are mutually non-isomorphic irreducible H-modules and n_i is the multiplicity of W_i in Q. Recall that $F=\pi_{cl}^{-1}(1)$. We set $M:=\mathrm{Mor}(F,\mathrm{GL}Q)^G$ which is the group of G-equivariant morphisms from F to $\mathrm{GL}Q$. Since $F\cong G/H$,

$$M=\mathrm{Mor}(F,\mathrm{GL}Q)^G\cong \mathrm{GL}(Q)^H\cong \prod_{i=1}^{q} GL_{n_i}.$$

Let $d:=\deg t$. Note that $d>0$ since $\mathcal{O}(X)$ is positively graded. The $\boldsymbol{C}^*$-action on X_{cl} induces a $\boldsymbol{C}^*$-action on $X_{cl}/\!/G=\boldsymbol{A}$. The induced $\boldsymbol{C}^*$-action on $\boldsymbol{A}$ is scalar multiplication with the d-th power. Let Γ be the group of d-th roots of unity. Then Γ acts trivially on $\boldsymbol{A}$, so $F=\pi_{cl}^{-1}(1)$ is invariant under the Γ-action. Let $\boldsymbol{B}=\mathrm{Spec}\,\boldsymbol{C}[s]$ where $t=s^d$. The group Γ acts on $\boldsymbol{B}$ by scalar multiplication and $\boldsymbol{B}/\Gamma\cong\boldsymbol{A}$. We define an action of $\gamma\in\Gamma$ on M by

$$(\gamma m)(f)=m(\gamma^{-1}f) \quad \text{for } m\in M,\ f\in F$$

and on $M(\boldsymbol{B}\times Y):=\mathrm{Mor}(\boldsymbol{B}\times Y, M)$ by

$$(\gamma\mu)(b,y)=\gamma(\mu(b\gamma,y)) \quad \text{for} \quad \mu\in M(\boldsymbol{B}\times Y),\ b\in\boldsymbol{B},\ y\in Y.$$

DEFINITION. A G-module Q is called *multiplicity free with respect to a reductive subgroup K* if $n_i=1$ for all i in the decomposition of Q as a direct sum of irreducible K-modules as above.

When Q is multiplicity free with respect to H, M is isomorphic to a q-dimensional torus. From now on, we assume that Q is multiplicity free with respect to H and identify M with $(\boldsymbol{C}^*)^q$ unless otherwise stated.

Lemma 2.1. *The group Γ acts on the torus $M\cong(\boldsymbol{C}^*)^q$ by permutation of $\boldsymbol{C}^*$s.*

Proof. Let $\gamma\in\Gamma$ be a generator. We make an observation about the isomorphisms between $M=\mathrm{Mor}(F,\mathrm{GL}Q)^G$ and a torus. Choose $f_0\in F$ whose isotropy group is H. Evaluating an element of $\mathrm{Mor}(F,\mathrm{GL}Q)^G$ at f_0 induces an isomorphism $M=\mathrm{Mor}(F,\mathrm{GL}Q)^G \to \mathrm{GL}(Q)^H$. Since the Γ-action on $F\cong G/H$ is G-equivariant and the isotropy group of f_0 is H, $\gamma^{-1}f_0=gf_0$ for some g in the normalizer of H in G. We fix such a $g\in G$. For $m\in M$ we have

$$(\gamma m)(f_0)=m(\gamma^{-1}f_0)=m(gf_0)=\rho(g)m(f_0)\rho(g)^{-1}$$

where $\rho:G\to\mathrm{GL}Q$ is the rational representation associated with Q. Hence the action of γ on M corresponds to conjugation by $\rho(g)\in\mathrm{GL}Q$ on $\mathrm{GL}(Q)^H$. Since g is in the normalizer of H in G, $\rho(g):\mathrm{Q}\to Q$ maps an H-submodule to an H-submodule (but $\rho(g)$ is not necessarily H-equivariant). Let $Q=Q_1\oplus\cdots\oplus Q_q$ where Q_i are mutually non isomorphic irreducible H-submodules. Since Q_i is an irreducible H-submodule, $\rho(g)Q_i$ is also an irreducible H-submodule and $Q=\oplus_{i=1}^q\rho(g)Q_i$ since $\rho(g)\in\mathrm{GL}Q$. From the assumption that irreducible H-submodules Q_i are mutually non isomorphic, it follows that irreducible H-submodules $\rho(g)Q_i$ are not isomorphic to each other. Hence the conjugation by $\rho(g)$ on $\mathrm{GL}(Q)^H=\Pi_i\mathrm{GL}(Q_i)^H$ is a permutation of $\mathrm{GL}(Q_i)^H\cong \boldsymbol{C}^*$. This shows that γ acts on $M\cong(\boldsymbol{C}^*)^q$ by permuting $\boldsymbol{C}^*$s. □

Let $\dot{\boldsymbol{B}}_Y:=\dot{\boldsymbol{B}}\times Y$ where $\dot{\boldsymbol{B}}=\boldsymbol{B}-\{0\}$. Since M is a torus, $M(\dot{\boldsymbol{B}}_Y)=\mathrm{Mor}(\dot{\boldsymbol{B}}_Y,M)$ is considered as a direct product of copies of $\mathcal{O}(\dot{\boldsymbol{B}}_Y)^*$ (the group of invertible elements in $\mathcal{O}(\dot{\boldsymbol{B}}_Y)$). Note that an element of $\mathcal{O}(\dot{\boldsymbol{B}}_Y)=\mathcal{O}(\dot{\boldsymbol{B}})\otimes_{\boldsymbol{C}}\mathcal{O}(Y)$ is a Laurent polynomial in s with coefficients in $\mathcal{O}(Y)$. Since Y is irreduceble, i.e. $\mathcal{O}(Y)$ is an integral domain, one easily sees that $\mathcal{O}(\dot{\boldsymbol{B}}_Y)^*=\mathcal{O}(\dot{\boldsymbol{B}})^*\mathcal{O}(Y)^*$. We denote by $H^1(\Gamma,M(\dot{\boldsymbol{B}}_Y))$ the group cohomology of Γ with values in $M(\dot{\boldsymbol{B}}_Y)$ (for the definition of a group cohomology, see [14] for example). For later use, we prove the next lemma.

Lemma 2.2

$$H^1(\Gamma,M(\dot{\boldsymbol{B}}_Y))=\{*\}.$$

Proof. Let $\gamma\in\Gamma$ be a generator. From Lemma 2.1, γ acts on the q-dimensional torus M by permuting components. It is sufficient to show that the cohomology group vanishes when M consists of a single Γ-orbit of one component $\boldsymbol{C}^*$. Hence we may assume that the action of γ on M is a cyclic permutation of q components. Note that $d=qk$ for some positive integer k since $\gamma^d=1$. Let $\{A(\gamma)\}_{\gamma\in\Gamma}$ be a 1-cocycle of Γ with values in $M(\dot{\boldsymbol{B}}_Y)$. It follows from the 1-cocycle condition that

$$I=A(\gamma^d)=A(\gamma^q)\cdot\gamma^qA(\gamma^q)\cdots\gamma^{q(k-1)}A(\gamma^q)$$

where I denotes the constant map to the identity element of M. Let $A(\gamma^q)(s,y)=(f_1(s,y),\cdots,f_q(s,y))$ where $f_i(s,y)\in\mathcal{O}(\dot{\boldsymbol{B}}_Y)^*=\mathcal{O}(\dot{\boldsymbol{B}})^*\mathcal{O}(Y)^*$. Since the action of γ^q on M is trivial, it follows from the above identity that

$$f_i(s,y)f_i(\gamma^q s,y)\cdots f_i(\gamma^{q(k-1)}s,y)=1 \quad \text{for } 1\le i\le q.$$

This implies that f_i is independent of s, so $f_i\in\mathcal{O}(Y)^*$ and $f_i^k=1$. Since $\mathcal{O}(Y)$ is an integral domain, f_i must be a k-th root of unity. Hence $A(\gamma^q)$ is a constant map to an element of M with entries of k-th roots of unity. Let $A(\gamma)(s,y)=(a_1(s,y),\cdots,a_q(s,y))$ where $a_i(s,y)\in\mathcal{O}(\dot{B})^*\mathcal{O}(Y)^*$. Since $A(\gamma)\cdot\gamma A(\gamma)\cdots\gamma^{q-1}A(\gamma)=A(\gamma^q)$ from the 1-cocycle condition, we obtain

$$(1)\qquad a_i(s,y)a_{i+1}(\gamma s,y)\cdots a_q(\gamma^{q-i}s,y)a_1(\gamma^{q-i+1}s,y)\cdots a_{i-1}(\gamma^{q-1}s,y)=\gamma^{qr_i}$$

for a positive integer r_i, $1\le i\le q$. Note that $a_i^{-1}(s,y)a_i(\gamma^q s,y)=\gamma^{q(r_{i+1}-r_i)}$ for $1\le i\le q-1$.

We will construct $\phi=(\phi_1(s,y),\cdots,\phi_q(s,y))\in M(\dot{B}_Y)$ such that $A(\gamma)=\phi^{-1}\cdot\gamma\phi$. The elements ϕ_i must satisfy

$$(2)\qquad \begin{aligned} a_i(s,y)&=\phi_i^{-1}(s,y)\phi_{i+1}(\gamma s,y) \quad \text{for} \quad 1\le i\le q-1\\ a_q(s,y)&=\phi_q^{-1}(s,y)\phi_1(\gamma s,y). \end{aligned}$$

We rewrite (1) using (2). Then the condition which ϕ_i must satisfy is

$$(3)\qquad \phi_i^{-1}(s,y)\phi_i(\gamma^q s,y)=\gamma^{qr_i} \quad 1\le i\le q.$$

Take $\phi_1(s,y)=s^{r_1}$ and define $\phi_j(s,y)=\phi_{j-1}(\gamma^{-1}s,y)a_{j-1}(\gamma^{-1}s,y)$ for $2\le j\le q$. Then ϕ_i satisfies (2) clearly, and (3) also since $a_i^{-1}(s,y)a_i(\gamma^q s,y)=\gamma^{q(r_{i+1}-r_i)}$. Hence $\phi=(\phi_1(s,y),\cdots,\phi_q(s,y))$ is the required element. □

3. Triviality over the principal stratum

Let $\dot{X}_{cl}:=X_{cl}-\pi_{cl}^{-1}(0)$. In this section, we show that for every $E\in\mathrm{Vec}_G(X_{cl}\times Y,Q)$, $E|_{\dot{X}_{cl}\times Y}$ is trivial when Y satisfies the assumptions in the Main Theorem in the introduction. Since E is trivial over a G-invariant open neighborhood of $\pi_{cl}^{-1}(0)\times Y$ by Lemma 1.1, it follows that $\mathrm{VEC}_G(X_{cl}\times Y,Q)$ is isomorphic to a double coset of a group of transition functions and $\mathrm{VEC}_G(X\times Y,Q)\cong\mathrm{VEC}_G(X_{cl}\times Y,Q)$ (Theorems 3.3 and 3.4).

We denote by $\boldsymbol{B}*^\Gamma F$ the quotient of $\boldsymbol{B}\times F$ by Γ where $\gamma\in\Gamma$ acts on $\boldsymbol{B}\times F$ by $(b,f)\gamma=(b\gamma,\gamma^{-1}f)$ for $b\in\boldsymbol{B}$, $f\in F$. The group G acts on $\boldsymbol{B}*^\Gamma F$ by $g[b,f]=[b,gf]$ for $g\in G$. There is a morphism $\dot{\boldsymbol{B}}*^\Gamma F\to X_{cl}$ mapping $[b,f]$ to bf where $\dot{\boldsymbol{B}}$ is identified with $\boldsymbol{C}^*$ so that bf makes sense. This morphism can be extended to a map $\varphi:\boldsymbol{B}*^\Gamma F\to X_{cl}$ by defining $\varphi([0,f])=x_0$.

Lemma 3.1 ([8, 3.1]). *The map $\varphi:\boldsymbol{B}*^\Gamma F\to X_{cl}$ is a G-morphism, and it restricts to an isomorphism from $\dot{\boldsymbol{B}}*^\Gamma F$ to $\dot{X}_{cl}$.*

Let $E\in\mathrm{Vec}_G(X_{cl}\times Y,Q)$. We denote by $\tilde{E}$ the pull-back of $E|_{\dot{X}_{cl}\times Y}$ under the map $\dot{\boldsymbol{B}}\times F\times Y\to(\dot{\boldsymbol{B}}*^{\Gamma}F)\times Y\overset{\varphi\times id}{\to}\dot{X}_{cl}\times Y$ where id denotes the identity map on Y.

Lemma 3.2. *If every vector bundle over $\dot{A}\times Y$ is trivial, then the $G\times\Gamma$-vector bundle $\tilde{E}$ is isomorphic to the product bundle $\dot{\boldsymbol{B}}\times F\times Y\times Q\to\dot{\boldsymbol{B}}\times F\times Y$ as a G-vector bundle.*

Proof. We identify F with G/H and set $E_0:=\tilde{E}|_{\dot{\boldsymbol{B}}\times\{eH\}\times Y}$. Then E_0 is isomorphic to a trivial H-vector bundle since the H-action on the base space is trivial and every vector bundle over $\dot{A}\times Y$ is trivial by assumption ([2, 2.1]). Since the fiber of E_0 is a G-module Q, $\tilde{E}\cong G*^{H}E_0$ is trivial as a G-vector bundle. □

The next theorem is the key fact to analyze $\mathrm{VEC}_G(X_{cl}\times Y,Q)$ and $\mathrm{VEC}_G(X\times Y,Q)$.

Theorem 3.3. *Let Q be a G-module which is multiplicity free with respect to H and Y be an irreducible affine variety such that every vector bundle over $\dot{A}\times Y$ is trivial.*

(1) *For every $E\in\mathrm{Vec}_G(X_{cl}\times Y,Q)$, $E|_{\dot{X}_{cl}\times Y}$ is trivial.*

(2) *Furthermore, if every vector bundle over Y is trivial, then the restriction map $\mathrm{VEC}_G(X\times Y,Q)\to\mathrm{VEC}_G(X_{cl}\times Y,Q)$ is a bijection.*

Proof. (1) By Lemma 3.2, we may assume that $\tilde{E}$ is the trivial G-vector bundle $\dot{\boldsymbol{B}}\times F\times Y\times Q$. From Lemma 3.1 and the fact that the Γ-action on $\dot{\boldsymbol{B}}\times F\times Y$ is free, it follows that $E|_{\dot{X}_{cl}\times Y}$ is isomorphic to the quotient of $\tilde{E}$ by the Γ-action.

The action of $\gamma\in\Gamma$ on $\tilde{E}=\dot{\boldsymbol{B}}\times F\times Y\times Q$ must be in the following form

$$(b,f,y,q)\gamma=(b\gamma,\gamma^{-1}f,y,\tilde{A}(\gamma)(b,f,y)(q))\qquad b\in\dot{\boldsymbol{B}},\ f\in F,\ y\in Y,\ q\in Q$$

where $\tilde{A}(\gamma)\in\mathrm{Mor}(\dot{\boldsymbol{B}}\times F\times Y,\mathrm{GL}Q)^G\cong M(\dot{\boldsymbol{B}}_Y)$. Set $A(\gamma):=\tilde{A}(\gamma)^{-1}$. Then one easily verifies that $\{A(\gamma)\}_{\gamma\in\Gamma}$ satisfies the 1-cocycle condition and gives rise to an element of $H^1(\Gamma,M(\dot{\boldsymbol{B}}_Y))$. Since $H^1(\Gamma,M(\dot{\boldsymbol{B}}_Y))=\{*\}$ by Lemma 2.2, there exists $\phi\in M(\dot{\boldsymbol{B}}_Y)$ such that $A(\gamma)=\phi^{-1}\cdot\gamma\phi$ for all $\gamma\in\Gamma$. Then the following map gives an isomorphism from $\tilde{E}$ to a trivial $G\times\Gamma$-vector bundle

$$\tilde{E}=\dot{\boldsymbol{B}}\times F\times Y\times Q\to\dot{\boldsymbol{B}}\times F\times Y\times Q$$

$$(b,f,y,q)\mapsto(b,f,y,(\phi(b,y)(f))(q)).$$

where the Γ-action on Q in the right hand side is trivial. This shows that $E|_{\dot{X}_{cl}\times Y}$ is isomorphic to a trivial G-vector bundle from the remark above.

(2) As noted in §1, the Equivariant Nakayama Lemma implies that the

restriction map $\mathrm{VEC}_G(X\times Y, Q)\to \mathrm{VEC}_G(X_{cl}\times Y, Q)$ is injective. We show its surjectivity. Let $E\in \mathrm{Vec}_G(X_{cl}\times Y, Q)$. From (1) and Lemma 1.1, E is trivial over $\dot{X}_{cl}\times Y$ and $(X_{cl}\times Y)_f$ for some $f\in \mathcal{O}(A\times Y)$ such that $f(0,y)=1$. Let $\dot{\psi}$ be the transition function of E with respect to trivializations over $\dot{X}_{cl}\times Y$ and $(X_{cl}\times Y)_f$. Note that $\dot{\psi}$ can be viewed as an equivariant vector bundle automorphism of a trivial bundle over $(\dot{X}_{cl}\times Y)\cap(X_{cl}\times Y)_f=(X_{cl}\times Y)_{tf}$ with fiber Q. Since $(X_{cl}\times Y)_{tf}$ is a closed G-subvariety of an affine variety $(X\times Y)_{tf}$ and contains all closed G-orbits in $(X\times Y)_{tf}$, $\dot{\psi}$ extends to an equivariant vector bundle automorphism $\bar{\psi}$ of a trivial bundle over $(X\times Y)_{tf}$ by the Equivariant Nakayama Lemma. Let $\bar{E}$ be the G-vector bundle over $X\times Y$ obtained from the transition function $\bar{\psi}$. Clearly $\bar{E}$ restricts to E, and this proves the surjectivity. □

REMARK. For $E\in \mathrm{Vec}_G(X\times Y, Q)$, $E|_{\dot{X}\times Y}$ is trivial since the restriction map $\mathrm{VEC}_G(\dot{X}\times Y, Q)\to \mathrm{VEC}_G(\dot{X}_{cl}\times Y, Q)$ is an injection from the Equivariant Nakayama Lemma.

By virtue of Theorem 3.3 (2), we will continue to study $\mathrm{VEC}_G(X_{cl}\times Y, Q)$ instead of $\mathrm{VEC}_G(X\times Y, Q)$ in the following. Set

$$\dot{A}_Y:=\dot{A}\times Y,\quad \tilde{\dot{A}}_Y:=\dot{A}_Y\times_{(A\times Y)}\tilde{A}_Y$$

where $\tilde{A}_Y$ is an affine scheme such that

$$\mathcal{O}(\tilde{A}_Y)=\{f(t,y)/g(t,y)\mid f(t,y), g(t,y)\in \mathcal{O}(A\times Y)\quad\text{and}\quad g(0,y)=1\}.$$

Note that $\mathcal{O}(\tilde{\dot{A}}_Y)=\mathcal{O}(\dot{A}_Y)\otimes_{\mathcal{O}(A\times Y)}\mathcal{O}(\tilde{A}_Y)$. Similar definition applies for $\boldsymbol{B}$. For a scheme Z together with a morphism $Z\to A\times Y$, we set

$$\mathfrak{P}(Z):=\mathrm{Mor}(Z\times_{A\times Y}(X_{cl}\times Y), \mathrm{GL}Q)^G.$$

Theorem 3.4. *Let Q be a G-module which is multiplicity free with respect to H. If Y is an irreducible affine variety and every vector bundle over Y and $\dot{A}\times Y$ is trivial, then there exists a bijection*

$$\mathrm{VEC}_G(X_{cl}\times Y, Q)\cong \mathfrak{P}(\dot{A}_Y)\backslash\mathfrak{P}(\tilde{\dot{A}}_Y)/\mathfrak{P}(\tilde{A}_Y).$$

Proof. Let $E\in \mathrm{Vec}_G(X_{cl}\times Y, Q)$. By Theorem 3.3 (1) and Lemma 1.1, there exist trivializations $\dot{\psi}: E|_{\dot{X}_{cl}\times Y}\cong \dot{X}_{cl}\times Y\times Q$ and $\tilde{\psi}: E|_{(X_{cl}\times Y)_f}\cong (X_{cl}\times Y)_f\times Q$ where $f\in\mathcal{O}(A\times Y)$ and $f(0,y)=1$. Then $\dot{\psi}\circ\tilde{\psi}^{-1}$ defines a transition function $\tilde{\dot{\alpha}}$ $\in \mathrm{Mor}((X_{cl}\times Y)_{tf}, \mathrm{GL}Q)^G$ by

$$\dot{\psi}\circ\tilde{\psi}^{-1}(x,y,q)=(x,y,\tilde{\dot{\alpha}}(x,y)q)$$

for $(x,y)\in(X_{cl}\times Y)_{tf}$, $q\in Q$. Note that an element of $\mathrm{Mor}((X_{cl}\times Y)_{tf}, \mathrm{GL}Q)$ is considered as an invertible matrix with entries in $\mathcal{O}((X_{cl}\times Y)_{tf})$. Since

$$\begin{aligned}\mathcal{O}((X_{cl}\times Y)_{tf}) &= \mathcal{O}((A\times Y)_{tf})\otimes_{\mathcal{O}(A\times Y)}\mathcal{O}(X_{cl}\times Y)\\ &= \mathcal{O}(A\times Y)_{tf}\otimes_{\mathcal{O}(A\times Y)}\mathcal{O}(X_{cl}\times Y)\end{aligned}$$

where $\mathcal{O}(A\times Y)_{tf}$ denotes the localization by tf, the canonical inclusion $\mathcal{O}(A\times Y)_{tf}\to\mathcal{O}(\tilde{A}_Y)$ induces an injection $\mathrm{Mor}((X_{cl}\times Y)_{tf},\mathrm{GL}Q)^G\to\mathfrak{P}(\tilde{A}_Y)$. We define a map $\Psi:\mathrm{VEC}_G(X_{cl}\times Y,Q)\to\mathfrak{P}(\dot{A}_Y)\backslash\mathfrak{P}(\tilde{\dot{A}}_Y)/\mathfrak{P}(\tilde{A}_Y)$ by $\Psi([E])=[\tilde{\dot{\alpha}}]$. Then the map Ψ is well-defined. In fact, let $E'\in\mathrm{Vec}_G(X_{cl}\times Y,Q)$ and $\phi:E'\to E$ be a G-vector bundle isomorphism. Let $\dot{\psi}'$ be a trivialization of $E'|_{\dot{X}_{cl}\times Y}$ and $\tilde{\psi}'$ a trivialization of $E'|_{(X_{cl}\times Y)_{f'}}$ where $f'\in\mathcal{O}(A\times Y)$, $f'(0,y)=1$. Then $\dot{\psi}'\circ\tilde{\psi}'^{-1}$ defines an element $\tilde{\dot{\alpha}}'\in\mathfrak{P}(\tilde{\dot{A}}_Y)$. The equivariant vector bundle automorphism $\tilde{\psi}\circ\phi\circ\tilde{\psi}'^{-1}$ of a trivial bundle over $(X_{cl}\times Y)_f\cap(X_{cl}\times Y)_{f'}=(X_{cl}\times Y)_{ff'}$ defines $\tilde{\alpha}\in\mathfrak{P}(\tilde{A}_Y)$. Similarly, $\dot{\psi}'\circ\phi^{-1}\circ\dot{\psi}^{-1}$ defines $\dot{\alpha}\in\mathrm{Mor}(\dot{X}_{cl}\times Y,\mathrm{GL}Q)^G=\mathfrak{P}(\dot{A}_Y)$. Since $\tilde{\dot{\alpha}}'=\dot{\alpha}\tilde{\dot{\alpha}}\tilde{\alpha}$, Ψ is well-defined. It is easy to see that Ψ is bijective. □

REMARK. There are two hypotheses on an irreducible affine variety Y: (1) every vector bundle over Y is trivial, and (2) every vector bundle over $\dot{A}\times Y$ is trivial. They are used in order to apply the glueing argument of Kraft-Schwarz; (1) is used in order to prove the bundle triviality over a neighborhood of $\pi_X^{-1}(0)\times Y$ (Lemma 1.1) and (2) is used in order to prove the bundle triviality over $\dot{X}\times Y$ (Theorem 3.3). If Y is smooth and satisfies (1), then every vector bundle over $A\times Y$ is trivial ([6]). However, the author does not know whether and when (1) implies (2).

Since $\varphi\times id:(\boldsymbol{B}*^{\Gamma}F)\times Y\to X_{cl}\times Y$ is an isomorphism over $\dot{A}_Y$ by Lemma 3.1, it induces an isomorphism:

$$(\varphi\times id)_*:\mathfrak{P}(\dot{A}_Y)\xrightarrow{\sim}M(\dot{\boldsymbol{B}}_Y)^{\Gamma}.$$

Lemma 3.5. *For any G-module Q and an irreducible affine variety Y, the morphism $\varphi\times id$ induces a bijection*

$$\mathfrak{P}(\dot{A}_Y)\backslash\mathfrak{P}(\tilde{\dot{A}}_Y)/\mathfrak{P}(\tilde{A}_Y)\cong M(\dot{\boldsymbol{B}}_Y)^{\Gamma}\backslash M(\tilde{\dot{\boldsymbol{B}}}_Y)^{\Gamma}/(\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y).$$

Proof. Note that $\mathcal{O}(\tilde{\boldsymbol{B}}_Y)\cong\mathcal{O}(\tilde{A}_Y)\otimes_{\mathcal{O}(A\times Y)}\mathcal{O}(\boldsymbol{B}\times Y)$. In fact, the product map $\mathcal{O}(\tilde{A}_Y)\otimes_{\mathcal{O}(A\times Y)}\mathcal{O}(\boldsymbol{B}\times Y)\to\mathcal{O}(\tilde{\boldsymbol{B}}_Y)$ defined by $h_1\otimes h_2\to h_1h_2$ is an isomorphism. It is obvious that the map is $\mathcal{O}(\tilde{A}_Y)$-algebra homomorphism and injective. We show that it is surjective. Let $f/g\in\mathcal{O}(\tilde{\boldsymbol{B}}_Y)$ where $f,g\in\mathcal{O}(\boldsymbol{B}\times Y)$ and $g(0,y)=1$. Set $\bar{g}:=\Pi_{\gamma\in\Gamma}\gamma g$. Then $\bar{g}\in\mathcal{O}(\boldsymbol{B}\times Y)^{\Gamma}=\mathcal{O}(A\times Y)$ and $\bar{g}(0,y)=1$. Hence $\bar{g}\in\mathcal{O}(\tilde{A}_Y)^*$ and f/g is the image of $\bar{g}^{-1}\otimes(f\bar{g}/g)\in\mathcal{O}(\tilde{A}_Y)\otimes_{\mathcal{O}(A\times Y)}\mathcal{O}(\boldsymbol{B}\times Y)$ by the product map. Thus

$$\begin{aligned}\mathcal{O}(\tilde{\dot{B}}_Y) &= \mathcal{O}(\tilde{B}_Y)\otimes_{\mathcal{O}(B\times Y)}\mathcal{O}(\dot{B}_Y)\\ &\cong \mathcal{O}(\tilde{A}_Y)\otimes_{\mathcal{O}(A\times Y)}\mathcal{O}(\dot{B}_Y)\\ &= \mathcal{O}(\tilde{\dot{A}}_Y)\otimes_{\mathcal{O}(\dot{A}_Y)}\mathcal{O}(\dot{B}_Y)\end{aligned}$$

i.e. $\tilde{\dot{B}}_Y \cong \tilde{\dot{A}}_Y \times_{\dot{A}_Y} \dot{B}_Y$. Since φ is G-equivariant, the isomorphism $\varphi\times id\colon \tilde{\dot{B}}_Y * {}^{\Gamma}F \cong \tilde{\dot{A}}_Y \times_{\dot{A}_Y}((\dot{B}*{}^{\Gamma}F)\times Y) \to \tilde{\dot{A}}_Y \times_{\dot{A}_Y}(\dot{X}_{cl}\times Y)$ induces an isomorphism $(\varphi\times \tilde{id})_*\colon \mathfrak{P}(\tilde{\dot{A}}_Y) \to M(\tilde{\dot{B}}_Y)^{\Gamma}$. It is easy to see that $\varphi\times id$ induces a bijection from $\mathfrak{P}(\dot{A}_Y)\backslash\mathfrak{P}(\tilde{\dot{A}}_Y)/\mathfrak{P}(\tilde{A}_Y)$ to $M(\dot{B}_Y)^{\Gamma}\backslash M(\tilde{\dot{B}}_Y)^{\Gamma}/(\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y)$. □

When Q is multiplicity free with respect to H, $M(\tilde{\dot{B}}_Y)^{\Gamma}$ is an abelian group since M is a torus. Hence we obtain from Theorem 3.4 and Lemma 3.5

Theorem 3.6. *Under the assumptions in Theorem* 3.4,

$$\mathrm{VEC}_G(X_{cl}\times Y, Q) \cong M(\tilde{\dot{B}}_Y)^{\Gamma}/(M(\dot{B}_Y)^{\Gamma}\ (\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y)).$$

By Theorem 3.6, we will analyze $M(\tilde{\dot{B}}_Y)^{\Gamma}/(M(\dot{B}_Y)^{\Gamma}\ (\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y))$ in the following sections.

4. The decomposition property

We set

$$\begin{aligned} M(\tilde{B}_Y)_1 &:= \{\mu\in M(\tilde{B}_Y) \mid \mu(0,y)=I\}\\ M(\tilde{B}_Y)_1^{\Gamma} &:= M(\tilde{B}_Y)_1 \cap M(\tilde{B}_Y)^{\Gamma}.\end{aligned}$$

Note that $M(\tilde{B}_Y)_1$ is considered as a direct product of copies of $\mathcal{O}(\tilde{B}_Y)_1 := \{f \in \mathcal{O}(\tilde{B}_Y) \mid f(0,y)=1\}$.

Lemma 4.1 (The decomposition property)

$$M(\tilde{\dot{B}}_Y)^{\Gamma} = M(\dot{B}_Y)^{\Gamma} M(\tilde{B}_Y)_1^{\Gamma}.$$

Proof. Every $0\neq h(s,y)\in\mathcal{O}(\tilde{\dot{B}}_Y)$ is written in the form

$$h(s,y) = s^r f(s,y)/g(s,y)$$

for $r\in\boldsymbol{Z}$, $f(s,y)$, $g(s,y)\in\mathcal{O}(B\times Y)$, $f(0,y)\neq 0$, $g(0,y)=1$. If h is invertible, then $f(0,y)\in\mathcal{O}(Y)^*$. In fact, there exists $h'=s^{r'}f'(s,y)/g'(s,y)$ such that $hh'=1$. Here, $r'\in\boldsymbol{Z}$ and f' and g' satisfy similar conditions to f and g, respectively. Thus $s^{r+r'}f(s,y)f'(s,y)=g(s,y)g'(s,y)$. Since the right hand side is a polynomial in s with constant term 1, $r+r'$ must not be positive. Suppose $r+r'<0$. Comparing the terms with the lowest degree in s in both sides of the above identity,

$f(0,y)f'(0,y)=0$. While $\mathcal{O}(Y)$ is an integral domain and neither $f(0,y)$ nor $f'(0,y)$ is zero, this is a contradiction. Thus $r+r'=0$ and $f(0,y)f'(0,y)=1$, i.e. $f(0,y)$ is invertible. Hence we obtain

$$h(s,y)=f(0,y)s^r\cdot f(0,y)^{-1}f(s,y)/g(s,y)\in \mathcal{O}(\dot{\boldsymbol{B}}_Y)^*\mathcal{O}(\tilde{\boldsymbol{B}}_Y)_1.$$

Thus $M(\tilde{\dot{\boldsymbol{B}}}_Y)=M(\dot{\boldsymbol{B}}_Y)M(\tilde{\boldsymbol{B}}_Y)_1$. Since $M(\dot{\boldsymbol{B}}_Y)\cap M(\tilde{\boldsymbol{B}}_Y)_1=I$, the decomposition of $M(\tilde{\dot{\boldsymbol{B}}}_Y)$ to a product of $M(\dot{\boldsymbol{B}}_Y)$ and $M(\tilde{\boldsymbol{B}}_Y)_1$ is unique. Let $\mu\in M(\tilde{\dot{\boldsymbol{B}}}_Y)^\Gamma$ and $\mu=\dot{\mu}\tilde{\mu}$ where $\dot{\mu}\in M(\dot{\boldsymbol{B}}_Y)$ and $\tilde{\mu}\in M(\tilde{\boldsymbol{B}}_Y)_1$. Since the Γ-action on $\mathcal{O}(\tilde{\dot{\boldsymbol{B}}}_Y)$ preserves the order at $s=0$ and Γ acts on M by permuting components (Lemma 2.1), it follows from the uniqueness of the decomposition of $M(\tilde{\dot{\boldsymbol{B}}}_Y)$ to a product of $M(\dot{\boldsymbol{B}}_Y)$ and $M(\tilde{\boldsymbol{B}}_Y)_1$ that $\dot{\mu}\in M(\dot{\boldsymbol{B}}_Y)^\Gamma$ and $\tilde{\mu}\in M(\tilde{\boldsymbol{B}}_Y)_1^\Gamma$. □

We denote by $\mathfrak{P}(\tilde{\boldsymbol{A}}_Y)_1$ the subgroup of $\mathfrak{P}(\tilde{\boldsymbol{A}}_Y)$ consisting of elements which are equal to the constant map to $I\in \mathrm{GL}Q$ on $\{x_0\}\times Y$.

Proposition 4.2

$$M(\tilde{\dot{\boldsymbol{B}}}_Y)^\Gamma/(M(\dot{\boldsymbol{B}}_Y)^\Gamma(\varphi\times id)_*\mathfrak{P}(\tilde{\boldsymbol{A}}_Y))\cong M(\tilde{\boldsymbol{B}}_Y)_1^\Gamma/(\varphi\times id)_*\mathfrak{P}(\tilde{\boldsymbol{A}}_Y)_1.$$

Proof. From Lemma 4.1 and the fact that $M(\dot{\boldsymbol{B}}_Y)^\Gamma\cap M(\tilde{\boldsymbol{B}}_Y)_1^\Gamma=I$, the projection $M(\tilde{\dot{\boldsymbol{B}}}_Y)^\Gamma\to M(\tilde{\dot{\boldsymbol{B}}}_Y)^\Gamma/M(\dot{\boldsymbol{B}}_Y)^\Gamma\cong M(\tilde{\boldsymbol{B}}_Y)_1^\Gamma$ induces an isomorphism

$$M(\tilde{\dot{\boldsymbol{B}}}_Y)^\Gamma/(M(\dot{\boldsymbol{B}}_Y)^\Gamma\ (\varphi\times id)_*\mathfrak{P}(\tilde{\boldsymbol{A}}_Y))\cong M(\tilde{\boldsymbol{B}}_Y)_1^\Gamma/(M(\tilde{\boldsymbol{B}}_Y)_1^\Gamma\cap(\varphi\times id)_*\mathfrak{P}(\tilde{\boldsymbol{A}}_Y)).$$

Since $M(\tilde{\boldsymbol{B}}_Y)_1^\Gamma\cap(\varphi\times id)_*\mathfrak{P}(\tilde{\boldsymbol{A}}_Y)=(\varphi\times id)_*\mathfrak{P}(\tilde{\boldsymbol{A}}_Y)_1$, the proposition follows. □

Let $\hat{\boldsymbol{B}}:=\mathrm{Spec}\,\boldsymbol{C}[[s]]$ where $\boldsymbol{C}[[s]]$ denotes the ring of formal power series in s. We set $\hat{\boldsymbol{B}}_Y=\hat{\boldsymbol{B}}\times Y$. The group $M(\hat{\boldsymbol{B}}_Y)$ has a natural grading induced from $\mathcal{O}(\hat{\boldsymbol{B}})=\boldsymbol{C}[[s]]$. For $r\geq 1$, we define

$$M(\hat{\boldsymbol{B}}_Y)_r:=\{\mu\in M(\hat{\boldsymbol{B}}_Y)\mid \mu=I+O(s^r)\}$$
$$M(\hat{\boldsymbol{B}}_Y)_r^\Gamma:=M(\hat{\boldsymbol{B}}_Y)_r\cap M(\hat{\boldsymbol{B}}_Y)^\Gamma.$$

We also define $\hat{\boldsymbol{A}}_Y=\hat{\boldsymbol{A}}\times Y$ where $\hat{\boldsymbol{A}}=\mathrm{Spec}\,\boldsymbol{C}[[t]]$ and $\mathfrak{P}(\hat{\boldsymbol{A}}_Y)_1$ in a similar way to $\mathfrak{P}(\tilde{\boldsymbol{A}}_Y)_1$. There exists a canonical map

$$M(\tilde{\boldsymbol{B}}_Y)_1^\Gamma/(\varphi\times id)_*\mathfrak{P}(\tilde{\boldsymbol{A}}_Y)_1\to M(\hat{\boldsymbol{B}}_Y)_1^\Gamma/(\varphi\times id)_*\mathfrak{P}(\hat{\boldsymbol{A}}_Y)_1.$$

In the following section, we will show that the above map is in fact a bijection. For preparation, we prove

Lemma 4.3. *For all* $r\geq 1$,

$$M(\hat{B}_Y)_1^\Gamma = M(\tilde{B}_Y)_1^\Gamma M(\hat{B}_Y)_r^\Gamma .$$

Proof. It is clear that $M(\hat{B}_Y)_1^\Gamma \supset M(\tilde{B}_Y)_1^\Gamma M(\hat{B}_Y)_r^\Gamma$. We show the opposite inclusion. Let $\mu=(h_1(s,y),\cdots,h_q(s,y))\in M(\hat{B}_Y)_1^\Gamma$ where $h_i(s,y)=1+\Sigma_{j=1}^{r-1}a_{ij}(y)s^j+O(s^r)$, and $a_{ij}(y)\in\mathcal{O}(Y)$ for $1\le i\le q$. Define $\tilde{\mu}=(\tilde{h}_1(s,y),\cdots,\tilde{h}_q(s,y))$ by $\tilde{h}_i(s,y):=1+\Sigma_{j=1}^{r-1}a_{ij}(y)s^j$ for $1\le i\le q$. Since the Γ-action preserves the grading on $M(\hat{B}_Y)_1$ (Lemma 2.1), $\tilde{\mu}\in M(\tilde{B}_Y)_1^\Gamma$ and $\tilde{\mu}^{-1}\cdot\mu\in M(\hat{B}_Y)_r^\Gamma$. □

5. Moduli of vector bundles over $X\times Y$

We define

$$\mathfrak{E}(\hat{A}_Y):=\mathrm{Mor}(\hat{A}_Y\times_{A\times Y}(X_{cl}\times Y),\ \mathrm{End}\,Q)^G.$$

Note that $\mathfrak{E}(\hat{A}_Y)\cong\mathcal{O}(\hat{A}_Y)\otimes_{\mathcal{O}(A)}\mathrm{Mor}(X_{cl},\mathrm{End}\,Q)^G$. Since $\mathrm{Mor}(X_{cl},\mathrm{End}\,Q)^G$ is a free module of rank $\dim\mathrm{End}(Q)^H$ over $\mathcal{O}(X_{cl})^G=\mathcal{O}(A)$ for any G-module Q ([10]), $\mathfrak{E}(\hat{A}_Y)$ is a free module of rank q over $\mathcal{O}(\hat{A}_Y)$.

Let $\mathfrak{m}$ be the Lie algebra of M, i.e.,

$$\mathfrak{m}:=\mathrm{Mor}(F,\mathrm{End}\,Q)^G\cong\mathrm{End}(Q)^H\cong C^q.$$

The map $\varphi:B*^\Gamma F\to X_{cl}$ induces an $\mathcal{O}(\hat{A}_Y)$-module homomorphism $(\varphi\times id)_\#:\mathfrak{E}(\hat{A}_Y)\to\mathfrak{m}(\hat{B}_Y)^\Gamma$. Setting Y to be a point, we obtain an $\mathcal{O}(\hat{A})$-module homomorphism $\varphi_\#:\mathfrak{E}(\hat{A})\to\mathfrak{m}(\hat{B})^\Gamma$ where $\mathfrak{E}(\hat{A}):=\mathrm{Mor}(\hat{A}\times_A X_{cl},\mathrm{End}\,Q)^G$. The morphism $\varphi_\#:\mathfrak{E}(\hat{A})\to\mathfrak{m}(\hat{B})^\Gamma$ is an injection of free $\mathcal{O}(\hat{A})$-modules and of full rank ([8, 6.1]). Through the canonical isomorphisms $\mathfrak{E}(\hat{A}_Y)\cong\mathfrak{E}(\hat{A})\otimes_C\mathcal{O}(Y)$ and $\mathfrak{m}(\hat{B}_Y)^\Gamma\cong\mathfrak{m}(\hat{B})^\Gamma\otimes_C\mathcal{O}(Y)$, $(\varphi\times id)_\#:\mathfrak{E}(\hat{A}_Y)\to\mathfrak{m}(\hat{B}_Y)^\Gamma$ agrees with $\varphi_\#\otimes id:\mathfrak{E}(\hat{A})\otimes_C\mathcal{O}(Y)\to\mathfrak{m}(\hat{B})^\Gamma\otimes_C\mathcal{O}(Y)$. Note that $\mathfrak{E}(\hat{A}_Y)$ inherits a grading induced from $\mathcal{O}(X_{cl})$. For $r\ge 1$, let $\mathfrak{E}(\hat{A}_Y)_r$ be the ideal of $\mathfrak{E}(\hat{A}_Y)$ generated by the homogeneous elements of degree r. We define

$$\mathfrak{P}(\hat{A}_Y)_r:=\{A\in\mathfrak{P}(\hat{A}_Y)\mid A-I\in\mathfrak{E}(\hat{A}_Y)_r\}$$
$$\mathfrak{m}(\hat{B}_Y)_r^\Gamma:=\{\mu\in\mathfrak{m}(\hat{B}_Y)^\Gamma\mid\mu=O(s^r)\}.$$

We have a commutative diagram

$$\begin{array}{ccc}\mathfrak{P}(\hat{A}_Y)_r & \xrightarrow{(\varphi\times id)_*} & M(\hat{B}_Y)_r^\Gamma\\ \exp\uparrow\wr & & \wr\uparrow\exp\\ \mathfrak{E}(\hat{A}_Y)_r & \xrightarrow[(\varphi\times id)_\#]{} & \mathfrak{m}(\hat{B}_Y)_r^\Gamma\end{array}$$

where the vertical maps are isomorphisms induced from $\exp:\mathrm{End}\,Q\to\mathrm{GL}Q$.

Lemma 5.1. *There exists a positive integer r_0 such that $(\varphi\times id)_*\mathfrak{P}(\hat{A}_Y)_r=M(\hat{B}_Y)^\Gamma_r$ for any $r\geq r_0$.*

Proof. Setting Y to be a point in $\mathfrak{E}(\hat{A}_Y)_r$, we also have $\mathfrak{E}(\hat{A})_r$ for $r\geq 1$. Then there exists a positive integer r_0 such that $\varphi_\#\mathfrak{E}(\hat{A})_r=\mathfrak{m}(\hat{B})^\Gamma_r$ for any $r\geq r_0$ ([8, 6.1]). Thus

$$\begin{aligned}(\varphi\times id)_\#\mathfrak{E}(\hat{A}_Y)_r&\cong\varphi_\#\mathfrak{E}(\hat{A})_r\otimes_{\boldsymbol{C}}\mathcal{O}(Y)\\&=\mathfrak{m}(\hat{B})^\Gamma_r\otimes_{\boldsymbol{C}}\mathcal{O}(Y)\\&\cong\mathfrak{m}(\hat{B}_Y)^\Gamma_r.\end{aligned}$$

Using the above commutative diagram, we have $(\varphi\times id)_*\mathfrak{P}(\hat{A}_Y)_r=M(\hat{B}_Y)^\Gamma_r$. □

Proposition 5.2. *The canonical map*

$$M(\tilde{B}_Y)^\Gamma_1/(\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y)_1\to M(\hat{B}_Y)^\Gamma_1/(\varphi\times id)_*\mathfrak{P}(\hat{A}_Y)_1$$

is a bijection.

Proof. The surjectivity follows from Lemmas 4.3 and 5.1. We show its injectivity. It is enough to show that $M(\tilde{B}_Y)^\Gamma_1\cap(\varphi\times id)_*\mathfrak{P}(\hat{A}_Y)_1\subset(\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y)_1$. Let $\mu\in M(\tilde{B}_Y)^\Gamma_1\cap(\varphi\times id)_*\mathfrak{P}(\hat{A}_Y)_1$. Since $M=\mathrm{Mor}(F,\mathrm{GL}\,Q)^G\subset\mathrm{Mor}(F,\mathrm{End}\,Q)^G=\mathfrak{m}$, we can consider $M(\tilde{B}_Y)^\Gamma$ as a subset of $\mathfrak{m}(\tilde{B}_Y)^\Gamma$. Similarly, we can consider $\mathfrak{P}(\hat{A}_Y)$ as a subset of $\mathfrak{E}(\hat{A}_Y)$. We regard μ as an element of $\mathfrak{m}(\tilde{B}_Y)^\Gamma\cap(\varphi\times id)_\#\mathfrak{E}(\hat{A}_Y)$ $\cong\mathcal{O}(\tilde{A}_Y)\otimes_{\mathcal{O}(A)}\mathfrak{m}(B)^\Gamma\cap\mathcal{O}(\hat{A}_Y)\otimes_{\mathcal{O}(A)}\varphi_\#\mathfrak{E}(A)$ where $\mathfrak{E}(A)=\mathrm{Mor}(X_{cl},\mathrm{End}\,Q)^G$. Since $\varphi_\#:\mathfrak{E}(A)\to\mathfrak{m}(B)^\Gamma$ is an injection of free $\mathcal{O}(A)$-modules and of full rank ([8, 6.1]), one sees that μ is an element of $\mathcal{O}(\tilde{A}_Y)\otimes_{\mathcal{O}(A)}\varphi_\#\mathfrak{E}(A)\cong(\varphi\times id)_\#\mathrm{Mor}(\tilde{A}_Y\times_{(A\times Y)}(X_{cl}\times Y),\mathrm{End}\,Q)^G$. Since $\mu\in(\varphi\times id)_*\mathfrak{P}(\hat{A}_Y)_1$, this implies that $\mu\in(\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y)_1$. Hence the injectivity follows. □

Now, we can describe $\mathrm{VEC}_G(X\times Y,Q)$.

Theorem 5.3. *Let X be a weighted G-cone with smooth one dimensional quotient and Y an irreducible affine variety such that every vector bundle over Y and $\hat{A}\times Y$ is trivial. When a G-module Q is multiplicity free with respect to a principal isotropy group of X, the map*

$$\begin{aligned}\Phi:\mathrm{Mor}(Y,\boldsymbol{C}^p)&\to\mathrm{VEC}_G(X\times Y,Q)\\f&\mapsto[(id_X\times f)^*\mathfrak{B}]\end{aligned}$$

is a bijection. Here p and $\mathfrak{B}$ are given in Theorem A in the introduction.

Proof. We have proved

$$\begin{aligned}\mathrm{VEC}_G(X\times Y,Q)&\cong \mathrm{VEC}_G(X_{cl}\times Y,Q) \quad \text{(by 3.3 (2))}\\ &\cong M(\tilde{\dot{B}}_Y)^\Gamma/(M(\dot{B}_Y)^\Gamma\,(\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y)) \quad \text{(by 3.6)}\\ &\cong M(\tilde{B}_Y)_1^\Gamma/(\varphi\times id)_*\mathfrak{P}(\tilde{A}_Y)_1 \quad \text{(by 4.2)}\\ &\cong M(\hat{B}_Y)_1^\Gamma/(\varphi\times id)_*\mathfrak{P}(\hat{A}_Y)_1 \quad \text{(by 5.2).}\end{aligned}$$

From the commutative diagram above Lemma 5.1, the exponential map induces an isomorphism

$$\begin{aligned}M(\hat{B}_Y)_1^\Gamma/(\varphi\times id)_*\mathfrak{P}(\hat{A}_Y)_1&\cong \mathfrak{m}(\hat{B}_Y)_1^\Gamma/(\varphi\times id)_\#\mathfrak{E}(\hat{A}_Y)_1\\ &\cong(\mathfrak{m}(\hat{B})_1^\Gamma/\varphi_\#\mathfrak{E}(\hat{A})_1)\otimes_{\boldsymbol{C}}\mathcal{O}(Y).\end{aligned}$$

Hence $\mathrm{VEC}_G(X\times Y,Q)\cong(\mathfrak{m}(\hat{B})_1^\Gamma/\varphi_\#\mathfrak{E}(\hat{A})_1)\otimes_{\boldsymbol{C}}\mathcal{O}(Y)$. In particular, when Y is a single point, we obtain a bijection $\mathrm{VEC}_G(X,Q)\cong\mathfrak{m}(\hat{B})_1^\Gamma/\varphi_\#\mathfrak{E}(\hat{A})_1$. By composing the bijection to the map $\boldsymbol{C}^p\ni z\mapsto[\mathfrak{B}|_{X\times\{z\}}]\in\mathrm{VEC}_G(X,Q)$, we have a bijection

$$\boldsymbol{C}^p\xrightarrow{\sim}\mathrm{VEC}_G(X,Q)\xrightarrow{\sim}\mathfrak{m}(\hat{B})_1^\Gamma/\varphi_\#\mathfrak{E}(\hat{A})_1.$$

We identify $\mathfrak{m}(\hat{B})_1^\Gamma/\varphi_\#\mathfrak{E}(\hat{A})_1$ with $\boldsymbol{C}^p$ through the above bijection. Using this identification we have a bijection

$$\begin{aligned}\mathrm{VEC}_G(X\times Y,Q)&\xrightarrow{\sim}(\mathfrak{m}(\hat{B})_1^\Gamma/\varphi_\#\mathfrak{E}(\hat{A})_1)\otimes_{\boldsymbol{C}}\mathcal{O}(Y)\\ &\cong\boldsymbol{C}^p\otimes_{\boldsymbol{C}}\mathcal{O}(Y)\\ &\cong\mathrm{Mor}(Y,\boldsymbol{C}^p)\end{aligned}$$

which we denote by $\Psi:\mathrm{VEC}_G(X\times Y,Q)\to\mathrm{Mor}(Y,\boldsymbol{C}^p)$. Note that when Y is a point, Ψ becomes $\Psi_0:\mathrm{VEC}_G(X,Q)\xrightarrow{\sim}\mathfrak{m}(\hat{B})_1^\Gamma/\varphi_\#\mathfrak{E}(\hat{A})_1\cong\boldsymbol{C}^p$ and it satisfies that $\Psi_0([\mathfrak{B}|_{X\times\{z\}}])=z$ for any $z\in\boldsymbol{C}^p$. Thus it follows from the way of constructing Ψ that

$$\begin{aligned}(\Psi\circ\Phi)(f)(y)&=\Psi([(id_X\times f)^*\mathfrak{B}])(y)\\ &=\Psi_0([\mathfrak{B}|_{X\times\{f(y)\}}])\\ &=f(y)\end{aligned}$$

for any $f\in\mathrm{Mor}(Y,\boldsymbol{C}^p)$ and $y\in Y$. Thus $\Psi\circ\Phi=id$ (in particular, Φ is an injection. cf. remark in the introduction). Since Ψ is a bijection, in particular, an injection, the above identity implies that Φ is a surjection. Hence Φ is bijective. □

As remarked in the introduction, if we take $Y=A^m$ the assumptions on Y in

Theorem 5.3 are satisfied.

Corollary 5.4. *Let X, Q, and p as in Theorem* 5.3. *Then*

$$\mathrm{VEC}_G(X \times A^m, Q) \cong \mathrm{Mor}(A^m, C^p).$$

Remark. There is a formula to compute the dimension p of $\mathrm{VEC}_G(X,Q)$ ([8, 6.5]), [5, VI]).

Let $Q \cong \oplus_{i=1}^{q} W_i$ where W_i $(1 \leq i \leq q)$ are irreducible H-modules. If every W_i is G-stable, then $\mathrm{VEC}_G(X,Q)$ is trivial (cf. [5, VII]). So we have

Corollary 5.5. *Let X and Q be as in Theorem* 5.3 *and W_i be as above. If every W_i is G-stable, then for any affine variety Y satisfying the assumptions in Theorem* 5.3, *$\mathrm{VEC}_G(X \times Y, Q)$ is trivial.*

For example, let $G = O(2) = C^* \rtimes Z/2Z$ and V_m $(m \geq 1)$ be a 2-dimensional G-module on which C^* acts with weights m and $-m$ and the generator of $Z/2Z$ acts by interchanging the weight spaces. It is easy to see that $V_m /\!/ G \cong A$ and the principal isotropy group of V_m is a dihedral group $D_m = Z/mZ \quad Z/2Z$. Note that V_l is an irreducible D_m-module when $m \nmid 2l$. Hence for any affine variety Y satisfying the assumptions in Theorem 5.3, $\mathrm{VEC}_G(V_m \times Y, V_l)$ is trivial for a positive integer l such that $m \nmid 2l$.

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