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UNIQUENESS IN THE CAUCHY PROBLEM FOR QUASI-HOMOGENEOUS OPERATORS WITH PARTIALLY HOLOMORPHIC COEFFICIENTS

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1. Introduction and main reults

The purpose of this work is to extend to the case of quasi-homogeneous symbols the recent results of Tataru [10], Hörmander [3] and Robbiano-Zuily [7] concerning the uniqueness of the Cauchy problem for operators with partially holomorphic coefficients. Even in the merely C^∞ coefficients case our results will be more general that those given in Isakov [4], Dehman [1] and Lascar-Zuily [6]. The method used here will be basically the same as in the proof given by [7], that is the use of the Sjöstrand theory of FBI transform to microlocalize the symbols and then symbolic calculus for anisotropic pseudo-differential operators and the Fefferman-Phong inequality.

Let us be more precise. Let n, d be two non negative integers with $n+d \geq 1$. We shall set $\mathbb{R}^{d+n} = \mathbb{R}^d \times \mathbb{R}^n$ and, for X or ζ in $\mathbb{R}^{d+n}$, $X = (x, y)$, $\zeta = (\xi, \tau)$. Here y will be the "C^∞ variables" and x the "analytic ones".

Let $m = (m_1, \dots, m_n)$, $\tilde{m} = (\tilde{m}_1, \dots, \tilde{m}_d)$ be multi-indices, such that

$$\text{(1.1)} \qquad \begin{cases} 0 < m_1 \leq \cdots \leq m_{q-1} < m_q = \cdots = m_n = M, \\ 0 < \tilde{m}_1 \leq \cdots \leq \tilde{m}_{p-1} < \tilde{m}_p = \cdots = \tilde{m}_d = \tilde{M} = M. \end{cases}$$

We set $h_j = M/m_j$, $\tilde{h}_j = M/\tilde{m}_j$. $\{\cdot, \cdot\}_0$ will denote the quasi-homogeneous Poisson bracket that is

$$\text{(1.2)} \qquad \{f, g\}_0 = \sum_{j=q}^{n} \left(\frac{\partial f}{\partial \tau_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial \tau_j} \right) + \sum_{j=p}^{d} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right).$$

If $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, we set

$$\text{(1.3)} \qquad |\alpha : \tilde{m}| = \sum_{j=1}^{d} \frac{\alpha_j}{\tilde{m}_j}, \quad |\beta : m| = \sum_{j=1}^{n} \frac{\beta_j}{m_j}.$$

Let $P = P(x, y, D_x, D_y)$ be the quasi-homogeneous differential operator

$$P = \sum_{|\alpha:\tilde{m}|+|\beta:m|\leq 1} a_{\alpha\beta}(x, y) D_x^\alpha D_y^\beta, \tag{1.4}$$

with symbol

$$p(x, y, \xi, \tau) = \sum_{|\alpha:\tilde{m}|+|\beta:m|\leq 1} a_{\alpha\beta}(x, y)\xi^\alpha \tau^\beta, \tag{1.5}$$

and quasi-homogeneous principal symbol

$$p_M(x, y, \xi, \tau) = \sum_{|\alpha:\tilde{m}|+|\beta:m|=1} a_{\alpha\beta}(x, y)\xi^\alpha \tau^\beta. \tag{1.6}$$

We shall assume that

$$\begin{cases} \text{the coefficients } (a_{\alpha\beta}) \text{ of } P \text{ are } C^\infty \text{ in } (x, y) \text{ and analytic in } x \\ \text{in a neighborhood of a point } (x_0, y_0) \in \mathbb{R}^{d+n}. \end{cases} \tag{1.7}$$

Let S be a C^2 hypersurface through (x_0, y_0) locally given by

$$S = \{(x, y) : \varphi(x, y) = \varphi(x_0, y_0)\}, \quad \nabla_{p,q}\varphi(x_0, y_0) \neq 0, \tag{1.8}$$

where

$$\nabla_{p,q}\varphi = \left(0, \dots, 0, \frac{\partial\varphi}{\partial x_p}, \dots, \frac{\partial\varphi}{\partial x_d}; 0, \dots, 0, \frac{\partial\varphi}{\partial y_q}, \dots, \frac{\partial\varphi}{\partial y_n}\right). \tag{1.9}$$

Our results are as follows.

Theorem A. *Let us assume*

(H.1) *transversal ellipticity*: $p_M(x_0, y_0; 0, \tau) \neq 0$, *for all* τ *in* $\mathbb{R}^n \setminus \{0\}$.

$$\text{(H.2)} \quad \begin{cases} \textit{quasi-homogeneous pseudo-convexity}: \\ \textit{let } \Xi = (x_0, y_0; (0, \tau) + i\lambda\nabla_{p,q}\varphi(x_0, y_0)), \quad \tau \in \mathbb{R}^n, \\ \textit{then } p_M(\Xi) = \{p_M, \varphi\}_0(\Xi) = 0 \textit{ implies} \\ \frac{1}{i}\left\{\overline{p}_M(X; \zeta - i\lambda\nabla_{p,q}\varphi(X)); p_M(X; \zeta + i\lambda\nabla_{p,q}\varphi(X))\right\}_0\Big|_{\substack{X=(x_0,y_0)\\ \xi=0}} > 0. \end{cases}$$

Let V be a neighborhood of (x_0, y_0) and $u \in C^\infty(V)$ be such that

$$\begin{cases} Pu = 0 \quad \textit{in } V \\ \operatorname{supp} u \subset \{X \in V : \varphi(X) \leq \varphi(X_0)\}. \end{cases}$$

Then there exists a neighborhood W of (x_0, y_0) in which $u \equiv 0$.

Theorem B. *Let us assume*

$$(\mathrm{H.1})' \quad \begin{cases} \textit{principal normality: } \left|\{\overline{p}_M; p_M\}(x,y;0,\tau)\right| \le C|\tau|_m^{M-1}|p_M(x,y;0,\tau)|, \\ \textit{for all } (x,y) \textit{ in a neighborhood of } (x_0,y_0) \textit{ and all } \tau \textit{ in } \mathbb{R}^n, \\ \textit{where } |\tau|_m^{2M} = \sum_{j=1}^n |\tau_j|^{2m_j}. \end{cases}$$

$$(\mathrm{H.2})' \quad \begin{cases} \quad \textit{quasi-homogeneous pseudo-convexity:} \\ \text{(i) } n = 0 \textit{ or } n \ge 1 \textit{ and, with } Z = (x_0, y_0; 0, \tau),\ \tau \in \mathbb{R}^n \setminus \{0\}, \textit{ then} \\ \quad p_M(Z) = \{p_M, \varphi\}_0(Z) = 0 \textit{ implies } \mathrm{Re}\{\overline{p}_M; \{p_M, \varphi\}_0\}_0(Z) > 0. \\ \text{(ii) } \textit{Let } W = (x_0, y_0; (0,\tau) + i\lambda\nabla_{p,q}\varphi(x_0,y_0)),\ \tau \in \mathbb{R}^n, \textit{ then} \\ \quad p_M(W) = \{p_M, \varphi\}_0(W) = 0 \textit{ implies} \\ \quad \frac{1}{i}\left\{\overline{p}_M(X; \zeta - i\lambda\nabla_{p,q}\varphi(X)); p_M(X; \zeta + i\lambda\nabla_{p,q}\varphi(X))\right\}_0\Big|_{\substack{X=(x_0,y_0)\\ \xi=0}} > 0. \end{cases}$$

$$(\mathrm{H.3})' \qquad \textit{On } \xi = 0,\ p_M \textit{ does not depend on } x.$$

Then the same conclusion, as in Theorem A, *holds.*

Let us make some comments on these results. The Theorems A and B contain the results of Tataru, Hörmander and Robbiano-Zuily for which we take $m = (M, \dots, M)$, $\tilde{m} = (M, \dots, M)$. In the C^∞ case ($d = 0$), the Theorems A and B extend the results of Lascar-Zuily ([6], thm 1.3) (take $m = (1, 2, \dots, 2)$), the Theorem 2.1 in Dehman [1] and contain the results of Isakov ([4], thm 1.1 and 1.2) who consider only elliptic or real symbols. Furthermore with slight modifications of notations (1.2), (1.9), Theorems A and B remain valid with $\tilde{M} < M$ or $\tilde{M} > M$ (see (1.1)).

1. Here is an application of Theorem A. Let us consider, in a neighborhood V of $(0,0)$ in $\mathbb{R}_x \times \mathbb{R}_y^n$ a second order parabolic symbol of the form

$$p(x,y;\xi,\tau) = \sum_{j,k=2}^{n} a_{jk}(x,y)\tau_j\tau_k + i\tau_1 + a(x,y)\xi^2,$$

where the coefficients (a_{jk}) are real-valued, belong to $C^\infty(\mathbb{R}_x \times \mathbb{R}_y^n)$ and are analytic in x with $a(0,0) \neq 0$. We assume that the following parabolicity condition is satisfied

$$\sum_{j,k=2}^{n} a_{jk}(x,y)\tau_j\tau_k \ge C(\tau_2^2 + \dots + \tau_n^2) \text{ for all } (x,y) \in V,\ (\tau_2, \dots, \tau_n) \in \mathbb{R}^{n-1}.$$

Then the conclusion of Theorem A holds with $S = \{(x,y) : y_n = 0\}$ (we take $\varphi(x,y) = \exp(-\lambda y_n) - 1$, for λ large).

2. Application of Theorem B. Let us consider the case where $(x,y) \in \mathbb{R} \times \mathbb{R}^n$, $S = \{\varphi(x,y) = y_1 = 0\}$ and

$$P = D_{y_1}^2 + \sum_{j,k=2}^{n-1} a_{jk}(y) D_{y_j} D_{y_k} + c(y) D_{y_n} + d(x,y) D_x^2.$$

Assume moreover that

- (a_{jk}), c are real-valued, C^∞ in y and $c(0) \neq 0$.
- d is C^∞ in (x, y), analytic in x and $d(0) \neq 0$ real.

Then, it follow that (H.1)$'$ is empty, (H.3)$'$ is trivially satisfied and $\nabla_{p,q}\varphi(0) \neq 0$. We show that (H.2)$'$ (i) is equivalent to

$$\forall(\tau_2, \dots, \tau_{n-1}) \in \mathbb{R}^{n-2}, \quad \sum_{j,k=2}^{n-1} \frac{\partial a_{jk}}{\partial y_1}(0)\tau_j\tau_k - \frac{\partial c/\partial y_1(0)}{c(0)} \sum_{j,k=2}^{n-1} a_{jk}(0)\tau_j\tau_k < 0.$$

For example, we can take, $P = D_{y_1}^2 - \sum_{j=2}^{n-1} D_{y_j}^2 + (1 - y_n)D_{y_n} + (1 + ix)D_x^2$.

The proofs follow from Carleman estimates with an exponential weight $e^{-\lambda\psi}$ and these estimates follow from Gårding type inequalities on the operator $P_\lambda = e^{\lambda\psi}Pe^{-\lambda\psi}$. The problem is that all our conditions are made on the set $\{\xi = 0\}$. So we have to microlocalize our symbol on this set; this is achieved by the use of Sjöstrand's theory of the FBI transform [8], [9]. We then use the C^∞-machinery (the Hörmander-Weyl calculus, the Fefferman-Phong inequality, see [2]) to prove a Carleman estimate using some techniques of Lerner [5].

Finally I would like to thank Professor C. Zuily for useful discussions during the preparation of this paper.

2. The partial FBI transformation

In this section we collect some material essentially taken from [9], [7]. We introduce the partial Fourier-Bros-Iagolnitzer (FBI) transformation. It is defined for u in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^n)$ by

$$Tu(z, y, \lambda) = C(\lambda) \int_{\mathbb{R}^d} e^{-(\lambda/2)(x-z)^2} u(x, y)dx \tag{2.1}$$

where $z \in \mathbb{C}^d$, $y \in \mathbb{R}^n$, $\lambda \geq 1$, $C(\lambda) = 2^{-d/2}(\lambda/\pi)^{3d/4}$ and $z^2 = \sum_{j=1}^d (z^j)^2$, $z = (z^j) \in \mathbb{C}^d$.

The function Tu is C^∞ on $\mathbb{R}^{2d} \times \mathbb{R}^n \times [1, \infty[$ and entire-holomorphic in $z \in \mathbb{C}^d$ for all (y, λ) in $\mathbb{R}^n \times [1, \infty[$. Let us set

$$\Phi(z) = \frac{1}{2}(\operatorname{Im} z)^2, \quad z \text{ in } \mathbb{C}^d, \tag{2.2}$$

$$\Lambda_\Phi = \left\{(z, \xi) \in \mathbb{C}^{2d} : \xi = \frac{2}{i}\frac{\partial \Phi}{\partial x}(z)\right\} = \left\{(z, \xi) \in \mathbb{C}^{2d} : \xi = -\operatorname{Im} z\right\}, \tag{2.3}$$

$$K_T(x, \xi) = (x - i\xi, \xi), \quad (x, \xi) \in T^*\mathbb{R}^d. \tag{2.4}$$

Then $K_T : T^*\mathbb{R}^d \to \Lambda_\Phi$ is a diffeomorphism.

In the sequel we shall also work with the partial FBI transformation T_η associated with the phase $(i/2)(1+\eta)(x-z)^2$ where η is a small non negative real number,

$$T_\eta u(z,y,\lambda) = C(\lambda)\int_{\mathbb{R}^d} e^{-(\lambda/2)(1+\eta)(x-z)^2} u(x,y)dx. \tag{2.5}$$

Let

$$K_{T_\eta}(x,\xi) = \Big(x - \frac{i\xi}{1+\eta};\xi\Big). \tag{2.6}$$

Let us introduce some notations. For $k \in \mathbb{N}$ we set

$$L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H^k(\mathbb{R}^n)) = L^2\Big((\mathbb{C}^d, e^{-2\lambda(1+\eta)\Phi(x)}L(dx)); H^k(\mathbb{R}^n)\Big) \tag{2.7}$$

where $L(dx)$ denotes the Lebesgue measure in $\mathbb{C}^d$ and $H^k(\mathbb{R}^n)$ the usual Sobolev space.

If $k = 0$ we shall set for short

$$L^2_{(1+\eta)\Phi}\big(\mathbb{C}^d, H^0(\mathbb{R}^n)\big) = L^2_{(1+\eta)\Phi}, \tag{2.8}$$

$$|||u|||_k^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^n} (\lambda + |\tau|_m)^{2k}|\hat{u}(\zeta)|^2 d\zeta. \tag{2.9}$$

Then we have

Proposition 2.1 (see [9]). i) *T_η is an isometry from $L^2(\mathbb{R}^d, H^k(\mathbb{R}^n))$ to $L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H^k(\mathbb{R}^n))$.*
ii) *$T_\eta^* T_\eta$ is the identity on $L^2(\mathbb{R}^n)$, where T_η^* is the adjoint of T_η.*
iii) *$T_\eta T_\eta^*$ is the projection from $L^2_{(1+\eta)\Phi}$ to $L^2_{(1+\eta)\Phi} \cap \mathcal{H}(\mathbb{C}^d)$ where $\mathcal{H}$ denotes the space of holomorphic functions. In particular $T_\eta T_\eta^* v = v$ if $v = Tw$ where w is in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^n)$.*

3. Transfer to the complex domain and the localization procedure

Let $p = \sum_{|\alpha:\tilde{m}|+|\beta:m|\le 1} a_{\alpha\beta}(x,y)\xi^\alpha\tau^\beta$, $(x,y) \in \mathbb{R}^d \times \mathbb{R}^n$, be a polynomial with coefficients in $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$.

Assume moreover that

$$\begin{cases} \text{there exists } C_0 > 0 \text{ such that if we set } \omega_1 = \{z \in \mathbb{C}^d : |z| < C_0\} \\ \text{and } \omega_2 = \{y \in \mathbb{R}^n : |y| < C_0\}, \text{ then for all } (\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^n, \\ |\alpha:\tilde{m}| + |\beta:m| \le 1, \text{ we have } a_{\alpha\beta} \in C^\infty(\omega_2, \mathcal{H}(\omega_1)). \end{cases} \tag{3.1}$$

Let $P = Op_\lambda^\omega(p)$ be the semi-classical Weyl quantized operator with symbol p, for $u \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$,

$$Pu(x,y) = \Big(\frac{\lambda}{2\pi}\Big)^{d+n}\iint e^{i\lambda(X-\tilde{X})\zeta} p\Big(\frac{X+\tilde{X}}{2};\lambda\zeta\Big)u(\tilde{X})d\tilde{X}d\zeta. \tag{3.2}$$

Let ψ be a real quadratic polynomial on $\mathbb{R}^d \times \mathbb{R}^n$. For any $\lambda \geq 1$, we shall denote P_λ the differential operator defined by

$$P_\lambda = e^{\lambda\psi} P e^{-\lambda\psi}. \tag{3.3}$$

It follows that

$$P_\lambda u(X) = \Big(\frac{\lambda}{2\pi}\Big)^{d+n} \iint e^{i\lambda(X-\tilde{X})\zeta} P\Big(\frac{X+\tilde{X}}{2}; \lambda\zeta + i\lambda\psi'\Big(\frac{X+\tilde{X}}{2}\Big)\Big) u(\tilde{X}) d\tilde{X} d\zeta. \tag{3.4}$$

Proposition 3.1 (see [7]). *For v in $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$, we have $TP_\lambda v = \tilde{P}_\lambda T v$ where*

$$\tilde{P}_\lambda T v(X, \lambda) = \Big(\frac{\lambda}{2\pi}\Big)^{d+n} \iint e^{i\lambda(y-\tilde{y})\tau} \Big(\iint_{\xi = -\operatorname{Im}((x+\tilde{x})/2)} \omega \Big) d\tilde{y} d\tau \tag{3.5}$$

where

$$\omega = e^{i\lambda(x-\tilde{x})\xi} p\Big(\frac{x+\tilde{x}}{2} + i\xi, \frac{y+\tilde{y}}{2}; \lambda\zeta + i\lambda\psi'\Big(\frac{x+\tilde{x}}{2} + i\xi; \frac{y+\tilde{y}}{2}\Big)\Big) \\ Tv(\tilde{x}, \tilde{y}, \lambda) d\tilde{x} \wedge d\xi. \tag{3.6}$$

Let δ is a positive real number such that $2\delta < C_0$ where C_0 is defined in (3.1) and v is a C^∞ function such that $\operatorname{supp} v \subset \{X \in \mathbb{R}^d \times \mathbb{R}^n : |X| \leq \delta\}$. Let $\tilde{P}_\lambda$ be defined in Proposition 3.1.

Case of Theorem A.

Theorem 3.2 (see [7]). *There exists $\chi \in C_0^\infty(\mathbb{C}^{2d})$, $\chi(x, \xi) = 1$ if $|x| + |\xi| \leq \delta$, $\chi(x, \xi) = 0$ if $|x| + |\xi| \geq 2\delta$ such that if we set, for $\eta \in]0, 1]$,*

$$\tilde{Q}_\lambda T v(X, \lambda) = \Big(\frac{\lambda}{2\pi}\Big)^{d+n} \iint e^{i\lambda(y-\tilde{y})\tau} \Big(\iint_{\xi = (1+\eta)\operatorname{Im}((x+\tilde{x})/2)} \chi\Big(\frac{x+\tilde{x}}{2}; \xi\Big) \omega \Big) d\tilde{y} d\tau \tag{3.7}$$

where ω is defined in (3.6), *then*

$$\tilde{P}_\lambda T v = \tilde{Q}_\lambda T v + \tilde{R}_\lambda T v + \tilde{g}_\lambda \tag{3.8}$$

where with, for any N in $\mathbb{N}$,

$$\|\tilde{R}_\lambda T v\|_{L^2_{(1+\eta)\Phi}} \leq \frac{C_N}{\lambda^N} \|Tv\|_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H^M_\lambda(\mathbb{R}^n))} \tag{3.9}$$

$$\|\tilde{g}_\lambda\|_{L^2_{(1+\eta)\Phi}} \leq C e^{-(\lambda/3)\eta\delta^2} \|v\|_{L^2(\mathbb{R}^d, H^M_\lambda(\mathbb{R}^n))} \tag{3.10}$$

where

$$\|w\|_{H_\lambda^M(\mathbb{R}^n)} = \sum_{\sum_{j=1}^n h_j\beta_j \le M} \lambda^{M-\sum_{j=1}^n h_j\beta_j} \|D^\beta w\|_{L^2(\mathbb{R}^n)}. \tag{3.11}$$

Case of Theorem B.

Recall that we have assumed

$$\text{on } \xi = 0, \ \ p_M \text{ does not depend on } x. \tag{3.12}$$

In the case we have

$$p_M(X; \lambda\zeta + i\lambda\psi'(X)) = p'_M(y,\tau) + p'_{M-1}(X,\zeta) \tag{3.13}$$

where p'_M is a polynomial of order M in τ and p'_{M-1} is a polynomial of order M in ζ, but of order $M-1$ in τ.

Writing $p(X,\zeta) = p_M(X,\zeta) + p''_M(X,\zeta)$ where

$$p''_M(X,\zeta) = \sum_{|\alpha:\tilde{m}|+|\beta:m| \le 1-1/M} a_{\alpha\beta}(X)\xi^\alpha\tau^\beta. \tag{3.14}$$

We have

Theorem 3.3 (see [7]). *There exists $\chi \in C_0^\infty(\mathbb{C}^{2d})$, $\chi(x,\xi) = 1$ if $|x| + |\xi| \le \delta$, $\chi(x,\xi) = 0$, if $|x| + |\xi| \ge 2\delta$, such that, if we set, for $\eta \in]0,1]$*

$$\tilde{Q}_\lambda T v(X,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{d+n} \iint e^{i\lambda(y-\tilde{y})\tau}\left(\iint_{\xi=-(1+\eta)\operatorname{Im}((x+\tilde{x})/2)} \tilde{\omega}\right) d\tilde{y}\,d\tau \tag{3.15}$$

where

$$\begin{aligned}\tilde{\omega} = e^{i\lambda(x-\tilde{x})\xi}\Big[p'_M(y,\tau) + \chi\Big(\frac{x+\tilde{x}}{2};\xi\Big)\Big[p'_{M-1}\Big(\frac{x+\tilde{x}}{2}+i\xi, \frac{y+\tilde{y}}{2};\zeta\Big)\\ + p''_M\Big(\frac{x+\tilde{x}}{2}+i\xi, \frac{y+\tilde{y}}{2}; \lambda\zeta + i\lambda\psi'\Big(\frac{x+\tilde{x}}{2}+i\xi;\frac{y+\tilde{y}}{2}\Big)\Big)\Big]\Big] Tv(\tilde{x},\tilde{y},\lambda)\,d\tilde{x}\wedge d\xi.\end{aligned} \tag{3.16}$$

Then we have, with $\tilde{P}_\lambda$ introduced in Proposition 3.1,

$$\tilde{P}_\lambda T v = \tilde{Q}_\lambda T v + \tilde{R}_\lambda T v + \tilde{g}_\lambda \tag{3.17}$$

with

$$\|\tilde{R}_\lambda T v\|_{L^2_{(1+\eta)\Phi}} \le \frac{C_N}{\lambda^N}\|Tv\|_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H_\lambda^{M-1}(\mathbb{R}^n))} \tag{3.18}$$

$$\|\tilde{g}_\lambda\|_{L^2_{(1+\eta)\Phi}} \le Ce^{-(\lambda/3)\eta\delta^2}\|v\|_{L^2(\mathbb{R}^d, H_\lambda^{M-1}(\mathbb{R}^n))}. \tag{3.19}$$

4. Back to the real domain

Let v be in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^n)$ and $w = T_\eta^* T v$, then it follows that

$$w = T_\eta^* T v \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^n), \quad T_\eta w = T v. \tag{4.1}$$

We deduce from Proposition 3.1

$$\tilde{Q}_\lambda T v = \tilde{Q}_\lambda T_\eta w = T_\eta Q_\lambda \omega, \tag{4.2}$$

where Q_λ is an operator on $\mathbb{R}^d \times \mathbb{R}^n$, pseudo-differential in x, differential in y.
Moreover denoting by σ^ω the Weyl symbol

$$\sigma^\omega(Q_\lambda)(x, \xi; y, \tau) = \sigma^\omega(\tilde{Q}_\lambda)(K_{T_\eta}(x, \xi); y, \tau), \tag{4.3}$$

where

$$\begin{cases} \sigma^\omega(Q_\lambda)(X, \zeta) - \chi\Big(x - \dfrac{i}{1+\eta}\xi; \xi\Big) p\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \lambda\zeta \\ \qquad\qquad + i\lambda\psi'\Big(x + \dfrac{i\eta}{1+\eta}\xi, y\Big)\Big) \quad \text{(thm A)} \\ \sigma^\omega(Q_\lambda)(X, \zeta) = p'_M(y, \tau) + \chi\Big(x - \dfrac{i}{1+\eta}\xi; \xi\Big)\Big[p'_{M-1}\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \zeta\Big) \\ \qquad + p''_M\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \lambda\zeta + i\lambda\psi'\Big(x + \dfrac{i\eta}{1+\eta}\xi, y\Big)\Big)\Big] \quad \text{(thm B)} \end{cases} \tag{4.4}$$

and

$$Q_\lambda u(X, \lambda) = \Big(\frac{\lambda}{2\pi}\Big)^{n+d} \iint e^{i\lambda(X-\tilde{X})\zeta} \sigma^\omega(Q_\lambda)\Big(\frac{X+\tilde{X}}{2}; \lambda\zeta\Big) u(\tilde{X}) d\tilde{X} d\zeta.$$

Moreover, we have

$$\sigma^\omega(Q_\lambda)(X, \zeta) = q_M(X, \zeta) + q_{M-1}(X, \zeta), \tag{4.5}$$

where

$$\begin{cases} q_M(X, \zeta) = \chi\Big(x - \dfrac{i}{1+\eta}\xi; \xi\Big) p_M\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \lambda\zeta \\ \qquad\qquad + i\lambda\psi'\Big(x + \dfrac{i\eta}{1+\eta}\xi, y\Big)\Big) \quad \text{(thm A)} \\ q_M(X, \zeta) = p'_M(y, \tau) + \chi\Big(x - \dfrac{i}{1+\eta}\xi, \xi\Big) p'_{M-1}\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \zeta\Big) \quad \text{(thm B)} \end{cases} \tag{4.6}$$

and

$$(4.7)\qquad q_{M-1}(X,\zeta) = \chi\Big(x - \frac{i}{1+\eta}\xi, \xi\Big) \times p''_M\Big(x + \frac{i\eta}{1+\eta}\xi, y; \lambda\zeta + i\lambda\psi'\Big(x + \frac{i\eta}{1+\eta}\xi, y\Big)\Big).$$

5. The estimates in case of Theorem A

We are now prepared to prove Carleman estimates for Q_λ. Without loss of generality we may assume that $(x_0, y_0) = 0$ and $\varphi(0) = 0$. Let, for $Z = (x_1, \dots, x_d; y_1, \dots, y_n)$,

$$(5.1)\qquad |Z|^{2M}_{(m,\tilde m)} = |x_1|^{2\tilde m_1} + \cdots + |x_d|^{2\tilde m_d} + |y_1|^{2m_1} + \cdots + |y_n|^{2m_n}.$$

Lemma 5.1. *There exist positive constants C, η_0 such that for all η in $]0, \eta_0]$ and if we set*

$$\psi(X) = \varphi'(0)X + \frac{1}{2}\varphi''(0)X \cdot X - \frac{1}{2C^2}|X|^2 + \frac{C}{2}(\varphi'(0)X)^2,$$

then

$$(5.2)\qquad C|q_M(X,\zeta)|^2 + \frac{1}{i}\{\overline{q}_M, q_M\}(X,\zeta) \ge \frac{1}{C}(\lambda + |\lambda\tau|_m)^{2M},$$

for $|X| + |\xi| \le 1/C^2$ and λ so large.

By homogeneity, (5.2) is still true with the same ψ if we replace ψ by $\rho\psi$ where ρ is a positive constant.

Proof. We first take C so large that $\chi = 1$ if $|X| + |\xi| \le 1/C^2$. It follows then from (4.6) that

$$q_M(X,\zeta) = p_M\Big(x + \frac{i\eta}{1+\eta}\xi, y; \lambda\zeta + i\lambda\psi'\Big(x + \frac{i\eta}{1+\eta}\xi, y\Big)\Big), = p_M\big(X; \lambda\zeta + i\lambda\psi'(X)\big) + \frac{\eta}{C^2}\mathcal{O}\big((\lambda + |\lambda\tau|_m)^M\big),$$

and

$$(5.3)\qquad \begin{cases} \{\overline{q}_M, q_M\}|_{\xi=0} = \big\{\overline{p}_M(X; \lambda\zeta - i\lambda\psi'(X)); p_M(X; \lambda\zeta + i\lambda\psi'(X))\big\}\big|_{\xi=0} \\ \qquad\qquad + \eta\mathcal{O}\big((\lambda + |\lambda\tau|_m)^{2M}\big). \end{cases}$$

We shall also write

$$(5.4)\qquad \{\overline{q}_M, q_M\}(X,\zeta) = \{\overline{q}_M, q_M\}|_{\xi=0}(X,\zeta) + \frac{1}{C^2}\mathcal{O}((\lambda + |\lambda\tau|_m)^{2M}),$$

and

$$p_M(X;\lambda\zeta+i\lambda\psi'(X)) = p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0} + \Big(\frac{1}{C^2}+\lambda^{-1/(M-1)}\Big)\mathcal{O}((\lambda+|\lambda\tau|_m)^M). \tag{5.5}$$

Then

$$q_M(X,\zeta) = p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0} + \Big(\frac{1}{C^2}+\lambda^{-1/(M-1)}\Big)\mathcal{O}((\lambda+|\lambda\tau|_m)^M), \tag{5.6}$$

and

$$\{\overline{q}_M,q_M\}(X,\zeta) = \Big\{\overline{p}_M(X;\lambda\zeta-i\lambda\nabla_{p,q}\psi(X)),\, p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))\Big\}\Big|_{\xi=0} + \Big(\eta+\frac{1}{C^2}+\lambda^{-1/(M-1)}\Big)\mathcal{O}((\lambda+|\lambda\tau|_m)^{2M}). \tag{5.7}$$

Furthermore, we have

$$\begin{aligned}&\frac{C}{4}\Big|p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\Big|^2\\&+\frac{1}{2i}\Big\{\overline{p}_M(X;\lambda\zeta-i\lambda\nabla_{p,q}\psi(X));\, p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))\Big\}\Big|_{\xi=0}\\&\geq \frac{1}{C}(\lambda+|\lambda\tau|_m)^{2M},\quad \text{for } |X|\leq\frac{1}{C^2} \text{ and } \tau \text{ in } \mathbb{R}^n.\end{aligned} \tag{5.8}$$

Indeed, (5.8) is equivalent to

$$\begin{aligned}&\frac{C}{4}\Big|p_M(X;\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\Big|^2\\&+\frac{\lambda}{2i}\Big\{\overline{p}_M(X;\zeta-i\lambda\nabla_{p,q}\psi(X));\, p_M(X;\zeta+i\lambda\nabla_{p,q}\psi(X))\Big\}\Big|_{\xi=0}\\&\geq \frac{1}{C}(\lambda+|\tau|_m)^{2M},\quad \text{for } |X|\leq\frac{1}{C^2}.\end{aligned}$$

We see, setting $\Gamma=\lambda/(\lambda+|\tau|_m)$, $W=(X,Z+i\Gamma\nabla_{p,q}\psi(X))$ and

$$Z=\Big(0,\ldots,0;\tau_1/(\lambda+|\tau|_m)^{h_1},\ldots,\tau_n/(\lambda+|\tau|_m)^{h_n}\Big)$$

that (5.8) is equivalent to

$$
\begin{aligned}
(5.9)\quad & \frac{C}{4}|p_M(W)|^2 + \Gamma \operatorname{Im}\Bigg(\sum_{j=1}^{n} (\lambda + |\tau|_m)^{1-h_j} \frac{\partial \overline{p}_M}{\partial \tau_j}(\overline{W}) \frac{\partial p_M}{\partial y_j}(W) \\
& + \sum_{k=1}^{d} (\lambda + |\tau|_m)^{1-\tilde{h}_k} \frac{\partial \overline{p}_M}{\partial \xi_k}(W) \frac{\partial p_M}{\partial x_k}(W) \Bigg) \\
& + \Gamma^2 \operatorname{Re}\Bigg(\sum_{j=1}^{n} \sum_{s=q}^{n} \frac{\partial^2 \psi}{\partial y_s \partial y_j}(X) \frac{\partial \overline{p}_M}{\partial \tau_j}(\overline{W}) \frac{\partial p_M}{\partial \tau_s}(W)(\lambda + |\tau|_m)^{1-h_j} \\
& + \sum_{j=1}^{n} \sum_{k=p}^{d} \frac{\partial^2 \psi}{\partial x_k \partial y_j}(X)(\lambda + |\tau|_m)^{1-h_j} \frac{\partial \overline{p}_M}{\partial \tau_j}(\overline{W}) \frac{\partial p_M}{\partial \xi_k}(W) \\
& + \sum_{j=1}^{d} \sum_{k=q}^{n} \frac{\partial^2 \psi}{\partial y_k \partial x_j}(X)(\lambda + |\tau|_m)^{1-\tilde{h}_j} \frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}) \frac{\partial p_M}{\partial \tau_k}(W) \\
& + \sum_{j=1}^{d} \sum_{k=p}^{d} \frac{\partial^2 \psi}{\partial x_k \partial x_j}(X)(\lambda + |\tau|_m)^{1-\tilde{h}_j} \frac{\partial \overline{p}_M}{\partial \xi_j}(W) \frac{\partial p_M}{\partial \xi_k}(W) \Bigg) \geq \frac{1}{C}, \quad \text{for } |X| \leq \frac{1}{C^2}.
\end{aligned}
$$

We prove (5.9) by contradiction. If it is false one can find sequences X_k, λ_k, τ_k, Γ_k with $|X_k| \leq 1/k^2$, $\lambda_k \geq e^k$ and τ_k in $\mathbb{R}^n$, such that, by definition ψ,

$$
\begin{aligned}
(5.10)\quad & \frac{k}{4}|p_M(W_k)|^2 + \Gamma_k \operatorname{Im}\Bigg(\sum_{j=q}^{n} \frac{\partial \overline{p}_M}{\partial \tau_j}(\overline{W}_k) \frac{\partial p_M}{\partial y_j}(W_k) + \sum_{j=p}^{d} \frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}_k) \frac{\partial p_M}{\partial x_j}(W_k) \Bigg) \\
& + \Gamma_k^2 \operatorname{Re}\Bigg(\sum_{s,j=q}^{n} \frac{\partial^2 \varphi}{\partial y_s \partial y_j}(0) \frac{\partial \overline{p}_M}{\partial \tau_j}(\overline{W}_k) \frac{\partial p_M}{\partial \tau_s}(W_k) + \sum_{j,s=p}^{d} \frac{\partial^2 \varphi}{\partial x_j \partial x_s}(0) \frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}_k) \frac{\partial p_M}{\partial \xi_s}(W_k) \\
& + 2 \sum_{s=q}^{n} \sum_{j=p}^{d} \frac{\partial^2 \varphi}{\partial y_s \partial x_j}(0) \frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}_k) \frac{\partial p_M}{\partial \tau_s}(W_k) \Bigg) \\
& + k\Gamma_k^2 \Bigg(\Bigg| \sum_{j=p}^{d} \frac{\partial \varphi}{\partial x_j}(0) \frac{\partial p_M}{\partial \xi_j}(W_k) \Bigg|^2 + \Bigg| \sum_{j=q}^{n} \frac{\partial \varphi}{\partial y_j}(0) \frac{\partial p_M}{\partial \tau_j}(W_k) \Bigg|^2 \Bigg) \\
& - \frac{\Gamma_k^2}{k^2} \Bigg(\sum_{j=q}^{n} \frac{\partial \overline{p}_M}{\partial \tau_j}(\overline{W}_k) \frac{\partial p_M}{\partial \tau_j}(W_k) + \sum_{j=p}^{d} \frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}_k) \frac{\partial p_M}{\partial \xi_j}(W_k) \Bigg) \\
& + 2k\Gamma_k^2 \operatorname{Re}\Bigg[\Bigg(\sum_{j=q}^{n} \frac{\partial \varphi}{\partial y_j}(0) \frac{\partial p_M}{\partial \tau_j}(W_k) \Bigg) \Bigg(\sum_{s=p}^{d} \frac{\partial \varphi}{\partial x_s}(0) \frac{\partial \overline{p}_M}{\partial \xi_s}(\overline{W}_k) \Bigg) \Bigg] + A_k \leq \frac{1}{k}
\end{aligned}
$$

where

$$(5.11)\qquad |A_k| \leq C_0 k \lambda_k^{-1/(M-1)} \leq C_0 k e^{-k/(M-1)}, \quad C_0 \text{ is independent of } k.$$

Since $\Gamma_k + |Z_k|_{(m,\tilde{m})} = 1$, taking subsequences, we may assume that

$$\Gamma_k \to \Gamma^0 \text{ and } Z_k \to Z^0 \text{ with } \Gamma^0 + |Z^0|_{(m,\tilde{m})} = 1. \tag{5.12}$$

CASE 1. $\Gamma^0 \neq 0$.

If we divide both members of (5.10) by k, we get with $W^0 = (0; Z^0 + i\Gamma\nabla_{p,q}\varphi(0))$

$$p_M(W^0) = \{p_M, \varphi\}_0(W^0) = 0.$$

Removing all positive terms in (5.10) and letting k go to $+\infty$, we get

$$\begin{aligned}
&\Gamma^0 \operatorname{Im}\left(\sum_{j=q}^{n} \frac{\partial \overline{p}_m}{\partial \tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial y_j}(W^0) + \sum_{j=p}^{d} \frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial x_j}(W^0) \right) \\
&+(\Gamma^0)^2 \operatorname{Re}\left(\sum_{s,j=q}^{n} \frac{\partial^2 \varphi}{\partial y_s \partial y_j}(0)\frac{\partial \overline{p}_M}{\partial \tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial \tau_s}(W^0) \right. \\
&+ \sum_{j,s=p}^{d} \frac{\partial^2 \varphi}{\partial x_j \partial x_s}(0)\frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial \xi_s}(W^0) \\
&\left. +2\sum_{s=q}^{n}\sum_{j=p}^{d} \frac{\partial^2 \varphi}{\partial y_s \partial x_j}(0)\frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial \tau_s}(W^0) \right) \leq 0
\end{aligned}$$

which contradicts the hypothesis (H.2) in theorem A.

CASE 2. $\Gamma^0 = 0$.

Since $\Gamma^0 + |Z^0|_{(m,\tilde{m})} = 1$, we have $Z^0 \neq 0$ and $W^0 = (0, Z^0)$. If we divide both members of (5.10) by k, we get $p_M(W^0) = 0$ which is contradiction with (H.1) in Theorem A.

Now (5.6), (5.7) and (5.8) imply (5.2) if η is small enough and C, λ so large. This ends the proof of Lemma 5.1. □

From now on C is fixed according to Lemma 5.1.

Let $\tilde{\theta}_0 \in C^\infty(\mathbb{C}^{2d})$ be such that $0 \leq \tilde{\theta}_0 \leq 1$ and

$$\begin{cases} \tilde{\theta}_0(x,\xi) = 1 & \text{if } |x| + |\xi| \leq \dfrac{\eta}{1+\eta} \cdot \dfrac{1}{4C^2}, \\ \tilde{\theta}_0(x,\xi) = 0 & \text{if } |x| + |\xi| \geq \dfrac{\eta}{1+\eta} \cdot \dfrac{1}{2C^2}, \\ \tilde{\theta}_0 \text{ is almost analytic on } \Lambda_{(1+\eta)\Phi}. \end{cases} \tag{5.13}$$

Let us set, with K_{T_η} defined in (2.6),

$$\theta_0 = \tilde{\theta}_0|_{\Lambda_{(1+\eta)\Phi}} \circ K_{T_\eta}. \tag{5.14}$$

It is easy to see that $\theta_0 \in C^\infty(\mathbb{R}^{2d})$ and there exists $\varepsilon_0 \in]0, 1/(2C^2)[$ such that

$$\theta_0(x,\xi) = \begin{cases} 1 & \text{if } |x| + |\xi| \le \varepsilon_0, \\ 0 & \text{if } |x| + |\xi| \ge \dfrac{1}{2C^2}. \end{cases} \tag{5.15}$$

Let $h \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le h \le 1$ and

$$h = \begin{cases} 1 & \text{if } |y| \le \dfrac{1}{4C^2}, \\ 0 & \text{if } |y| \ge \dfrac{1}{2C^2}. \end{cases} \tag{5.16}$$

Finally let us set

$$\theta(X,\xi) = h(y)\theta_0(x,\xi). \tag{5.17}$$

Then

$$\theta(X,\xi) = \begin{cases} 1 & \text{if } |X| + |\xi| \le \varepsilon_0, \\ 0 & \text{if } |X| + |\xi| \ge \dfrac{1}{C^2}. \end{cases} \tag{5.18}$$

Lemma 5.2. *Let $Q = Op_\lambda^w(q_M)$. There exist positive constants C_0, C_1, λ_0 such that for every u in $\mathcal{S}(\mathbb{R}^{d+n})$ and $\lambda \ge \lambda_0$, we have*

$$\frac{C_1}{\lambda}\big(Op_\lambda^w((1-\theta)(\lambda + |\lambda\tau|_m)^{2M})u, u\big)_{L^2} + \|Qu\|_{L^2}^2 \ge \frac{C_0}{\lambda}|||u|||_M^2. \tag{5.19}$$

Proof. We write $Q = Q_R + iQ_I$ where $Q_R = Op_\lambda^w(\operatorname{Re} q_M)$, $Q_I = Op_\lambda^w(\operatorname{Im} q_M)$. Then writing $\|\cdot\|$ for the $L^2(\mathbb{R}^{d+n})$-norm

$$\|Qu\|^2 = \|Q_R u\|^2 + \|Q_I u\|^2 + \frac{1}{2}([Q^*, Q]u, u). \tag{5.20}$$

Now the semiclassical principal symbols of $[Q^*, Q]$ and $Q_K^* Q_K$ are $(1/i)\{\overline{q}_M, q_M\}$ and q_K^2 where $q_R = \operatorname{Re} q_M$, $q_I = \operatorname{Im} q_M$. We claim that one can find a positive constant B such that

$$\begin{aligned} &B(1-\theta)(\lambda + |\lambda\tau|_m)^{2M} + C|q_M(X,\zeta)|^2 + \frac{1}{i}\{\overline{q}_M, q_M\}(X,\zeta) \\ &\ge \frac{1}{C}(\lambda + |\lambda\tau|_m)^{2M}, \quad \text{for all } (X,\zeta) \in \mathbb{R}^{2(d+n)}. \end{aligned} \tag{5.21}$$

Indeed Lemma 5.1 implies (5.21) if $|X|+|\xi| \le 1/C^2$, since $0 \le \theta \le 1$, and if $|X|+|\xi| \ge 1/C^2$ then, by (5.18), $\theta = 0$ and $|q_M|^2 + |\{\overline{q}_M, q_M\}| \le C_1(\lambda + |\lambda\tau|_m)^{2M}$, thus (5.21) is true if B is large enough.

Then we can apply the Gårding inequality in the following context. Let

$$g = dx^2 + dy^2 + d\xi^2 + \sum_{j=1}^{n} \frac{\lambda^2 d\tau_j^2}{(\lambda + |\lambda\tau|_m)^{2h_j}}.$$

This is a metric which is temperate and slowly varying in the sense of Hörmander [2]. Let $a \in S((\lambda + |\lambda\tau|_m)^k, g)$, $k \in \mathbb{N}$, be a symbol such that $\operatorname{Re} a \geq \delta(\lambda + |\lambda\tau|_m)^{2k}$, and $A = Op_\lambda^w(a)$. Then there exists $\lambda_0 > 0$ such that for every u in $\mathcal{S}(\mathbb{R}^{d+n})$ and every $\lambda \geq \lambda_0$

$$\operatorname{Re}(Au, u)_{L^2} \geq \frac{\delta}{2} |||u|||_k^2. \tag{5.22}$$

Thus we may apply (5.22) with, for a, the left hand side of (5.21). It follows that for $\lambda \geq \lambda_0$

$$\begin{aligned} &B\big(Op_\lambda^w((1-\theta)(\lambda + |\lambda\tau|_m)^{2M})u, u\big) + C\|Q_R u\|^2 + C\|Q_I u\|^2 \\ &+\lambda([Q^*, Q]u, u) \geq \frac{1}{2C}|||u|||_M^2. \end{aligned}$$

Now, we deduce from (5.20) that

$$2\lambda\|Qu\|_{L^2}^2 \geq C\big(\|Q_R u\|^2 + \|Q_I u\|^2 + \lambda([Q^*, Q]u, u)\big) \quad \text{if } 2\lambda \geq C,$$

and Lemma 5.2 follows. □

Proposition 5.3. *Let Q_λ be defined in* (4.4). *Then one can find positive constants C_0, C_1, λ_0 such that for u in $\mathcal{S}(\mathbb{R}^{d+n})$ and $\lambda \geq \lambda_0$*

$$\frac{C_1}{\lambda}\big(Op_\lambda^w((1-\theta)(\lambda + |\lambda\tau|_m)^{2M})u, u\big)_{L^2} + \|Q_\lambda u\|_{L^2}^2 \geq \frac{C_0}{\lambda}|||u|||_M^2. \tag{5.23}$$

Proof. Writing $Q_\lambda = Q + Q_{M-1}$ where $Q_{M-1} = Op_\lambda^w(q_{M-1})$ defined in (4.7), then

$$\|Qu\|_{L^2}^2 \leq 2\|Q_\lambda u\|_{L^2}^2 + 2\|Q_{M-1}u\|_{L^2}^2,$$

and

$$Q_{M-1} \in Op_\lambda^w\big(S((\lambda + |\lambda\tau|_m)^{M-1}, g)\big),$$

we deduce that

$$\|Qu\|_{L^2}^2 \leq 2\|Q_\lambda u\|_{L^2}^2 + \frac{C}{\lambda^2}|||u|||_M^2. \tag{5.24}$$

It follows from Lemma 5.2 and (5.24)

$$\frac{C_1}{\lambda}\big(Op_\lambda^w((1-\theta)(\lambda+|\lambda\tau|_m)^{2M})u,u\big)_{L^2}+2\|Q_\lambda u\|_{L^2}^2+\frac{C}{\lambda^2}|||u|||_M^2\geq\frac{C_0}{\lambda}|||u|||_M^2,$$

and Proposition 5.3 follows. □

We are now ready to prove the following estimate.

Proposition 5.4 (see [7]). *Let $\tilde{Q}_\lambda$ be defined in* Theorem 3.2. *Then there exist positive constants C_1, C_2, λ_0, such that for $v\in C_0^\infty(\mathbb{R}^{d+n})$,* $\operatorname{supp} v\subset\big\{X:|X|\leq 1/(4C^2)\big\}$ *and $\lambda\geq\lambda_0$*

$$\|Tv\|^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d,H^M_\lambda(\mathbb{R}^n))}\leq C_1\lambda\|\tilde{Q}_\lambda Tv\|^2_{L^2_{(1+\eta)\Phi}}+C_2e^{-\lambda\sigma}|||v|||_M^2,\tag{5.25}$$

where $\sigma>0$ depends only on η and C.

Proof. We apply Proposition 5.3 to $u=T_\eta^*Tv$ which is in $\mathcal{S}(\mathbb{R}^{d+n})$. It follows from Proposition 2.1

$$|||u|||_M=\|T_\eta u\|_{L^2_{(1+\eta)\Phi}(H^M_\lambda)}=\|Tv\|_{L^2_{(1+\eta)\Phi}(H^M_\lambda)},\tag{5.26}$$

$$\|Q_\lambda u\|_{L^2}=\|T_\eta Q_\lambda T_\eta^*Tv\|_{L^2_{(1+\eta)\Phi}}=\|\tilde{Q}_\lambda Tv\|_{L^2_{(1+\eta)\Phi}}.\tag{5.27}$$

Let us set $R=Op_\lambda^w((1-\theta)(\lambda+|\lambda\tau|_m)^{2M})$. Then Proposition 4.6 in [7] show that for any integer N one can find a positive constant C_N such that

$$|(Ru,u)_{L^2}|\leq\frac{C_N}{\lambda^N}\|Tv\|^2_{L^2_{(1+\eta)\Phi}(H^M_\lambda)}+\mathcal{O}(e^{-\lambda\sigma}|||v|||_M^2),\quad\sigma>0.\tag{5.28}$$

It follows from (5.23), (5.26), (5.27) and (5.28) that Proposition 5.4 is proved. □

Theorem 5.5. *Let $\tilde{P}_\lambda$ be the operator occuring in* Proposition 3.1. *One can find positive constants C_1, C_2, λ_0, σ such that for $v\in C_0^\infty(\mathbb{R}^{d+n})$,* $\operatorname{supp} v\subset\{X:|X|\leq 1/(4C^2)\}$ *and $\lambda\geq\lambda_0$ we have*

$$\|Tv\|^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d,H^M_\lambda(\mathbb{R}^n))}\leq C_1\lambda\|\tilde{P}_\lambda Tv\|^2_{L^2_{(1+\eta)\Phi}}+C_2e^{-\lambda\sigma}|||v|||_M^2.\tag{5.29}$$

Proof. This follows from Proposition 5.4 and Theorem 3.2. □

6. The estimates in case of Theorem B

Let $Q_M=Op_\lambda^w(q_M)$ where q_M is defined in (4.5). We have

$$\|Q_Mu\|_{L^2}^2=\|Q_Ru\|_{L^2}^2+\|Q_Iu\|_{L^2}^2+\frac{1}{2}([Q_M^*,Q_M]u,u),\tag{6.1}$$

where $Q_M = Q_R + iQ_I$, $Q_R^* = Q_R$ and $Q_I^* = Q_I$.

Let us introduce the following Hörmander's metrics

$$(6.2)\qquad \begin{cases} g_1 = dx^2 + dy^2 + \sum_{j=1}^{d} \dfrac{\lambda^2 d\xi_j^2}{(\lambda + |\lambda\tau|_m)^{2\tilde{h}_j}} + \sum_{j=1}^{n} \dfrac{\lambda^2 d\tau_j^2}{(\lambda + |\lambda\tau|_m)^{2h_j}}, \\ g_2 = dx^2 + dy^2 + d\xi^2 + \sum_{j=1}^{n} \dfrac{\lambda^2 d\tau_j^2}{(\lambda + |\lambda\tau|_m)^{2h_j}}. \end{cases}$$

Then it is easy to see from (4.5) that

$$(6.3)\qquad q_M(X, \zeta) = p'_M(y, \tau) + \tilde{\chi}(x, \xi)(r_{M-1}(X, \zeta) + \eta s_{M-1}(X, \zeta)),$$

where

$$(6.4)\qquad \begin{cases} \tilde{\chi}(x, \xi) = \chi\Big(x - \dfrac{i}{1+\eta}\xi, \xi\Big); r_{M-1}(X, \zeta) = p'_{M-1}(X, \zeta), \\ r_{M-1} \in S\big(\lambda(\lambda + |\lambda\tau|_m)^{M-1}, g_2\big), \ s_{M-1} \in S\big(\lambda(\lambda + |\lambda\tau|_m)^{M-1}, g_2\big), \\ p'_M \in S\big((\lambda + |\lambda\tau|_m)^M, g_1\big). \end{cases}$$

We shall write $Q_M = P'_M + R_{M-1} + \eta S_{M-1}$ where $\sigma^\omega(P'_M) = p'_M(y, \tau)$, $\sigma^\omega(R_{M-1}) = \tilde{\chi} r_{M-1}$, and $\sigma^\omega(S_{M-1}) = \tilde{\chi} s_{M-1}$. Let us set

$$(6.5)\qquad L = P'_M + R_{M-1}.$$

Since R_{M-1} and S_{M-1} belong to $Op_\lambda^w\big(S(\lambda(\lambda + |\lambda\tau|_m)^{M-1}, g_2)\big)$ and p'_M depends only on (y, τ), it is easy to see that

$$(6.6)\qquad [Q_M^*, Q_M] - [L^*, L] \in \frac{\eta}{\lambda} Op_\lambda^w\big(S(\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2}, g_2)\big).$$

We shall set $\sigma^\omega(L) = \ell_1 + \ell_2 = \ell$ where

$$(6.7)\qquad \begin{cases} \ell_1 = p'_M(y, \tau) + (\tilde{\chi} r_{M-1})|_{\xi=0}, \\ \ell_2 = \tilde{\chi} r_{M-1} - (\tilde{\chi} r_{M-1})|_{\xi=0}. \end{cases}$$

Then

$$(6.8)\qquad \ell_1 \in S\big((\lambda + |\lambda\tau|_m)^M, g_1\big), \ \ell_2 \in S\big(\lambda(\lambda + |\lambda\tau|_m)^{M-1}, g_2\big).$$

We shall also write

$$(6.9)\qquad \sigma^\omega([L^*, L]) = \frac{1}{\lambda}(d_1 + d_2) \text{ where } d_1 = \frac{1}{i}\{\bar{\ell}, \ell\}|_{\xi=0}.$$

Then since p'_M depends only on (y,τ), we have

$$d_1 \in S\big(\lambda(\lambda+|\lambda\tau|_m)^{2M-1}, g_1\big),\ \ d_2 \in S\big(\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2}, g_2\big). \tag{6.10}$$

Lemma 6.1. *There exists a positive constant C such that if we set*

$$\psi(X) = \varphi'(0)X + \frac{1}{2}\varphi''(0)X\cdot X - \frac{1}{2C^2}|X|^2 + \frac{C}{2}(\varphi'(0)X)^2$$

then

$$C^3|\ell_1(X,\tau)|^2 + d_1(X,\tau) \geq \frac{1}{C}\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2}, \tag{6.11}$$

for $|X| \leq 1/C^2$ and τ in $\mathbb{R}^n$. Moreover, by homogeneity, (6.11), with possibly other constants, is still true with the same ψ if we replace ψ by $\rho\psi$ where ρ is a positive constant.

Proof. We first take C so large that $\tilde{\chi} = 1$ if $|x|+|\xi| \leq 1/C^2$. Then from (6.7) and (6.9), we have

$$\begin{cases} \ell_1(X,\tau) = p_M(X;\lambda\zeta + i\lambda\psi'(X))|_{\xi=0}, \\ d_1(X,\tau) = \dfrac{1}{i}\{\overline{p}_M(X,\lambda\zeta - i\lambda\psi'(X)); p_M(X,\lambda\zeta+i\lambda\psi'(X))\}|_{\xi=0}. \end{cases}$$

Now, we write

$$\begin{cases} \ell_1(X,\tau) = p_M(X;\lambda\zeta + i\lambda\nabla_{p,q}\psi(X))|_{\xi=0} + s_\lambda(\xi,\tau), \\ d_1(X,\tau) = \dfrac{1}{i}\{\overline{p}_M(X;\lambda\zeta - i\lambda\nabla_{p,q}\psi(X)); p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))\}|_{\xi=0} \\ \qquad\qquad\qquad + r_\lambda(X,\tau), \end{cases} \tag{6.12}$$

where

$$\begin{cases} s_\lambda \in S\big(\lambda(\lambda+|\lambda\tau|_m)^{M-1-1/(M-1)}, g_1\big) \\ \text{and} \\ r_\lambda \in S\big(\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2-1/(M-1)}, g_1\big). \end{cases} \tag{6.13}$$

First, we shall

$$\begin{aligned} &\frac{C^3}{4}\left|p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\right|^2 \\ &+\frac{1}{2i}\{\overline{p}_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X)); p_M(X,\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))\}\big|_{\xi=0} \\ &\geq \frac{1}{C}\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2} \text{ for } |X| \leq \frac{1}{C^2} \text{ and } \tau \text{ in } \mathbb{R}^n. \end{aligned} \tag{6.14}$$

(6.14) is equivalent to

$$\begin{aligned}&\frac{C^3}{4\lambda^2}\left|p_M(X;\zeta+i\lambda\nabla_{p,q}\psi(X)|_{\xi=0}\right|^2\\&+\frac{1}{2i\lambda}\left\{\overline{p}_M(X;\zeta-i\lambda\nabla_{p,q}\psi(X));p_M(X,\zeta+i\lambda\nabla_{p,q}\psi(X))\right\}|_{\xi=0}\\&\geq\frac{1}{C}(\lambda+|\tau|_m)^{2M-2}\text{ for }|X|\leq\frac{1}{C^2}.\end{aligned}$$

We see (6.14), setting $\Gamma=\lambda/(\lambda+|\tau|_m)$, $W=(X,Z+i\Gamma\nabla_{p,q}\psi(X))$,

$$Z=\left(0,\ldots,0;\frac{\tau_1}{(\lambda+|\tau|_m)^{h_1}},\ldots,\frac{\tau_n}{(\lambda+|\tau|_m)^{h_m}}\right)$$

that (6.14) is equivalent to

$$\begin{aligned}(6.15)\quad&\frac{C^3}{4\Gamma^2}|p_M(W)|^2+\frac{1}{\Gamma}\operatorname{Im}\Bigg(\sum_{j=1}^{d}(\lambda+|\tau|_m)^{1-\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W})\frac{\partial p_M}{\partial x_j}(W)\\&+\sum_{j=1}^{n}(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial y_j}(W)\Bigg)\\&+\operatorname{Re}\Bigg(\sum_{j=1}^{d}\sum_{k=p}^{d}\frac{\partial^2\psi}{\partial x_k\partial x_j}(X)(\lambda+|\tau|_m)^{1-\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W})\frac{\partial p_M}{\partial x_j}(W)\\&+\sum_{j=1}^{d}\sum_{k=q}^{n}\frac{\partial^2\psi}{\partial y_k\partial x_j}(X)(\lambda+|\tau|_m)^{1-\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W})\frac{\partial p_M}{\partial\tau_k}(W)\\&+\sum_{j=1}^{n}\sum_{k=p}^{d}\frac{\partial^2\psi}{\partial x_k\partial y_j}(X)(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial\xi_k}(W)\\&+\sum_{j=1}^{n}\sum_{k=q}^{n}\frac{\partial^2\psi}{\partial y_k\partial y_j}(X)(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial\tau_k}(W)\Bigg)\geq\frac{1}{C},\text{ for }|X|\leq\frac{1}{C^2}.\end{aligned}$$

We prove (6.15) by contradiction. If it is false one can find sequences X_k, λ_k, τ_j, Γ_k with $|X_k|\leq 1/k^2$, $\lambda_k\geq e^k$ and τ_k in $\mathbb{R}^n$, such that

$$\begin{aligned}(6.16)\quad&\frac{k^3}{4\Gamma_k^2}|p_M(W_k)|^2+\frac{1}{\Gamma_k}\operatorname{Im}\Bigg(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial y_j}(W_k)\\&+\sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial x_j}(W_k)\Bigg)+\operatorname{Re}\Bigg(\sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_s}(W_k)\\&+\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)+2\sum_{s=q}^{M}\sum_{j=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)\Bigg)\end{aligned}$$

$$+k\Big(\Big|\sum_{j=p}^{d}\frac{\partial\varphi}{\partial x_j}(0)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big|^2+\Big|\sum_{j=q}^{n}\frac{\partial\varphi}{\partial y_j}(0)\frac{\partial p_M}{\partial\tau_j}(W_k)\Big|^2$$

$$+2\operatorname{Re}\Big[\Big(\sum_{j=q}^{n}\frac{\partial\varphi}{\partial\partial y_j}(0)\frac{\partial p_M}{\partial\tau_j}(W_k)\Big)\Big(\sum_{s=p}^{d}\frac{\partial\varphi}{\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_s}(\overline{W}_k)\Big)\Big]\Big)$$

$$-\frac{1}{k^2}\Big(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_j}(W_k)+\sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big)+B_k\le\frac{1}{k}$$

where

$$(6.17)\qquad |B_k|\le\frac{C_1k}{\Gamma_k}\lambda_k^{-1/(M-1)},\ C_1\ \text{ independent of } k.$$

Since $\Gamma_k+|Z_k|_{(m,\tilde{m})}=1$, taking subsequences, we may assume that

$$(6.18)\qquad \Gamma_k\to\Gamma^0 \text{ and } Z_k\to Z^0 \text{ with } \Gamma^0+|Z^0|_{(m,\tilde{m})}=1.$$

CASE 1. $\Gamma^0\neq 0$.

If we divide both members of (6.16) by k^3, we get

$$(6.19)\qquad p_M(W^0)=\{p_M,\varphi\}_0(W^0)=0,$$

with $W^0=(0;Z^0+i\Gamma^0\nabla_{p,q}\varphi(0))$.

Removing all positive terms in (6.16) and letting k go to $+\infty$, we get

$$\frac{1}{\Gamma^0}\operatorname{Im}\Big(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial y_j}(W^0)+\sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial x_j}(W^0)\Big)$$

$$+\operatorname{Re}\Big(\sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial\xi_s}(W^0)+\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial\tau_s}(W^0)$$

$$+2\sum_{s=q}^{n}\sum_{j=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial\tau_s}(W^0)\Big)\le 0$$

which contradicts the hypothesis (H.2)′ ii) in Theorem B.

CASE 2. $\Gamma^0=0$.

Since $\Gamma^0+|Z^0|_{(m,\tilde{m})}=1$, we have $Z^0\neq 0$. In this case, we write

$$(6.20)\qquad B_k=\frac{1}{\Gamma_k}\operatorname{Im}\Big(\sum_{j=1}^{d}(\lambda_k+|\tau_k|_m)^{1+\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial x_j}(W_k)$$

$$+\sum_{j=1}^{n}(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial y_j}W_k)\Big)+D_k$$

where

$$|D_k| \leq C_2 k \lambda_k^{-1/(M-1)}, \ C_2 \quad \text{independent of } k.$$

Therefore

$$(6.21) \quad B_k = \frac{1}{2i\Gamma_k}(\lambda_k + |\tau_k|_m)^{1-2M}\{\overline{p}_M, p_M\}(X_k; 0, \tau_k)$$
$$+ \operatorname{Re}\Big(\sum_{s,j=q}^{n} \frac{\partial\psi}{\partial y_s}(X_k)\Big(\frac{\partial \overline{p}_M}{\partial \tau_j}(X_k, Z_k)\frac{\partial^2 p_M}{\partial \tau_j \partial y_j}(X_k, Z_k)$$
$$- \frac{\partial p_M}{\partial y_j}(X_k, Z_k)\frac{\partial^2 \overline{p}_M}{\partial \tau_s \partial \tau_j}(X_k, Z_k)\Big) + \sum_{s,j=p}^{d} \frac{\partial \psi}{\partial x_s}(X_k)$$
$$\Big(\frac{\partial \overline{p}_M}{\partial \xi_j}(X_k, Z_k)\frac{\partial^2 \overline{p}_M}{\partial \xi_s \partial x_j}(X_k, Z_k) - \frac{\partial p_M}{\partial x_j}(X_k, Z_k)\frac{\partial^2 \overline{p}_M}{\partial \xi_s \partial \xi_j}(X_k, Z_k)\Big)\Big) + D_k'$$

where

$$|D_k'| \leq C_3\Big(k\lambda_k^{-1/(M-1)} + \Gamma_k\Big), \ C_3 \text{ independent of } k.$$

We use then the assumptions (H.1)$'$ in Theorem B. We get

$$\Big|(\lambda_k + |\tau_k|_m)^{1-2M}\{\overline{p}_M, p_M\}(X_k, 0, \tau_k)\Big| \leq C'|p_m(X_k, 0, \tau_k)|(\lambda_k + |\tau_k|_m)^{-M}$$
$$\leq C'|p_M(X_k, Z_k)| \leq C'|p_M(W_k)| + C'\Gamma_k\Big(\Big|\sum_{j=q}^{n}\frac{\partial\psi}{\partial y_j}(X_k)\frac{\partial p_M}{\partial \tau_j}(W_k)\Big|$$
$$+\Big|\sum_{j=p}^{d}\frac{\partial\psi}{\partial x_j}(X_k)\frac{\partial p_M}{\partial \xi_j}(W_k)\Big|\Big) + \mathcal{O}(\Gamma_k^2).$$

Therefore

$$(6.22) \quad \Big|\frac{1}{2i}(\lambda_k + |\tau_k|_m)^{1-2M}\{\overline{p}_M, p_M\}(X_k; 0, \tau_k)\Big| \leq \frac{k^{3/2}}{4\Gamma_k}|p_M(W_k)|^2 + \frac{4(C')^2\Gamma_k}{k^{3/2}}$$
$$+C'\Gamma_k\Big(\Big|\sum_{j=q}^{n}\frac{\partial\psi}{\partial y_j}(X_k)\frac{\partial p_M}{\partial \tau_j}(W_k)\Big| + \Big|\sum_{j=p}^{d}\frac{\partial\psi}{\partial x_j}(X_k)\frac{\partial p_M}{\partial \xi_j}(W_k)\Big|\Big) + \mathcal{O}(\Gamma_k^2).$$

It follows from (6.21), (6.22) that (6.16) is equivalent to

$$(6.23) \quad \frac{1}{4}\Big(\frac{k^3}{\Gamma_k^2} - \frac{k^{3/2}}{\Gamma_k^2}\Big)|p_M(W_k)|^2$$
$$+\operatorname{Re}\Big(\sum_{s,j=q}^{m}\frac{\partial\psi}{\partial y_s}(X_k)\Big(\frac{\partial \overline{p}_M}{\partial \tau_j}(X_k, Z_k)\frac{\partial^2 p_M}{\partial \tau_j \partial y_j}(X_k, Z_k) - \frac{\partial p_M}{\partial y_j}(X_k, Z_k)\frac{\partial^2 \overline{p}_M}{\partial \tau_s \partial \tau_j}(X_k, Z_k)\Big)$$

$$+\sum_{s,j=p}^{d}\frac{\partial\psi}{\partial x_s}(X_k)\Big(\frac{\partial\overline{p}_M}{\partial\xi_j}(X_k,Z_k)\frac{\partial^2\overline{p}_M}{\partial\xi_s\partial x_j}(X_k,Z_k)-\frac{\partial p_M}{\partial x_j}(X_k,Z_k)\frac{\partial^2\overline{p}_M}{\partial\xi_s\partial\xi_j}(X_k,Z_k)\Big)\Big)$$

$$+k\Big(\Big|\sum_{j=q}^{n}\frac{\partial\varphi}{\partial y_j}(0)\frac{\partial p_M}{\partial\partial\tau_j}(W_k)\Big|^2+\Big|\sum_{j=p}^{d}\frac{\partial\varphi}{\partial x_j}(0)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big|^2$$

$$+2\,\mathrm{Re}\Big[\Big(\sum_{j=q}^{n}\frac{\partial\varphi}{\partial y_j}(0)\frac{\partial p_M}{\partial\tau_j}(W_k)\Big)\Big(\sum_{s=p}^{d}\frac{\partial\varphi}{\partial x_s}(0)\frac{\partial p_M}{\partial\xi_s}(W_k)\Big)\Big]\Big)$$

$$-C'\Big(\Big|\sum_{j=q}^{n}\frac{\partial\psi}{\partial y_j}(X_k)\frac{\partial p_M}{\partial\tau_j}(W_k)\Big|+\Big|\sum_{j=p}^{d}\frac{\partial\psi}{\partial x_j}(X_k)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big|\Big)$$

$$-\frac{1}{k^2}\Big(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_j}(W_k)+\sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big)$$

$$+\mathrm{Re}\Big(\sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_s}(W_k)+\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)$$

$$+2\sum_{s=q}^{n}\sum_{j=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)\Big)+\mathcal{O}\Big(k\lambda_k^{-1/(M-1)}+\Gamma_k+\frac{1}{k^{3/2}}\Big)\le\frac{1}{k}.$$

Dividing both members by k^3/Γ_k^2, we get, since $\Gamma_k\to 0$, $k\to+\infty$,

$$p_M(W^0)=0 \text{ with } W^0=(0,Z^0),\quad Z^0\neq 0. \tag{6.24}$$

Now since, $(k^3/\Gamma_k^2-k^{3/2}/\Gamma_k^2)|p_M(W_k)|^2\ge 0$, dividing (6.23) by k, we get

$$\{p_M,\varphi\}_0(W^0)=0. \tag{6.25}$$

Removing all positive terms in (6.23) and letting k go to $+\infty$, we get

$$\mathrm{Re}\Big[\sum_{s,j=q}^{n}\frac{\partial\varphi}{\partial y_s}(0)\Big(\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}^0)\frac{\partial^2 p_M}{\partial\tau_s\partial y_j}(W^0)-\frac{\partial p_M}{\partial y_j}(W^0)\frac{\partial^2\overline{p}_M}{\partial\tau_s\partial\tau_j}(\overline{W}^0)\Big)$$

$$+\sum_{s,j=p}^{d}\frac{\partial\varphi}{\partial\partial x_s}(0)\Big(\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial^2 p_M}{\partial\xi_s\partial x_j}(W^0)-\frac{\partial p_M}{\partial x_j}(W^0)\frac{\partial^2\overline{p}_M}{\partial\xi_j\partial\xi_s}(\overline{W}^0)\Big)$$

$$+\sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial\xi_s}(W^0)+\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial\tau_s}(W^0)$$

$$+2\sum_{s=q}^{n}\sum_{s=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial\tau_s}(W^0)\Big]\le 0$$

which is contradiction with (H.2)′ i) in Theorem B.

It follows from (6.12), (6.13) and (6.14) that

$$\frac{C^3}{4}\left|p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\right|^2+\frac{1}{2}d_1(X,\tau)\geq\frac{1}{C}\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2}+\frac{1}{2i}r_\lambda(X,\tau).$$

But we have

$$\begin{cases}\left|p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\right|^2\leq 2|\ell_1(X,\tau)|^2+C'\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2-2/(M-1)}\\ \left|\frac{1}{2i}r_\lambda(X,\tau)\right|\leq C''\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2-1/(M-1)}.\end{cases}$$

Il follows that

$$\frac{C^3}{2}|\ell_1(X,\tau)|^2+\frac{1}{2}d_1(X,\tau)\geq\frac{1}{2C}\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2},$$

for large λ and Lemma 6.1 follows. □

Lemma 6.2. *We have*

$$\left(\frac{C^3+1}{\lambda^2}\right)\left(\|Op_\lambda^w(\mathrm{Re}\,\ell_1)u\|_{L^2}^2+\|Op_\lambda^w(\mathrm{Im}\,\ell_1)u\|_{L^2}^2\right)+\frac{1}{\lambda^2}(Op_\lambda^w(d_1)u,u)\geq\frac{1}{2C}|||u|||_{M-1}^2, \tag{6.26}$$

where $|||\cdot|||_{M-1}$ *is defined* (2.9), *and for large* λ.

Proof. Let us $a=(C^3/\lambda^2)|\ell_1|^2+d_1/\lambda^2$ and $a_0=a|_{x=0}$. Let $h_0\in C_0^\infty(\mathbb{R}^d)$ be such that $h_0=1$ if $|x|\leq 1/(4C^2)$, $h_0=0$ if $|x|\geq 1/(2C^2)$ and $0\leq h_0\leq 1$. Then we have

$$a+(1-h_0)(a_0-a)\geq\frac{1}{C}(\lambda+|\lambda\tau|_m)^{2M-2},\ \text{ if } |y|\leq\frac{1}{2C^2}. \tag{6.27}$$

Indeed, if $|x|\leq 1/(2C^2)$, then by Lemma 6.1, a and a_0 satisfy (6.11) thus (6.27) is true. If $|x|\geq 1/(2C^2)$ then $h_0=0$ and a_0 satisfies (6.11) and (6.27) is also true.

Now denoting by t_k a symbol in the class $S((\lambda+|\lambda\tau|_m)^k,g_2)$, by (6.8) and (6.9), we have

$$a=\frac{C^3}{\lambda^2}|p'_M(y,\tau)|^2+\frac{2}{\lambda^2}\,\mathrm{Im}\left(\frac{\partial}{\partial\tau}(p'_M(y,\tau))\frac{\partial}{\partial y}(p'_M(y,\tau))\right)+\frac{1}{\lambda}\,\mathrm{Re}(\ell_1\cdot t_{M-1})+t_{2M-2}.$$

Thus $a-a_0=(1/\lambda)\,\mathrm{Re}(\ell_1\cdot t_{M-1})+t_{2M-2}$ so

$$|a-a_0|\leq\frac{|\ell_1|^2}{\lambda^2}+C'(\lambda+|\lambda\tau|_m)^{2M-2}. \tag{6.28}$$

It follows from (6.11), (6.27) and (6.28) that if $|y| \leq 1/(2C^2)$

$$\frac{(C^3+1)}{\lambda^2}|\ell_1|^2 + \frac{1}{\lambda^2}d_1 + C'(1-h_0)(\lambda+|\lambda\tau|_m)^{2M-2} \geq \frac{1}{C}(\lambda+|\lambda\tau|_m)^{2M-2}. \tag{6.29}$$

Let $h_1 \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq h_1 \leq 1$, $h_1 = 0$ if $|y| \geq 1/(2C^2)$ and $h_1 = 1$ if $|y| \leq 1/(4C^2)$. Thus we have, from (6.29)

$$\Big(\frac{(C^3+1)}{\lambda^2}|\ell_1|^2 + \frac{1}{\lambda^2}d_1 + C'(1-h_0)(\lambda+|\lambda\tau|_m)^{2M-2} - \frac{1}{C}(\lambda+|\lambda\tau|_m)^{2M-2}\Big)\lambda^2 h_1^2(y) \geq 0$$

for any (X,τ) in $\mathbb{R}^{d+n}\times\mathbb{R}^n$, and this symbol belongs to $S((\lambda+|\lambda\tau|_m)^{2M}, g_1)$. Therefore we can apply the Fefferman-Phong inequality and get

$$\begin{aligned}(6.30)\qquad &\Big(Op_\lambda^w\Big(\frac{(C^3+1)}{\lambda^2}|\ell_1|^2h_1^2\Big)u,u\Big) + \Big(Op_\lambda^w\Big(\frac{d_1}{\lambda^2}h_1^2\Big)u,u\Big)\\ &\geq \frac{1}{C}\Big(Op_\lambda^w(h_1^2(\lambda+|\lambda\tau|_m)^{2M-2})u,u\Big)\\ &\qquad -C'\Big(Op_\lambda^w(h_1^2(1-h_0)(\lambda+|\lambda\tau|_m)^{2M-2})u,u\Big) - \frac{C''}{\lambda^2}|||u|||_{M-1}^2.\end{aligned}$$

We can use the symbolic calculus in $S(\cdot, g_1)$. We get

$$\begin{aligned}J = \Big(Op_\lambda^w\Big(\frac{(C^3+1)}{\lambda^2}|\ell_1|^2h_1^2\Big)u,u\Big) &= \frac{(C^3+1)}{\lambda^2}\Big(\Big(Op_\lambda^w(\ell_1^R h_1)^* Op_\lambda^w(\ell_1^R h_1)\\ &\quad + Op_\lambda^w(\ell_1^I h_1)^* Op_\lambda^w(\ell_1^I h_1)\Big)u,u\Big) + \frac{1}{\lambda^2}\mathcal{O}(|||u|||_{M-1}^2)\end{aligned}$$

where $\ell_1^R = \operatorname{Re}\ell_1$ and $\ell_1^I = \operatorname{Im}\ell_1$. Thus

$$J = \frac{(C^3+1)}{\lambda^2}\Big(\|Op_\lambda^w(\ell_1^R)u\|_{L^2}^2 + \|Op_\lambda^w(\ell_1^I)u\|_{L^2}^2\Big) + \frac{1}{\lambda^2}\mathcal{O}(|||u|||_{M-1}^2) \tag{6.31}$$

because

$$Op_\lambda^w(\ell_1^K)h_1 = Op_\lambda^w(\ell_1^K h_1) + Op_\lambda^w\big(S((\lambda+|\lambda\tau|_m)^{M-1}, g_1)\big)$$

for $K = R$ or I and $h_1 u = u$ since $\operatorname{supp} u \subset \big\{|y| \leq 1/(4C^2)\big\}$. By the same way

$$Op_\lambda^w(d_1h_1^2) = Op_\lambda^w(d_1)h_1^2 + Op_\lambda^w\big(S(\lambda(\lambda+|\lambda\tau|_m)^{2M-2}, g_1)\big)$$

thus

$$(Op_\lambda^w(d_1h_1^2)u,u) = (Op_\lambda^w(d_1)u,u) + \lambda\mathcal{O}(|||u|||_{M-1}^2). \tag{6.32}$$

We have also

$$(6.33)\qquad \big(Op_\lambda^w(h_1^2(\lambda+|\lambda\tau|_m)^{2M-2})u,u\big) = |||u|||_{M-1}^2 + \frac{1}{\lambda}\mathcal{O}(|||u|||_{M-1}^2),$$

$$(6.34)\qquad \big(Op_\lambda^w(h_1^2(1-h_0)(\lambda+|\lambda\tau|_m)^{2M-2})u,u\big) = |||(1-h_0)u|||_{M-1}^2 + \frac{1}{\lambda}\mathcal{O}(|||u|||_{M-1}^2),$$

and

$$(6.35)\qquad |||(1-h_0)u|||_{M-1}^2 \le \frac{C_N}{\lambda^N}|||u|||_{M-1}^2, \text{ for any } N \text{ in } \mathbb{N}.$$

Thus (6.26) follows from (6.30) to (6.35). □

Lemma 6.3. *Let ℓ_2 and d_2 be defined in* (6.7) *and* (6.9). *Then there exists $\sigma > 0$ such that for any $\varepsilon > 0$ one can find a positive constant C_ε such that*

$$(6.36)\qquad \|Op_\lambda^w(\ell_2)u\|_{L^2(\mathbb{R}^{d+n})} \le \lambda\varepsilon|||u|||_{M-1} + \sqrt{\lambda}C_\varepsilon|||u|||_{M-1} + \mathcal{O}(e^{-\lambda\sigma}|||v|||_{M-1}),$$

and

$$(6.37)\qquad |(Op_\lambda^w(d_2)u,u)| \le \lambda^2\Big(\varepsilon|||u|||_{M-1}^2 + \frac{C_\varepsilon}{\sqrt{\lambda}}|||u|||_{M-1}^2\Big) + \mathcal{O}(e^{-\lambda\sigma}|||v|||_{M-1}^2),$$

for any $u = T_\eta^* T v$, $v \in C_0^\infty(\mathbb{R}^{n+d})$.

Proof. Given $\varepsilon > 0$, let $\chi(X,\xi)$ in C^∞ with $0 \le \chi \le 1$ and $\operatorname{supp}\chi \subset \{|X|+|\xi| \le \varepsilon\}$. We claim that one can find $C_\varepsilon > 0$ such that

$$(6.38)\qquad \frac{1}{\lambda}\|Op_\lambda^w(\ell_2\chi)u\|_{L^2} \le \varepsilon|||u|||_{M-1} + \frac{C_\varepsilon}{\sqrt{\lambda}}|||u|||_{M-1}.$$

This follows from the sharp Gårding inequality in the class $S(1, g_2)$. Indeed, we have $\varepsilon^2(\lambda+|\lambda\tau|_m)^{2M-2} - \xi^2\chi^2(\lambda+|\lambda\tau|_m)^{2M-2} \ge 0$. Thus

$$(6.39)\qquad \varepsilon^2\big(Op_\lambda^w((\lambda+|\lambda\tau|_m)^{2M-2})u,u\big) - \big(Op_\lambda^w(\xi^2\chi^2(\lambda+|\lambda\tau|_m)^{2M-2})u,u\big) \ge -\frac{C_\varepsilon}{\lambda}|||u|||_{M-1}^2.$$

Since $\ell_2 \in S(\lambda(\lambda+|\lambda\tau|_m)^{M-1}, g_2)$ and $\ell_2|_{\xi=0}$, we have

$$(6.40)\qquad \|Op_\lambda^w(\ell_2\chi)u\|_{L^2} \le C\lambda\|Op_\lambda^w(\xi\chi(\lambda+|\lambda\tau|_m)^{M-1})u\|_{L^2}.$$

We deduce (6.38) from (6.39) and (6.40).

Therefore taking $\chi = \theta(x, \xi)g(y)$, such that $\chi = 1$ if $|X| + |\xi| \leq \varepsilon/2$, we write

$$\|Op^w_\lambda(\ell_2)u\|_{L^2} \leq \|Op^w_\lambda(\ell_2\chi)u\|_{L^2} + \|Op^w_\lambda((1-\chi)\ell_2)u\|_{L^2}.$$

It follows from Proposition 4.6 in [7] that

$$\|Op^w_\lambda((1-\chi)\ell_2)u\|_{L^2} \leq \frac{C_N}{\lambda^N}|||u|||_{M-1} + \mathcal{O}(e^{-\lambda\sigma}|||v|||_{M-1}). \tag{6.41}$$

Then we deduce (6.36) from (6.40) and (6.41). This gives the first part of the lemma. For the second part, we observe that $d_2 \in S(\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2}, g_2)$. Therefore from (6.39) and Proposition 4.6 in [7], we deduce (6.37). □

We are now ready to prove the Carleman estimate for Q_M.

Proposition 6.4. *Let $Q_M = Op^w_\lambda(q_M)$ be defined in* (4.6). *Then one can find positive constants C_0, C_1, λ_0, σ such that, for any $u = T^*_\eta T v$, $v \in C^\infty_0$,* $\operatorname{supp} v \subset \{|X| \leq 1/(4C^2)\}$ *and $\lambda \geq \lambda_0$, we have*

$$C_0|||u|||^2_{M-1} \leq \frac{C_1}{\lambda}\|Q_M u\|^2_{L^2} + \mathcal{O}(e^{-\lambda\sigma}|||v|||^2_{M-1}). \tag{6.42}$$

Proof. It follows from (6.3), (6.5) and (6.7) that

$$\|Op^w_\lambda(\ell^R_1)u\|_{L^2} \leq \|Q_R u\|_{L^2} + \|Op^w_\lambda(\ell^R_2)u\|_{L^2} + \eta\|Op^w_\lambda(\tilde{\chi}s^R_{M-1})u\|_{L^2}.$$

Therefore, applying Lemma 6.3, we deduce

$$\begin{aligned}\|Op^w_\lambda(\ell^K_1)u\|_{L^2} &\leq \|Q_K u\|_{L^2} + C_1\lambda\Big(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + C_2\eta\Big)|||u|||_{M-1} \\ &+\mathcal{O}\big(e^{-\lambda\sigma}|||v|||_{M-1}\big), \quad \text{for } K = R, I.\end{aligned} \tag{6.43}$$

Using (6.6), (6.9) and Lemma 6.3, we get

$$\begin{aligned}&\Big|\big((Op^w_\lambda(d_1) - \lambda[Q^*_M, Q_M])u, u\big)\Big| \\ &= \Big|\big((Op^w_\lambda(d_2) - \eta Op^w_\lambda(S(\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2}, g_2))u, u\big)\Big|, \\ &\leq |(Op^w_\lambda(d_2)u, u)| + \eta\lambda^2\big|(Op^w_\lambda(S((\lambda + |\lambda\tau|_m)^{2M-2}, g_2))u, u)\big|, \\ &\leq C_1\lambda^2\Big(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + C_2\eta\Big)|||u|||^2_{M-1} + \mathcal{O}\big(e^{-\lambda\sigma}|||v|||^2_{M-1}\big).\end{aligned} \tag{6.44}$$

It follows from (6.43), (6.44) and Lemma 6.2 that

$$\frac{1}{2C}|||u|||_{M-1}^2 \leq \frac{2}{\lambda^2}(C^3+1)\Big(\|Q_I u\|_{L^2}^2 + \|Q_I u\|_{L^2}^2 + \frac{\lambda}{2}\big([Q_M^*, Q_M]u, u\big)\Big)$$
$$+\tilde{C}_1\Big(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + \tilde{C}_2\eta\Big)|||u|||_{M-1}^2 + \mathcal{O}\big(e^{-\lambda\sigma}|||v|||_{M-1}^2\big).$$

Taking ε and η small, then λ large, we get, by (6.1), proposition 6.4. □

Theorem 6.5. *Let $\tilde{P}_\lambda$ the operator occuring in* Proposition 3.1. *One can find positive constants C_1, C_2, λ_0, ε_2, σ such that for $v \in C_0^\infty(\mathbb{R}^{d+n})$, $\operatorname{supp} v \subset \{|X| \leq \varepsilon_2\}$ and $\lambda \geq \lambda_0$ we have*

$$\lambda\|Tv\|^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H_\lambda^{M-1}(\mathbb{R}^n))} \leq C_1\|\tilde{P}_\lambda T v\|^2_{L^2_{(1+\eta)\Phi}} + C_2 e^{-\lambda\sigma}|||v|||^2_{M-1}. \tag{6.45}$$

Proof. By Theorem 3.2, (6.45) will follow from the same estimate for $\tilde{Q}_\lambda$. Now

$$\|\tilde{Q}_\lambda T v\|_{L^2_{(1+\eta)\Phi}} = \|Q_\lambda u\|_{L^2}$$

and by (4.5) we have $\sigma^w(Q_\lambda) = \sigma^w(Q_M) + \sigma^w(Q'_{M-1})$ where

$$Q'_{M-1} \in Op_\lambda^w(S((\lambda + |\lambda\tau|_m)^{M-1}, g_2)).$$

Thus (6.45) follows from Proposition 6.4 if λ is large enough. □

7. End of the proof of the Theorems A and B

The Theorems 5.5 and 6.5 ensure that one can find $\sigma > 0$ such that

$$\lambda^{2M-1}\|Tv\|^2_{L^2_{(1+\eta)\Phi}} \leq C_1\|\tilde{P}_\lambda T v\|^2_{L^2_{(1+\eta)\Phi}} + C_2 e^{-\lambda\sigma}|||v|||^2_M. \tag{7.1}$$

The end of the proof, *i.e.* the passage from Carleman's inequality (7.1) to uniqueness of the Cauchy problem for the operator P, is the same as the one in Robbiano-Zuily [7].

The proof of Theorems A and B is complete.

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