Kawanaka, N.
Osaka J. Math.
10 (1973), 1–13

A THEOREM ON FINITE CHEVALLEY GROUPS

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(Received June 28, 1972)

Introduction

In 1955, J.A. Green [5] gave a description of all the irreducible complex characters of the general linear group $GL_n(k)$ with coefficients in a finite field k. In particular, he proved an interesting formula for the inner product of two class functions on $GL_n(k)$. This formula is very suggestive and may be regarded as an analogue of Weyl's integration formula used in the character theory of compact semisimple Lie groups. However, there exists a disadvantage in the Green's method. Namely, it is too combinatorial theoretic and there seems no direct way of relating it to the structure of GL_n as a linear algebraic group defined over k. The purpose of this paper is to prove a general inner product formula for certain type of class functions on a finite Chevalley group $G(k)$ and to show, when $G=GL_n$, that this provides a new interpretation and proof of the Green's formula.

The main contents of the paper are as follows. In section 1 and 2 we recall some known results on class functions on finite groups and reductive linear algebraic groups. In section 3 we prove the main theorem (Theorem 3.1) mentioned above. Section 4 is devoted to prove a key lemma (Lemma 3.3). In the proof, the following theorem due to R. Steinberg [13] plays an important role:

Let G be a connected reductive linear algebraic group defined over k. Then the number of maximal tori of G defined over k equals to that of unipotent elements of G defined over k.

In sectoin 5 we consider the special case $G=GL_n$ and show that the inner product formula of Green follows easily from Theorem 3.1.

It can be conjectured (see, for example, [6] [7] [8]) that a similar formula exists for general G. Our main theorem in section 3 may be considered as a first step in this direction.

Notations

For a group G and a subset X of G, $Z_G(X)$, $N_G(X)$ denote the centralizer and normalizer of X in G and $C_G(X)$ the G-conjugacy class of X. If G is a linear algebraic group defined over a finite field k, $G(k)$ denotes the finite group of its k-rational elements and G_0 the identity component of G. $Z_{G(k)}(X)$, $N_{G(k)}(X)$ denote the centralizer and normalizer of X in $G(k)$ and $C_{G(k)}(X)$ the $G(k)$-conjugacy class of X. Finally, if S is a finite set, $|S|$ denotes the number of its elements.

1. Class functions on finite groups

Let G be a finite group. $A(G)$ denotes the complex vector space of all complex valued class functions on G. The inner product in $A(G)$ is defined by

$$(\chi_1, \chi_2)_G = \frac{1}{|G|} \sum_{x \in G} \chi_1(x) \overline{\chi_2(x)} \qquad (\chi_1, \chi_2 \in A(G)) .$$

Let H be a subgroup of G. We define the induction map $\psi \to i_{H \to G}[\psi]$ of $A(H)$ into $A(G)$ in the usual manner:

$$(1.1) \qquad (i_{H \to G}[\psi])(x) = \frac{|Z_G(x)|}{|H|} \sum_{x'} \psi(x') \qquad (x' \in C_G(x) \cap H) .$$

2. Reductive linear algebraic groups

Until the end of section 4, unless otherwise stated, G denotes a connected reductive linear algebraic group defined over a finite field k. All the lemmas in this section are well known.

Lemma 2.1. *Let G be as above. Then G has a Borel subgroup B defined over k and B always contains a maximal torus T_1 of G defined over k. Any two such couples (B, T_1) are conjugate by an element of $G(k)$.*

This is proved in [9; 2.9].

DEFINITION 2.1. G is said to be of *Chevalley type* if T_1 in Lemma 2.1 splits over k.

Lemma 2.2. *Let C be Cartan subgroup of G, i.e. the centralizer group $Z_G(T)$ of a maxiaml torus T of G. Then $C = T$.*

For a proof see [1; p.316].

Lemma 2.2. *Let s be a semisimple element of G. Then a maximal torus T contains s if and only if T is contained in $Z_G(s)_0$.*

Proof. By Lemma 2.2, $Z_G(T) = T$. Hence T contains s if and only if $Z_G(T) \ni s$. This means $Z_G(s) \supset T$. By the connectedness of T, this is equivalent to $Z_G(s)_0 \supset T$.

A closed subgroup P of G is called parabolic if it contains a Borel subgroup. As is well known P is connected and

$$(2.1) \qquad N_G(P) = P ,$$

Lemma 2.4. *Let P_1 and P_2 be parabolic subgroups of G defined over k. Then $P_1 \cap P_2$ is a connected subgroup of G defined over k and contains a maximal torus of G.*

This is proved in [2; 4.5, 4.6].

Lemma 2.5. *Let G be a connected semisimple group defined over k and s a semisimple element of $G(k)$.*

(a) *The identity component $Z_G(s)_0$ of the centralizer group of s in G is a connected reductive group defined over k.*

(b) *$Z_G(s)_0$ contains all of the unipotent elements of $Z_G(s)$.*

These are proved in [13; 9.4, 15.4].

Lemma 2.6. *Let G be a connencted semisimple group defined over k. Let x be an element of $G(k)$, and s and u its semisimple and unipotent parts. If P is a parabolic k-subgroup of G containing x, $P \cap Z_G(s)_0$ is a parabolic k-subgroup of the connected reductive k-group $Z_G(s)_0$ containing u.*

Proof. It is sufficient to prove the assertion for the case when P is a Borel subgroup. This case is proved in [11; 3.6].

The number of $G(k)$-conjugacy classes of maximal tori defined over k is finite. In particular, if G is of Chevalley type there is a bijection between the $G(k)$-conjugacy classes of maximal tori and the conjugacy classes of the Weyl group $W = N_G(T)/T$ (T a maximal torus) of G. See [9] and [10] for the details. Here we quote only the following result.

Lemma 2.7. *Let G be of Chevalley type and T a maximal torus of G defined over k and W_T the finite group defined by $W_T = N_{G(k)}(T)/T(k)$. If c_T is the conjugacy class of W corresponding to T, $W_T \cong Z_W(w)$ ($w \in c_T$). In particular, if T_1 is as in Lemma 2.1, $W_{T_1} \cong W$.*

The number of $G(k)$-conjugacy classes of parabolic k-subgroups is also finite. Let G be of Chevalley type and B, T_1 be as in Lemma 2.1. B determines a simple roots system of G with respect to T_1. Then, as is well known, the parabolic k-subgroups containing B are in one-to-one correspondence with the subgroups of $W = W_{T_1}$ generated by simple reflections. Moreover, an arbitrary parabolic k-subgroup is $G(k)$-conjugate to one and only one such parabolic subgroup.

3. An inner product formula

The main purpose of this section is to prove Theorem 3.1 and discuss briefly the connections with the works of other authors'. Before stating the theorem, we first introduce some notations. For $x \in G$, x_s and x_u denote its semisimple and unipotent parts, respectively. Let P be a parabolic k-subgroup of G and ϕ an element of $A(P(k))$, i.e. a class function on $P(k)$. For $Q \in C_{G(k)}(P)$ we define $\phi^Q \in \mathrm{A}(Q(k))$ by

$$\phi^Q(x) = \phi(gxg^{-1}) \qquad (x \in Q(k)) \tag{3.1}$$

if $Q = g^{-1}Pg (g \in G(k))$. By (2.1) this is well defined. Define a subspace of $A(P(k))$ by

$$B(P(k)) = \{\phi \in A(P(k));\ \phi(x) = \phi(x_s)\ (x \in P(k))\} .$$

Let ϕ be an element of $B(P(k))$ and T a maximal torus of G defined over k. Then we define a function on T(k):

$$\phi_T^P(t) = \textstyle\sum_{Q \supset T} \phi^Q(t), \tag{3.2}$$

where the summation is over the set of all parabolic k-subgroups $Q \in C_{G(k)}(P)$ containing T. If there is no such parabolic subgroup ϕ_T^P is defined to be identically zero. Clearly, ϕ_T^P is invariant under the action of $W_T = N_{G(k)}(T)/T(k)$. Let $T_1 \cdots, T_m$ be a set of representatives of $G(k)$-conjugacy classes of maximal tori of G defined over k. For brevity, we write W_i for W_{T_i}.

Theorem 3.1. *Let P and P' be parabolic k-subgroups of G and ϕ and ϕ' be elements of $B(P(k))$ and $B(P'(k))$, respectively. Then*

$$(i_{P(k)\to G(k)}[\phi],\ i_{P'(k)\to G(k)}[\phi'])_{G(k)} = \textstyle\sum_{i=1}^m \dfrac{1}{|W_i|} (\phi_{T_i}^P, \phi'^{P'}_{T_i})_{T_i(k)} , \tag{3.3}$$

where $\phi_{T_i}^P$'s are W_i-invariant functions on T_i (k) defined by (3.2).

Let P be a fixed parabolic k-subgroup of G and let $\mathcal{P}$ be the subset of $C_{G(k)}(P) \times G(k)$ consisting of all pairs (Q, x) such that $Q \ni x$. For $a \in G(k)$ we define the following subsets of $\mathcal{P}$:

$$\mathcal{P}(a) = \{(Q, a) \in \mathcal{P}\}$$

and

$$\mathcal{P}([a]) = \{(Q, x) \in \mathcal{P};\ x \in \mathrm{C}_{G(k)}(a)\} .$$

If (Q, x) is an element of $\mathcal{P}$, (Q, x_s) also belongs to $\mathcal{P}$. Thus we can consider the mapping:

$$f\colon (Q, x) \to (Q, x_s) \tag{3.4}$$

from $\mathcal{P}$ to itself. $G(k)$ acts on $\mathcal{P}([a])$ (hence also on $\mathcal{P}$) by $(g, (Q, x)) \to (gQg^{-1}, gxg^{-1})$.

Lemma 3.1. *Let ϕ be an element of $A(P(k)$. Then*

$$i_{P(k)\to G(k)}[\phi](x) = \sum_{Q\ni x}\phi^Q(x) \qquad (x\in G(k)),$$

where the sum is over the set of all $Q\in C_{G(k)}(P)$ which contain x.

Proof. Let a be an element of $G(k)$. By the definitions of $\mathcal{P}(a)$ and $\mathcal{P}([a])$ we have

$$\begin{aligned}(3.5)\qquad \mathcal{P}([a]) &= \cup_x\mathcal{P}(x) \qquad (x\in C_{G(k)}(a)) \\ &= \cup_Q X(Q, a)(Q\in C_{G(k)}(P)),\end{aligned}$$

where $X(Q, a)=\{(Q, x);\ x\in Q(k)\cap C_{G(k)}(a)\}$. Consider the function:

$$(3.6)\qquad (Q, x) \to \phi^Q(x)$$

defined on $\mathcal{P}$. By (2.1) this function is constant on each $G(k)$-orbit in $\mathcal{P}$. Hence if $x\in C_{G(k)}(a)$

$$\sum_{(Q,x)\in\mathcal{P}(x)}\phi^Q(x) = \sum_{(Q,a)\in\mathcal{P}(a)}\phi^Q(a)$$

and if $Q\in C_{G(k)}(P)$

$$\sum_{(Q,x)\in X(Q,a)}\phi^Q(x) = \sum_{(P,x)\in X(P,a)}\phi^P(x).$$

Hence by (3.5) we see that

$$\begin{aligned}&\sum_{(Q,x)\in\mathcal{P}([a])}\phi^Q(x) \\ &= \frac{|G(k)|}{|Z_{G(k)}(a)|}\{\sum_{(Q,a)\in\mathcal{P}(a)}\phi^Q(a)\} \\ &= \frac{|G(k)|}{|P(k)|}\{\sum_{(P,x)\in X(P,a)}\phi^P(x)\}.\end{aligned}$$

Thus by (1.1)

$$\sum_{(Q,a)\in\mathcal{P}(a)}\phi^Q(a) = i_{P(k)\to G(k)}[\phi](a)$$

as required.

$G(k)$ acts on $f(\mathcal{P})$. Let O be the set of all $G(k)$-orbits in $f(\mathcal{P})$. For a semisimple elements of $G(k)$, $O([s])$ denotes the set of all $G(k)$-orbits in $\mathcal{P}([s])$. Clearly $O=\cup_s O([s])$. Let ϕ be an element of $B(P(k))$. Then the function on $\mathcal{P}$ defined by (3.6) is constant on $f^{-1}(o)$ ($o\in O$). We denote this constant value by ϕ^o. This combined with Lemma 3.1 gives the following lemma.

Lemma 3.2. *Let ϕ be an element of $B(P(k))$. Then*

$$i_{P(k)\to G(k)}[\phi](x) = \sum_{o\in O([x_S])}N(P, x, O)\phi^o,$$

where $N(P, x, o)=|\mathcal{P}(x)\cap f^{-1}(o)|$.

REMARK 3.1. Let ϕ be as above and define $\phi_o \in B(P(k)$ by

$$\phi_o(x) = \begin{cases} \phi(x) & \text{if } (P, x) \in f^{-1}(o) \\ 0 & \text{if } (P, x) \notin f^{-1}(o) . \end{cases}$$

Then $N(P, x, o)\phi^o = \mathrm{i}_{P(k)\to G(k)}[\phi_o](x)$.

Let P' be another parabolic k-subgroup of G. Replacing P with P', we define $\mathcal{P}'$, $\mathcal{P}'(x)$, $\mathcal{P}'([x])$, f', O' and O' ($[s]$) as above.

Lemma 3.3. *Let o and o' be elements of O and O', respectively. Then*

$$\begin{aligned}&\textstyle\sum_{x\in G(k)} N(P, x, o)N(P', x, o')\\ &= \sum_{i=1}^m \frac{|G(k)|}{|W_i|\,|T_i(k)|}\{\textstyle\sum_{t\in T_i(k)} L_o(P, T_i, t)L_{o'}(P', T_i, t)\},\end{aligned}$$

where $L_0(P, T, t) = |\{(Q, t) \in o\,;\ Q \supset T \ni t\}|$.

A proof of this key lemma will be given in the next section. Assuming this, we can now prove Theorem 3.1. By Lemma 3.2 and Lemma 3.3 we see that the left hand side of (3.3) equals to

$$\begin{aligned}&\frac{1}{|G(k)|}\textstyle\sum_{o\in O, o'\in O'}\{\sum_{x\in G(k)} N(P, x, o)N(P', x, o')\}\phi^o\bar{\phi}^{o'} .\\ &= \sum_{i=1}^m \frac{1}{|W_i|\,|T_i(k)|}\cdot\textstyle\sum_{t\in T_i(k)}\{\sum_o L_o(P, T_i, t)\phi^o\}\{\sum_o L_o(P, T_i, t)\}\bar{\phi}'^{o'}\} .\end{aligned}$$

This is the right hand side of (3.3). In fact, by the definitions of $L_o(P, T_i, t)$ and ϕ^o we have

$$\textstyle\sum_{o\in O} L_o(P, T_i, t)\phi^o = \phi_{T_i}^P(t) .$$

In the remainder of this section, we consider some special cases of Theorem 3.1. Let B and T_1 be as in Lemma 2.1 and U the maximal unipotent k-subgroup. Let θ be a character of $T_1(k)$ and extend it to a linear character of $B(k)$ by putting

$$\theta(tu) = \theta(t) \qquad (t \in T_1(k),\ u \in U(k)) .$$

By Lemma 2.1 we see that

$$\theta_{T_1}^B = \textstyle\sum_w \theta^w \qquad (w \in W_{T_1}),$$

where θ^w is the character of $T_1(t)$ defined by $\theta^w(t) = \theta(w^{-1}tw)$. Thus by Theorem 3.1,

$$(i_{B(k)\to G(k)}[\theta],\ i_{B(k)\to G(k)}[\theta])_{G(k)} = \frac{1}{|W_{T_1}|}(\textstyle\sum_w \theta^w,\ \sum_w \theta^w)_{T_1(k)} .$$

This equals to 1 if and only if $\theta \neq \theta^w$ for all $w \in W_{T_1}$. Hence $i_{B(k)\to G(k)}[\theta]$ is irreducible if and only if $\theta \neq \theta^w$ for all $w \in W_{T_1}$. See [12; section 14], where a different proof is given.

Let G be of Chevalley type and P, P' its parabolic k-subgroups. $W(P)$ and $W(P')$ denote the corresponding subgroups of the Weyl group $W = W_{T_1}$ in the sense of section 2. Then

$$(i_{P(k)\to G(k)}[1_P], i_{P'(k)\to G(k)}[1_{P'}])_{G(k)} = (i_{W(P)\to W}[1_{W(P)}], i_{W(P')\to W}[1_{W(P')}])_W,$$

where 1_P, $1_{W(P)}, \cdots$ are functions on P, $W(P), \cdots$ which are identically 1. In fact, this easily follows from Theorem 3.1 and Lemma 3.4(b) given below. More generally a similar formula holds ([3]) for an arbitrary finite group with a (B, N)-pair. Recently, Curtis, Iwahori and Kilmoyer [4] proved deeper results in this direction.

Lemma 3.4. *Let P and T be a parabolic subgroup and a maximal torus of G defined over k. Put*

$$L(P, T) = |\{Q \in C_{G(k)}(P);\ Q \supset T\}|.$$

(a) $$L(P, T) = \frac{|N_{G(k)}(T)|\,|\{S \in C_{G(k)}(\mathrm{T});\ S \subset P\}|}{|P(k)|}$$

(b) *If G is of Chevalley type, the $G(k)$-conjugacy class of P and T determine a subgroup $W(P)$ and a conjugacy class c_T of $W = W_{T_1}$, respectively.* (*See section* 2.) *Then*

$$L(P, T) = i_{W(P)\to W}[1_{W(P)}](w) \qquad (w \in c_T).$$

Proof. (a) can be proved easily. To prove (b), we first note that

$$|\{S \in C_{G(k)}(T);\ S \subset P\}| = \sum_{S_i} \frac{|P(k)|}{|N_{P(k)}(S_i)|},$$

where the sum is over a set of representatives of $P(k)$-conjugacy classes of maximal tori of G contained in $\{S \in C_{G(k)}(T);\ S \subset P\}$. By Lemma 2.7 and its proof given in [10], it is easy to see that this expression is equal to

$$\sum_{v_i} \frac{|P(k)|}{|T(k)|} \frac{1}{|Z_{W(P)}(v_i)|},$$

where the sum is over a set of representatives of the conjugacy classes of $W(P)$ which intersect with c_T. Hence by (a) and Lemma 2.7 we have

$$L(P,\ T)=\frac{|N_{G(k)}(T)|}{|P(k)|}\sum_{v_i}\frac{|P(k)|}{|T(k)|}\frac{1}{|Z_{W(P)}(v_i)|}$$
$$=|Z_W(w)|\frac{|C_W(w)\cap W(P)|}{|W(P)|}\qquad (w\in c_T)$$
$$=i_{W(P)\to W}[1_{W(P)}](w)\,.$$

4. Proof of Lemma 3.3.

As mentioned in the introduction, the proof of Lemma 3.3 depends on the following theorem due to R. Steinberg [13; 14.16, 15.3].

Theorem 4.1. *Let H be a connected reductive linear algebraic group defined over a finite field k of q elements. Let n be the dimension of H, and s and r the dimensions of a Cartan subgroup and a maximal torus, respectively.*

(a) *The number of maximal tori of G defined over k is q^{n-s}.*

(b) *The number of unipotent elements of $G(k)$ is q^{n-r}.*

REMARK 4.1. In [13], part (a) of the above theorem is proved by a combinatorial calculation and part (b) using the values of the Steinberg character. It is desirable to prove these two assertions by a uniform principle.

Lemma 4.1. *Let G be a connected reductive group defined over a finite field k and P_1, P_2 parabolic k-subgroups of G. Then the number of maximal tori of $P_1\cap P_2$ defined over, k equals to that of unipotent elements of $(P_1\cap P_2)$ (k).*

Proof. By Lemma 2.2 and Lemma 2.4, a Cartan subgroup of $P_1\cap P_2$ is a maximal torus of $P_1\cap P_2$. Hence the lemma follows from Theorem 4.1.

Lemma 4.2. *Let G, P, P', O, O' f, f' be as in section* 3. *Fix $o\in O$ and $o'\in O'$. Then*

$$\begin{aligned}(4.1)\qquad &|\{y\in G(k);\ (P,y)\in f^{-1}(o),\ (P',y)\in f'^{-1}(o')\}|\\ &=|\{(T,t)\in \boldsymbol{T}\times G(k);\ P\cap P'\supset T\ni t,\ (P,t)\in o,\ (P',t)\in o'\}|,\end{aligned}$$

where $\boldsymbol{T}$ is the set of all maximal tori of G defined over k.

Proof. By the decomposition of a connected reductive group into the product of its semisimple part and the central torus, we can easily reduce the lemma to the case when G is semisimple. Let t be a semisimple element of $(P\cap P')$ (k). Then by the uniqueness of the Jordan decompositions we have

$$\begin{aligned}&|\{y\in G(k);\ y\in P\cap P',\ y_s=t\}|\\ &=|\{\text{unipotent elements of }(Z_G(t)\cap \mathrm{P}\cap \mathrm{P}')(k)\}|.\end{aligned}$$

This equals to

(*) $|\{\text{unipotent elements of } (Z_G(t)_0 \cap P \cap P'(k)\}|$

by Lemma 2.5(b). By Lemma 2.5 (a) and Lemma 2.6 we can apply Lemma 4.1 to the connected reductive group $Z_G(t)_0$ and its parabolic subgroups $Z_G(t)_0 \cap P$, $Z_G(t)_0 \cap P'$. Then we see that the number (*) is also the number of maximal tori of G which are defined over k and are contained in $Z_G(t)_0 \cap P \cap P'$. Hence using Lemma 2.3 we have

$$(4.2) \qquad |\{y \in G(k); y \in P \cap P', y_s = t\}| = |\{T \in \boldsymbol{T}; P \cap P' \supset T \ni t\}|.$$

The left hand side of (4.1) equals to

$$\textstyle\sum_t |\{y \in G(k): y \in P \cap P', y_s = t\}|,$$

where the sum is over the set of all semisimple elements t for which $(P, t) \in o$ and $(P', t) \in o'$. By (4.2) this equals to the right hand side of (4.1). The lemma is proved.

We can now prove Lemma 3.3. We first remark that

$$\begin{aligned} &N(P, x, o)N(P', x, o') \\ &= |\{(Q, Q', x); (Q, x) \in f^{-1}(o) \cap \mathcal{P}(x), (Q', x) \in f'^{-1}(o') \cap \mathcal{P}'(x)\}| \end{aligned}$$

and

$$\begin{aligned} &L_o(P, T, t)L_{o'}(P', T, t) \\ &= |\{(Q, Q' T, t); Q \cap Q' \supset T \ni t, (Q, t) \in o \cap \mathcal{P}(t), (Q', t) \in o' \cap \mathcal{P}(t)\}|. \end{aligned}$$

Hence

$$\begin{aligned} (4.3) \qquad &\textstyle\sum_{x \in G(k)} \mathrm{N}\,(P, x, o)N(P', x, o') \\ &= |\{(Q, Q', y); (Q, y) \in f^{-1}(o), (Q', y) \in f'^{-1}(o'))\}| \end{aligned}$$

and

$$\begin{aligned} (4.4) \quad &\sum_{i=1}^{m} \frac{|G(k)|}{|W_i|\,|T_i(k)|} \{\textstyle\sum_{t \in T_i(k)} L_o(P, T_i, t)L_{o'}(P, T_i, t)\} \\ &= |\{(Q, Q', S, s); \boldsymbol{T} \ni S \ni s, Q \cap Q' \supset S, (Q, s) \in o, (Q', s) \in o'\}|. \end{aligned}$$

The right hand side of (4.3) equals to that of (4.4) by Lemma 4.2. This proves the lemma.

5. The finite general linear groups $GL_n(k)$

In this section, G denotes the general linear group GL_n considered as a connected reductive algebraic group defined over a finite field k. G is of Chevalley type. In fact, the group T_1 of all diagonal elements in G is a maximal torus of G which splits over k. B denotes the group of all upper triangular elements in G. This is a Borel k-subgroup of G containing T_1. $W=W_{T_1}$ is naturally isomorphic to the symmetric group S_n of degree n.

DEFINITION 4.1. A sequence $\mu=(n_1, n_2, \cdots, n_r)$ of positive integers is called *a partition of n* if $n_1+n_2+\cdots+n_r=n$ and $n_1\geqq n_2\geqq\cdots\geqq n_r>0$. We denote the set of all partitions of n by M.

Let $\mu=(n_1, n_2, \cdots, n_r)$ be a partition of n. P_μ denotes the subgroup of G consisting of all elements

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & A_{22} & \cdots & A_{2r} \\ \cdots & & & \\ 0 & 0 & \cdots & A_{rr} \end{pmatrix} \in GL_n$$

for which $A_{ii}\in GL_{n_i}(i=1, 2, \cdots, r)$. This is a parabolic k-subgroup containing B. Let $W(P_\mu)$ be the subgroup of W which corresponds to P_μ in the sense of section 2. Then

$$W(P_\mu)\cong S_\mu=S_{n_1}\times S_{n_2}\times\cdots\times S_{n_r}\subset S_n\,.$$

The conjugacy classes of $W=S_n$ are parametrized by the partitions of n. Using a standard notation to describe elements of S_n, a set of representatives of conjugacy classes of S_n is given by $\{w_\mu\}_\mu$, where

$$w_\mu=(1, 2, \cdots, n_1)(n_1+1, \cdots, n_1+n_2)\cdots(n_1+\cdots+n_{r-1}+1, \cdots, n)\,.$$

$\boldsymbol{T}$ denotes the set of all maximal tori of G defined over k. Let T_μ be a representatives of the $G(k)$-conjugacy classes of $\boldsymbol{T}$ corresponding to w_μ. In the following, we write $i_\mu[\phi]$, $\phi_\nu^\mu(\phi\in B(P_\mu(k)))$, W_μ for $i_{P_\mu(k)\to G(k)}[\phi]$, $\phi_{T\nu}^{P\mu}$, W_{T_μ}.

Let $I(T_\mu(k))$ be the space of W_μ-invariant functions on $T_\mu(k)$. Put $I(G(k))=\oplus_{\mu\in M}I(T_\mu k))$ and define an inner product on $I(G(k))$:

$$(\alpha, \beta)_{I(G(K))}=\sum_{\mu\in M}|W_\mu|^{-1}(\alpha_\mu, \beta_\mu)_{T_\mu(k)},$$

where $\alpha=(\alpha_\mu)$ and $\beta=(\beta_\mu)$ are elements of $I(G(k))$.

In section 3 we defined an element $(\phi_\mu^\xi)_\mu\in I(G(k))$ for each $\phi\in B(P_\xi(k))$ ($\xi\in M$). Moreover, by Theorem 3.1 we have

$$(\mathrm{i}_\xi[\phi], \mathrm{i}_\eta[\psi])_{G(k)}=\sum_{\mu\in M}|W_\mu|^{-1}(\phi_\mu^\xi, \psi_\mu^\eta)_{T_\mu(k)}$$

for $\phi\in B(P_\xi(k))$ and $\psi\in B(P_\eta(k))$ ($\xi, \eta\in M$). Hence the mapping $\wedge=(\wedge_\mu)_{\mu\in M}$ from a subset of $A(G(k))$ into $I(G(k)$ defined by

$$(5.1) \qquad \{i_\xi[\phi]^\wedge = (i_\xi[\phi]^{\wedge\mu})_{\mu\in M} = (\phi^\xi_\mu)_{\mu\in M} \qquad (\xi\in M,\ \phi\in B(P_\xi(k)))$$

can be extended to a linear isometric transformation from a subspace of $A(G(k))$ into I(G(k)).

Theorem 5.1. *Let the notations be above. Then $\wedge$ is a unitary transformation from $A(G(lk))$ onto $I(G(k))=\oplus_{\mu\in M}I(T_\mu(k))$, that is,*

$$(5.2) \qquad (\chi_1, \chi_2)_{G(k)} = \textstyle\sum_{\mu\in M} |W_\mu|^{-1}(\chi_1^{\wedge\mu}, \chi_2^{\wedge\mu})_{T_\mu(k)}\,.$$

Corollary.

(a) *Let χ be an element of $B(G(k))$. Then*

$$\chi^{\wedge\mu} = \chi\,|\,T_\mu(k) \qquad (\mu\in M)\,.$$

(Cf. [5; Theorem 8].)

(b) *Let χ be an element of $A(G(k))$ and $T'_\mu(k)$ be the set of all regular elements (see* [9; *p.* 216]) *contained in $T_\mu(k)$.* Then

$$\chi^{\wedge\mu}\,|\,T'_\mu(k) = \chi\,|\,T'_\mu(k) \qquad (\mu\in M)\,.$$

(Cf. [5; p. 423, Examples 1].)

REMARK 5.1. As mentioned in the introduction, it is interesting to compare the formula (5.2) with the Weyl's integration formula:

$$\int_G f_1(x)\overline{f_2(x)}dx = \frac{1}{|W|}\int_T f_1(t)\overline{f_2(t)}\,|D(t)|^2 dt\,.$$

Here G denotes a compact semisimple Lie group and T its maximal torus, which is uniquely determined up to G-conjugacy. dx and dt are the Haar measures on G and T normalized by $\int_G dx=\int_T dt=1$. W is the Weyl group of G. f_1 and f_2 are arbitrary class functions on G and $D(t)$ is a function on T independent of f_1 and f_2.

To prove the theorem we need the following

Lemma 5.1.

(a) *T_μ can be so chosen that $P_\mu\supset T_\mu$.*

(b) *Let α be an element of $I(T_\mu(k))$. If T_μ is chosen as in (a), α can be extended to an element of $B(P_\mu(k))$ by*

$$\alpha(x) = \alpha(t) \qquad \textit{if } x_s\in C_{P(k)}(t)\ (t\in T_\mu(k))$$

$$\textit{and} \quad \alpha(x) = 0 \qquad \textit{otherwise.}$$

(c) *Let* $\alpha\in T_\mu(k)$ *be extended to an element of* $B(P_\mu(k))$. *Then*

$$\alpha_\mu^\mu(t) = L(P_\mu, T_\mu)\alpha(t) \qquad (t\ni T_\mu(k)),$$

where $L(P_\mu, T_\mu)=|\{Q\in C_{G(k)}(P_\mu);\ Q\supset T_\mu\}|$.

Proof. We use the explicit form of T_μ given in [8; p. 126]. (a) is easy. (b) follows from the fact that two elements of $T_\mu(k)$ are $P_\mu(k)$-conjugate to each other if and only if they are conjugate by an element of $N_{G(k)}(T_\mu)\cap P_\mu(k)$. To prove (c) we first note that the number of $P_\mu(k)$-conjugacy classes of $\{S\in C_{G(k)}(T_\mu);\ S\subset P_\mu\}$ equals to that of $W(P_\mu)$-conjugacy classes of $C_W(\mathrm{w}_\mu)\cap W(P_\mu)$ by the theroy given in [10; section 2]. The latter number is easily seen to be 1. Thus any two elements of $\{S\in C_{G(k)}(T_\mu);\ S\subset P_\mu\}$ are P_μ-conjugate to each other. Hence if $Q\in C_{G(k)}(P_\mu)$ contains T_μ, (Q, t) is $G(k)$-conjugate to (P_μ, t') for some elements $t'\in T_\mu(k)$ which is W_μ-conjuagte to t. This implies that $\phi^Q(t)=\phi^{P_\mu}(t)$ for $\phi\in I(T_\mu(k))$. (c) follows from this.

(Proof of Theorem 5.1)

We first note that dim $A(G(k))=$dim $I(G(k))$ as complex vector spaces. (See [8; p.127].) Hence it is sufficient to show that $\wedge$ is surjective. We write $\mu>\nu\,(\mu, \nu\in M)$ if $P_\mu\supsetneqq P_\nu$. By induction on this order we shall show that for an arbitrary $\phi\in I(\mathrm{T}_\mu(k))$ there exists an element $\chi\in A(G(k))$ such that $\chi^{\wedge\mu}=\phi$ and $\chi^{\wedge\mu'}=0$ $(\mu'\neq\mu)$. When $\mu=\{1, \cdots, 1\}$, i.e. $T_\mu=T_1$ and $P_\mu=B$, put $\chi=i_\mu[\phi/|W|]$. Then by (5.1) and Lemma 5.1(c) the assertion follows. For general μ put $\chi_1=\mathrm{i}_\mu[\phi/L(P_\mu, T_\mu)]$. Then $\chi_1^{\wedge\mu}=\phi$ and $\chi_1^{\wedge\mu'}=0$ for $\mu'>\mu$. By the induction hypothesis there is an element χ_2 such that $\chi_2^{\wedge\nu}=\chi_1^{\wedge\nu}$ for $\nu<\mu$ and $\chi_2^{\wedge\nu}=0$ for. $\nu\geqq\mu$. Put $\chi=\chi_1-\chi_2$. Then χ satisfies the conditions. This proves the theorem. Corollary (a) is trivial by (5.1). Let r be a regular semisimple element of $G(k)$ contained in $T_\mu(k)$. Then a parabolic subgroup Q contains r if and only if Q contains T_μ. In fact, if Q contains r, Q contains a maximal torus S containing r. By a property of regular semisimple elements ([9; 1.7(c)]) $S=T_\mu$. Hence Q contains T_μ. The converse is obvious. From this fact and (3.2) and Lemma 3.1, we have

$$i_\xi[\phi]^{\wedge\mu}(r) = \phi_\mu^\xi(r) \qquad (\phi\in B(P_\xi(k)),\ \xi\in M)\,.$$

Hence Corollary (b) follows from Theorem 5.1.

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