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PERTURBATION OF IRREGULAR WEYL-HEISENBERG WAVE PACKET FRAMES IN $L^2(\mathbb{R})$

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Abstract
In this paper, we consider the perturbation problem of irregular Weyl-Heisenberg wave packet frame $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ about dilation, translation and modulation parameters. We give a method to determine whether the perturbation systems is a frame for wave packet functions whose Fourier transforms have small support and prove the stability about dilation parameter on Paley-Wiener space. For a wave packet function, we give a definite answer to the stability about translation parameter b.

1. Introduction and preliminaries

Duffin and Schaeffer [10] introduced frames for Hilbert spaces, while addressing some deep problems in nonharmonic Fourier series. Later, in 1986, Daubechies, Grossmann and Meyer [9] found new applications to wavelet and Gabor transforms in which frames played an important role. The basic theory of frames can be found in [3].

The wave packet systems were introduced and studied by Cordoba and Fefferman [6] by applying certain collections of dilations, modulations and translations to the Gaussian function in the study of some classes of singular integral operators. Lebate et al. [13] adopted the same expression to describe any collections of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R})$. More precisely, Gabor systems, wavelet systems and the Fourier transform of wavelet systems are special cases of wave packet systems. Wave packet systems have recently been successfully applied to some problems in harmonic analysis and operator theory. The wave packet systems have been studied by several authors, see [5, 7, 12, 14].
We now recall basic notations and definitions.

For $1 \leq p < \infty$, let $L^p(\mathbb{R})$ denote the Banach space of complex-valued Lebesgue integrable functions f on $\mathbb{R}$ satisfying

$$||f||_p = \left(\int_{\mathbb{R}} |f(t)|^p dt\right)^{\frac{1}{p}} < \infty.$$

For $p = 2$, an inner product on $L^p(\mathbb{R})$ is given by

$$\langle f, g\rangle = \int_R f\overline{g}dt,$$

where $\overline{g}$ denotes the complex conjugate of g.

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We consider the unitary operators on $L^2(\mathbb{R})$ which are given by :
Translation $\leftrightarrow T_a f(t) = f(t-a),\ a \in \mathbb{R}$.
Modulation $\leftrightarrow E_b f(t) = e^{2\pi i b t} f(t),\ b \in \mathbb{R}$.
Dilation $\leftrightarrow D_a f(t) = \sqrt{|a|} f(at),\ a \in \mathbb{R}$.
One can easily verify that for $g \in L^2(\mathbb{R})$, $E_{mb}T_{na}g(t) = e^{2\pi i m b t} g(t-na)$. If $a, b > 0$ and $(g, a, b) = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$, then we call (g, a, b) a *Gabor frame* or a *Weyl-Heisenberg frame* for $L^2(\mathbb{R})$. Casazza [2] introduced and studied irregular Weyl-Heisenberg frames for $L^2(\mathbb{R})$. Let $(x_m, y_n) \in \mathbb{R}^2$ and let $g \in L^2(\mathbb{R})$. A system of the form $\{E_{x_m}T_{y_n}g(t)\}_{m,n\in\mathbb{Z}}$ is called an *irregular Weyl-Heisenberg frame* (or *IWH frame*) for $L^2(\mathbb{R})$, if $\{E_{x_m}T_{y_n}g(t)\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$.
Today stability and perturbation of frames are very useful in many areas of mathematics, physics and engineering, that is, Harmonic analysis, quantum mechanics, scattering theory, signal and image processing. Many results in this direction have been established during last two decades. Favier and Zalik [11] and Zhang [16] considered stability of wavelets frames and Riesz bases. Balan [1] studies the perturbation of translation parameter b. This problem was first considered by Daubechies [8] for Meyer orthogonal wavelet basis. For more details on stability and perturbation one may refer to [3, 4].

The objective of this paper is to investigate the perturbation problem of irregular Weyl-Heisenberg wave packet frame $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ about dilation, translation and modulation parameters. We give a method to determine whether the perturbation systems is a frame for wave packet functions whose Fourier transforms have small support and prove the stability about dilation parameter on Paley-Wiener space. For a nice deal of wave packet functions, we give a definite answer to the stability about translation parameter b. Moreover, for a given wave packet functions, we can estimate the frame bounds about the perturbation of translation and dilation parameter.

We need the following lemmas and theorem; for more details, see [4, 11, 15].

Lemma 1.1. [4, 11] *Let $\{f_j\}$ be a frame (Riesz basis) for Hilbert space $\mathcal{H}$ with frame bounds A and B. Assume $\{g_j\} \subset \mathcal{H}$ and $\{f_j - g_j\}$ is a Bessel sequence with bound $M < A$. Then $\{g_j\}$ is a frame (Riesz basis) with frame bounds $A\left[1 - \sqrt{\frac{M}{A}}\right]^2$ and $B\left[1 + \sqrt{\frac{M}{B}}\right]^2$.*

Using the triangle inequality, the following lemma obviously holds.

Lemma 1.2. *Let $\{f_j\}_{j\in\mathbb{N}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B. If*

$$\left|\sum_j |\langle f, f_j\rangle|^2 - \sum_j |\langle f, g_j\rangle|^2\right| \le M\|f\|^2 < A\|f\|^2,$$

then $\{g_j\}_{j\in\mathbb{N}}$ is a frame with frame bounds $A - M$ and $B + M$.

Theorem 1.3. [15] *Let $\psi \in L^2(\mathbb{R})$, $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$, $\{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ and $b > 0$. Suppose that the wave packet system $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A and B. Then $\hat{\psi}$ satisfies*

$$A \le \frac{1}{b}\sum_{j\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} |\hat{\psi}(a_j^{-1}\xi - c_m)|^2 \le B, \quad a.e.\ \xi.$$

2. Perturbation of IWH Wave Packet Frames

DEFINITION 2.1. Let $\psi \in L^2(\mathbb{R})$, $\{a_j\}_{j\in\mathbb{Z}} \subset \mathbb{R}^+$, $\{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}$ and $b \neq 0$. A system of the form $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is called an *irregular Weyl-Heisenberg wave packet system* (or IWH wave packet system).

DEFINITION 2.2. If IWH wave packet system $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ constitutes a frame for $L^2(\mathbb{R})$, i.e., if there exist positive constants A_0 and B_0 such that

$$A_0\|f\|^2 \leq \sum_{j,k,m\in\mathbb{Z}} |\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 \leq B_0\|f\|^2, \text{ for all } f \in L^2(\mathbb{R}),$$

then we say that $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is an *irregular Weyl-Heisenberg wave packet frame* (or *IWH wave packet frame*) for $L^2(\mathbb{R})$. Then, the function ψ is called a frame wave packet function.

EXAMPLE 2.3. Let $\psi = \chi_{[0,1]}$ and let $a_j = 2^j$, $j \in \Lambda = \{1, 2, ..., n\}$. Choose $b = 1$ and $c_m = m$, for all $m \in \mathbb{Z}$.
Then, for $f \in L^2(\mathbb{R})$ we have

$$\begin{aligned}\sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 &= \sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle f, D_{2^j}T_kE_m\psi\rangle|^2 \\ &= \sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle f, D_{2^j}T_kE_m\chi_{[0,1]}\rangle|^2 \\ &= \sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle D_{2^{-j}}f, T_kE_m\chi_{[0,1]}\rangle|^2.\end{aligned}$$

Now, $\{E_mT_k\chi_{[0,1]}\}_{m,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$ (see [3], p. 71). Therefore,

$$\sum_{j\in\Lambda,k,m\in\mathbb{Z}} |\langle D_{2^{-j}}f, T_kE_m\chi_{[0,1]}\rangle|^2 = \sum_{j\in\Lambda} \|D_{2^{-j}}f\|^2 = n\|f\|^2.$$

Hence, $\left\{D_{a_j}T_{bk}E_{c_m}\psi\right\}_{j\in\Lambda,k,m\in\mathbb{Z}}$ is a wave packet frame for $L^2(\mathbb{R})$.

EXAMPLE 2.4. Let $\psi = \chi_{[0,1]}$ and let $a_j = 2^j$, $j \in \mathbb{Z}$. Choose $b = 1$ and $c_m = m$, for all $m \in \mathbb{Z}$.
Then, $\left\{D_{a_j}T_{bk}E_{c_m}\psi\right\}_{j,k,m\in\mathbb{Z}}$ is not a wave packet frame for $L^2(\mathbb{R})$. Indeed, choose $f_o = \chi_{[0,1]}$ we compute

$$\begin{aligned}\sum_{j,k,m\in\mathbb{Z}} |\langle f_o, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 &= \sum_{j,k,m\in\mathbb{Z}} |\langle f_o, D_{2^j}T_kE_m\psi\rangle|^2 \\ &= \sum_{j,k,m\in\mathbb{Z}} |\langle f_o, D_{2^j}T_kE_m\chi_{[0,1]}\rangle|^2 \\ &= \sum_{j,k,m\in\mathbb{Z}} |\langle D_{2^{-j}}f_o, T_kE_m\chi_{[0,1]}\rangle|^2 \\ &= \sum_{j\in\mathbb{Z}} \|D_{2^{-j}}f_o\|^2 > n\|f_o\|^2, \text{ for any } n.\end{aligned}$$

Hence, $\left\{D_{a_j}T_{bk}E_{c_m}\psi\right\}_{j,k,m\in\mathbb{Z}}$ does not satisfy the upper frame condition for $L^2(\mathbb{R})$.

We consider the perturbation about translation parameter b in Theorem 2.5 and Theorem 2.7.

Theorem 2.5. *Let Λ be a finite set, $\psi \in L^2(\mathbb{R})$, $\{c_m\}_{m\in\Lambda} \subset \mathbb{R}^-$, If $\{D_{a_o^j} T_{b_o k} E_{c_m}\psi\}_{j,k\in\mathbb{Z},m\in\Lambda}$ is a wave packet frame for $L^2(\mathbb{R})$ with bounds A and B, $\hat{\psi}$ is continuous, vanish in the neighborhood of each $-c_m$ and bounded by*

$$|\hat{\psi}(\xi)| \le C\frac{|\xi|^\alpha}{(1+|\xi|)^{1+\gamma}},$$

where $\gamma > \alpha > 0$. Then there exists a $\delta > 0$ such that for any b with $|b - b_o| < \delta$, $\{D_{a_o^j} T_{bk} E_{c_m}\psi\}_{j,k\in\mathbb{Z},m\in\Lambda}$ is a frame for $L^2(\mathbb{R})$.

Proof. Without loss of generality we assume that the scaling parameter $a_o > 1$. Define a unitary operator

$$U_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$
$$(U_b\psi)(x) = \left(\frac{b}{b_o}\right)^{1/2} \psi\left(\frac{b}{b_o}x\right) = \phi(x)$$

we see that

$$\hat{\phi}(\xi) = \left(\frac{b}{b_o}\right)^{-1/2} \hat{\psi}\left(\frac{b_o}{b}\xi\right),$$
$$U_b(D_{a_o^j} T_{bk} E_{c_m}\psi) = D_{a_o^j} T_{b_o k} E_{c_m}\phi.$$

Therefore, $\{D_{a_o^j} T_{bk} E_{c_m}\psi\}_{j,k\in\mathbb{Z},m\in\Lambda}$ is a frame if and only if $\{D_{a_o^j} T_{b_o k} E_{c_m}\phi\}_{j,k\in\mathbb{Z},m\in\Lambda}$ is a frame for $L^2(\mathbb{R})$.

For all $f, g \in L^2(\mathbb{R})$, we compute

$$\begin{aligned}
\sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} |\langle f, D_{a^j} T_{bk} E_{c_m} g\rangle|^2 &= \sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} |\langle \mathcal{F}f, \mathcal{F}D_{a^j} T_{bk} E_{c_m} g\rangle|^2 \\
&= \sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} |\langle \mathcal{F}f, D_{a^{-j}} E_{-bk} T_{c_m} \hat{g}\rangle|^2 \\
&= \sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} |\langle \mathcal{F}f, E_{-kba^{-j}} D_{a^{-j}} T_{c_m} \hat{g}\rangle|^2 \\
&= \sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} \left|\int \hat{f}(\xi)\overline{E_{-kba^{-j}} D_{a^{-j}} T_{c_m} \hat{g}}(\xi) d\xi\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} \left|\int \hat{f}(\xi) a^{-j/2}\overline{\hat{g}(a^{-j}\xi - c_m)} e^{2\pi i a^{-j} b\xi k} d\xi\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} \left|\frac{a^{j/2}}{b}\int \hat{f}(\frac{a^j}{b}\xi)\overline{\hat{g}(\frac{\xi}{b} - c_m)} e^{2\pi i\xi k} d\xi\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} \frac{a^j}{b^2}\left|\int \hat{f}(\frac{a^j}{b}\xi)\overline{\hat{g}(\frac{\xi}{b} - c_m)} e^{2\pi i\xi k} d\xi\right|^2 \\
&= \sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda} \frac{a^j}{b^2}\left|\int_{-\pi}^{\pi} \sum_{l\in\mathbb{Z}} \hat{f}(\frac{a^j(\xi + 2\pi l)}{b})\overline{\hat{g}(\frac{\xi + 2\pi l}{b} - c_m)} e^{2\pi i\xi k} d\xi\right|^2
\end{aligned}$$

$$= \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \frac{a^j}{b^2} \int_{-\pi}^{\pi} \Big| \sum_{l\in\mathbb{Z}} \hat{f}(\frac{a^j(\xi+2\pi l)}{b}) \overline{\hat{g}(\frac{\xi+2\pi l}{b} - c_m)} \Big|^2 d\xi$$

$$= \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \frac{a^j}{b^2} \int_{-\infty}^{\infty} \hat{f}(\frac{a^j\xi}{b}) \overline{\hat{g}(\frac{\xi}{b} - c_m)}$$

$$\times \sum_{l\in\mathbb{Z}} \overline{\hat{f}(\frac{a^j(\xi+2\pi l)}{b})} \hat{g}(\frac{\xi+2\pi l}{b} - c_m) d\xi$$

$$= \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \sum_{l\in\mathbb{Z}} \frac{1}{b} \int \hat{f}(\xi) \overline{\hat{g}(a^{-j}\xi - c_m)}$$

$$\times \overline{\hat{f}(\xi + \frac{2\pi l a^j}{b})} \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) d\xi.$$

Thus,

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\Lambda} |\langle f, D_{a^j} T_{bk} E_{c_m} g\rangle|^2 = \sum_{j\in\mathbb{Z}} \sum_{m\in\Lambda} \sum_{l\in\mathbb{Z}} \frac{1}{b} \int \hat{f}(\xi) \overline{\hat{g}(a^{-j}\xi - c_m)}$$

$$\times \overline{\hat{f}(\xi + \frac{2\pi l a^j}{b})} \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) d\xi.$$

By the Cauchy-Schwarz inequality we have

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\Lambda} |\langle f, D_{a^j} T_{bk} E_{c_m} g\rangle|^2$$

$$\leq \frac{1}{b} \sum_{j\in\mathbb{Z}, m\in\Lambda} \sum_{l\in\mathbb{Z}} \Big(\int |\hat{f}(\xi)|^2 \Big| \hat{g}(a^{-j}\xi - c_m) \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) \Big| d\xi \Big)^{1/2}$$

$$\times \Big(\int \Big| \hat{f}(\xi + \frac{2\pi l a^j}{b}) \Big|^2 \Big| \hat{g}(a^{-j}\xi - c_m) \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) \Big| d\xi \Big)^{1/2}$$

$$\leq \frac{1}{b} \Big(\sum_{j\in\mathbb{Z}, m\in\Lambda} \sum_{l\in\mathbb{Z}} \int |\hat{f}(\xi)|^2 \Big| \hat{g}(a^{-j}\xi - c_m) \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) \Big| d\xi \Big)^{1/2}$$

$$\times \Big(\sum_{j\in\mathbb{Z}, m\in\Lambda} \sum_{l\in\mathbb{Z}} \int |\hat{f}(\xi)|^2 \Big| \hat{g}(a^{-j}\xi - c_m) \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) \Big| d\xi \Big)^{1/2}$$

$$= \frac{1}{b} \sum_{j\in\mathbb{Z}, m\in\Lambda} \sum_{l\in\mathbb{Z}} \int |\hat{f}(\xi)|^2 \Big| \hat{g}(a^{-j}\xi - c_m) \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) \Big| d\xi$$

$$\leq \frac{1}{b} \sup_{\xi\in\mathbb{R}} \sum_{j\in\mathbb{Z}, m\in\Lambda} \sum_{l\in\mathbb{Z}} \Big| \hat{g}(a^{-j}\xi - c_m) \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) \Big| \|f\|^2$$

$$= \frac{1}{b} \sup_{1\leq|\xi|\leq a} \sum_{j\in\mathbb{Z}, m\in\Lambda} \sum_{l\in\mathbb{Z}} \Big| \hat{g}(a^{-j}\xi - c_m) \hat{g}(a^{-j}\xi + \frac{2\pi l}{b} - c_m) \Big| \|f\|^2.$$

Let $g = \psi - \phi$, we have

$$\sum_{j,k\in\mathbb{Z}} \sum_{m\in\Lambda} |\langle f, D_{a_o^j} T_{b_o k} E_{c_m} (\psi - \phi)\rangle|^2$$

$$\leq \frac{1}{b_o} \sup_{1\leq|\xi|\leq a_o} \sum_{j\in\mathbb{Z},m\in\Lambda} \sum_{l\in\mathbb{Z}} |\hat{\psi}(a_o^{-j}\xi - c_m) - \hat{\phi}(a_o^{-j}\xi - c_m)|$$

$$\times \left|\hat{\psi}(a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m) - \hat{\phi}(a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m)\right| \|f\|^2$$

$$= \frac{1}{b_o} \sup_{1\leq|\xi|\leq a_o} \sum_{j\in\mathbb{Z},m\in\Lambda} |\hat{\psi}(a_o^{-j}\xi - c_m) - \hat{\phi}(a_o^{-j}\xi - c_m)|$$

$$\times \sum_{l\in\mathbb{Z}} \left|\hat{\psi}(a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m) - \hat{\phi}(a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m)\right| \|f\|^2.$$

For all m, j and ξ, and by hypothesis, we have

$$\begin{aligned}
\sup_{1\leq|\xi|\leq a_o} \sum_{l\in\mathbb{Z}} \left|\hat{\psi}(a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m)\right| &\leq C \sup_{1\leq|\xi|\leq a_o} \sum_{l\in\mathbb{Z}} \frac{|a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m|^\alpha}{(1 + |a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m|)^{1+\gamma}} \\
&\leq C \sup_{1\leq|\xi|\leq a_o} \sum_{l\in\mathbb{Z}} \frac{1}{(1 + |a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m|)^{1+\gamma-\alpha}} \\
&\leq C \sup_{1\leq|\xi|\leq a_o} \sum_{l\in\mathbb{Z}} \frac{1}{(1 + |a_o^{-j}\xi + \frac{2\pi l}{b_o}|)^{1+\gamma-\alpha}} \\
&\leq C \sum_{l\in\mathbb{Z}} \frac{1}{(1 + |l|)^{1+\gamma-\alpha}} \\
&\leq C < \infty.
\end{aligned}$$

For the same reason, the second term is bounded uniformly, that is,

$$\sup_{1\leq|\xi|\leq a_o} \sum_{l\in\mathbb{Z}} \left|\hat{\phi}(a_o^{-j}\xi + \frac{2\pi l}{b_o} - c_m)\right| \leq C < \infty.$$

Now for all $J \in \mathbb{N}$, we have

$$\begin{aligned}
&\sup_{1\leq|\xi|\leq a_o} \sum_{j\in\mathbb{Z},m\in\Lambda} |\hat{\psi}(a_o^{-j}\xi - c_m) - \hat{\phi}(a_o^{-j}\xi - c_m)| \\
&\leq \sup_{1\leq|\xi|\leq a_o} \sum_{\substack{|j|\leq J \\ m\in\Lambda}} \left|\hat{\psi}(a_o^{-j}\xi - c_m) - (\frac{b}{b_o})^{-1/2}\hat{\psi}(\frac{b_o}{b}(a_o^{-j}\xi - c_m))\right| \\
&\quad + \sup_{1\leq|\xi|\leq a_o} \sum_{\substack{j\leq -J \\ m\in\Lambda}} |\hat{\psi}(a_o^{-j}\xi - c_m) - \hat{\phi}(a_o^{-j}\xi - c_m)| \\
&\quad + \sup_{1\leq|\xi|\leq a_o} \sum_{\substack{j>J \\ m\in\Lambda}} |\hat{\psi}(a_o^{-j}\xi - c_m) - \hat{\phi}(a_o^{-j}\xi - c_m)| \\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}$$

For every $\epsilon > 0$, choose J such that $a_o^{-J} < \epsilon$. Since $1 \leq |\xi| \leq a_o$, $|j| \leq J$ and $\hat{\psi}(a_o^{-j}\xi - c_m)$ is uniformly continuous on ξ, we can choose δ small enough such that if $|b - b_o| < \delta$, then

$$|\hat{\psi}(a_o^{-j}\xi - c_m) - \hat{\psi}(\frac{b_o}{b}(a_o^{-j}\xi - c_m))| < \epsilon, \text{ for all } |j| \leq J.$$

Therefore,

$$\mathrm{I} = \sup_{1\le|\xi|\le a_o} \sum_{\substack{|j|\le J\\ m\in\Lambda}} \Big|\hat{\psi}(a_o^{-j}\xi - c_m) - (\frac{b}{b_o})^{-1/2}\hat{\psi}(a_o^{-j}\xi - c_m)$$

$$+(\frac{b}{b_o})^{-1/2}\hat{\psi}(a_o^{-j}\xi - c_m) - (\frac{b}{b_o})^{-1/2}\hat{\psi}(\frac{b_o}{b}(a_o^{-j}\xi - c_m))\Big|$$

$$\le \sup_{1\le|\xi|\le a_o} \sum_{\substack{|j|\le J\\ m\in\Lambda}} \left(\left|1 - (\frac{b}{b_o})^{-1/2}\right| \left|\hat{\psi}(a_o^{-j}\xi - c_m)\right|\right.$$

$$\left.+(\frac{b}{b_o})^{-1/2}\left|\hat{\psi}(a_o^{-j}\xi - c_m) - \hat{\psi}(\frac{b_o}{b}(a_o^{-j}\xi - c_m))\right|\right)$$

$$\le C(2J+1)\left(\left|1 - (\frac{b}{b_o})^{-1/2}\right| + \frac{b}{b_o}\epsilon\right) = o(1),\ b \to b_o.$$

We now turn to II. We will just estimate the first term in the series, since the other term can be handled similarly.

$$\sup_{1\le|\xi|\le a_o} \sum_{\substack{j<-J\\ m\in\Lambda}} |\hat{\psi}(a_o^{-j}\xi - c_m)| \le \sup_{1\le|\xi|\le a_o} C \sum_{\substack{j<-J\\ m\in\Lambda}} \frac{1}{(1+|a_o^{-j}\xi - c_m|)^{1+\gamma-\alpha}}$$

$$\le \sup_{1\le|\xi|\le a_o} C \sum_{\substack{j<-J\\ m\in\Lambda}} \frac{1}{(1+|a_o^{-j}\xi|)^{1+\gamma-\alpha}}$$

$$\le C \sum_{\substack{j<-J\\ m\in\Lambda}} a_o^{j(1+\gamma-\alpha)}$$

$$\le C \sum_{m\in\Lambda} a_o^{-J(1+\gamma-\alpha)} = o(1), \quad J \to \infty.$$

Finally, the last part is also small, since if J is large,

$$\sup_{1\le|\xi|\le a_o} \sum_{\substack{j>J\\ m\in\Lambda}} |\hat{\psi}(a_o^{-j}\xi - c_m)| \le \sup_{1\le|\xi|\le a_o} C \sum_{\substack{j>J\\ m\in\Lambda}} \frac{|a_o^{-j}\xi - c_m|^{\alpha}}{(1+|a_o^{-j}\xi - c_m|)^{1+\gamma}}$$

$$\le \sup_{1\le|\xi|\le a_o} C \sum_{\substack{j>J\\ m\in\Lambda}} \frac{1}{(1+|a_o^{-j}\xi - c_m|)^{1+\gamma-\alpha}}$$

$$\le \sup_{1\le|\xi|\le a_o} C \sum_{\substack{j>J\\ m\in\Lambda}} \frac{1}{(1+|a_o^{-j}\xi|)^{1+\gamma-\alpha}}$$

$$\le C \sum_{\substack{j>J\\ m\in\Lambda}} (|a_o^{-j}a_o|)^{\alpha}$$

$$\le C \sum_{m\in\Lambda} (a_o^{-J\alpha}) = o(1), \quad J \to +\infty.$$

Thus, for every $\epsilon > 0$ there exist $\delta > 0$ such that for all $|b - b_0| < \delta$, we have $\sum_{j,k\in\mathbb{Z}} \sum_{m\in\Lambda} |\langle f, D_{a_o^j} T_{b_o k} E_{c_m}(\psi - \phi)\rangle|^2 \le \epsilon \|f\|^2$. Hence, $\{D_{a_o^j} T_{b_o k} E_{c_m}\phi\}$ is a wave packet frame for $L^2(\mathbb{R})$ for sufficiently close to b_0 by Lemma 1.1.
This complete the proof. □

REMARK 2.6. In Theorem 2.5 we can not relax the condition of $\hat{\psi}$ vanish in the neighborhood of each $-c_m$ and finiteness of Λ both, if we take $\Lambda = \mathbb{Z}$, then $\{D_{a_o^j} T_{b_o k} E_{c_m}\psi\}$ is not a wave packet frame for $L^2(\mathbb{R})$. Indeed, choose $c_m = m$, the sum in the second and third terms II, III in the proof of the Theorem 2.5 with respect to m diverges.

Theorem 2.7. *Suppose that* $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ *is a wave packet frame for* $L^2(\mathbb{R})$ *with frame bounds A and B. Then there exists a* $\delta > 0$ *such that for any* b' *with* $|b-b'| < \delta$ *and* $\operatorname{supp}\hat{\psi} \subset [-\pi/(b'\vee b), \pi/(b'\vee b)]$, $\{D_{a_j}T_{b'k}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ *is a wave packet frame with frame bounds* $A-M$ *and* $B+M$, *where* $M = |1-(\frac{b}{b'})|B < A$, $b'\vee b = \max(b', b)$.

Proof. Since $\operatorname{supp}\hat{\psi} \subset [-\pi/(b'\vee b), \pi/(b'\vee b)]$, we have

$$\begin{aligned}
&\left|\sum_{j,k,m\in\mathbb{Z}} |\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 - \sum_{j,k,m\in\mathbb{Z}} |\langle f, D_{a_j}T_{b'k}E_{c_m}\psi\rangle|^2\right| \\
&= \left|\sum_{j,k,m\in\mathbb{Z}} |\langle \mathcal{F}f, \mathcal{F}D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2 - \sum_{j,k,m\in\mathbb{Z}} |\langle \mathcal{F}f, \mathcal{F}D_{a_j}T_{b'k}E_{c_m}\psi\rangle|^2\right| \\
&= \left|\sum_{j,k,m\in\mathbb{Z}} |\langle \hat{f}, D_{a_j^{-1}}E_{-bk}T_{c_m}\hat{\psi}\rangle|^2 - \sum_{j,k,m\in\mathbb{Z}} |\langle \hat{f}, D_{a_j^{-1}}E_{-b'k}T_{c_m}\hat{\psi}\rangle|^2\right| \\
&= \left|\sum_{j,k,m\in\mathbb{Z}} |\langle \hat{f}, E_{-bka_j^{-1}}D_{a_j^{-1}}T_{c_m}\hat{\psi}\rangle|^2 - \sum_{j,k,m\in\mathbb{Z}} |\langle \hat{f}, E_{-b'ka_j^{-1}}D_{a_j^{-1}}T_{c_m}\hat{\psi}\rangle|^2\right| \\
&= \Bigg|\sum_{j,k,m\in\mathbb{Z}} \left|\int \hat{f}(\xi)a_j^{-1/2}\overline{\hat{\psi}(a_j^{-1}\xi - c_m)}e^{2\pi ikba_j^{-1}\xi}d\xi\right|^2 \\
&\qquad - \sum_{j,k,m\in\mathbb{Z}} \left|\int \hat{f}(\xi)a_j^{-1/2}\overline{\hat{\psi}(a_j^{-1}\xi - c_m)}e^{2\pi ikb'a_j^{-1}\xi}d\xi\right|^2\Bigg| \\
&= \left|\frac{1}{b}\int |\hat{f}(\xi)|^2 \sum_{j,m\in\mathbb{Z}} |\hat{\psi}(a_j^{-1}\xi - c_m)|^2 d\xi - \frac{1}{b'}\int |\hat{f}(\xi)|^2 \sum_{j,m\in\mathbb{Z}} |\hat{\psi}(a_j^{-1}\xi - c_m)|^2 d\xi\right| \\
&= |b^{-1} - b'^{-1}| \int |\hat{f}(\xi)|^2 \sum_{j,m\in\mathbb{Z}} |\hat{\psi}(a_j^{-1}\xi - c_m)|^2 d\xi.
\end{aligned}$$

By Theorem 1.3 the last term above is dominated by

$$|b^{-1} - b'^{-1}|bB\|f\|^2 = |1 - \frac{b}{b'}|B\|f\|^2.$$

Choosing those b such that $|1-\frac{b}{b'}|B < A$. Thus by using Lemma 1.2, $\{D_{a_j}T_{b'k}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a wave packet frames for $L^2(\mathbb{R})$ with desired bounds. □

The following theorem gives perturbation with respect to dilation parameter in terms of series.

Theorem 2.8. *Suppose that* $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ *is a wave packet frame for* $L^2(\mathbb{R})$ *with bounds A and B,* $\operatorname{supp}\hat{\psi} \subset [-\pi/b, \pi/b]$, *and*

$$\lambda = \operatorname{essup}\left|\sum_{j,m\in\mathbb{Z}} |\hat{\psi}(a_j^{-1}\xi - c_m)|^2 - \sum_{j,m\in\mathbb{Z}} |\hat{\psi}(a_j'^{-1}\xi - c_m)|^2\right| < bA.$$

Then $\{D_{a'_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ *is a wave packet frame for* $L^2(\mathbb{R})$ *with frame bounds* $A - b^{-1}\lambda$ *and* $B + b^{-1}\lambda$.

Proof. We can compute similar to Theorem 2.7, we have

$$\begin{aligned}
&\Big|\sum_{j,k,m\in\mathbb{Z}}|\langle f, D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2-\sum_{j,k,m\in\mathbb{Z}}|\langle f, D_{a'_j}T_{bk}E_{c_m}\psi\rangle|^2\Big| \\
&=\frac{1}{b}\int|\hat{f}(\xi)|^2(\sum_{j,m\in\mathbb{Z}}|\hat{\psi}(a_j^{-1}\xi-c_m)|^2-\sum_{j,m\in\mathbb{Z}}|\hat{\psi}(a_j'^{-1}\xi-c_m)|^2)d\xi \\
&\le\frac{\lambda}{b}\|f\|^2.
\end{aligned}$$

Hence, by Lemma 1.2, $\{D_{a'_j}T_{bk}E_{c_m}\psi\}$ is a wave packet frame for $L^2(\mathbb{R})$ with bounds $A-b^{-1}\lambda$ and $B+b^{-1}\lambda$.
This completes the proof. □

The following theorem shows that perturbation of wave packet frame about dilation parameter a on Paley-Wiener space.

Theorem 2.9. *Let Λ be a finite set, $\psi\in L^2(\mathbb{R})$, $\{c_m\}_{m\in\Lambda}\subset\mathbb{R}^-$. If $\{D_{a^j}T_{bk}E_{c_m}\psi\}_{j,k\in\mathbb{Z},m\in\Lambda}$ is a wave packet frame for $L^2(\mathbb{R})$,* supp $\hat{\psi}\subset[-\pi/b,\pi/b]$*, $\hat{\psi}$ is continuous, vanish in the neighborhood of each $-c_m$ and*

$$|\hat{\psi}(\xi)|\le C\frac{|\xi|^\alpha}{(1+|\xi|)^\gamma},\ \gamma>\alpha>0.$$

Then $\{D_{a'^j}T_{bk}E_{c_m}\psi\}_{j,k\in\mathbb{Z},m\in\Lambda}$ is a wave packet frame on Paley-Wiener space for all a' in some neighborhood of a.

Proof. Without loss of generality we assume that $a>1$. We compute

$$\begin{aligned}
&\Big|\sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda}|\langle f, D_{a^j}T_{bk}E_{c_m}\psi\rangle|^2-\sum_{j,k\in\mathbb{Z}}\sum_{m\in\Lambda}|\langle f, D_{a'^j}T_{bk}E_{c_m}\psi\rangle|^2\Big| \\
&=\frac{1}{b}\Big|\int|\hat{f}(\xi)|^2(\sum_{j\in\mathbb{Z}}\sum_{m\in\Lambda}|\hat{\psi}(a^{-j}\xi-c_m)|^2-\sum_{j\in\mathbb{Z}}\sum_{m\in\Lambda}|\hat{\psi}(a'^{-j}\xi-c_m)|^2)d\xi\Big|.
\end{aligned}$$

Now, if f is a function in Paley-Wiener space, more precisely, there exists two constant $0<n<N<\infty$, such that supp $\hat{f}\subset\{\xi:n\le|\xi|\le N\}$, then for big enough J, if $|j|>J$, we have

$$\begin{aligned}
\sum_{j<-J}\sum_{m\in\Lambda}|\hat{\psi}(a^{-j}\xi-c_m)|^2 &\le C\sum_{j<-J}\sum_{m\in\Lambda}\frac{|a^{-j}\xi-c_m|^{2\alpha}}{(1+|a^{-j}\xi-c_m|)^{2\gamma}} \\
&\le C\sum_{j<-J}\sum_{m\in\Lambda}\frac{1}{(1+|a^{-j}\xi-c_m|)^{2(\gamma-\alpha)}} \\
&\le C\sum_{j<-J}\sum_{m\in\Lambda}|a^{-j}\xi|^{-2(\gamma-\alpha)} \\
&\le C\sum_{j<-J}\sum_{m\in\Lambda}|a^{-j}n|^{-2(\gamma-\alpha)} \\
&\le C\sum_{m\in\Lambda}a^{-2J(\gamma-\alpha)}=o(1),\quad J\to\infty.
\end{aligned}$$

Also,

$$\begin{aligned}\sum_{j>J}\sum_{m\in\Lambda}|\hat{\psi}(a^{-j}\xi-c_m)|^2 &\leq C\sum_{j>J}\sum_{m\in\Lambda}\frac{|a^{-j}\xi-c_m|^{2\alpha}}{(1+|a^{-j}\xi-c_m|)^{2\gamma}}\\ &\leq C\sum_{j>J}\sum_{m\in\Lambda}\frac{1}{(1+|a^{-j}\xi-c_m|)^{2(\gamma-\alpha)}}\\ &\leq C\sum_{j>J}\sum_{m\in\Lambda}\frac{1}{(1+|a^{-j}\xi|)^{2(\gamma-\alpha)}}\\ &\leq C\sum_{j>J}\sum_{m\in\Lambda}|a^{-j}N|^{2\alpha}\\ &\leq C\sum_{m\in\Lambda}a^{-2J\alpha}=o(1),\quad J\to\infty.\end{aligned}$$

If $|j|\leq J$, $n\leq|\xi|<N$, then $\hat{\psi}(a^{-j}\xi-c_m)$ is uniformly continuous. Thus for every $\epsilon>0$, there exist δ, such that for all $|a-a'|<\delta$, $|\hat{\psi}(a^{-j}\xi-c_m)-\hat{\psi}(a'^{-j}\xi-c_m)|<\epsilon$. Hence,

$$\begin{aligned}&\Big|\sum_{|j|\leq J}\sum_{m\in\Lambda}(|\hat{\psi}(a^{-j}\xi-c_m)|^2-|\hat{\psi}(a'^{-j}\xi-c_m)|^2)\Big|\\ &\leq\sum_{|j|\leq J}\sum_{m\in\Lambda}(|\hat{\psi}(a^{-j}\xi-c_m)|+|\hat{\psi}(a'^{-j}\xi-c_m)|)|\hat{\psi}(a^{-j}\xi-c_m)-\hat{\psi}(a'^{-j}\xi-c_m)|\\ &\leq C(2J+1)\epsilon=o(1).\end{aligned}$$

This completes the proof. □

Remark 2.10. In the Theorem 2.9, if $a=1$, then $a^j=1$ for all $j\in\mathbb{Z}$, hence, the system $\{D_{a^j}T_{bk}E_{c_m}\psi\}_{j,k\in\mathbb{Z},m\in\Lambda}$ is not a wave packet frames for $L^2(\mathbb{R})$. Indeed, the sum in the Theorem 1.3 diverges.

Finally, the following theorem shows that perturbation about modulation parameter in terms of series.

Theorem 2.11. *Let $\{D_{a_j}T_{bk}E_{c_m}\psi\}_{j,k,m\in\mathbb{Z}}$ be a wave packet frame for $L^2(\mathbb{R})$ with bounds A and B,* supp $\hat{\psi}\subset[-\pi/b,\pi/b]$, *and*

$$\mu=\operatorname{essup}\Big|\sum_{j,m\in\mathbb{Z}}|\hat{\psi}(a_j^{-1}\xi-c_m)|^2-\sum_{j,m\in\mathbb{Z}}|\hat{\psi}(a_j^{-1}\xi-c'_m)|^2\Big|<bA.$$

Then $\{D_{a_j}T_{bk}E_{c'_m}\psi\}_{j,k,m\in\mathbb{Z}}$ is a wave packet frame for $L^2(\mathbb{R})$.

Proof. By the same computation as in the proof of Theorem 2.7, we have

$$\Big|\sum_{j,k,m\in\mathbb{Z}}|\langle f,D_{a_j}T_{bk}E_{c_m}\psi\rangle|^2-\sum_{j,k,m\in\mathbb{Z}}|\langle f,D_{a_j}T_{bk}E_{c'_m}\psi\rangle|^2\Big|\leq\frac{\mu}{b}\|f\|^2.$$

This completes the proof by using Lemma 1.2. □

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