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SELF MINI-INJECTIVE RINGS

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Let R be a ring. We have studied rings whose projective modules have the extending property of simple modules in [3] and [5]. In this note, we shall further study those rings when R is an artinian ring and give some relations between those rings and mini-injectivity (see §1).

If R is a QF-ring [8], every projective has the extending property of direct decompositions of the socle [3]. In order to characterize artinian rings with above property, we have defined the condition (**2) in [3]. We shall introduce new concepts: (weakly) mini-injective module and (weakly) uni-injective module. We shall show, for a left and right artinian ring R, that R is a QF-ring if and only if R is mini-injective as a both left and right R-module and if and only if R is uni-injective as a right R-module and right QF-2. When R is right artinian, we shall show that the above extending property for right R-projectives is valid if and only if R is right QF-2 and right R mini-injective.

We can consider the dual property, namely the lifting property of simple modules. However, when R is right artinian, every R-projective P has the lifting property of simple modules and further the lifting property of direct decompositions of P/J(P) [5], where J(P) is the Jacobson radical of P.

1 Definitions

Throughout this note, R is a ring with identity and every module M is a unitary right R-module. We shall denote the Jacobson radical, an injective envelope and the socle of M by J(M), E(M) and S(M), respectively. If for any simple (resp. uniform) submodule A of M there exists a (completely indecomposable) direct summand M_1 of M such that $S(M_1)=A$ (resp. A is an essential submodule of M_1), then we say that M has the extending property of simple modules (resp. uniform submodules). Futhermore, if for any direct decomposition of S(M): $S(M)=\sum_I \oplus A_\alpha$ (resp. any independent set of uniform submodules B_α such that $\sum_I \oplus B_\alpha$ is essential in M) there exists a direct decomposition M= $\sum_I \oplus M_\alpha$ of M such that $S(M_\alpha)=A_\alpha$ (resp. B_α is an essential submodule in M_α) for all $\alpha \in I$, then we say that M has the extending property of direct decompositions M. HARADA

of S(M) (resp. direct sum of uniform submodules).

In this note, we consider only artinian rings and so from now on we understand that a ring R is always right artinian. We note that most results in this note are true for left and right perfect rings. Let

$$R = \sum_{i=1}^{n} \sum_{i=1}^{p(i)} \bigoplus e_{ij} R$$

be the standard decomposition, namely the e_{ij} are primitive idempotents and $e_{ij}R \approx e_{i1}R$, $e_{j1}R \approx e_{i1}R$ if $i \neq j$. If $S(e_{i1}R)$ is simple for each *i*, then we say *R* is right QF-2 [3] and [9]. If E(R) is right *R*-pojective, *R* is called a right QF-3 ring [7] and [9]. Finally if $e_{i1}R$ is a serial module for each *i*, we call *R* a right generalized uniserial ring [8] and [5].

First we shall generalize the concept of injectivity. Let M be an R-module and I a right ideal in R. We take an R-homomorphism f of I to M. Put $M_1 = \inf f$ and consider a diagram:



We shall introduce two conditions.

(I) There exists $h \in \text{Hom}_R(R, M)$ such that hi = f.

(II) There exists either $h \in \operatorname{Hom}_R(R, M)$ or $h' \in \operatorname{Hom}_R(M, R)$ such that hi=f or $if^{-1}=h' | M_1$ provided f is an monomorphism.

If M satisfies (I) (resp. (II)) for every minimal right ideal I in R and any f in Hom_R(I, M), we say R is *right* (resp. *weakly*) *mini-injective*. Similarly if M satisfies (I) (resp. (II)) for every uniform right ideal I in R and any f in Hom_R(I, M), then we say M is right (resp. *weakly*) *uni-injective*.

It is clear that every injective is uni-injective and uni-injective is miniinjective. The converse is not true in general (see Example 5 below). Every semi-simple module is weakly mini-injective, but not mini-injective. If R is a right QF-2 ring, every uni-injective is injective (see the proof of 7) \rightarrow 1) in Theorem 13 below).

2 Mini-injective modules

We shall study some elementary properties of the mini-injective modules. From the definitions and the standard argument [1], we have

Proposition 1. Let M be an R-module and $M=M_1\oplus M_2$. Then

1) M is mini-injective (resp. uni-injective) if and only if so is each M_i .

2) If M is weakly mini-injective (resp. weakly uni-injective), then so is each M_i .

Theorem 2. Let R be a right artinian ring and M an R-module. Then M is mini-injective (resp. uni-injective) if and only if any minimal (resp. uniform) right ideal I in e_iR and any f in $\operatorname{Hom}_R(I,M)$, f is extendable to an element in $\operatorname{Hom}_R(e_iR, M)$, where e_i runs through all primitive idempotents.

Proof. "If" part. First we take a minimal right ideal I in $R = \sum_{i=1}^{n} \bigoplus e_i R$. Let f be in $\operatorname{Hom}_R(I, M)$ and $\pi_i \colon R \to e_i R$ projection. We may assume $I_i = \pi_i(I) \neq 0$ for $i \leq \text{some } t$ and $I_j = 0$ for j > t. Since $\pi_1 \mid I$ is an monomorphism, put $f_1 = f(\pi_1 \mid I)^{-1}$. Then there exists F_1 in $\operatorname{Hom}_R(e_1 R, M)$ such that $F_1 \mid I_1 = f_1$ by the assumption. Put $F_j = 0$ ($\in \operatorname{Hom}_R(e_j R, M)$) for $j \neq 1$ and $F = \sum F_i$. Let x be in I and $x = \sum_{i=1}^{t} \pi_i(x)$. Then $F(x) = \sum F_i \pi_1(x) = f(\pi_1 \mid I)^{-1} \pi_1(x) = f(x)$. If I is uniform, $\bigcap_i \ker(\pi_i \mid I) = 0$ implies that some $\pi_i \mid I$ is an monomorphism. Hence, we can use the same argument in this case, too.

3 Self mini-injective rings

Let R be a right artinian ring. We assume that every idempotent in this note is always primitive and we denote it by e. We put $R/J = \overline{R}$ and \overline{e} means the residue class of e in \overline{R} , where J = J(R).

First we shall study the extending property for R-projectives.

Theorem 3. Let R be right artinian. Then

1) Every projective has the extending property of simple modules if and only if R is right QF-2 and R is weakly mini-injective as a right R-module (cf. [3], Theorem 2).

2) Every projective has the extending property of direct decompositions of the socle if and only if R is right QF-2 and mini-injective as a right R-module.

Proof. 1) We assume that every projective has the extending property of simple modules. Then R is right QF-2. Let $R = \sum_{i=1}^{m} \bigoplus e_i R$ with e_i primitive and let $\pi_i \colon R \to e_i R$ be the projection. We take two minimal right ideals K_1 and K_2 and assume $f \colon K_1 \to K_2$ is an isomorphism. We assume $K_i \subseteq \sum_{j=1}^{l_i} \bigoplus I_{i,j(i)}$, where $I_{ij(i)} = \pi_{j(i)}(K_i) \neq 0$. Since $I_{ij(i)} \approx K_1$ for all i, j, from [6], Corollary 8 we can find minimal one among $e_{j(i)}R$ with respect to the order $<^*$ in [6], say $e_{j(i)}R = e_1R$ and i=1. We consider $p_k = \pi_k f \pi_1^{-1} \colon I_{11} \to e_k R$. If $k \in \{2(1), 2(2), \cdots, 2(t_2)\}$, $p_k = 0$. Hence, since e_1R is minimal, there exists $F_k \in \text{Hom}_R(e_1R, e_kR)$.

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such that $F_k | I_{11} = p_k$ by [6]. Corollary 8. Put $h = (\sum_{k=1}^m F_k) \pi_1 \in \operatorname{Hom}_R(R, R)$. Then $h | K_1 = ((\sum_k F_k) \pi_1) | K_1 = (\sum_k \pi_k f) | K_1 = f$. If the minimal one above $e_{j(i)}R$ is equal to $e_{j(2)}R$, we take f^{-1} in the above. Then we can find $h' \in \operatorname{Hom}_R(R, R)$ such that $h' | K_2 = f^{-1}$. The converse is clear from [6], Corollary 8.

2) We can similarly show it by making use of [6], Corollary 20 instead of Corollary 8.

Let $S(R) = \sum_{i=1}^{k} \bigoplus S_i$ and the S_i simple. If $S_1 \approx S_j$ for any $j \neq 1$, S_1 is called *isolated*. From the similar argument to the above we have

Theorem 3' Let R be as above. Then R has the extending property of direct decompositions of the socle (resp. of simple modules) as a right R-module if and only if R is right QF-2 and (I) (resp. (II)) is satisfied for non-isolated minimal right ideals.

For the uni-injective case, we have

Theorem 4. Let R be right artinian. Then

1) Every projective has the extending property of uniform submodules if and only if R is right QF-2 and weakly uni-injective as a right R-module.

2) Every projective has the extending property of direct sums of uniform submodules if and only if R is right QF-2 and uni-injective as a right R-module.

Proof. First we note that every uniform submodule in a projective module P is finitely generated. Let $P = \sum_{i} \bigoplus P_{\alpha}$ and $P_{\alpha} \approx e_{i(\alpha)}R$ and U a uniform submodule. Let $x \neq 0$ be in U. Then $x = \sum_{i=1}^{n} p_{\alpha_i}; p_{\alpha_i} \in P_{\alpha_i}$. Hence, $U \cap \sum_{i=1}^{n} \bigoplus P_{\alpha_i} \neq 0$ and so $U \cap \sum_{I = \lfloor \alpha_i \rfloor} \bigoplus P_{\beta} = 0$. Accordingly, U is isomorphic to a submodule of $\sum_{i=1}^{n} \bigoplus P_{\alpha_i}$. Furthermore, $U \approx \pi_i(U)$ for some i, where $\pi_i: P \rightarrow P_{\alpha_i}$ is the projection. Therefore, we can apply the same argument given in the proof of Theorem 3 by making use of [6], Theorems 10 and 22.

Next we shall study self (resp. weakly) mini-injective rings.

Theorem 5. Let R be right artinian and mini-injective as a right R-module. Then

1) If $e_1R \approx e_2R$, no minimal submodule in e_1R is isomorphic to any minimal one in e_2R .

2) $S(e_1R) = e_1J^k$ and every minimal submodule in e_1R is isomorphic to one another.

3) $r(J) \supseteq 1(J)$ and J = Z(R). Where J = J(R), the e_i are primitive idempotents, $r(J) = \{x \in R \mid Jx = 0\}$ and

 $1(J) = \{x \in R \mid xJ = 0\}$. Z(R) is the right singular ideal of R.

Proof. Let $e_1R \approx e_2R$ and I_i a minimal right ideal in e_iR for i=1, 2. If $I_1 \approx I_2$, there exist y in $e_2Re_1=e_2Je_1$ and z in e_1Je_2 such that $I_2=yI_1$, $I_1=zI_2$ by the assumption. Hence, $I_1=zyI_1$ and $zy \in J$, which is a contradiction. Therefore, $\{I_i\}_{i=1}^{n}$ is the representative set of minimal R-modules. Let S be a minimal right ideal in e_1R . Then S must be isomorphic to I_1 from the above. Let $e_1J^k \neq 0$ and $e_1J^{k+1}=0$. We take a minimal right ideal K in e_1J^k . Since $K \approx S$, there exists x in e_1Re_1 such that $S=xK \subseteq e_1J^k$. Hence, $S(e_1R)=e_1J^k$. We have obtained 1) and 2).

3) We take I_1 in $S(e_1R)$. Let $I_1 = xR$ and $x \in e_1R$. Now $Jx \subseteq \sum_{i=1}^{m} e_i Jx = \sum_{i=1}^{m} e_i Je_1 x$. If $e_j R \approx e_1 R$, $e_j Je_1 x R = 0$ by 1). If $e_j R \approx e_1 R$, we take z in $e_1 Re_j$ which induces an isomorphism of $e_j R$ to $e_1 R$. Then $ze_j Je_1 x R \subseteq e_1 Je_1 x R = 0$ by 2). Hence, $e_j Je_1 x R = 0$. Therefore, Jx = 0 and $1(J) = S(R_R) \subseteq r(J)$. Furthermore, $Z(R) = \{x \subseteq R \mid x 1(J) = 0\} \supseteq J$ and so Z(J) = J, since every ideal properly containing J contains a projective submodule.

Proposition 6. Let R be a right artinian ring. Then R is mini-injective as a right R-module if and only if R is weakly mini-injective as a right R-module and $1(J) \subseteq r(J)$.

Proof. "If" part. We assume $I_1 \approx I_2$ for minimal right ideals I_i in $e_i R$. Then there exists an element x in either $e_1 R e_2$ or $e_2 R e_1$ which induces an isomorpism between I_1 and I_2 . Hence, $x \notin J$ by the assumption. Therefore, x induces an isomorphism between $e_1 R$ and $e_2 R$. Accordingly, R is mini-injective for $\operatorname{Hom}_R(e_i R, e_j R) = e_j R e_i$. The converse is clear from Theorem 5.

Similarly to the above

Proposition 7. Let R be right artinian. Then R is uni-injective as a right R-module if and only if R is weakly uni-injective as a right R-module and $1(J) \subseteq r(J)$.

Proof. Since uni-injective is mini-injective, the "Only if" part is clear from Theorem 5. Let U_i be a uniform submodule of e_iR and $f: U_1 \rightarrow U_2$ a homomorphism. If ker $f \neq 0$, f is extendable to an element in $\operatorname{Hom}_R(e_1R, e_2P)$ by the assumption. We assume ker f=0. We know from Proposition 6 that R is mini-injective as a right R-module. Hence, $e_1R \approx e_2R$ by Theorem 5. Therefore, f and f^{-1} are extendable to elements in $\operatorname{Hom}_R(e_1R, e_2R)$ and $\operatorname{Hom}_R(e_2R, e_1R)$, respectively. Thus R is uni-injective by Theorem 2.

The author can not find an artinian ring which is self mini-injective but not self uni-injective

We consider algebras over a field.

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Proposition 8. Let K be a field and R a K-algebra with finite dimension. If R is mini-injective as a right R-module, then R is right QF-2.

Proof. Let I_1 be a minimal right ideal in e_1R , where e_1 is primitive. We assume $I_1 \approx \overline{e_2'R}$. Since $I_1 \subseteq e_1 J^{k_1}$ and $e_1 J^{k_1+1} = 0$, each element in $\overline{e_1Re_1}$ gives an element in $\operatorname{Hom}_{R}(I_{1}, e_{1}R) (= \operatorname{Hom}_{R}(I_{1}, S(e_{1}R)))$ via the left multiplication and $\operatorname{Hom}_{R}(I_{1}, e_{1}R) = \overline{e_{1}Re_{1}}$ as a K-module by the assumption. Put $I_{1} = x\overline{e_{2}}'R$ and consider an isomorphism f of I_1 by setting f(x) = xa for $a \in \overline{e_2'Re_2'}$. Then f is extendable to an element in $\operatorname{Hom}_{\mathbb{R}}(e_1\mathbb{R}, e_1\mathbb{R})$ by the assumption. Hence, xa=bxfor some b in e_1Re_1 . This relation gives us a K-monomorphism of $e_2'Re_2'$ to $\overline{e_1Re_1}$. Hence, $[\overline{e_1Re_1}:K] \ge [\overline{e_2'Re_2'}:K]$. Repeating those arguments, we obtain a chain of primitive idempotents $e_1, e_2', \dots, e_i', \dots$ such that a minimal right ideal I_i in $e_i'R$ is isomorphic to $\overline{e_{i+1}'R}$ and $[\overline{e_i'Re_i'}:K] \ge [\overline{e_{i+1}'Re_{i+1}'}:K]$. We may assume $e_i'R \approx e_j'R$ for some i < j. Then $I_{i-1} \approx \overline{e_i'R} \approx \overline{e_i'R} \approx I_{j-1}$. Hence, $e_{i-1}'R \approx I_{i-1}$. $e_{j-1}'R$ by Theorem 5. Therefore, $e_1R \approx e_k'R$ for some k. Accordingly, $[\overline{e_1Re_1}:K] = [\overline{e_2'Re_2'}:K] = [\overline{e_k'Re_k'}:K]. \text{ Hence, } \operatorname{Hom}_R(I_1, e_1R) = \overline{e_2'Re_2'}. \text{ Let } S \text{ be}$ a minimal right ideal in e_1R . Then there exists b in $\overline{e_1Re_1}$ such that $bI_1 = S$ by the assumption. However, since $\operatorname{Hom}_{\mathbb{R}}(I_1, e_1\mathbb{R}) = \overline{e_2' R e_2'}$ as above, there exists a in $e_2'Re_2'$ such that bx = xa. Hence, $S = bI_1 = bxR = xaR \subseteq I_1$. Therefore, $S(e_1R)$ is simple.

REMARK. If $\operatorname{End}_{R}(eR)$ is given by the multiplication of the central elements in R for each idempotent e, Proposition 8 is valid for such artinian rings from the above proof.

Proposition 9. Let R be a K-algebra as above. We assume [eRe: K] = [e'Re': K] for any primitive idempotents e and e'. Then every projective has the extending property of simple modules (resp. direct decompositions of the socle) if and only if R is right QF-2 and if $S(e_1R) \approx S(e_2R)$, either $e_2RS(e_1R) = S(e_2R)$ or $e_1RS(e_2R) = S(e_1R)$ (resp. $e_2RS(e_1R) = S(e_2R)$), where the e_i are primitive.

Proof. "If" part. Since R is right QF-2, $e_1Je_1S(e_1R)=0$. Hence, $I_1=$ $S(e_1R)$ is a left $\overline{e_1Re_1}$ -module. We assume $I_1 \approx e_2\overline{R}$ and so $\operatorname{End}_R(I_1) \approx \overline{e_2Re_2}$. Since I_1 is a left $\overline{e_1Re_1}$ -module, each element x in $\overline{e_1Re_1}$ induces an element in $\operatorname{End}_R(I_1)$ by the left multiplication. Now, $[\overline{e_1Re_1}:K]=[\overline{e_2Re_2}:K]$ from the assumption. Hence, we may assume $\operatorname{End}_R(I_1)=\overline{e_1Re_1}$. Let $I_3=S(e_3R)$ and $I_3\approx I_1$. If $e_3RI_1=I_3$, $yI_1=I_3$ for some $y\in e_2Re_1$. Then $g:I_1\to I_3$ given by setting $g(x)=yx; x\in I_1$ is an isomorphism. Let f be any isomorphism of I_1 to I_3 . Then $g^{-1}f\in \operatorname{End}_R(I_1)=\overline{e_1Re_1}$. Hence, f(x)=yzx for some z in e_1Re_1 . Therefore, f is extendable to an element in $\operatorname{Hom}_R(e_1R, e_3R)$. Thus, every projective has the extending property of simple modules (resp. direct decompositions of the socle)

by [3], Theorem 2 (resp. [6], Corollary 20).

Since the extending property is preserved by Morita equivalence, if R/J is a simple ring, we may assume R is a local ring.

Proposition 10. Let R be a right artinian and local ring. Then every projective has the extending property of uniform submodules if and only if R is a QFring.

Proof. If R has the extending property, R is right QF-2. Since every projective is a direct sum of copies of R, R is a QF-ring by [6], Theorem 10.

Proposition 11. Let R be a right artinian and local ring. We assume that every monomorphism of R/J into itself as a field is an isomorphism. Then every projective has the extending property of simple modules (and hence of direct decompositions of the socle) if and only if R is right QF-2.

Proof. "If" part. Since R is local QF-2, S(R)=I is a unique minimal right ideal and a left ideal in R. Let I=xR. Then since JI=0, for any element a in \overline{R} , there exists b in \overline{R} such that ax=xb. Hence, the correspondence $\sigma: a \rightarrow b$ gives us a monomorphism of \overline{R} into \overline{R} . Therefore, σ is onto by the assumption, which means that R is right mini-injective. Accordingly, every projective has the extending property of direct decompositions of the socle by Theorem 3.

Finally we shall give an additional result to [5].

Proposition 12. Let R be a right artinian, generalized uniserial and right QF-3 ring. Then every R-projective module has the extending property of simple modules.

Proof. Let $S(R) = \sum_{i=1}^{m} \bigoplus S_i$ and $S_i = S(e_i R)$. We assume $S_1 \approx S_2 \approx \cdots \approx S_i$ and $S_j \approx S_1$ for j > i. Since R is right QF-3, $E(S_1)$ is isomorphic to some $e_k R$. Hence, $e_p R$ is isomorphic to some submodule of $e_k R$ for $p \leq i$. Now $e_k R$ is serial and injective by the assumption. Hence, each submodule of $e_k R$ is a character submodule and $End_R(S_k)$ is extendable to $End_R(e_k R)$. Therefore, every R-projective has the extending property of simple modules by [3], Theorem 2.

4 QF-rings

We shall give some characterizations of QF-rings in terms of extending property of projectives.

Theorem 13. Let R be left and right artinian. Then the following condi-

tions are equivalent.

1) R is a QF-ring.

2) Every right (and left) R-projective has the extending property of direct decompositions of the socle.

3) Every right R-projective has the extending property of direct decompositions of the socle and $r(J) \subseteq 1(J)$.

4) Every right R-injective E has the lifting property of direct decompositions of E|J(E) and R is a right QF-2 (see [4]).

5) R is right and left QF-2 and mini-injective as a right R-module.

6) R is mini-injective as a left and right R-module.

7) R is uni-injective as a right R-module and right QF-2.

Proof. $1 \rightarrow 2 \rightarrow 7$, $2 \rightarrow 1$ and $5 \rightarrow 1$. They are clear from Theorems 3 and 5, [2], Theorem 3, [3], Theorem 2 and [8].

3) \rightarrow 1). It is sufficient to show that R is left QF-2, since R is right QF-2 and R-mini-injective by Theorem 3. We take a unique minimal right ideal x_1R in e_1R . We may assume $x_1 \in e_1Re_2'$ as the proof of Proposition 8. Since $r(J) \supseteq 1(J)$, $Jx_1=0$. Hence, Rx_1 is semi-simple. On the other hand, since $Rx_1=Re_1x_1$, Rx_1 is a minimal left ideal in Re_2' . Let Rx_2 be another minimal one in Re_2' and $x_2 \in e_3'Re_2'$. Then $S(e_1R) = x_1R \approx \overline{e_2'R} \approx x_2R = S(e_3'R)$ since $r(J) \subseteq 1(J)$ by the assumption. Hence, $e_1R \approx e_3'R$ by Theorem 5. Noting that x_1R is minimal, we obtain an isomorphism $f: x_1R \rightarrow x_2R$ with $f(x_1) = x_2$. f is extendable to an element $y \in \text{Hom}_R(e_1R, e_3'R)$ by [6], Corollary 20. Hence, $x_2 = yx_1$ and so $Rx_2 = Rx_1$. The above correspondence $e_1 \rightarrow e_2'$ gives a permutation of the set $\{e_{i1}\}_{i=1}^{n-1}$ by Theorem 5. Hence, R is left QF-2.

4) \rightarrow 1). We know from [2], Theorem 3 that there exists the representative set $\{e_{i1}R/e_{i1}A_{ij}\}_{i=1}^{k} \stackrel{\kappa(i)}{j=1}$ of indecomposable injectives. Since R is artinian, $\kappa(i)=1$ for all *i* by [4], Theorem 2. $e_{i1}R$ is uniform by the assumption. Hence, $E(e_{i1}R)\approx e_{j1}R/e_{j1}A_{j1}$ for some *j*. We consider a diagram, where $e_k=e_{k1}$, $A_k=A_{k1}$ and φ is the natural epimorphism:

$$0 \longrightarrow e_i R \longrightarrow E(e_i R) \approx e_j R/e_j A_j$$

$$\downarrow \varphi$$

$$e_i R/e_i A_i \qquad \qquad h$$

Since $e_i R/e_i A_i$ is injective, we obtain an epimorphism $h: e_j R/e_j A_j \rightarrow e_i R/e_i A_i$. Hence, i = j and $e_i A_i = 0$. Since $\kappa(i) = 1$ for all i, p = n. Therefore, $R = \sum_{i=1}^{n} \sum_{j=1}^{p(i)} \bigoplus e_{ij}R$ is self injective as a right *R*-module. $(0) \rightarrow 1$). We assume that *R* is self mini-injective. Let *xR* be a minimal right ideal in $e_i R$, where e_1 is primitive. Then $xR = xe_2'R$ and $x \in e_1Re_2'$. Since Jx = 0by Theorem 5, *Rx* is minimal in Re_2' as above. Therefore, for any element *b*

in $\overline{e_1Re_1}$ there exists a in $\overline{e_2'Re_2'}$ such that bx=xa as the proof of Proposition 8 for R is left mini-injective. Again using the same argument, we know $xR=S(e_1R)$. Hence, R is QF-2. Therefore, R is a QF-ring by Theorem 5 and [8].

7) \rightarrow 1). We shall show that *R* is self-injective as a right *R*-module. We can use the standard argument [1]. Let *I* be a right ideal in *R* and $f \in \operatorname{Hom}_R(I, R)$. We can find a maximal one among the set of extensions of *f* by Zorn's Lemma, say $(I_0, f_0: I_0 \rightarrow R)$. We assume $I_0 \neq R$. Then there exists a primitive idempotent *e* such that $e \notin I_0$. Put $K = eR \cap I_0$ and $I_1 = I_0 + eR$. We take an extension f_1 of $f_0 \mid K$ from the assumption. We put $g(x) = f_0(x_1) + f_1(er)$, where $x_1 \in I_0$ and $r \in R$. Then $g \in \operatorname{Hom}_R(I_1, R)$, which contradicts the assumption of I_0 . Hence, $I_0 = R$ and *R* is self-injective.

Theorem 14. Let R be a K-algebra with $[R:K] < \infty$. Then the following conditions are equivalent.

1) R is a QF-ring.

2) R is mini-injective as a right R-module and r(J)=1(J).

3) R is uni-injective as a right R-module.

Proof. It is clear from Proposition 8 and Theorem 13.

5 Examples

Let K be a field.

1. Put

$$R = \begin{pmatrix} K & 0 & 0 \\ K & K & 0 \\ K & 0 & K \end{pmatrix}.$$

Then $\operatorname{Hom}_R(S(e_{22}R), S(e_{33}R))$ is not extendable to $\operatorname{Hom}_R(e_{22}R, e_{33}R)$. Hence, R is right generalized uniserial, but does not have the extending property of simple modules as a right R-module (cf. Proposition 12).

2. We shall give an example, where artinian and right self mini-injective rings are not right QF-2 in general. Let x be an indeterminate and Q a field. Put L=Q(x) and $K=Q(x^2)$. Then we have an isomorphism σ of L onto K and [L: K]=2. Let $R=L1\oplus Lu$ be a left vector space over L. We put $(Lu)^2=0$ and $ul=\sigma(l)u$ for $l\in L$. Then R is a ring and [R: L]=2 as a left L-module and [R: L]=3 as a right L-module. Hence, R is a left and right artinian ring. J=Lu contains minimal right ideals Ku and xKu. Let I be a minimal right ideal in J. Then I=aL; a=lu and $End_R(Ku)=K$. Therefore, R is self right mini-injective (and uni-injective). We note that $End_R(J)$ as a left R-module \cong {the right multiplications of R} and R is left QF-2. Furthermore, R satisfies the conditions in Theorem 5 as a left R-module. However, R is not left miniinjective (cf. Theorems 13 and 14).

In case of QF-rings, right artinian and right self-injective rings satisfy the same conditions on the left side. However, this fact is not true for self mini-injective rings from this example.

3. Let K and L be as in Example 2. Put

$$R = \begin{pmatrix} L & L \\ 0 & L \end{pmatrix}.$$

Then R is right weakly mini-injective. However R is not right QF-2 and hence not right mini-injective. $e_{22}R$ is weakly uni-injective, but not mini-injective. (cf. Proposition 8).

4. Put

$$R = \left\{ \begin{pmatrix} a & b & c \\ o & d & e \\ o & o & a \end{pmatrix} \middle| a \sim e \in K \right\}.$$

Then R is weakly mini-injective but not weakly uni-injective for $f: e_{11}R \rightarrow e_{11}J^2$ is not extendable.

5. Put

$$R = \begin{pmatrix} K & uK + vK & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$$

and $e_{12}(uk_1+vk_2)e_{23}k_3=e_{13}(k_1k_3+k_2k_3)$ for $k_i \in K$. Then $e_{11}R$ is mini-injective. On the other hand, $e_{11}R$ contains two isomorphic uniform modules (0, uK, K), (0, vK, K). The above isomorphism is not extendable to an element in $\operatorname{Hom}_R(e_{11}R, e_{11}R)$. Hence $e_{11}R$ is not uni-injectcetve.

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