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THE SCHUR INDICES OF THE CUSPIDAL UNIPOTENT CHARACTERS OF THE FINITE CHEVALLEY GROUPS $E_7(q)$

Dedicated to Professor Herbert Pahlings on his 65th birthday

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Abstract

We show that the two cuspidal unipotent characters of a finite Chevalley group $E_7(q)$ have Schur index 2, provided that q is an even power of a (sufficiently large) prime number p such that $p \equiv 1 \bmod 4$. The proof uses a refinement of Kawanaka's generalized Gelfand–Graev representations and some explicit computations with the *CHEVIE* computer algebra system.

1. Introduction

Throughout this paper, let G be a simple algebraic group of adjoint type E_7. Assume that G is defined over the finite field $\mathbb{F}_q$, with corresponding Frobenius map $F\colon G \to G$. There are precisely two cuspidal unipotent characters of G^F denoted by $E_7[\pm\xi]$ where $\xi = \sqrt{-q}$; see the table in [2], §13.9.

The purpose of this paper is to determine the Schur index of $E_7[\pm\xi]$, at least if the characteristic of $\mathbb{F}_q$ is large enough. Modulo this condition on the characteristic, this completes the determination of the Schur indices of the unipotent characters of finite groups of Lie type; see [12], [5] and the references there.

By [4], Table 1, the character values of $E_7[\pm\xi]$ generate the field $\mathbb{Q}(\xi)$. Furthermore, by [4], Example 6.4, we already know that the Schur index is 1 if $p \not\equiv 1 \bmod 4$ or if q is not a square, where p is the characteristic of $\mathbb{F}_q$. Thus, the remaining task is to determine the Schur index when q is a square and $p \equiv 1 \bmod 4$.

Theorem 1.1. *Assume that q is an even power of a* (*sufficiently large*) *prime p such that $p \equiv 1 \bmod 4$. Then the characters $E_7[\pm\xi]$ have Schur index* 2.

Here, p is "sufficiently large" if Lusztig's results [11] on generalized Gelfand–Graev characters hold; it is conjectured that this is the case if p is good for G.

The idea of the proof is as follows. We have already seen in [5], §4, that $E_7[\pm\xi]$ occur with multiplicity 1 in a generalized Gelfand–Graev character Γ_u, where u is a

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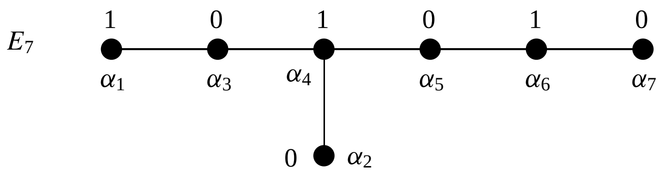
Table 1. The weighted Dynkin diagram for the unipotent support of the cuspidal unipotent characters in type E_7



certain unipotent element in G. Here, we shall use a refinement of the construction of Γ_u to show that, under the given assumptions on p and q, the characters $E_7[\pm\xi]$ occur with odd multiplicity in an induced character which cannot be realized over $\mathbb{Q}(\xi)$. By standard arguments on Schur indices, this implies that $E_7[\pm\xi]$ cannot be realized over $\mathbb{Q}(\xi)$. At some stage, the proof relies on the fact that, in Lusztig's parametrization of the irreducible characters of G^F, the function Δ occurring in [10], Main Theorem 4.23, takes value -1 on the labels corresponding to $E_7[\pm\xi]$.

Furthermore, we rely on some explicit computations in G^F. However, we shall only use computations with the root system and the irreducible characters of the Weyl group of G, for which the *CHEVIE* system [6] is a convenient tool.

2. Generalized Gelfand–Graev characters for type E_7

A short summary of the construction of generalized Gelfand–Graev characters is given in [5], §2. Assume that q is a power of a "good" prime $p \neq 2, 3$. Let Φ be the root system of G with respect to a fixed maximally split torus T. Let C be the unipotent class of G whose weighted Dynkin diagram $d\colon \Phi \to \mathbb{Z}$ is given in Table 1. (The notation in that table also defines a labelling of the simple roots in the root system of G.) The class C is the "unipotent support" of the two cuspidal unipotent characters of G^F; see [5], §4, and the references there.

Given the weight function $d\colon \Phi \to \mathbb{Z}$ specified by the diagram in Table 1, we define unipotent subgroups

$$U_{d,2} := \prod_{\substack{\alpha\in\Phi^+\\ d(\alpha)\geq 2}} X_\alpha \qquad \text{and} \qquad U_{d,1} := \prod_{\substack{\alpha\in\Phi^+\\ d(\alpha)\geq 1}} X_\alpha,$$

where X_α is the root subgroup in G corresponding to the root α. (It is understood that the products are taken in some fixed order.) The generalized Gelfand–Graev character associated with an element in C^F is obtained by inducing a certain linear character from $U_{d,2}^F$. We have $C_G(u)/C_G(u)^\circ \cong \mathbb{Z}/2\mathbb{Z}$ for $u \in C$. Thus, C^F splits into two classes in the finite group G^F. By Mizuno [13], Lemma 28, representatives of these two G^F-classes are given by

$$\begin{aligned} y_{74} &= x_{20}(1)x_{21}(1)x_{23}(1)x_{28}(1)x_{31}(1), \\ y_{75} &= x_{20}(1)x_{21}(1)x_{28}(1)x_{24}(1)x_{23}(1)x_{25}(1)x_{36}(\zeta), \end{aligned}$$

where ζ is a generator for the multiplicative group of $\mathbb{F}_q$ and where the subscripts correspond to the following roots in Φ^+:

$$\begin{aligned} &20 : \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, && 21 : \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \\ &23 : \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, && 24 : \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \\ &25 : \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, && 28 : \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \\ &31 : \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, && 36 : \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7. \end{aligned}$$

(Attention: Here, we use the labelling of the roots as given by the *CHEVIE* system [6], which is slightly different from that of Mizuno.) We note that both y_{74} and y_{75} lie in $C \cap U_{d,2}^F$. Now let us fix

$$u \in \{y_{74}, y_{75}\} \subseteq C \cap U_{d,2}^F;$$

the above expressions show that

$$u = \prod_{\substack{\alpha\in\Phi^+ \\ d(\alpha)=2}} x_\alpha(\eta_\alpha) \qquad \text{where } \eta_\alpha \in \mathbb{F}_q.$$

Then we define a linear character $\varphi_u \colon U_{d,2}^F \to \mathbb{C}^\times$ by the formula

$$\varphi_u\Bigg(\prod_{\substack{\alpha\in\Phi^+ \\ d(\alpha)\geq 2}} x_\alpha(\xi_\alpha)\Bigg) = \chi\Bigg(\sum_{\substack{\alpha\in\Phi^+ \\ d(\alpha)=2}} c_\alpha\, \eta_\alpha\, \xi_\alpha\Bigg) \qquad \text{for all } \xi_\alpha \in \mathbb{F}_q,$$

where $c_\alpha \in \mathbb{F}_q$ are certain fixed constants (independent of the η_α and ξ_α) and where $\chi \colon \mathbb{F}_q^+ \to \mathbb{C}^\times$ is a fixed non-trivial character of the additive group of $\mathbb{F}_q$; see [5], Definition 2.1, for more details. It will actually be convenient to choose χ in the following special way. Let $\chi_0 \colon \mathbb{F}_p^+ \to \mathbb{C}^\times$ be a fixed non-trivial character of the additive group of $\mathbb{F}_p$. Then we take χ to be

$$\chi := \chi_0 \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$$

where $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} \colon \mathbb{F}_q^+ \to \mathbb{F}_p^+$ is the trace map. Now we have

$$\mathrm{Ind}_{U_{d,2}^F}^{G^F}\big(\varphi_u\big) = [U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u,$$

where Γ_u is the generalized Gelfand–Graev character associated with u. We have seen in [5], Corollary 4.3, that

$$\big\langle E_7[\pm\xi], \Gamma_u\big\rangle_{G^F} = 1 \qquad \text{for suitable } u \in \{y_{74}, y_{75}\}.$$

(Here, and throughout the paper, we denote by $\langle\ ,\ \rangle_A$ the standard inner product on the character ring of a finite group A.)

We will now refine the construction of Γ_u. The strategy for doing this has already been outlined in [5], §4. For this purpose, we shall assume from now on that

q is an even power of p.

Since G is simple of adjoint type, we have an $\mathbb{F}_q$-isomorphism

$$h\colon \underbrace{k^\times \times \cdots \times k^\times}_{7\ \text{factors}} \to T, \qquad (x_1, \dots, x_7) \mapsto h(x_1, \dots, x_7),$$

such that $\alpha_i(h(x_1, \dots, x_7)) = x_i$ for $1 \leq i \leq 7$. In particular, we have $T^F = \{h(x_1, \dots, x_7) \mid x_i \in \mathbb{F}_q^\times\}$. We shall set

$$t := h\big(\nu^{1/2}, 1, 1, \nu^{1/2}, 1, \nu^{1/2}, 1\big) \in T^F$$

as in the proof of [5], Lemma 4.1, where ν is a generator for the multiplicative group of $\mathbb{F}_p \subset \mathbb{F}_q$ and $\nu^{1/2}$ is a square root of ν in $\mathbb{F}_q$. (The square root exists since q is an even power of p.) Then t has the property that $\alpha(t) = \nu$ for all roots α involved in the expressions for y_{74} or y_{75} as products of root subgroup elements; furthermore, we have $\alpha(t) = 1$ for all roots α such that $d(\alpha) = 0$. The element t has order $2(p-1)$ and $H := \langle t \rangle$ normalizes $U_{d,2}$. We set

$$s_1 := h(-1, 1, 1, -1, 1, -1, 1) = t^{p-1} \in T^F.$$

Note that $\alpha(s_1) = 1$ for all roots $\alpha \in \Phi^+$ which are involved in the expressions of y_{74} and y_{75} as products of root subgroup elements. Thus, s_1 fixes the character φ_u and so we can extend φ_u to $U_{d,2}^F.\langle s_1 \rangle$. Actually, there are two such extensions which we denote by $\tilde{\varphi}_u$ and $\tilde{\varphi}'_u$. Their values are determined by

$$\tilde{\varphi}_u(xs_1) = \varphi_u(x) \qquad \text{and} \qquad \tilde{\varphi}'_u(xs_1) = -\varphi_u(x) \qquad \text{for all } x \in U_{d,2}^F.$$

DEFINITION 2.1. Let $u \in \{y_{74}, y_{75}\}$. Then we set

$$\psi_u := \mathrm{Ind}_{U_{d,2}^F.\langle s_1 \rangle}^{U_{d,2}^F.H}\big(\tilde{\varphi}_u\big) \qquad \text{and} \qquad \psi'_u := \mathrm{Ind}_{U_{d,2}^F.\langle s_1 \rangle}^{U_{d,2}^F.H}\big(\tilde{\varphi}'_u\big).$$

Thus, we have

$$\mathrm{Ind}_{U_{d,2}^F}^{U_{d,2}^F.H}\big(\varphi_u\big) = \psi_u + \psi'_u \qquad \text{and} \qquad [U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u = \tilde{\Gamma}_u + \tilde{\Gamma}'_u,$$

where

$$\tilde{\Gamma}_u := \mathrm{Ind}_{U_{d,2}^F.\langle s_1 \rangle}^{G^F}\big(\tilde{\varphi}_u\big) = \mathrm{Ind}_{U_{d,2}^F.H}^{G^F}\big(\psi_u\big),$$

$$\tilde{\Gamma}'_u := \operatorname{Ind}_{U_{d,2}^F.\langle s_1\rangle}^{G^F}\big(\tilde{\varphi}'_u\big) = \operatorname{Ind}_{U_{d,2}^F.H}^{G^F}\big(\psi'_u\big).$$

The following result provides some crucial information concerning ψ_u and ψ'_u.

Proposition 2.2. *Recall that q is an even power of p. Then, with the above notation, the following hold.*
(a) *Both ψ_u and ψ'_u are irreducible characters of $U_{d,2}^F.H$.*
(b) *ψ_u can be realized over $\mathbb{Q}$.*
(c) *ψ'_u is rational-valued but cannot be realized over $\mathbb{Q}$. In fact, ψ'_u has non-trivial local Schur indices at ∞ and at the prime p.*

Proof. (see also the argument of Ohmori [14], p. 154.) Let

$$x := \prod_{\substack{\alpha\in\Phi^+\\ d(\alpha)\geq 2}} x_\alpha(\xi_\alpha) \in U_{d,2}^F \qquad \text{and} \qquad \gamma_x := \sum_{\substack{\alpha\in\Phi^+\\ d(\alpha)=2}} c_\alpha\eta_\alpha\xi_\alpha, \tag{1}$$

where $\xi_\alpha \in \mathbb{F}_q$. Then, as in the proof of [5], Proposition 2.3, we have

$$\varphi_u(t^ixt^{-i}) = \chi(\nu^i\gamma_x) \qquad \text{for } 1\leq i\leq 2(p-1).$$

In particular, this implies $\operatorname{Stab}_H(\varphi_u) = \langle s_1\rangle$. Hence, by Clifford theory, the induced character

$$\operatorname{Ind}_{U_{d,2}^F}^{U_{d,2}^F.H}\big(\varphi_u\big) = \psi_u + \psi'_u$$

has inner product 2. Thus, we ψ_u and ψ'_u must be irreducible, proving (a).

Next we prove (b). Using Mackey's formula and relation (1), we have that

$$\begin{aligned}\operatorname{Ind}_{U_{d,2}^F}^{U_{d,2}^F.H}\big(\varphi_u\big)(x) &= \sum_{i=1}^{2(p-1)} \varphi_u(t^ixt^{-i}) = \sum_{i=1}^{2(p-1)} \chi(\nu^i\gamma_x)\\ &= \sum_{i=1}^{2(p-1)} \chi_0\big(\nu^i \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\gamma_x)\big) = \begin{cases} 2(p-1) & \text{if } \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\gamma_x)=0,\\ -2 & \text{if } \operatorname{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(\gamma_x)\neq 0.\end{cases}\end{aligned}$$

In particular, this shows that the values are rational integers. Thus, $\psi_u+\psi'_u$ is rational-valued. Now assume, if possible, that ψ_u is not rational-valued. Then the characters ψ_u and ψ'_u must be algebraically conjugate. Consequently, ψ_u and ψ'_u occur with the same multiplicity in every rational-valued character. Now, by the Mackey formula and Frobenius reciprocity, we have

$$\begin{aligned}\Big\langle \psi'_u, \operatorname{Ind}_H^{U_{d,2}^F.H}\big(\mathbf{1}_H\big)\Big\rangle_{U_{d,2}^F.H} &= \Big\langle \operatorname{Ind}_{U_{d,2}^F.\langle s_1\rangle}^{U_{d,2}^F.H}\big(\tilde{\varphi}'_u\big), \operatorname{Ind}_H^{U_{d,2}^F.H}\big(\mathbf{1}_H\big)\Big\rangle_{U_{d,2}^F.H}\\ &= \Big\langle \operatorname{Res}_{\langle s_1\rangle}^{U_{d,2}^F}\big(\tilde{\varphi}'_u\big), \mathbf{1}_{\langle s_1\rangle}\Big\rangle_{U_{d,2}^F}\end{aligned}$$

$$= 0,$$

since $\tilde{\varphi}'_u(s_1) = -1$. (Here, the symbol $\mathbf{1}$ stands for the unit character.) By a similar argument, since $\tilde{\varphi}_u(s_1) = 1$, we also have

$$\left\langle \psi_u, \operatorname{Ind}_H^{U_{d,2}^F.H}\big(\mathbf{1}_H\big)\right\rangle_{U_{d,2}^F.H} = 1.$$

Thus, ψ_u and ψ'_u do not occur with the same multiplicity in some rational-valued character, a contradiction. Thus, our assumption was wrong and so both ψ_u and ψ'_u are rational-valued. But then the above multiplicity 1 formula implies that ψ_u can be realized over $\mathbb{Q}$, by a standard argument concerning Schur indices (see Isaacs [8], Corollary 10.2).

Finally, we prove (c). We begin by showing that the local Schur index at ∞ is non-trivial. In other words, we must show that ψ'_u cannot be realized over $\mathbb{R}$. For this purpose, by a well-known criterion due to Frobenius and Schur (see Isaacs [8], Chapter 4), it is enough to show that

$$\frac{1}{|U_{d,2}^F.H|} \sum_{g \in U_{d,2}^F.H} \psi'_u(g^2) = -1.$$

Now, in order to evaluate the above sum, we note that

$$\frac{1}{|U_{d,2}^F.H|} \sum_{g \in U_{d,2}^F.H} \psi_u(g^2) = 1,$$

since ψ_u can be realized over $\mathbb{Q}$. Thus, it will be enough to show that

$$\frac{1}{|U_{d,2}^F.H|} \sum_{g \in U_{d,2}^F.H} \operatorname{Ind}_{U_{d,2}}^{U_{d,2}^F.H}\big(\varphi_u\big)\big(g^2\big) = \frac{1}{|U_{d,2}^F.H|} \sum_{g \in U_{d,2}^F.H} \big(\psi_u + \psi'_u\big)\big(g^2\big) = 0.$$

Let $g \in U_{d,2}^F.H$ and write $g = xh$ where $x \in U_{d,2}^F$ and $h \in H$. Now the value of the above induced character on g^2 is zero unless $g^2 \in U_{d,2}^F$. Thus, we only need to consider elements $g = xh$ where $h = 1$ or $h = s_1$. So we must show that

$$\sum_{x \in U_{d,2}^F} \operatorname{Ind}_{U_{d,2}}^{U_{d,2}^F.H}\big(\varphi_u\big)\big(x^2\big) + \sum_{x \in U_{d,2}^F} \operatorname{Ind}_{U_{d,2}}^{U_{d,2}^F.H}\big(\varphi_u\big)(xs_1xs_1) = 0.$$

Now, since $U_{d,2}^F$ has odd order, the map $x \mapsto x^2$ defines a bijection of $U_{d,2}^F$ onto itself. Hence the first sum evaluates to

$$\sum_{x \in U_{d,2}^F} \operatorname{Ind}_{U_{d,2}}^{U_{d,2}^F.H}\big(\varphi_u\big)\big(x^2\big) = |U_{d,2}^F.H| \cdot \left\langle \operatorname{Ind}_{U_{d,2}}^{U_{d,2}^F.H}\big(\varphi_u\big), \mathbf{1}_{U_{d,2}^F.H}\right\rangle_{U_{d,2}^F.H} = 0.$$

Now consider the second sum. For this purpose, we note that $\alpha(s_1) = 1$ for all roots $\alpha \in \Phi^+$ which are involved in the expressions of y_{74} and y_{75} as products of root subgroup elements. Thus, if x and γ_x are as in (1), then we have

$$\gamma_{(xs_1)^2} = \sum_{\substack{\alpha\in\Phi^+\\ d(\alpha)=2}} c_\alpha \eta_\alpha \big(\alpha(s_1)+1\big)\xi_\alpha = 2\gamma_x = \gamma_{x^2}.$$

Using once more Mackey's formula as at the beginning of this proof, we see that

$$\begin{aligned}\operatorname{Ind}_{U_{d,2}}^{U_{d,2}^F.H}(\varphi_u)(x^2) &= \sum_{i=1}^{2(p-1)} \chi\big(\nu^i\gamma_{x^2}\big) = \sum_{i=1}^{2(p-1)} \chi\big(\nu^i\gamma_{(xs_1)^2}\big)\\ &= \operatorname{Ind}_{U_{d,2}}^{U_{d,2}^F.H}(\varphi_u)(xs_1xs_1)\end{aligned}$$

for all $x \in U_{d,2}^F$. Consequently, the second sum also equals 0. Thus, we have shown that ψ_u' cannot be realized over $\mathbb{R}$. We shall now use some general properties of Schur indices; see Feit [3], §2, for references. First, since ψ_u' is rational-valued but ψ_u' cannot be realized over $\mathbb{R}$, the Schur index of ψ_u' is 2 (by the Brauer–Speiser theorem; see [3], 2.4). Furthermore, there exists at least one prime number l such that the l-local Schur index of ψ_u' is 2 (by the Hasse sum formula; see [3], 2.15). Thus, it will be enough to show that the l-local Schur index of ψ_u' is 1, for every prime $l \neq p$. Let l be such a prime. If $l \neq 2$, then ψ_u' is a character of l-defect 0 of $U_{d,2}^F.H$. So the l-local Schur index is 1 by [3], 2.10. Finally, if $l = 2$, then ψ_u' is a character of 2-defect 1 and, hence, lies in a block with a cyclic defect group of order 2. Consequently, that block contains only two irreducible characters and so ψ_u' remains irreducible as a 2-modular Brauer character. This implies again that the local Schur index is 1; see [3], 2.10. □

3. A subgroup of type $D_6 \times A_1$

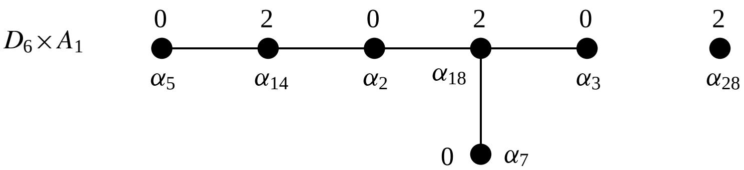
Our next aim is to compute the multiplicity of $E_7[\pm\xi]$ in $\tilde{\Gamma}_u$ and $\tilde{\Gamma}_u'$; see Definition 2.1. We already know that the multiplicity of $E_7[\pm\xi]$ in the sum $\tilde{\Gamma}_u + \tilde{\Gamma}_u'$ equals $[U_{d,1}^F : U_{d,2}^F]^{1/2}$, for suitable $u \in \{y_{74}, y_{75}\}$. We shall now try to compute the multiplicity in the difference $\tilde{\Gamma}_u - \tilde{\Gamma}_u'$. For this purpose, we take a closer look at the semisimple element s_1 and its centralizer. Let

$$G_1 := \langle T, X_\alpha \mid \alpha \in \Phi_1\rangle \qquad \text{where} \qquad \Phi_1 := \{\alpha \in \Phi \mid \alpha(s_1) = 1\}.$$

Using the *CHEVIE* function *ReflectionSubgroup*, we check that the root system Φ_1 has type $D_6 \times A_1$; a system of simple roots in Φ_1 is given by

$$\Pi_1 = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_{14}, \alpha_{18}, \alpha_{28}\}$$

Table 2. The restriction of d to the subsystem of type $D_6 \times A_1$



where

$$\alpha_{14} := \alpha_1 + \alpha_3 + \alpha_4, \quad \alpha_{18} := \alpha_4 + \alpha_5 + \alpha_6, \quad \alpha_{28} := \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5.$$

(Here, the numbering of the roots is the same as that given by *CHEVIE*.) The corresponding Dynkin diagram and the restriction of the weight function d to Π_1 are given in Table 2. Furthermore, one can check, using *CHEVIE* (for example), that

$$N_W(W_1) = \{w \in W \mid w(\Phi_1) \subseteq \Phi_1\} = W_1$$

where $W_1 := \langle w_\alpha \mid \alpha \in \Phi_1 \rangle \subset W$ is the Weyl group of G_1 (and where we denote by w_β the reflection with root β, for any root $\beta \in \Phi$).

Lemma 3.1. *We have $C_G(s_1) = G_1$; in particular, $C_G(s_1)$ is connected.*

Proof. By Carter [2], §3.5, we have $C_G(s_1)^\circ = G_1$. Hence, G_1 is a normal subgroup in $C_G(s_1)$. So it is enough to show that $N_G(G_1) = G_1$. Let $g \in N_G(G_1)$. Then gTg^{-1} is a maximal torus in G_1 and so there exists some $g_1 \in G_1$ such that $gTg^{-1} = g_1Tg_1^{-1}$. Thus, we have $g_1^{-1}g \in N_G(T)$ and so $g \in G_1.N_G(T)$. Hence, we may assume without loss of generality that $g \in N_G(T) \cap N_G(G_1)$. Now, for any $g \in N_G(T) \cap N_G(G_1)$ and any $\alpha \in \Phi_1$, we have $gX_\alpha g^{-1} = X_{w(\alpha)} \subseteq G_1$, where w is the image of g in $W = N_G(T)/T$. Thus, we have $w(\Phi_1) \subseteq \Phi_1$ and so $w \in W_1$ (see the above remarks). This implies $g \in G_1$, as required. □

Let C_1 be the conjugacy class of y_{74} in G_1 and denote by $d_1 \colon \Phi_1 \to \mathbb{Z}$ the corresponding weighted Dynkin diagram. Using the identification results in [1], Theorem 11.3.2, it is straightforward to check that, under the natural matrix representation of a group of type $D_6 \times A_1$, the elements y_{74} and y_{75} correspond to matrices with Jordan blocks of size 1, 1, 2, 5, 5 (where the block of size 2 comes from the A_1-factor). Hence, using [2], §13.1, we see that d_1 is given by the restriction of d to Φ_1, as specified in Table 2. Furthermore, we notice that the above roots can all be written as sums of roots in Π_1. Thus, we have

$$y_{74}, \; y_{75} \in C_1 \cap U_{d_1,2}^F,$$

where $U_{d_1,2}$ is the unipotent subgroup of G_1 defined with respect to d_1.

Lemma 3.2. *Let* $u \in \{y_{74}, y_{75}\}$. *Then we have* $\dim \mathfrak{B}_u^1 = 4$ (*where* $\mathfrak{B}_u^1$ *denotes the variety of Borel subgroups of* G_1 *containing* u) *and*

$$C_{G_1}(u)/C_{G_1}(u)^\circ \cong C_G(u)/C_G(u)^\circ \cong \mathbb{Z}/2\mathbb{Z}.$$

Proof. Let $u := y_{74}$. The formula for $\dim \mathfrak{B}_u^1$ follows from [2], §13.1. To prove the remaining statements, we note that

$$s_1 \in S := \left\{h(x, x^{-2}, x^{-2}, x^3, x^{-2}, x, 1) \mid x \in k^\times\right\} \subseteq C_{G_1}(u).$$

Furthermore, one checks that $Z(G_1) = \{t \in T \mid \alpha(t) = 1 \text{ for all } \alpha \in \Phi_1\} = \langle s_1 \rangle$. Thus, since S is connected, we have $Z(G_1) \subseteq C_{G_1}(u)^\circ$.

Now let $\pi \colon G_1 \to H_1$ be the adjoint quotient of G_1, where H_1 is a semisimple group of adjoint type $D_6 \times A_1$. Let $\bar{u}$ be the image of u in H_1. Then, by Carter [2], §13.1, we know that $C_{H_1}(\bar{u})/C_{H_1}(\bar{u})^\circ \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, π induces a surjective homomorphism

$$C_{G_1}(u)/C_{G_1}(u)^\circ \twoheadrightarrow C_{H_1}(\bar{u})/C_{H_1}(\bar{u})^\circ \cong \mathbb{Z}/2\mathbb{Z}$$

with kernel given by the image of $Z(G_1)$ in $C_{G_1}(u)/C_{G_1}(u)^\circ$. Since $Z(G_1) \subseteq C_{G_1}(u)^\circ$, that image is trivial and so the above surjective map is also injective. □

Proposition 3.3. *Let* $u \in \{y_{74}, y_{75}\} \subseteq C \cap U_{d,2}^F$. *Then, as we already noted, we have* $u \in C_1 \cap U_{d_1,2}^F$ *and so the corresponding generalized Gelfand–Graev character* Γ_u^1 *of* G_1^F *is well-defined. We have*

$$\tilde{\Gamma}_u(ys_1) - \tilde{\Gamma}'_u(ys_1) = \Gamma_u^1(y) \qquad \textit{for all } y \in G_1^F \textit{ unipotent}.$$

Proof. By the Mackey formula, we have

$$\begin{aligned}\tilde{\Gamma}_u(ys_1) &= \operatorname{Res}_{G_1^F}^{G^F}\big(\tilde{\Gamma}_u\big)(ys_1) = \operatorname{Res}_{G_1^F}^{G^F}\Big(\operatorname{Ind}_{U_{d,2}^F.\langle s_1\rangle}^{G^F}\big(\tilde{\varphi}_u\big)\Big)(ys_1)\\ &= \sum_z \operatorname{Ind}_{(U_{d,2}^F.\langle s_1\rangle)^z\cap G_1^F}^{G_1^F}\Big(\operatorname{Res}_{(U_{d,2}^F.\langle s_1\rangle)^z\cap G_1^F}^{(U_{d,2}^F.\langle s_1\rangle)^z}\big(\tilde{\varphi}_u^z\big)\Big)(ys_1),\end{aligned}$$

where z runs over a set of representatives of the $(U_{d,2}^F.\langle s_1\rangle, G_1^F)$-double cosets of G^F. Let us fix such a double coset representative, z say. Assume that the value at ys_1 of the corresponding induced character in the above sum is non-zero. Then ys_1 must be G_1^F-conjugate to an element in the subgroup $(U_{d,2}^F.\langle s_1\rangle)^z \cap G_1^F$. Consequently, s_1 must be G_1^F-conjugate to an element in that subgroup. Since $\langle s_1\rangle$ is a Sylow 2-subgroup of $U_{d,2}^F.\langle s_1\rangle$, we conclude that all elements of order 2 in $U_{d,2}^F.\langle s_1\rangle$ are of the form xs_1x^{-1}

where $x \in U_{d,2}^F$. Thus, we have $c^{-1}s_1c = z^{-1}xs_1x^{-1}z$ for some $c \in G_1^F$ and some $x \in U_{d,2}^F$. Consequently, $x^{-1}zc^{-1} \in C_G(s_1)^F = G_1^F$ and so $z \in xG_1^Fc \in U_{d,2}^F.G_1^F$. Thus, z represents the trivial double coset and so we can take $z = 1$. Using the fact that

$$U_{d,2}^F.\langle s_1\rangle \cap G_1^F = U_{d_1,2}^F \times \langle s_1\rangle$$

(where $U_{d_1,2} \subseteq G_1$ is the unipotent subgroup defined with respect to the weighted Dynkin diagram $d_1\colon \Phi_1 \to \mathbb{Z}$) we find that

$$\tilde{\Gamma}_u(ys_1) = \mathrm{Ind}_{U_{d_1,2}^F\times\langle s_1\rangle}^{G_1^F}\big(\varphi_u^1 \boxtimes \mathbf{1}_{\langle s_1\rangle}\big)(ys_1)$$

where φ_u^1 denotes the restriction of φ_u to $U_{d_1,2}^F$. Since s_1 is in the center of G_1, it is readily checked that

$$\tilde{\Gamma}_u(ys_1) = \frac{1}{2}\,\tilde{\varphi}_u(s_1)\ \mathrm{Ind}_{U_{d_1,2}^F}^{G_1^F}\big(\varphi_u^1\big)(y) = \frac{1}{2}\ \mathrm{Ind}_{U_{d_1,2}^F}^{G_1^F}\big(\varphi_u^1\big)(y).$$

By a completely analogous argument, we also obtain that

$$\tilde{\Gamma}_u'(ys_1) = \frac{1}{2}\,\tilde{\varphi}_u'(s_1)\ \mathrm{Ind}_{U_{d_1,2}^F}^{G_1^F}\big(\varphi_u^1\big)(y) = -\frac{1}{2}\ \mathrm{Ind}_{U_{d_1,2}^F}^{G_1^F}\big(\varphi_u^1\big)(y).$$

Thus, it remains to check that

$$\Gamma_u^1 = \mathrm{Ind}_{U_{d_1,2}^F}^{G_1^F}\big(\varphi_u^1\big).$$

For this purpose, we must show that φ_u^1 indeed is the linear character of $U_{d_1,2}^F$ required in the definition of Γ_u^1. Now, the definition of Γ_u^1 requires the choice of a non-degenerate bilinear form and of an opposition automorphism on the Lie algebra of G_1. However, the Lie algebra of G_1 is naturally contained in the Lie algebra of G, with compatible Cartan decompositions. Thus, the chosen bilinear form and the chosen opposition automorphism restrict to the Lie algebra of G_1, and this implies that φ_u^1 is the required linear character of $U_{d_1,2}^F$. □

A formula of this kind has been stated (without proof) by Kawanaka in [9], Lemma 2.3.5; see also the Ph. D. thesis of Wings [16], §3.2.1.

REMARK 3.4. Let $g \in G^F$ and write $g = g_sg_u = g_ug_s$ where $g_s \in G^F$ is semisimple and $g_u \in G^F$ is unipotent. Assume that g_s is not conjugate to s_1 in G^F. Then we have

$$(\tilde{\Gamma}_u - \tilde{\Gamma}_u')(g) = 0.$$

Indeed, if the value is non-zero, then g must be G^F-conjugate to an element in $U_{d,2}^F.\langle s_1\rangle$. But then g_s will also be G^F-conjugate to an element in that subgroup. Using a Sylow argument as in the above proof, we see that either $g_s = 1$ or g_s is

G^F-conjugate to s_1, as claimed. Furthermore, if $g_s = 1$, then it is readily checked that $\tilde{\Gamma}_u(g) = \tilde{\Gamma}'_u(g)$.

Thus, in order to compute the scalar product of $E_7[\pm\xi]$ with $\tilde{\Gamma}_u - \tilde{\Gamma}'_u$, it will be enough to know the values of $E_7[\pm\xi]$ on elements of the form ys_1 where $y \in G_1^F$ is unipotent. Furthermore, since $E_7[\xi]$ and $E_7[-\xi]$ are complex conjugate and since $\tilde{\Gamma}_u$ and $\tilde{\Gamma}'_u$ are rational-valued, it will actually be enough to consider the sum $E_7[\xi] + E_7[-\xi]$. Now, by Lusztig [10], Main Theorem 4.23, we have

$$E_7[\xi] + E_7[-\xi] = R_{512_a} - R_{512'_a}.$$

(Note that the function Δ occurring in [10], 4.23, takes value -1 on the labels corresponding to the characters $E_7[\pm\xi]$.) Here, 512_a, $512'_a$ are the two irreducible characters of W of degree 512 and R_{512_a}, $R_{512'_a}$ are the corresponding "almost characters", as defined by Lusztig [10], (3.7). For any $\phi \in \mathrm{Irr}(W)$, we have

$$R_\phi := \frac{1}{|W|} \sum_{w \in W} \phi(w)\, R_{T_w,1};$$

here, $T_w \subseteq G$ is an F-stable maximal torus obtained from T by twisting with w and $R_{T_w,1}$ is the Deligne–Lusztig generalized character associated with the trivial character of T_w^F. Similarly, for any $\psi \in \mathrm{Irr}(W_1)$, we denote by R_ψ^1 the corresponding almost character of G_1^F.

Lemma 3.5. *Let $\phi \in \mathrm{Irr}(W)$ and write*

$$\mathrm{Res}_{W_1}^W(\phi) = \sum_{\psi \in \mathrm{Irr}(W_1)} m(\phi,\psi)\,\psi \qquad \textit{where } m(\phi,\psi) \in \mathbb{Z}_{\geq 0}.$$

Let $y \in G_1^F$ be a unipotent element. Then we have

$$R_\phi(ys_1) = \sum_{\psi \in \mathrm{Irr}(W_1)} m(\phi,\psi)\, R_\psi^1(y).$$

Proof. The character formula for $R_{T_w,1}$ (see [2], Theorem 7.2.8) shows that

$$R_{T_w,1}(ys_1) = \frac{|C_W(w)|}{|W_1|} \sum_{\substack{w_1 \in W_1 \\ w \sim w_1}} R^1_{T_{w_1},1}(y)$$

where the relation $\sim$ means conjugacy in W. (Here, $R^1_{T_{w_1},1}$ denotes a Deligne–Lusztig generalized character of G_1^F.) Thus, we have

$$R_\phi(ys_1) = \frac{1}{|W|} \sum_{\substack{w \in W, w_1 \in W_1 \\ w \sim w_1}} \frac{|C_W(w)|}{|W_1|}\, \phi(w)\, R^1_{T_{w_1},1}(y)$$

$$= \frac{1}{|W_1|} \sum_{w_1 \in W_1} \Big(\frac{1}{|W|} \sum_{\substack{w \in W \\ w \sim w_1}} |C_W(w)|\, \phi(w) \Big) R^1_{T_{w_1},1}(y)$$

Now, we have $\phi(w) = \phi(w_1)$ and $|C_W(w)| = |C_W(w_1)|$ for all $w_1 \in W_1$ such that $w \sim w_1$. Thus, we have

$$\frac{1}{|W|} \sum_{\substack{w \in W \\ w \sim w_1}} |C_W(w)|\, \phi(w) = \frac{|C_W(w_1)|}{|W|} \phi(w_1) \sum_{\substack{w \in W \\ w \sim w_1}} 1 = \phi(w_1).$$

Writing $\phi(w_1) = \sum_\psi m(\phi, \psi)\psi(w_1)$, we obtain the desired expression. □

Corollary 3.6. *With the notation of* Proposition 3.3 *and* Lemma 3.5, *we have*

$$\langle R_\phi, \tilde{\Gamma}_u - \tilde{\Gamma}'_u \rangle_{G^F} = \sum_{\psi \in \mathrm{Irr}(W_1)} m(\phi, \psi) \langle R_\psi, \Gamma^1_u \rangle_{G_1^F},$$

for any $\phi \in \mathrm{Irr}(W)$ *and* $u \in \{y_{74}, y_{75}\} \subseteq C_1 \cap U^F_{d_1,2}$.

Proof. Immediate from Proposition 3.3, Remark 3.4 and Lemma 3.5. □

We now need some explicit information concerning the restriction of characters from W to W_1. Using the *CHEVIE* function *InductionTable*, we compute that

$$\mathrm{Res}^W_{W_1}(512_a) \otimes \varepsilon = ([21, 3] \boxtimes \mathbf{1}) + \text{sum of } \psi \text{ where } \psi \in \mathrm{Irr}(W_1) \text{ and } a_\psi > 4,$$
$$\mathrm{Res}^W_{W_1}(512'_a) \otimes \varepsilon = ([2, 31] \boxtimes \mathbf{1}) + \text{sum of } \psi \text{ where } \psi \in \mathrm{Irr}(W_1) \text{ and } a_\psi > 4.$$

Here, $\mathbf{1}$ denotes the unit character on the A_1-factor of W_1 and ε denotes the sign character of W_1. The characters of the D_6-factor are denoted by $[\lambda, \mu]$ where λ and μ are partitions such that $|\lambda| + |\mu| = 6$. The a-invariant of a character is defined as in Lusztig [10], (4.1); in *CHEVIE*, these a-invariants are obtained by the function *LowestPowerGenericDegrees*. We have

$$a_\psi = 4 \qquad \text{for } \psi = [21, 3] \boxtimes \mathbf{1} \text{ and } \psi = [2, 31] \boxtimes \mathbf{1}.$$

With these explicit formulas, we can now prove the following result.

Proposition 3.7. *Assume that the characteristic p is large enough, such that Lusztig's formula in* [11], Theorem 7.5, *for the values of a generalized Gelfand–Graev holds for* Γ^1_u. *By* [5], Corollary 4.3, *there exists some* $u \in \{y_{74}, y_{75}\}$ *such that* $\langle E_7[\pm\xi], \Gamma_u \rangle_{G^F} = 1$. *For this element u, we have*

$$\langle E_7[\pm\xi], \tilde{\Gamma}_u - \tilde{\Gamma}'_u \rangle_{G^F} = -1.$$

Proof. We have already mentioned in the remarks preceding Lemma 3.5 that

$$E_7[\xi] + E_7[-\xi] = R_{512_a} - R_{512'_a}.$$

Since $\tilde{\Gamma}_u$ and $\tilde{\Gamma}'_u$ are rational-valued (see Proposition 2.2), we have

$$\begin{aligned}\left\langle E_7[\pm\xi], \tilde{\Gamma}_u - \tilde{\Gamma}'_u\right\rangle_{G^F} &= \frac{1}{2}\left\langle E_7[\xi] + E_7[-\xi], \tilde{\Gamma}_u - \tilde{\Gamma}'_u\right\rangle_{G^F} \\ &= \frac{1}{2}\left\langle R_{512_a} - R_{512'_a}, \tilde{\Gamma}_u - \tilde{\Gamma}'_u\right\rangle_{G^F}.\end{aligned}$$

Now let $\psi \in \mathrm{Irr}(W_1)$ be a constituent in the restriction of 512_a or $512'_a$ from W to W_1. Then, by Corollary 3.6, we must compute the scalar product $\langle R_\psi, \Gamma^1_u\rangle_{G_1^F}$. Let D denote the Alvis–Curtis–Kawanaka duality operation on the character ring of G_1^F; see Lusztig [10], (6.8). We have $D(R_\psi) = R_{\psi\otimes\varepsilon}$ and so

$$\langle R_\psi, \Gamma^1_u\rangle_{G_1^F} = \left\langle D(R_\psi), D(\Gamma^1_u)\right\rangle_{G_1^F} = \left\langle R_{\psi\otimes\varepsilon}, D(\Gamma^1_u)\right\rangle_{G_1^F}.$$

Now, in order to evaluate the above scalar product, it is enough to know the values of $R_{\psi\otimes\varepsilon}$ on the unipotent elements of G_1^F. By Shoji's algorithm [15] and by [11], Corollary 10.9, we know that $R_{\psi\otimes\varepsilon}(y) = 0$ if $\dim \mathfrak{B}^1_y < a_{\psi\otimes\varepsilon}$. On the other hand, we have $D(\Gamma^1_u)(y) = 0$ if $\dim \mathfrak{B}^1_u < \dim \mathfrak{B}^1_y$. (This follows from [11]; see the remarks in [4], (2.4).) Thus, the above scalar product is zero if $a_{\psi\otimes\varepsilon} > \dim \mathfrak{B}^1_u = 4$. Taking into account the explicit information concerning the restrictions of 512_a and $512'_a$ from W to W_1, we conclude that

$$\left\langle E_7[\pm\xi], \tilde{\Gamma}_u - \tilde{\Gamma}'_u\right\rangle_{G^F} = \frac{1}{2}\left\langle R_{[21,3]\boxtimes\mathbf{1}} - R_{[2,31]\boxtimes\mathbf{1}}, D(\Gamma^1_u)\right\rangle_{G_1^F}. \tag{1}$$

Now $[21, 3] \boxtimes \mathbf{1}$ and $[2, 31] \boxtimes \mathbf{1}$ lie in the same family of characters of W_1; see [10], Chapter 4. The Fourier matrix (which has size 4×4) for that family shows that

$$R_{[21,3]\boxtimes\mathbf{1}} - R_{[2,31]\boxtimes\mathbf{1}} = -\rho_1 - \rho_2$$

where ρ_1 and ρ_2 are unipotent characters of G_1^F. Now, we can explicitly compute the unipotent support of these two characters; see [11], §11, or [7], §3.C. This involves the knowledge of the Springer correspondence for G_1. Using the description of that correspondence in [2], §13.3, we find that ρ_1 and ρ_2 have unipotent support C_1. Thus, by the formula in [7], Remark 3.8, we have

$$\left\langle \rho_i, D(\Gamma^1_{y_{74}}) + D(\Gamma^1_{y_{75}})\right\rangle_{G_1^F} = \left\langle D(\rho_i), \Gamma^1_{y_{74}} + \Gamma^1_{y_{75}}\right\rangle_{G_1^F} = 1 \quad \text{for } i = 1,\, 2. \tag{2}$$

Note that $C_{G_1}(y_{74})/C_{G_1}(y_{74})^\circ \cong \mathbb{Z}/2\mathbb{Z}$ by Lemma 3.2 and that $D(\rho_1)$, $D(\rho_2)$ are actual characters in the present situation; see [10], (6.8.2). Now we have $u \in \{y_{74}, y_{75}\}$ and we would like to show that

$$(3) \qquad \big\langle D(\rho_i), \Gamma_u^1 \big\rangle_{G_1^F} = \big\langle \rho_i, D(\Gamma_u^1) \big\rangle_{G_1^F} = 1 \quad \text{for } i = 1,\ 2.$$

This can be seen as follows. Fix $i \in \{1, 2\}$. Since $D(\rho_i)$ is an actual character, we certainly have $\langle D(\rho_i), \Gamma_u^1 \rangle_{G_1^F} \geq 0$. Hence, using (2), the latter scalar product equals 0 or 1. Assume, if possible, that the scalar product is zero. Then the scalar product of $-\rho_1 - \rho_2$ with $D_G(\Gamma_u^1)$ would be -1 or 0. Consequently, the scalar product in (1) would be $-1/2$ or 0. Thus, the only possibility is that the scalar product in (1) equals 0. But this would mean that

$$\big\langle E_7[\pm\xi], \tilde{\Gamma}_u + \tilde{\Gamma}'_u \big\rangle_{G^F} = [U_{d,1}^F : U_{d,2}^F]^{1/2} \big\langle E_7[\pm\xi], \Gamma_u \big\rangle_{G^F} = [U_{d,1}^F : U_{d,2}^F]^{1/2}$$

is an even number, which is not true. So, our assumption was wrong and (3) holds. Inserting this into (1), we obtain the desired result. □

4. Proof of Theorem 1.1

By [5], Corollary 4.3, the Schur index of $E_7[\pm\xi]$ is at most 2. Hence, we only need to show that $E_7[\pm\xi]$ cannot be realized over $\mathbb{Q}(\xi)$. Now, we have

$$\langle E_7[\pm\xi], \Gamma_u \rangle_{G^F} = 1 \qquad \text{for suitable } u \in \{y_{74}, y_{75}\}.$$

So, using the formulas in Definition 2.1, we obtain that

$$\big\langle E_7[\pm\xi], \tilde{\Gamma}_u + \tilde{\Gamma}'_u \big\rangle_{G^F} = [U_{d,1}^F : U_{d,2}^F]^{1/2} = q^m \quad \text{for some } m \geq 1.$$

Combining this with Proposition 3.7 and using Frobenius reciprocity, this yields

$$\Big\langle \operatorname{Res}^{G^F}_{U_{d,2}^F.H}\big(E_7[\pm\xi]\big), \psi'_u \Big\rangle_{U_{d,2}^F.H} = \big\langle E_7[\pm\xi], \tilde{\Gamma}'_u \big\rangle_{G^F} = \frac{1}{2}(q^m + 1).$$

Since $p \equiv 1 \bmod 4$, we also have $q \equiv 1 \bmod 4$ and so the above scalar product is an odd number. Now assume, if possible, that $E_7[\pm\xi]$ can be realized over $\mathbb{Q}(\xi)$. Then the restriction of $E_7[\pm\xi]$ to $U_{d,2}^F.H$ can also be realized over $\mathbb{Q}(\xi)$. Thus, by a standard argument on Schur induces ([8], Corollary 10.2), the Schur index of ψ'_u over $\mathbb{Q}(\xi)$ divides the above odd number. Since the Schur index of ψ'_u over $\mathbb{Q}(\xi)$ is at most 2 (see Proposition 2.2), it must be one. Thus, ψ'_u can be realized over $\mathbb{Q}(\xi)$. Now, since q is a square, we have $\xi = \sqrt{-1}$. Furthermore, since $p \equiv 1 \bmod 4$, we have $\sqrt{-1} \in \mathbb{Q}_p$ (the field of p-adic numbers). Hence ψ'_u can be realized over $\mathbb{Q}_p$, contradicting Proposition 2.2(c). Thus, our assumption was wrong and so $E_7[\pm\xi]$ cannot be realized over $\mathbb{Q}(\xi)$.

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