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Osaka University
THE SCHUR INDICES OF THE CUSPIDAL UNIPOTENT CHARACTERS OF THE FINITE CHEVALLEY GROUPS $E_7(q)$

Dedicated to Professor Herbert Pahlings on his 65th birthday

MEINOLF GECK

(Received July 8, 2003)

Abstract

We show that the two cuspidal unipotent characters of a finite Chevalley group $E_7(q)$ have Schur index 2, provided that $q$ is an even power of a (sufficiently large) prime number $p$ such that $p \equiv 1 \mod 4$. The proof uses a refinement of Kawanaka’s generalized Gelfand–Graev representations and some explicit computations with the CHEVIE computer algebra system.

1. Introduction

Throughout this paper, let $G$ be a simple algebraic group of adjoint type $E_7$. Assume that $G$ is defined over the finite field $\mathbb{F}_q$, with corresponding Frobenius map $F: G \to G$. There are precisely two cuspidal unipotent characters of $G^F$ denoted by $E_7[\pm \xi]$ where $\xi = \sqrt{-q}$; see the table in [2], §13.9.

The purpose of this paper is to determine the Schur index of $E_7[\pm \xi]$, at least if the characteristic of $\mathbb{F}_q$ is large enough. Modulo this condition on the characteristic, this completes the determination of the Schur indices of the unipotent characters of finite groups of Lie type; see [12], [5] and the references there.

By [4], Table 1, the character values of $E_7[\pm \xi]$ generate the field $\mathbb{Q}(\xi)$. Furthermore, by [4], Example 6.4, we already know that the Schur index is 1 if $p \not\equiv 1 \mod 4$ or if $q$ is not a square, where $p$ is the characteristic of $\mathbb{F}_q$. Thus, the remaining task is to determine the Schur index when $q$ is a square and $p \equiv 1 \mod 4$.

**Theorem 1.1.** Assume that $q$ is an even power of a (sufficiently large) prime $p$ such that $p \equiv 1 \mod 4$. Then the characters $E_7[\pm \xi]$ have Schur index 2.

Here, $p$ is “sufficiently large” if Lusztig’s results [11] on generalized Gelfand–Graev characters hold; it is conjectured that this is the case if $p$ is good for $G$.

The idea of the proof is as follows. We have already seen in [5], §4, that $E_7[\pm \xi]$ occur with multiplicity 1 in a generalized Gelfand–Graev character $\Gamma'_u$, where $u$ is a

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Table 1. The weighted Dynkin diagram for the unipotent support of the cuspidal unipotent characters in type $E_7$

![Dynkin diagram](image)

certain unipotent element in $G$. Here, we shall use a refinement of the construction of $\Gamma_u$ to show that, under the given assumptions on $p$ and $q$, the characters $E_7[\pm \xi]$ occur with odd multiplicity in an induced character which cannot be realized over $\mathbb{Q}(\xi)$. By standard arguments on Schur indices, this implies that $E_7[\pm \xi]$ cannot be realized over $\mathbb{Q}(\xi)$. At some stage, the proof relies on the fact that, in Lusztig’s parametrization of the irreducible characters of $G^F$, the function $\Delta$ occurring in [10], Main Theorem 4.23, takes value $-1$ on the labels corresponding to $E_7[\pm \xi]$.

Furthermore, we rely on some explicit computations in $G^F$. However, we shall only use computations with the root system and the irreducible characters of the Weyl group of $G$, for which the CHEVIE system [6] is a convenient tool.

2. Generalized Gelfand–Graev characters for type $E_7$

A short summary of the construction of generalized Gelfand–Graev characters is given in [5], §2. Assume that $q$ is a power of a “good” prime $p \neq 2, 3$. Let $\Phi$ be the root system of $G$ with respect to a fixed maximally split torus $T$. Let $C$ be the unipotent class of $G$ whose weighted Dynkin diagram $d: \Phi \rightarrow \mathbb{Z}$ is given in Table 1. (The notation in that table also defines a labelling of the simple roots in the root system of $G$.) The class $C$ is the “unipotent support” of the two cuspidal unipotent characters of $G^F$; see [5], §4, and the references there.

Given the weight function $d: \Phi \rightarrow \mathbb{Z}$ specified by the diagram in Table 1, we define unipotent subgroups

$$U_{d,2} := \prod_{\alpha \in \Phi^+ \atop d_\alpha > 2} X_\alpha$$

and

$$U_{d,1} := \prod_{\alpha \in \Phi^+ \atop d_\alpha \geq 1} X_\alpha,$$

where $X_\alpha$ is the root subgroup in $G$ corresponding to the root $\alpha$. (It is understood that the products are taken in some fixed order.) The generalized Gelfand–Graev character associated with an element in $C^F$ is obtained by inducing a certain linear character from $U_{d,2}^F$. We have $C_G(u)/C_G(u)^2 \cong \mathbb{Z}/2\mathbb{Z}$ for $u \in C$. Thus, $C^F$ splits into two classes in the finite group $G^F$. By Mizuno [13], Lemma 28, representatives of these
two $G^F$-classes are given by
\[
\begin{align*}
\gamma_{74} &= x_{20}(1)x_{21}(1)x_{23}(1)x_{28}(1)x_{31}(1), \\
\gamma_{75} &= x_{20}(1)x_{21}(1)x_{28}(1)x_{24}(1)x_{23}(1)x_{25}(1)x_{36}(\xi),
\end{align*}
\]
where $\xi$ is a generator for the multiplicative group of $\mathbb{F}_q$ and where the subscripts correspond to the following roots in $\Phi^+$:
\[
\begin{align*}
20 &: \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \\
23 &: \alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \\
25 &: \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
31 &: \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, \\
36 &: \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7.
\end{align*}
\]
(Attention: Here, we use the labelling of the roots as given by the CHEVIE system [6], which is slightly different from that of Mizuno.) We note that both $\gamma_{74}$ and $\gamma_{75}$ lie in $C \cap U_{d,2}^F$. Now let us fix
\[
\mathbf{u} \in \{\gamma_{74}, \gamma_{75}\} \subseteq C \cap U_{d,2}^F;
\]
the above expressions show that
\[
\mathbf{u} = \prod_{\alpha_0 \in \Phi^+} x_{\alpha}(\eta_{\alpha}) \quad \text{where } \eta_{\alpha} \in \mathbb{F}_q.
\]
Then we define a linear character $\varphi_{\mathbf{u}} : U_{d,2}^F \to \mathbb{C}^\times$ by the formula
\[
\varphi_{\mathbf{u}} \left( \prod_{\alpha_0 \in \Phi^+} x_{\alpha}(\xi_{\alpha}) \right) = \chi \left( \sum_{\alpha_0 \in \Phi^+} c_{\alpha} \eta_{\alpha} \xi_{\alpha} \right) \quad \text{for all } \xi_{\alpha} \in \mathbb{F}_q,
\]
where $c_{\alpha} \in \mathbb{F}_q$ are certain fixed constants (independent of the $\eta_{\alpha}$ and $\xi_{\alpha}$) and where $\chi : \mathbb{F}_q^* \to \mathbb{C}^\times$ is a fixed non-trivial character of the additive group of $\mathbb{F}_q$; see [5], Definition 2.1, for more details. It will actually be convenient to choose $\chi$ in the following special way. Let $\chi_0 : \mathbb{F}_p^+ \to \mathbb{C}^\times$ be a fixed non-trivial character of the additive group of $\mathbb{F}_p$. Then we take $\chi$ to be
\[
\chi := \chi_0 \circ \text{Tr}_{\mathbb{F}_q / \mathbb{F}_p}
\]
where $\text{Tr}_{\mathbb{F}_q / \mathbb{F}_p} : \mathbb{F}_q^+ \to \mathbb{F}_p^+$ is the trace map. Now we have
\[
\text{Ind}_{U_{d,2}^F}^{G^F} (\varphi_{\mathbf{u}}) = \left[ U_{d,1}^F : U_{d,2}^F \right]^{1/2} \cdot \Gamma_{\mathbf{u}},
\]
where $\Gamma_{\mathbf{u}}$ is the generalized Gelfand–Graev character associated with $\mathbf{u}$. We have seen in [5], Corollary 4.3, that
\[
\langle E_7[\pm \xi], \Gamma_{\mathbf{u}} \rangle_{G^F} = 1 \quad \text{for suitable } \mathbf{u} \in \{\gamma_{74}, \gamma_{75}\}.
\]
(Here, and throughout the paper, we denote by $\langle \ , \ \rangle_A$ the standard inner product on the character ring of a finite group $A$.)

We will now refine the construction of $\Gamma_u$. The strategy for doing this has already been outlined in [5], §4. For this purpose, we shall assume from now on that

$q$ is an even power of $p$.

Since $G$ is simple of adjoint type, we have an $\mathbb{F}_q$-isomorphism

$$h: k^x \times \cdots \times k^x \to T, \quad (x_1, \ldots, x_7) \mapsto h(x_1, \ldots, x_7),$$

such that $\alpha_i(h(x_1, \ldots, x_7)) = x_i$ for $1 \leq i \leq 7$. In particular, we have $T^F = \{h(x_1, \ldots, x_7) \mid x_i \in \mathbb{F}_q^x\}$. We shall set

$$t := h(v^{1/2}, 1, 1, v^{1/2}, 1, v^{1/2}, 1) \in T^F$$

as in the proof of [5], Lemma 4.1, where $v$ is a generator for the multiplicative group of $\mathbb{F}_p \subset \mathbb{F}_q$ and $v^{1/2}$ is a square root of $v$ in $\mathbb{F}_q$. (The square root exists since $q$ is an even power of $p$.) Then $t$ has the property that $\alpha(t) = v$ for all roots $\alpha$ involved in the expressions for $\gamma_4$ or $\gamma_5$ as products of root subgroup elements; furthermore, we have $\alpha(t) = 1$ for all roots $\alpha$ such that $d(\alpha) = 0$. The element $t$ has order $2(p - 1)$ and $H := \{t\}$ normalizes $U_{d,2}$. We set

$$s_1 := h(-1, 1, 1, -1, 1, -1, 1) = t^{p-1} \in T^F.$$

Note that $\alpha(s_1) = 1$ for all roots $\alpha \in \Phi^+$ which are involved in the expressions of $\gamma_4$ and $\gamma_5$ as products of root subgroup elements. Thus, $s_1$ fixes the character $\varphi_u$ and so we can extend $\varphi_u$ to $U_{d,2}^{F\cdot\langle s_1 \rangle}$. Actually, there are two such extensions which we denote by $\varphi_u$ and $\varphi_u'$. Their values are determined by

$$\varphi_u(xs_1) = \varphi_u(x) \quad \text{and} \quad \varphi_u'(xs_1) = -\varphi_u(x) \quad \text{for all} \ x \in U_{d,2}^F.$$

\textbf{Definition 2.1.} Let $u \in \{\gamma_4, \gamma_5\}$. Then we set

$$\psi_u := \text{Ind}_{U_{d,2}^{F\cdot\langle s_1 \rangle}}^{U_{d,2}^{F\cdot\langle s_1 \rangle}}(\varphi_u) \quad \text{and} \quad \psi_u' := \text{Ind}_{U_{d,2}^{F\cdot\langle s_1 \rangle}}^{U_{d,2}^{F\cdot\langle s_1 \rangle}}(\varphi_u').$$

Thus, we have

$$\text{Ind}_{U_{d,2}^{F\cdot\langle s_1 \rangle}}^{U_{d,2}^{F\cdot\langle s_1 \rangle}}(\varphi_u) = \psi_u + \psi_u' \quad \text{and} \quad [U_{d,1}^F : U_{d,2}^F]^{1/2} \cdot \Gamma_u = \hat{\Gamma}_u + \hat{\Gamma}_u',$$

where

$$\hat{\Gamma}_u := \text{Ind}_{U_{d,2}^{F\cdot\langle s_1 \rangle}}^{U_{d,2}^{F\cdot\langle s_1 \rangle}}(\varphi_u) = \text{Ind}_{U_{d,2}^{F\cdot\langle s_1 \rangle}}^{U_{d,2}^{F\cdot\langle s_1 \rangle}}(\psi_u),$$

and

$$\hat{\Gamma}_u' := \text{Ind}_{U_{d,2}^{F\cdot\langle s_1 \rangle}}^{U_{d,2}^{F\cdot\langle s_1 \rangle}}(\varphi_u') = \text{Ind}_{U_{d,2}^{F\cdot\langle s_1 \rangle}}^{U_{d,2}^{F\cdot\langle s_1 \rangle}}(\psi_u').$$
\[ \tilde{\Gamma}' := \text{Ind}_{U_{d,2}(G_2)}^{U_{d,2}(H)}(\tilde{\psi}'_{\bar{u}}) = \text{Ind}_{U_{d,2}(H)}^{U_{d,2}(H)}(\psi'_{\bar{u}}). \]

The following result provides some crucial information concerning \( \psi_u \) and \( \psi'_u \).

**Proposition 2.2.** Recall that \( q \) is an even power of \( p \). Then, with the above notation, the following hold.

(a) Both \( \psi_u \) and \( \psi'_u \) are irreducible characters of \( U_{d,2}(H) \).

(b) \( \psi_u \) can be realized over \( \mathbb{Q} \).

(c) \( \psi'_u \) is rational-valued but cannot be realized over \( \mathbb{Q} \). In fact, \( \psi'_u \) has non-trivial local Schur indices at \( \infty \) and at the prime \( p \).

Proof. (see also the argument of Ohmori [14], p. 154.) Let

\[(1) \quad x := \prod_{\alpha, \beta \in F_{d,2}^*} x_{\alpha}(\xi_{\alpha}) \in U_{d,2} \quad \text{and} \quad \gamma_x := \sum_{\alpha, \beta \in F_{d,2}^*} \zeta_\alpha \eta_\beta \xi_\alpha \text{,} \]

where \( \xi_\alpha \in \mathbb{F}_q \). Then, as in the proof of [5], Proposition 2.3, we have

\[ \varphi_u(t^i x t^{-i}) = \chi(\nu^i \gamma_x) \quad \text{for } 1 \leq i \leq 2(p - 1). \]

In particular, this implies \( \text{Stab}_H(\varphi_u) = \langle s_1 \rangle \). Hence, by Clifford theory, the induced character

\[ \text{Ind}_{U_{d,2}^{'H}}^{U_{d,2}H}(\varphi_u) = \psi_u + \psi'_u \]

has inner product 2. Thus, we \( \psi_u \) and \( \psi'_u \) must be irreducible, proving (a).

Next we prove (b). Using Mackey’s formula and relation (1), we have that

\[ \text{Ind}_{U_{d,2}^{'H}}^{U_{d,2}H}(\varphi_u)(x) = \sum_{i=1}^{2(p-1)} \varphi_u(t^i x t^{-i}) = \sum_{i=1}^{2(p-1)} \chi(\nu^i \gamma_x) \]

\[ = \sum_{i=1}^{2(p-1)} \chi_0(\nu^i \text{Tr}_{\mathbb{F}_q / \mathbb{F}_p}(\gamma_x)) = \begin{cases} 2(p - 1) & \text{if } \text{Tr}_{\mathbb{F}_q / \mathbb{F}_p}(\gamma_x) = 0, \\ -2 & \text{if } \text{Tr}_{\mathbb{F}_q / \mathbb{F}_p}(\gamma_x) \neq 0. \end{cases} \]

In particular, this shows that the values are rational integers. Thus, \( \psi_u + \psi'_u \) is rational-valued. Now assume, if possible, that \( \psi_u \) is not rational-valued. Then the characters \( \psi_u \) and \( \psi'_u \) must be algebraically conjugate. Consequently, \( \psi_u \) and \( \psi'_u \) occur with the same multiplicity in every rational-valued character. Now, by the Mackey formula and Frobenius reciprocity, we have

\[ \left\langle \psi'_u, \text{Ind}_{U_{d,2}^{'H}}^{U_{d,2}H}(1_H) \right\rangle_{U_{d,2}^{'H}} = \left\langle \text{Ind}_{U_{d,2}^{'H}}^{U_{d,2}H}(\varphi'_u), \text{Ind}_{U_{d,2}^{'H}}^{U_{d,2}H}(1_H) \right\rangle_{U_{d,2}^{'H}} \]

\[ = \left\langle \text{Res}_{(s_1)}^{U_{d,2}^{'H}}(\varphi'_u), 1_{(s_1)} \right\rangle_{U_{d,2}^{'H}} \]
since $\tilde{\psi}_u'(s_1) = -1$. (Here, the symbol 1 stands for the unit character.) By a similar argument, since $\tilde{\psi}_u(s_1) = 1$, we also have
\[
\left( \psi_u, \text{Ind}^{U_{d^2}^F H}_{U_{d^2}^F H} (1) \right)_{U_{d^2}^F H} = 1.
\]
Thus, $\psi_u$ and $\psi_u'$ do not occur with the same multiplicity in some rational-valued character, a contradiction. Thus, our assumption was wrong and so both $\psi_u$ and $\psi_u'$ are rational-valued. But then the above multiplicity 1 formula implies that $\psi_u$ can be realized over $\mathbb{Q}$, by a standard argument concerning Schur indices (see Isaacs [8], Corollary 10.2).

Finally, we prove (c). We begin by showing that the local Schur index at $\infty$ is non-trivial. In other words, we must show that $\psi_u'$ cannot be realized over $\mathbb{R}$. For this purpose, by a well-known criterion due to Frobenius and Schur (see Isaacs [8], Chapter 4), it is enough to show that
\[
\frac{1}{|U_{d^2}^F H|} \sum_{g \in U_{d^2}^F H} \psi_u'(g^2) = -1.
\]
Now, in order to evaluate the above sum, we note that
\[
\frac{1}{|U_{d^2}^F H|} \sum_{g \in U_{d^2}^F H} \psi_u(g^2) = 1,
\]
since $\psi_u$ can be realized over $\mathbb{Q}$. Thus, it will be enough to show that
\[
\frac{1}{|U_{d^2}^F H|} \sum_{g \in U_{d^2}^F H} \text{Ind}^{U_{d^2}^F H}_{U_{d^2}^F H} (\phi_u)(g^2) = \frac{1}{|U_{d^2}^F H|} \sum_{g \in U_{d^2}^F H} (\psi_u + \psi_u')(g^2) = 0.
\]
Let $g \in U_{d^2}^F H$ and write $g = xh$ where $x \in U_{d^2}^F$ and $h \in H$. Now the value of the above induced character on $g^2$ is zero unless $g^2 \in U_{d^2}^F$. Thus, we only need to consider elements $g = xh$ where $h = 1$ or $h = s_1$. So we must show that
\[
\sum_{x \in U_{d^2}^F} \text{Ind}^{U_{d^2}^F H}_{U_{d^2}^F H} (\phi_u)(x^2) + \sum_{x \in U_{d^2}^F} \text{Ind}^{U_{d^2}^F H}_{U_{d^2}^F H} (\phi_u)(x_1 x s_1) = 0.
\]
Now, since $U_{d^2}^F$ has odd order, the map $x \mapsto x^2$ defines a bijection of $U_{d^2}^F$ onto itself. Hence the first sum evaluates to
\[
\sum_{x \in U_{d^2}^F} \text{Ind}^{U_{d^2}^F H}_{U_{d^2}^F H} (\phi_u)(x^2) = |U_{d^2}^F H| \cdot \left( \text{Ind}^{U_{d^2}^F H}_{U_{d^2}^F H} (\phi_u), 1_{U_{d^2}^F H} \right)_{U_{d^2}^F H} = 0.
\]
Now consider the second sum. For this purpose, we note that \( \alpha(s_1) = 1 \) for all roots \( \alpha \in \Phi^+ \) which are involved in the expressions of \( \gamma_{74} \) and \( \gamma_{75} \) as products of root subgroup elements. Thus, if \( \chi \) and \( \gamma_\chi \) are as in (1), then we have

\[
\gamma_{\chi(\chi)^2} = \sum_{\omega \in \Phi^+} c_{\omega} \eta_{\omega}(\alpha(s_1)) + 1) \xi_\omega = 2 \gamma_\chi = \gamma_\chi^2.
\]

Using once more Mackey’s formula as at the beginning of this proof, we see that

\[
\text{Ind}_{\mathcal{U}_{d,2}^{F,H}} \left( \psi_\mu \right)(x^2) = \sum_{i=1}^{2(p-1)} \chi(\psi_i \gamma^x) = \sum_{i=1}^{2(p-1)} \chi(\psi_i \gamma_{\chi(\chi)^2}) = \text{Ind}_{\mathcal{U}_{d,2}^{F,H}} \left( \psi_\mu \right)(\chi \gamma_1 \chi \gamma_1)
\]

for all \( x \in \mathcal{U}_{d,2}^{F} \). Consequently, the second sum also equals 0. Thus, we have shown that \( \psi_\mu \) cannot be realized over \( \mathbb{R} \). We shall now use some general properties of Schur indices; see Feit [3], §2, for references. First, since \( \psi_\mu \) is rational-valued but \( \psi_\mu ' \) cannot be realized over \( \mathbb{R} \), the Schur index of \( \psi_\mu ' \) is 2 (by the Brauer–Speiser theorem; see [3], 2.4). Furthermore, there exists at least one prime number \( l \) such that the \( l \)-local Schur index of \( \psi_\mu ' \) is 2 (by the Hasse sum formula; see [3], 2.15). Thus, it will be enough to show that the \( l \)-local Schur index of \( \psi_\mu ' \) is 1, for every prime \( l \neq p \).

Let \( l \) be such a prime. If \( l \neq 2 \), then \( \psi_\mu ' \) is a character of \( l \)-defect 0 of \( \mathcal{U}_{d,2}^{F,H} \). So the \( l \)-local Schur index is 1 by [3], 2.10. Finally, if \( l = 2 \), then \( \psi_\mu ' \) is a character of 2-defect 1 and, hence, lies in a block with a cyclic defect group of order 2. Consequently, that block contains only two irreducible characters and so \( \psi_\mu ' \) remains irreducible as a 2-modular Brauer character. This implies again that the local Schur index is 1; see [3], 2.10.

\[\Box\]

3. A subgroup of type \( D_6 \times A_1 \)

Our next aim is to compute the multiplicity of \( \mathcal{E}_1[\pm x] \) in \( \Gamma_u \) and \( \Gamma_u' \); see Definition 2.1. We already know that the multiplicity of \( \mathcal{E}_1[\pm x] \) in the sum \( \Gamma_u + \Gamma_u' \) equals \( [U_{d,1}^F : U_{d,2}^F]^{1/2} \), for suitable \( u \in \{\gamma_{74}, \gamma_{75}\} \). We shall now try to compute the multiplicity in the difference \( \Gamma_u - \Gamma_u' \). For this purpose, we take a closer look at the semisimple element \( s_1 \) and its centralizer.

Let

\[
G_1 := \langle T, X_\alpha \mid \alpha \in \Phi_1 \rangle \quad \text{where} \quad \Phi_1 := \{\alpha \in \Phi \mid \alpha(s_1) = 1\}.
\]

Using the CHEVIE function \( \text{ReflectionSubgroup} \), we check that the root system \( \Phi_1 \) has type \( D_6 \times A_1 \); a system of simple roots in \( \Phi_1 \) is given by

\[
\Pi_1 = \{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_{14}, \alpha_{18}, \alpha_{28}\}
\]
The restriction of $d$ to the subsystem of type $D_6 \times A_1$

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<td>$a_5$</td>
<td>$a_{14}$</td>
<td>$a_2$</td>
<td>$a_{18}$</td>
<td>$a_3$</td>
<td>$a_{28}$</td>
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where

$a_{14} := a_1 + a_3 + a_4$, \quad $a_{18} := a_4 + a_5 + a_6$, \quad $a_{28} := a_2 + a_3 + 2a_4 + a_5$.

(Here, the numbering of the roots is the same as that given by CHEVIE.) The corresponding Dynkin diagram and the restriction of the weight function $d$ to $\Pi_1$ are given in Table 2. Furthermore, one can check, using CHEVIE (for example), that

$$N_W(W_1) = \{w \in W \mid w(\Phi_1) \subseteq \Phi_1\} = W_1$$

where $W_1 := \langle w_\alpha \mid \alpha \in \Phi_1 \rangle \subseteq W$ is the Weyl group of $G_1$ (and where we denote by $w_\beta$ the reflection with root $\beta$, for any root $\beta \in \Phi$).

**Lemma 3.1.** We have $C_{G}(s_1) = G_1$; in particular, $C_{G}(s_1)$ is connected.

Proof. By Carter [2], §3.5, we have $C_{G}(s_1)^G = G_1$. Hence, $G_1$ is a normal subgroup in $C_{G}(s_1)$. So it is enough to show that $N_G(G_1) = G_1$. Let $g \in N_G(G_1)$. Then $gTg^{-1}$ is a maximal torus in $G_1$ and so there exists some $g_1 \in G_1$ such that $gTg^{-1} = g_1Tg_1^{-1}$. Thus, we have $g_1^{-1}g \in N_G(T)$ and so $g \in G_1N_G(T)$. Hence, we may assume without loss of generality that $g \in N_G(T) \cap N_G(G_1)$. Now, for any $g \in N_G(T) \cap N_G(G_1)$ and any $\alpha \in \Phi_1$, we have $gX_{\alpha}g^{-1} = X_{w(\alpha)} \subseteq G_1$, where $w$ is the image of $g$ in $W = N_G(T)/T$. Thus, we have $w(\Phi_1) \subseteq \Phi_1$ and so $w \in W_1$ (see the above remarks). This implies $g \in G_1$, as required.

Let $C_1$ be the conjugacy class of $y_{74}$ in $G_1$ and denote by $d_i : \Phi_1 \rightarrow \mathbb{Z}$ the corresponding weighted Dynkin diagram. Using the identification results in [1], Theorem 11.3.2, it is straightforward to check that, under the natural matrix representation of a group of type $D_6 \times A_1$, the elements $y_{74}$ and $y_{75}$ correspond to matrices with Jordan blocks of size 1, 1, 2, 5, 5 (where the block of size 2 comes from the $A_1$-factor). Hence, using [2], §13.1, we see that $d_i$ is given by the restriction of $d$ to $\Phi_1$, as specified in Table 2. Furthermore, we notice that the above roots can all be written as sums of roots in $\Pi_1$. Thus, we have

$$y_{74}, y_{75} \in C_1 \cap U_{d_i,2},$$
where \( U_{d_1,2} \) is the unipotent subgroup of \( G_1 \) defined with respect to \( d_1 \).

**Lemma 3.2.** Let \( u \in \{y_74, y_75\} \). Then we have \( \dim \mathfrak{B}_u^1 = 4 \) (where \( \mathfrak{B}_u^1 \) denotes the variety of Borel subgroups of \( G_1 \) containing \( u \)) and

\[
C_{G_1}(u)/C_{G_1}(t)^o \cong C_{G_1}(u)/C_{G_1}(t)^o \cong \mathbb{Z}/2\mathbb{Z}.
\]

Proof. Let \( u := y_74 \). The formula for \( \dim \mathfrak{B}_u^1 \) follows from [2], §13.1. To prove the remaining statements, we note that

\[
s_1 \in S := \{ h(x, x^{-2}, x^{-2}, x^3, x^{-2}, x, 1) \mid x \in k^\times \} \subseteq C_{G_1}(u).
\]

Furthermore, one checks that \( Z(G_1) = \{ t \in T \mid \alpha(t) = 1 \text{ for all } \alpha \in \Phi_1 \} = \langle s_1 \rangle \). Thus, since \( S \) is connected, we have \( Z(G_1) \subseteq C_{G_1}(u)^o \).

Now let \( \pi : G_1 \to H_1 \) be the adjoint quotient of \( G_1 \), where \( H_1 \) is a semisimple group of adjoint type \( D_5 \times A_1 \). Let \( \bar{u} \) be the image of \( u \) in \( H_1 \). Then, by Carter [2], §13.1, we know that \( C_{H_1}(\bar{u})/C_{H_1}(\bar{u})^o \cong \mathbb{Z}/2\mathbb{Z} \). Furthermore, \( \pi \) induces a surjective homomorphism

\[
C_{G_1}(u)/C_{G_1}(t)^o \twoheadrightarrow C_{H_1}(\bar{u})/C_{H_1}(\bar{u})^o \cong \mathbb{Z}/2\mathbb{Z}
\]

with kernel given by the image of \( Z(G_1) \) in \( C_{G_1}(u)/C_{G_1}(t)^o \). Since \( Z(G_1) \subseteq C_{G_1}(u)^o \), that image is trivial and so the above surjective map is also injective.

**Proposition 3.3.** Let \( u \in \{y_74, y_75\} \subseteq C \cap U_{d_1,2}^F \). Then, as we already noted, we have \( u \in C_1 \cap U_{d_1,2}^F \) and so the corresponding generalized Gelfand–Graev character \( \Gamma_u^1 \) of \( G_1^F \) is well-defined. We have

\[
\hat{\Gamma}_u(yS_1) - \hat{\Gamma}_u'(yS_1) = \Gamma_u^1(y) \quad \text{for all } y \in G_1^F \text{ unipotent},
\]

Proof. By the Mackey formula, we have

\[
\hat{\Gamma}_u(yS_1) = \text{Res}^{G_1}_F(\hat{\Gamma}_u)(yS_1) = \text{Res}^{G_1}_F\left(\text{Ind}^{G_1}_F(U_{d_1,2},\langle s_1 \rangle)(\hat{\phi}_u)\right)(yS_1)
\]

\[
= \sum \text{Ind}^{G_1}_F(U_{d_1,2,\langle s_1 \rangle}(\bar{s}_1)\cap G_1^F(\hat{\phi}_{\bar{s}_1})(yS_1),
\]

where \( z \) runs over a set of representatives of the \((U_{d_1,2}^F, \langle s_1 \rangle, G_1^F)\)-double cosets of \( G_1^F \). Let us fix such a double coset representative, \( z \) say. Assume that the value at \( yS_1 \) of the corresponding induced character in the above sum is non-zero. Then \( yS_1 \) must be \( G_1^F \)-conjugate to an element in the subgroup \((U_{d_1,2}^F, \langle s_1 \rangle)^e \cap G_1^F \). Consequently, \( s_1 \) must be \( G_1^F \)-conjugate to an element in that subgroup. Since \( \langle s_1 \rangle \) is a Sylow 2-subgroup of \( U_{d_1,2}^F, \langle s_1 \rangle \), we conclude that all elements of order 2 in \( U_{d_1,2}^F, \langle s_1 \rangle \) are of the form \( xS_1x^{-1} \).
where \( x \in U_{d,2}^F \). Thus, we have \( c^{-1}x_1c = z^{-1}x_1x^{-1}z \) for some \( c \in G_1^F \) and some \( x \in U_{d,2}^F \). Consequently, \( x^{-1}zc^{-1} \in C_G(s_1)^F = G_1^F \) and so \( z \in xG_1^F \subset U_{d,2}^F \). Thus, \( z \) represents the trivial double coset and so we can take \( z = 1 \). Using the fact that

\[
U_{d,2}^F \cap G_1^F = U_{d,2}^F \times \langle s_1 \rangle
\]

(where \( U_{d,2} \subset G_1 \) is the unipotent subgroup defined with respect to the weighted Dynkin diagram \( d_1 : \Phi_1 \to \mathbb{Z} \)) we find that

\[
\tilde{\Gamma}_u(y s_1) = \text{Ind}_{U_{d,2}^F \times \langle s_1 \rangle}^{G_1^F} (\varphi_u^1 \boxtimes 1_{s_1})(y s_1)
\]

where \( \varphi_u^1 \) denotes the restriction of \( \varphi_u \) to \( U_{d,2}^F \). Since \( s_1 \) is in the center of \( G_1 \), it is readily checked that

\[
\tilde{\Gamma}_u(y s_1) = \frac{1}{2} \varphi_u(s_1) \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi_u^1)(y) = \frac{1}{2} \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi_u^1)(y).
\]

By a completely analogous argument, we also obtain that

\[
\tilde{\Gamma}_u'(y s_1) = \frac{1}{2} \varphi_u'(s_1) \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi_u^1)(y) = -\frac{1}{2} \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi_u^1)(y).
\]

Thus, it remains to check that

\[
\Gamma_u^1 = \text{Ind}_{U_{d,2}^F}^{G_1^F} (\varphi_u^1).
\]

For this purpose, we must show that \( \varphi_u^1 \) indeed is the linear character of \( U_{d,2}^F \) required in the definition of \( \Gamma_u^1 \). Now, the definition of \( \Gamma_u^1 \) requires the choice of a non-degenerate bilinear form and of an opposition automorphism on the Lie algebra of \( G_1 \). However, the Lie algebra of \( G_1 \) is naturally contained in the Lie algebra of \( G \), with compatible Cartan decompositions. Thus, the chosen bilinear form and the chosen opposition automorphism restrict to the Lie algebra of \( G_1 \), and this implies that \( \varphi_u^1 \) is the required linear character of \( U_{d,2}^F \). \( \Box \)

A formula of this kind has been stated (without proof) by Kawanaka in [9], Lemma 2.3.5; see also the Ph. D. thesis of Wings [16], §3.2.1.

**Remark 3.4.** Let \( g \in G^F \) and write \( g = g_u g_s = g_u g_s \) where \( g_s \in G^F \) is semisimple and \( g_u \in G^F \) is unipotent. Assume that \( g_s \) is not conjugate to \( s_1 \) in \( G^F \). Then we have

\[
(\tilde{\Gamma}_u - \tilde{\Gamma}_u')(g) = 0.
\]

Indeed, if the value is non-zero, then \( g \) must be \( G^F \)-conjugate to an element in \( U_{d,2}^F \langle s_1 \rangle \). But then \( g_s \) will also be \( G^F \)-conjugate to an element in that subgroup. Using a Sylow argument as in the above proof, we see that either \( g_s = 1 \) or \( g_s \) is
$G^F$-conjugate to $s_1$, as claimed. Furthermore, if $g_s = 1$, then it is readily checked that $\tilde{\Gamma}'_u \equiv \tilde{\Gamma}'_u = \Gamma'_u$. Thus, in order to compute the scalar product of $E_7[\pm \xi]$ with $\tilde{\Gamma}_u - \tilde{\Gamma}'_u$, it will be enough to know the values of $E_7[\pm \xi]$ on elements of the form $y s_1$ where $y \in G_1^F$ is unipotent. Furthermore, since $E_7[\xi]$ and $E_7[\xi]$ are complex conjugate and since $\tilde{\Gamma}_u$ and $\tilde{\Gamma}'_u$ are rational-valued, it will actually be enough to consider the sum $E_7[\xi] + E_7[\xi]$. Now, by Lusztig [10], Main Theorem 4.23, we have

$$E_7[\xi] + E_7[\xi] = R_{512_a} - R_{512_a}.$$ 

(Note that the function $\Delta$ occurring in [10], 4.23, takes value $-1$ on the labels corresponding to the characters $E_7[\pm \xi]$.) Here, $512_a$, $512_a'$ are the two irreducible characters of $W$ of degree $512$ and $R_{512_a}$, $R_{512_a'}$ are the corresponding “almost characters”, as defined by Lusztig [10], (3.7). For any $\phi \in \text{Irr}(W)$, we have

$$R_\phi := \frac{1}{|W|} \sum_{w \in W} \phi(w) R_{T_w,1};$$

here, $T_w \subseteq G$ is an $F$-stable maximal torus obtained from $T$ by twisting with $w$ and $R_{T_w,1}$ is the Deligne–Lusztig generalized character associated with the trivial character of $T_w^F$. Similarly, for any $\psi \in \text{Irr}(W_1)$, we denote by $R^1_\psi$ the corresponding almost character of $G_1^F$.

**Lemma 3.5.** Let $\phi \in \text{Irr}(W)$ and write

$$\text{Res}^W_{W_1}(\phi) = \sum_{\psi \in \text{Irr}(W_1)} m(\phi, \psi) \psi \quad \text{where} \quad m(\phi, \psi) \in \mathbb{Z}_{\geq 0}.$$ 

Let $y \in G_1^F$ be a unipotent element. Then we have

$$R_\phi(y s_1) = \sum_{\psi \in \text{Irr}(W_1)} m(\phi, \psi) R^1_\psi(y).$$

Proof. The character formula for $R_{T_w,1}$ (see [2], Theorem 7.2.8) shows that

$$R_{T_w,1}(y s_1) = \left[ \frac{C_W(w)}{|W_1|} \right] \sum_{y w = w_1} R^1_{T_{w_1},1}(y)$$

where the relation $\sim$ means conjugacy in $W$. (Here, $R^1_{T_{w_1},1}$ denotes a Deligne–Lusztig generalized character of $G_1^F$.) Thus, we have

$$R_\phi(y s_1) = \frac{1}{|W|} \sum_{w \in W \cap W_1} \left[ \frac{C_W(w)}{|W_1|} \right] \phi(w) R^1_{T_{w_1},1}(y)$$
\[
\frac{1}{|W_1|} \sum_{w_1 \in W_1} \left( \frac{1}{|W|} \sum_{w \in W} |C_W(w)| \phi(w) \right) R_{\Gamma_{w_1},1}^1(y) = \phi(w_1).
\]

Now, we have \( \phi(w) = \phi(w_1) \) and \( |C_W(w)| = |C_W(w_1)| \) for all \( w_1 \in W_1 \) such that \( w \sim w_1 \). Thus, we have

\[
\frac{1}{|W|} \sum_{w \in W} |C_W(w)| \phi(w) = \frac{|C_W(w_1)|}{|W|} \phi(w_1) \sum_{w \in W} 1 = \phi(w_1).
\]

Writing \( \phi(w_1) = \sum_{\psi} m(\phi, \psi) \psi(w_1) \), we obtain the desired expression. \( \square \)

**Corollary 3.6.** With the notation of Proposition 3.3 and Lemma 3.5, we have

\[
\{R_{\phi}, \Gamma_u - \Gamma_{u}^{d_1}\}_{GF} = \sum_{\psi \in \text{Irr}(W_1)} m(\phi, \psi) \{R_{\psi}, \Gamma_{u}^{d_1}\}_{GF},
\]

for any \( \phi \in \text{Irr}(W) \) and \( u \in \{y_{74}, y_{75}\} \subseteq C_1 \cap U_{d_1,2}^F \).

**Proof.** Immediate from Proposition 3.3, Remark 3.4 and Lemma 3.5. \( \square \)

We now need some explicit information concerning the restriction of characters from \( W \) to \( W_1 \). Using the CHEVIE function InductionTable, we compute that

\[
\text{Res}^W_{W_1}(512_d) \otimes \varepsilon = \left( [21, 3] \boxtimes 1 \right) + \text{sum of } \psi \text{ where } \psi \in \text{Irr}(W_1) \text{ and } d_\psi > 4,
\]

\[
\text{Res}^W_{W_1}(512_d^2) \otimes \varepsilon = \left( [2, 31] \boxtimes 1 \right) + \text{sum of } \psi \text{ where } \psi \in \text{Irr}(W_1) \text{ and } d_\psi > 4.
\]

Here, \( 1 \) denotes the unit character on the \( A_1 \)-factor of \( W_1 \) and \( \varepsilon \) denotes the sign character of \( W_1 \). The characters of the \( D_6 \)-factor are denoted by \( [\lambda, \mu] \) where \( \lambda \) and \( \mu \) are partitions such that \( |\lambda| + |\mu| = 6 \). The \( a \)-invariant of a character is defined as in Lusztig [10], (4.1); in CHEVIE, these \( a \)-invariants are obtained by the function LowestPowerGenericDegrees. We have

\[
a_\psi = 4 \quad \text{for } \psi = [21, 3] \boxtimes 1 \text{ and } \psi = [2, 31] \boxtimes 1.
\]

With these explicit formulas, we can now prove the following result.

**Proposition 3.7.** Assume that the characteristic \( p \) is large enough, such that Lusztig’s formula in [11], Theorem 7.5, for the values of a generalized Gelfand–Graev holds for \( \Gamma_u^d \). By [5], Corollary 4.3, there exists some \( u \in \{y_{74}, y_{75}\} \) such that \( \{E_7[\pm \xi], \Gamma_u\}_{GF} = 1 \). For this element \( u \), we have

\[
\{E_7[\pm \xi], \Gamma_u - \Gamma_u^{d_1}\}_{GF} = -1.
\]
Proof. We have already mentioned in the remarks preceding Lemma 3.5 that
\[ E_7[\xi] + E_7[-\xi] = R_{312_a} - R_{312_a}. \]

Since \( \tilde{\Gamma}_u \) and \( \tilde{\Gamma}_u' \) are rational-valued (see Proposition 2.2), we have
\[ \langle E_7[\pm \xi], \tilde{\Gamma}_u - \tilde{\Gamma}_u' \rangle_{GF} = \frac{1}{2} \langle E_7[\xi] + E_7[-\xi], \tilde{\Gamma}_u - \tilde{\Gamma}_u' \rangle_{GF} \]
\[ = \frac{1}{2} \langle R_{312_a} - R_{312_a'}, \tilde{\Gamma}_u - \tilde{\Gamma}_u' \rangle_{GF} \cdot \]

Now let \( \psi \in \text{Irr}(W) \) be a constituent in the restriction of \( 512_a \) or \( 512_a' \) from \( W \) to \( W_1 \). Then, by Corollary 3.6, we must compute the scalar product \( \langle R_\psi, \Gamma_u \rangle_{GF} \).

Let \( D \) denote the Alvis-Curtis–Kawanaka duality operation on the character ring of \( G_1^f \); see Lusztig [10], (6.8). We have \( D(R_\psi) = R_\psi \otimes \psi \) and so
\[ \langle R_\psi, \Gamma_u \rangle_{GF} = \langle D(R_\psi), D(\Gamma_u) \rangle_{GF} = \langle R_\psi \otimes \psi, D(\Gamma_u) \rangle_{GF}. \]

Now, in order to evaluate the above scalar product, it is enough to know the values of \( R_\psi \otimes \psi \) on the unipotent elements of \( G_1^f \). By Shoji’s algorithm [15] and by [11], Corollary 10.9, we know that \( R_\psi \otimes \psi(y) = 0 \) if \( \dim \mathfrak{B}_u^1 < a_\psi \otimes \psi \). On the other hand, we have \( D(\Gamma_u^1)(y) = 0 \) if \( \dim \mathfrak{B}_u^1 < \dim \mathfrak{B}_u^1 \). This follows from [11]; see the remarks in [4], (2.4.) Thus, the above scalar product is zero if \( a_\psi \otimes \psi > \dim \mathfrak{B}_u^1 = 4 \). Taking into account the explicit information concerning the restrictions of \( 512_a \) and \( 512_a' \) from \( W \) to \( W_1 \), we conclude that

\[ \langle E_7[\pm \xi], \tilde{\Gamma}_u - \tilde{\Gamma}_u' \rangle_{GF} = \frac{1}{2} \langle R_{2[1,3]} \mathbb{2} \Gamma_1 - R_{2[3,1]} \mathbb{2} \Gamma_1, D(\Gamma_u^1) \rangle_{GF}. \]

Now \([21,3] \boxtimes 1\) and \([23,1] \boxtimes 1\) lie in the same family of characters of \( W_1 \); see [10], Chapter 4. The Fourier matrix (which has size \( 4 \times 4 \)) for that family shows that
\[ R_{2[1,3]} \mathbb{2} \Gamma_1 - R_{2[3,1]} \mathbb{2} \Gamma_1 = -\rho_1 - \rho_2 \]
where \( \rho_1 \) and \( \rho_2 \) are unipotent characters of \( G_1^f \). Now, we can explicitly compute the unipotent support of these two characters; see [11], §11, or [7], §3.C. This involves the knowledge of the Springer correspondence for \( G_1 \). Using the description of that correspondence in [2], §13.3, we find that \( \rho_1 \) and \( \rho_2 \) have unipotent support \( C_1 \). Thus, by the formula in [7], Remark 3.8, we have

\[ \langle \rho_i, D(\Gamma_{y_i}) + D(\Gamma_{y_i}) \rangle_{GF} = \langle D(\rho_i), \Gamma_{y_i} + \Gamma_{y_i} \rangle_{GF} = 1 \quad \text{for } i = 1, 2. \]

Note that \( C_{G_1}(y_{74})/C_{G_1}(y_{74})^0 \cong \mathbb{Z}/2\mathbb{Z} \) by Lemma 3.2 and that \( D(\rho_1), D(\rho_2) \) are actual characters in the present situation; see [10], (6.8.2). Now we have \( u \in \{y_{74}, y_{75}\} \) and
we would like to show that
\[ \langle D(\rho_1), \Gamma_{u_1}^\Gamma \rangle_{u_1}\Gamma = \langle \rho_1, D(\Gamma_{u_1}^\Gamma) \rangle_{u_1}\Gamma = 1 \quad \text{for } i = 1, 2. \]

This can be seen as follows. Fix \( i \in \{1, 2\} \). Since \( D(\rho_i) \) is an actual character, we
certainly have \( \langle D(\rho_i), \Gamma_{u_i}^\Gamma \rangle_{u_i}\Gamma \geq 0 \). Hence, using (2), the latter scalar product equals 0
or 1. Assume, if possible, that the scalar product is zero. Then the scalar product of
\( -\rho_1 - \rho_2 \) with \( D(\Gamma_{u_i}^\Gamma) \) would be \( -1 \) or 0. Consequently, the scalar product in (1)
would be \( -1/2 \) or 0. Thus, the only possibility is that the scalar product in (1)
equals 0. But this would mean that
\[ \langle E_7[\pm \xi], \Gamma_{u}^\Gamma \rangle_{u}\Gamma = [U_{d_1}^F : U_{d_2}^F]^{1/2} \langle E_7[\pm \xi], \Gamma_{u}^\Gamma \rangle_{u}\Gamma = [U_{d_1}^F : U_{d_2}^F]^{1/2} \]
is an even number, which is not true. So, our assumption was wrong and (3) holds.
Inserting this into (1), we obtain the desired result. \( \square \)

4. **Proof of Theorem 1.1**

By [5], Corollary 4.3, the Schur index of \( E_7[\pm \xi] \) is at most 2. Hence, we only
need to show that \( E_7[\pm \xi] \) cannot be realized over \( \mathbb{Q}(\xi) \). Now, we have
\[ \langle E_7[\pm \xi], \Gamma_{u}^\Gamma \rangle_{u}\Gamma = 1 \quad \text{for suitable } u \in \{y_4, y_5\}. \]
So, using the formulas in Definition 2.1, we obtain that
\[ \langle E_7[\pm \xi], \Gamma_{u}^\Gamma \rangle_{u}\Gamma = [U_{d_1}^F : U_{d_2}^F]^{1/2} = q^m \quad \text{for some } m \geq 1. \]
Combining this with Proposition 3.7 and using Frobenius reciprocity, this yields
\[ \left\langle \text{Res}_{U_{d_2}^F, H}^{U_{d_1}^F} \langle E_7[\pm \xi], \psi_{u}^\Gamma \rangle_{U_{d_1}^F, H}, \psi_{u}^\Gamma \right\rangle_{U_{d_1}^F, H} = \langle E_7[\pm \xi], \Gamma_{u}^\Gamma \rangle_{u}\Gamma = \frac{1}{2}(q^m + 1). \]
Since \( p \equiv 1 \mod 4 \), we also have \( q \equiv 1 \mod 4 \) and so the above scalar product is an odd number.
Now assume, if possible, that \( E_7[\pm \xi] \) can be realized over \( \mathbb{Q}(\xi) \). Then the restriction of \( E_7[\pm \xi] \) to \( U_{d_2}^F, H \) can also be realized over \( \mathbb{Q}(\xi) \). Thus, by
a standard argument on Schur induces ([8], Corollary 10.2), the Schur index of \( \psi_{u}^\Gamma \)
over \( \mathbb{Q}(\xi) \) divides the above odd number. Since the Schur index of \( \psi_{u}^\Gamma \) over \( \mathbb{Q}(\xi) \) is
at most 2 (see Proposition 2.2), it must be one. Thus, \( \psi_{u}^\Gamma \) can be realized over \( \mathbb{Q}(\xi) \).
Now, since \( q \) is a square, we have \( \xi = -1. \) Furthermore, since \( p \equiv 1 \mod 4 \), we have \( \sqrt{-1} \in \mathbb{Q}_p \) (the field of \( p \)-adic numbers). Hence \( \psi_{u}^\Gamma \) can be realized over \( \mathbb{Q}_p \),
contradicting Proposition 2.2(c). Thus, our assumption was wrong and so \( E_7[\pm \xi] \) cannot
be realized over \( \mathbb{Q}(\xi) \).

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References


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