Isozaki, H.
Osaka J. Math.
26 (1989), 399–410

SINGULAR LIMITS FOR THE COMPRESSIBLE EULER EQUATION IN AN EXTERIOR DOMAIN, II —BODIES IN A UNIFORM FLOW

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(Received May 25, 1988)

1. Introduction

1.1. Bodies in a uniform flow. Suppose an unbounded domain Ω in $\boldsymbol{R}^3$ exterior to a bounded obstacle with compact smooth boundary S is occupied by an ideal gas. Let P be its pressure and V the velocity. The entropy is assumed to be constant. Then the compressible Euler equation in a suitable non-dimensional form is written as

$$(1.1)\qquad \begin{cases} \partial_t P+(V\cdot\nabla)P+\gamma P\nabla\cdot V=0\,, \\ \partial_t V+(V\cdot\nabla)V+\lambda^2 P^{-1/\gamma}\nabla P=0\,, \\ \langle n, V\rangle=0 \quad \text{on} \quad S\,, \end{cases}$$

Where $\partial_t=\partial/\partial t$, γ is a constant >1, n is the outer unit normal to S. $\langle\ ,\ \rangle$ denotes the standard inner product in $\boldsymbol{R}^3$, and λ is a large parameter propotional to the inverse of the Mach number (see [5], p. 52). We shall explain the derivation of the above equation in §5 of this paper. Let $H^m=H^m(\Omega)$ be the usual Sobolev space of order m. In our previous works [1], [2], we have already shown that the solution of the above equation converges to that of the incompressible Euler equation as $\lambda\to\infty$ under the main condition that the initaial pressure $P^\lambda(0)=\text{Const.}+O(\lambda^{-1})$ and the initial velocity $V^\lambda(0)\in H^{N+1}$, $N\geq 4$. The assumption that the initial velocity belongs to $L^2(\Omega)$ is rather restrictive, since it excludes physically important flows which are both solenoidal and irrotational. In fact, if a vector field $V(x)\in L^2(\Omega)$ satisfies div $V=0$, curl $V=0$ and the boundary condition, it must be identically equal to 0. In this article, we consider the flow constant at infinity as an important example of such a non L^2 flow.

Let a constant non zero vector $\xi\in\boldsymbol{R}^3$ be fixed, which is the velocity at infinity. Take $w_0(x)\in\mathcal{B}(\Omega)$=the space of smooth functions with bounded derivatives such that

$$
\text{(1.2)}\qquad \begin{cases} \operatorname{div} w_0 = 0 \quad \text{in} \quad \Omega\,, \\ \langle n, w_0\rangle = 0 \quad \text{on} \quad S\,, \\ |\partial^\alpha(w_0(x)-\xi)| \leq C\langle x\rangle^{-\varepsilon-|\alpha|}, \ |\alpha| = 0, 1\,, \\ (w_0\cdot\nabla)\, w_0 \in L^2(\Omega)\,, \end{cases}
$$

where C and ε are positive constants, $\langle x\rangle=(1+|x|^2)^{1/2}$ and for a multi-index $\alpha=(\alpha_1, \alpha_2, \alpha_3)$, $\partial^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$, $\partial_j=\partial/\partial x_j$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$.

Examples. For instance, one can take $w_0(x)=\xi+\nabla\varphi(x)$, where φ solves the following Neumann problem:

$$
\text{(1.3)}\qquad \begin{cases} \Delta\varphi = 0 \quad \text{in} \quad \Omega\,, \\ \dfrac{\partial\varphi}{\partial n} = -\langle n, \xi\rangle \quad \text{on} \quad S\,. \end{cases}
$$

Indeed, $\varphi(x)$ is written by a single layer potential on the boundary

$$
\varphi(x) = \int_S \mu(y)\frac{1}{|x-y|}\, dS_y\,,
$$

whence $|\partial^\alpha(w_0(x)-\xi)| \leq C_\alpha\langle x\rangle^{-2-|\alpha|}$, for a constant C_α. Note that this w_0 is a stationary solution to the incompressible Euler equation.

One can also construct other examples by setting $w_0(x)=\xi+ \operatorname{curl} A(x)$.

To make (1.1) easier to handle, we transform P into $Q=\dfrac{\gamma}{\gamma-1}P^{1-1/\gamma}$. Then (1.1) is rewritten as

$$
\begin{aligned} &\partial_t Q+(V\cdot\nabla)Q+(\gamma-1)Q\nabla\cdot V = 0\,, \\ &\partial_t V+(V\cdot\nabla)V+\lambda^2\nabla Q = 0\,. \end{aligned}
$$

We set $\gamma=2$ for the sake of simplicity. We are going to assume that the initial pressure behaves like Const. $+O(\lambda^{-1})$. Thus, we set, without loss of generality, $Q=1+p/\lambda$. Then we arrive at

$$
\begin{aligned} &\partial_t p+(V\cdot\nabla)p+p\nabla\cdot V+\lambda\nabla\cdot V = 0\,, \\ &\partial_t V+(V\cdot\nabla)V+\qquad\quad \lambda\nabla p = 0\,. \end{aligned}
$$

Since we consider the velocity close to w_0 specified by (1.2), we set $V=v+w_0$ and obtain the following equation

$$
\text{(1.4)}\qquad \begin{cases} \partial_t p+(v\cdot\nabla)p+p\nabla\cdot v+(w_0\cdot\nabla)p+\lambda\nabla\cdot v = 0\,, \\ \partial_t v+(v\cdot\nabla)v+(v\cdot\nabla)w_0+(w_0\cdot\nabla)v+\lambda\nabla p = -(w_0\cdot\nabla)w_0\,, \\ \langle v, n\rangle = 0 \quad \text{on} \quad S\,. \end{cases}
$$

1.2. Main results. The following assumptions are imposed on the initial data p_0^λ, v_0^λ:

(A-1) $\{(p_0^\lambda, v_0^\lambda); \lambda>0\}$ *is a bounded set in* $H^{N+1}(\Omega)\cap L^1(\Omega)$, *where* N *is an integer* ≥ 4.
(A-2) *The compatibility condition is satisfied up to order* $N+1$.
(A-3) $P_S v_0^\lambda \to v_0^\infty$ *in* $H^N(\Omega)$ *as* $\lambda\to\infty$, P_S *being the projection onto the soleniodal fields.*

Let us remark that for a real uniform flow, one should slightly change the formulation, which will be discussed in §5.

Let $||\cdot||_m$ be the norm of H^m. Our main results are the following theorems.

Theorem A. *There exist constants* $T>0$ *and* $\Lambda>0$ *such that for any* $\lambda>\Lambda$, *there exists a unique solution* $p^\lambda(t)$, $v^\lambda(t)\in\bigcap_{k=0}^{N} C^k(I; H^{N-k}(\Omega))$, $I=[0, T]$, *of the above equation (1.4). Moreover, it satisfies the following uniform estimate*

$$\sup_{\lambda\in\Lambda, t\in I}(||p^\lambda(t)||_N+||v^\lambda(t)||_N)<\infty .$$

Theorem B. *For* $0<t\leq T$, $p^\lambda(t)\to 0$ *and* $v^\lambda(t)\to v^\infty(t)$ *in* $H^{N-1}_{\mathrm{loc}}(\bar{\Omega})$ *as* $\lambda\to\infty$. *Furthermore,* $u^\infty(t)=v^\infty(t)+w_0$ *satisfies the following incompressible Euler equation*

$$\begin{aligned}
&\partial_t u^\infty+(u^\infty\cdot\nabla)\,u^\infty+\nabla q^\infty = 0\,, \quad in \quad \Omega, \ t\in I\,,\\
&div\, u^\infty = 0\,, \quad in \quad \Omega, \ t\in I\,,\\
&\langle n, u^\infty\rangle = 0 \quad on \quad S\,,\\
&u^\infty(0) = v_0^\infty+w_0\,,\\
&u^\infty(t)-w_0\in H^{N-1}(\Omega)\,,
\end{aligned}$$

where $q^\infty=q^\infty(t)\in H^{N-1}_{\mathrm{loc}}(\bar{\Omega})$ *is calculated from* $u^\infty(t)$.

The above theorems serve as basic steps for the low Mach number expansion of compressibel fluids (see e.g. [8], p. 19) and also hold in any exterior domain in $\boldsymbol{R}^n$, $n\geq 2$, by a slight modification of the proof. To fix the idea, however, we consider the 3-dimensional case in this paper. We also point out that the regularity assumption (A-1) on the initial data can be relaxed so that they are bounded set in $H^3(\Omega)\cap L^1(\Omega)$. But we adopt this stronger one to economize technical details.

1.3. Methods of the proof. The proof of Theorem A is almost the same as in [1], §5. That is, we derive the energy estimate for the linearized equation of (1.4), from which the non-linear equation (1.4) is solved by iteration. To prove Theorem B, we study in §3 an asymptotic property as $\lambda\to\infty$ of the equation

$$(1.5)\qquad \begin{cases} \partial_t p+(w_0\cdot\nabla)p+\lambda\nabla\cdot v=0\,, \\ \partial_t v+(w_0\cdot\nabla)v+\lambda\nabla p=0\,, \\ \langle v,n\rangle=0 \quad \text{on} \quad S\,, \end{cases}$$

by utilizing the results obtained in [1] on the spectral properties of the linearized operator of acoustics (Theorem 3.2). Using Theorem 3.2, we shall prove Theorem B in §4 by employing the arguments in [2], §3.

1.4. Remaining problems. So for we have studied the equation (1.1) in an exterior domain, but many problems are left unsolved. For instance it is not easy to relate the pressure $q^\infty(t)$ to the original $p^\lambda(t)$. Even in the case $w_0(x)=0$, some geometric assumptions on the boundary (e.g. non trapping conditions) seem to be necessary. If we consider the non-isentropic fluid, in order to prove Theorem B, we have to study spectral and scattering problems of the linearized operator of acoustics with coefficients depending on time, which seems to be a delicate problem. Theorem A also holds for the interior domain (we take $w_0=0$). In this case, the incompressible limit (Theorem B) is derived under the additional assumption that, roughly, $p_0^\lambda=O(\lambda^{-1})$, div $v_0^\lambda=O(\lambda^{-1})$, and the initial layer does not appear. But what occurs when we consider the limit $\lambda\to\infty$ under our original assumptions? The incompressible limit for the boundary value problem of the compressible Navier-Stokes equation is also an interesting problem. For the stationary case, this was studied by [4]. But the non stationary problem is yet unsolved. A good explanation of these problems of singular limits in non-linear equations of fluid is given in [5], Chapter 2.

2. Non-linear compressible Euler equation

In this section, we shall briefly explain the outline of the proof of Theorem A. Let $C_0^\infty(\Omega)$ be the space of smooth functions with compact support in Ω and

$$C_{0,\sigma}^\infty(\Omega)=\{w\in C_0^\infty(\Omega);\ \text{div}\ w=0\}\ .$$

$S(\Omega)$ is the closure of $C_{0,\sigma}^\infty(\Omega)$ in $L^2(\Omega)^3$ and $G(\Omega)$ is the orthogonal complement of $S(\Omega)$ in $L^2(\Omega)^3$. Let P_G and P_S be the orthogonal projections onto $G(\Omega)$ and $S(\Omega)$, respectively. Then we have

Lemma 2.1 ([2], Lemma 2.1).
(1) *If* $w\in S(\Omega)$, div $w=0$ *in distribution sense and* $\langle n,w\rangle=0$ *in* $H^{-1/2}(S)$.
(2) *If* $v\in G(\Omega)$, *there exists a* $\varphi\in H_{\text{loc}}^1(\bar{\Omega})$ *such that* $v=\nabla\varphi$.

Let L be the linearized operator of acoustics in $L^2(\Omega)$. More precisely, L is the self-adjoint realization in $L^2(\Omega)^4$ of the differential operator

$$L=-i\begin{pmatrix}0 & \nabla \\ {}^t\nabla & 0\end{pmatrix},\ \nabla=(\partial_1, \partial_2, \partial_3)\,,$$

with the boundary condition $\langle n, v\rangle=0$ on S (see [1], Definition 1.8). Let Γ_0 and Γ be the orthogonal projections onto the null space of L and its orhtogonal complement, respectively. Then for $f={}^t(p, v)$,

$$\Gamma_0 f={}^t(0, P_S v)\,,\quad \Gamma f={}^t(p, P_G v)\,. \tag{2.1}$$

For $g={}^t(q, w)$, we define the differential operator $A(g)$ by

$$A(g)f=\begin{pmatrix}(w\cdot\nabla)p+q\nabla\cdot v \\ (w\cdot\nabla)v+(v\cdot\nabla)w_0\end{pmatrix},\quad f={}^t(p, v)\,. \tag{2.2}$$

Then, letting $F={}^t(0, -(w_0\cdot\nabla)w_0)$, one can rewrite (1.4) as

$$\partial_t f+A(f)f+(w_0\cdot\nabla)f+i\lambda Lf=F\,. \tag{2.3}$$

The linearized equation for (2.3) is

$$\partial_t f+A(g)f+(w_0\cdot\nabla)f+i\lambda Lf=G\,. \tag{2.4}$$

The treatment of the equation (2.4) is essentially the same as the one given in [1], §5. We split the solution f of (2.4) into two parts: $f=\Gamma_0 f+\Gamma f$. The part $\Gamma_0 f$ satisfies the linearized incompressible Euler equation

$$\begin{aligned}\partial_t\Gamma_0 f+\Gamma_0 A(g)\,\Gamma_0 f+\Gamma_0(w_0\cdot\nabla)\,\Gamma_0 f \\ =\Gamma_0 G-\Gamma_0 A(g)\,\Gamma f-\Gamma_0(w_0\cdot\nabla)\,\Gamma f\,,\end{aligned}$$

which can be studied separately by the method of Agemi [3], p.p. 180, 181. To obtain the regularity of Γf, we use the following coerciveness estimate ([1], Lemma 1.12): Let $f\in D(L)$. Then for any $m\geq 0$, there exists a constant $C_m>0$ such that

$$\|\Gamma f\|_{m+1}\leq C_m(\|\Gamma f\|+\|L\Gamma f\|_m)\,. \tag{2.5}$$

Invoking these two facts, one can prove the following energy estimate. Let $\|f(t)\|_{X^m}$ be defined by

$$\|f(t)\|_{X^m}=\sum_{k=0}^{m-1}\left\|\left(\frac{1}{\lambda}\,\partial_t\right)^k f(t)\right\|_{m-k}\quad(m\geq 1)\,, \tag{2.6}$$

and $\bar N$ be an integer ≥ 3. Let

$$\|w_0\|_{\mathscr{B}^{\bar N}}=\sum_{|\alpha|\leq\bar N}\|\partial^\alpha w_0\|_{L^\infty(\Omega)}\,.$$

For an interval $I=[0,T]$, we set

$$\gamma = \sup_{t\in I} \|g(t)\|_{X^{\bar N}} + \|w_0\|_{\mathcal{B}^{\bar N}} .$$

Then there exists a constant $C>0$ independent of λ, T, γ and a non-negative, non-decreasing function $C(\cdot)$ such that for the solution $f(t)$ of (2.4)

$$\|f(t)\|_{X^m} \le C\, e^{tC(\gamma)} \Big(\|f(0)\|_{X^m} + \int_0^t \|G(s)\|_{X^m}\, ds + \frac{1}{\lambda}\|G(t)\|_{X^m}\Big) \tag{2.7}$$

holds if $\lambda > C(\gamma)\, e^{C(\gamma)}$, $1\le m\le \bar N$, $t\in I$ (see [1], Lemma 5.5).

Once we have established the energy estimate (2.7), Theorem A readily follows by the standard method of iteration.

3. An asymptotic property for the linearized equation

We study in this section an asymptotic property as $\lambda\to\infty$ of the equation

$$\partial_t f + (w_0\cdot\nabla) f + i\lambda L f = 0 \quad \text{in} \quad \Omega \tag{3.1}$$

with the boundary condition $\langle v, n\rangle = 0$. It is easy to see by (2.7) and the assumption div $w_0=0$ that (3.1) generates a unitary group which we denote by $U_\lambda(t)$.

Let $\varphi_t(x)$ be the solution of the differential equation

$$\frac{d}{dt}\varphi_t(x) = w_0(\varphi_t(x)) , \quad \varphi_0(x) = x \in \Omega . \tag{3.2}$$

Then one can easily show the following lemma.

Lemma 3.1. (1) *$\varphi_t(x)$ defines a volume preserving* 1*-parameter group of diffeomorphism in Ω.*
(2) *For any $T>0$, there is a constant $C>0$ such that*

$$C^{-1}\langle x\rangle \le \langle \varphi_t(x)\rangle \le C\langle x\rangle,\ x\in\Omega,\ |t|\le T .$$

(3) *Let $d\varphi_t(x)$ be the differential of φ_t. Then for any $T>0$ there exists a constant $C>0$ such that*

$$|d\varphi_t(x) - I_3| \le C|t|\langle x\rangle^{-1-\varepsilon},\ x\in\Omega,\ |t|\le T ,$$

I_3 being the 3×3 identity matrix.

We define a unitary gorup $\Phi(t)$ by

$$(\Phi(t) f)(x) = f(\varphi_t(x)) , \quad f\in L^2(\Omega) . \tag{3.3}$$

The following theorem reduces the asymptotic properties of $U_\lambda(t)\Gamma$ to those of $e^{-it\lambda L}\Gamma$.

Theorem 3.2. *$U_\lambda(t)\Gamma - \Phi(-t)\, e^{-it\lambda L}\Gamma \to 0$ strongly in $L^2(\Omega)$ as $\lambda\to\infty$ for* *any* $t \in \boldsymbol{R}^1$.

Proof. Let $f(t)=U_\lambda(t)f$ and $g(t)=\Phi(t)f(t)$. Since $f(t)$ satisfies (3.1), $g(t)$ is the solution of the equation

$$\partial_t g + iL(t)g = 0,\ g(0) = f, \tag{3.4}$$

where $L(t)$ is obtained from L by the change of variables. Letting $M(t)=L(t)-L=\sum_{j=1}^{3} M_j(t, x)\,\partial_j$, we have by Lemma 3.1

$$|M_j(t, x)| \leq C|t|\langle x\rangle^{-1-\varepsilon},\ |t| \leq T. \tag{3.5}$$

Let $V_\lambda(t, s)$ be the evolution operator for (3.4). Obviously

$$V_\lambda(t, 0) = \Phi(t)\,U_\lambda(t). \tag{3.6}$$

Integrating the relation

$$\frac{d}{ds} V_\lambda(t, s)\,e^{-is\lambda L} = i\lambda V_\lambda(t, s)\,M(s)\,e^{-is\lambda L},$$

we have

$$V_\lambda(t, 0)\Gamma - e^{-it\lambda L}\,\Gamma = -i\lambda \int_0^t V_\lambda(t, s)\,M(s)\,e^{-is\lambda L}\,\Gamma ds. \tag{3.7}$$

In view of (3.6), we have only to show that the right-hand side of (3.7) tends to 0 strongly in $L^2(\Omega)$ as $\lambda \to \infty$. Using (3.5), we have for $t>0$

$$\begin{aligned} &\|\lambda \int_0^t V_\lambda(t, s)\,M(s)\,e^{-is\lambda L}\,\Gamma f ds\| \\ &\leq \lambda \int_0^t \|M(s)\,e^{-is\lambda L}\,\Gamma f\| ds \\ &\leq C\lambda \sum_{j=1}^{3} \int_0^t s\|\langle x\rangle^{-1-\varepsilon}\,\partial_j\,e^{-is\lambda L}\,\Gamma f\| ds. \end{aligned} \tag{3.8}$$

Here we note the following facts: For any $\varphi(\lambda) \in C_0^\infty(\boldsymbol{R}^1-\{0\})$,

$$\|\langle x\rangle^{-s}\,e^{-itL}\,\varphi(L)\,\langle x\rangle^{-s}\| \leq C_{s,\varepsilon}(1+|t|)^{-s+\varepsilon} \tag{3.9}$$

for any $s, \varepsilon>0$, where $\|\cdot\|$ denotes the operator norm in $L^2(\Omega)$ and $C_{s,\varepsilon}$ is a constant independent of t,

$$\langle x\rangle^{-s}\,\partial_j\,\varphi(L)\,\langle x\rangle^{s} \in \boldsymbol{B}(L^2(\Omega);\, L^2(\Omega)) \quad \text{for any} \quad s \in \boldsymbol{R}^1, \tag{3.10}$$

where for Banach spaces X and Y, $\boldsymbol{B}(X;\, Y)$ denotes the totality of bounded operators from X to Y,

Granting (3.9) and (3.10) for the moment, we continue the proof of Theorem 3.2. Take $\varphi(\lambda)$, $\psi(\lambda) \in C_0^\infty(\boldsymbol{R}^1-\{0\})$ such that $\psi(\lambda)=1$ on supp φ. Let $f \in$

$C_0^\infty(\Omega)$. Then, since $\varphi(L)=\psi(L)\varphi(L)$, we have

$$\begin{aligned}&\langle x\rangle^{-1-\varepsilon}\,\partial_j\, e^{-is\lambda L}\,\varphi(L) f\\ &\quad=\langle x\rangle^{-1-\varepsilon}\,\partial_j\,\psi(L)\,\langle x\rangle^{1+\varepsilon}\cdot\langle x\rangle^{-1-\varepsilon}\,e^{-is\lambda L}\,\varphi(L) f\,.\end{aligned}$$

Invoking (3.8), (3.9) and (3.10) we have

$$\begin{aligned}&\|\lambda\int_0^t V_\lambda(t,s)\,M(s)\,e^{-is\lambda L}\,\varphi(L) f ds\|\\ &\quad\leq C\int_0^t \lambda s(1+\lambda s)^{-1-\varepsilon/2}\,ds\,,\end{aligned}$$

which tends to 0 as $\lambda\to\infty$. To complete the proof, we have only to note that the set $\{\varphi(L)f;\, \varphi\in C_0^\infty(\boldsymbol{R}^1-\{0\}), f\in C_0^\infty(\Omega)\}$ is dense in $\Gamma L^2(\Omega)$.

It remains to prove (3.9) and (3.10). (3.9) follows from [1], Theorem 3.3. To prove (3.10) we introduce a function space $H^{m,s}$ by

$$H^{m,s}=\{f;\, \|f\|_{m,s}^2=\sum_{|\alpha|\leq m}\|\langle x\rangle^s\,\partial^\alpha f\|_{L^2(\Omega)}^2<\infty\}\,.$$

Then by [1], Corollary 1.13,

$$\varphi(L)\in\boldsymbol{B}(H^{0,0};\,H^{m,0})\quad\text{for any}\quad m\geq 0\,.$$

[1], Lemma 4.4 implies that

$$\varphi(L)\in\boldsymbol{B}(H^{0,s};\,H^{0,s})\quad\text{for any}\quad s\in\boldsymbol{R}^1\,.$$

By an interpolation, we have

$$\varphi(L)\in\boldsymbol{B}(H^{0,s};\,H^{m,s})\quad\text{for any}\quad m>0,\, s\in\boldsymbol{R}^1\,,$$

which proves (3.10).

4. Proof of Theorem B

For the solution $p^\lambda(t)$, $v^\lambda(t)$ of (2.3), we set $f^\lambda(t)={}^t(p^\lambda(t), v^\lambda(t))$ and rewrite (2.3) into the integral equation

$$\begin{aligned}&f^\lambda(t)=U_\lambda(t)f_0^\lambda-\int_0^t U_\lambda(t-s)\,A(f^\lambda)f^\lambda\,ds+\int_0^t U_\lambda(t-s)\,F ds\,,\\ &f_0^\lambda={}^t(p_0^\lambda, v_0^\lambda)\,.\end{aligned}\tag{4.1}$$

We begin by showing

Step 1. For $t>0$, $\Gamma f^\lambda(t)\to 0$ weakly in $L^2(\Omega)$ as $\lambda\to\infty$.

Taking the inner product of (4.1) with $g\in L^2(\Omega)$, we have

$$\begin{aligned}(\Gamma f^\lambda(t), g)=(f_0^\lambda, U_\lambda(-t)\,\Gamma g)&-\int_0^t (A(f^\lambda)f^\lambda, U_\lambda(s-t)\,\Gamma g)\,ds\\ &+\int_0^t (F, U_\lambda(s-t)\,\Gamma g)\,ds\,.\end{aligned}\tag{4.2}$$

We show for $t-s>0$, $(A(f^\lambda)f^\lambda, U_\lambda(s-t)\Gamma g)\to 0$ as $\lambda\to\infty$. In fact, by Theorem 3.2,

$$\begin{aligned}(A(f^\lambda)f^\lambda, U_\lambda(s-t)\Gamma g) &= (A(f^\lambda)f^\lambda, \Phi(t-s)\, e^{i(t-s)\lambda L}\Gamma g)+o(1)\\ &= (\Phi(s-t)\, A(f^\lambda)f^\lambda, e^{i(t-s)\lambda L}\Gamma g)+o(1)\,.\end{aligned}$$

We split $A(f^\lambda)f^\lambda$ into two parts, $A_1(f^\lambda)f^\lambda+A_2f^\lambda$, where

$$\begin{aligned}A_1(f^\lambda)f^\lambda &= {}^t((v^\lambda\cdot\nabla)\,p^\lambda+p^\lambda\nabla\cdot v^\lambda, (v^\lambda\cdot\nabla)\,v^\lambda)\,,\\ A_2f^\lambda &= {}^t(0, (v^\lambda\cdot\nabla)\,w_0)\,.\end{aligned}$$

By Theorem A, $\{A_1(f^\lambda)f^\lambda\}_{\lambda>\Lambda}$ is a bounded set in $L^1(\Omega)$. Lemma 3.1 implies that $\Phi(t)$ is isometric in $L^1(\Omega)$. Therefore

$$|(\Phi(s-t)\,A_1(f^\lambda)f^\lambda, e^{i(t-s)\lambda L}\Gamma g)|\le C\|e^{i(t-s)\lambda L}\Gamma g\|_{L^\infty(\Omega)} \tag{4.3}$$

for a constant C independent of λ. Lemma 3.1 also implies that $\langle x\rangle^{1+\varepsilon}\,\Phi(t)\,\langle x\rangle^{-1-\varepsilon}$ is a bounded operator in $L^2(\Omega)$. Hence again using Theorem A and the third condition in (1.2), we have

$$\begin{aligned}&|(\Phi(s-t)\,A_2f^\lambda, e^{i(t-s)\lambda L}\Gamma g)|\\ &\le C\|\langle x\rangle^{-1-\varepsilon}\, e^{i(t-s)\lambda L}\Gamma g\|_{L^2(\Omega)}\,.\end{aligned} \tag{4.4}$$

Now we assume that $g=\varphi(L)\,h$, where $\varphi\in C_0^\infty(\boldsymbol{R}^1-\{0\})$ and $h\in C_0^\infty(\Omega)$. Such $g's$ are dense in $\Gamma L^2(\Omega)$. Then in view of (4.3) and (4.4) we have for a constant $C>0$

$$\begin{aligned}&|(\Phi(s-t)\,A(f^\lambda)f^\lambda, e^{i(t-s)\lambda L}\Gamma g|\\ &\le C(\|e^{i(t-s)\lambda L}\,\varphi(L)\,h\|_{L^\infty(\Omega)}+\|\langle x\rangle^{-1-\varepsilon}\, e^{i(t-s)\lambda L}\,\varphi(L)\,h\|_{L^2(\Omega)})\,.\end{aligned} \tag{4.5}$$

The first term of the right-hand side tends to 0 as $\lambda\to\infty$ by virtue of [1], Lemma 4.8 and so does the second term by (3.9). Similarly, all the rerms of the right-hand side of (4.2) are shown to converge to 0 as $\lambda\to\infty$. To treat the third term, we approximate F by a function of compact support and apply the same arguments as above.

Once we have proved Step 1, one can follow the arguments of [2] §3, with no essential change. Indeed, by [2] Lemma 3.1, we have

Step 2. For $t>0$, $\Gamma f^\lambda(t)\to 0$ in $H^{N-1}_{\text{loc}}(\bar\Omega)$ as $\lambda\to\infty$.

Applying the Rellich and the Ascoli-Arzela theorems and also an interpolation theorem, we have

Step 3. There exists a subsequence $\{\lambda_\nu\}$ such that $\Gamma_0 f^{\lambda_\nu}(t)$ is convergent in $C(I; H^{N-1}_{\text{loc}}(\bar\Omega))$.

Step 4. Let $f^{\infty}(t)$ be the limit of $\Gamma_0 f^{\lambda_\nu}(t)$. Step 2 and Step 3 imply that $f^{\infty}(t)={}^t(0, v^{\infty}(t))$, $v^{\infty}(t)\in C(I; H^{N-1}_{\mathrm{loc}}(\bar{\Omega}))$, $\sup_{t\in I} \|v^{\infty}(t)\|_{N-1}<\infty$. Furthermore, multiplying (2.3) by Γ_0 and letting $\lambda=\lambda_\nu$ tend to infinity, we can easily show

$$\begin{aligned} &\partial_t v^{\infty}+P_S(v^{\infty}\cdot\nabla)v^{\infty}+P_S(w_0\cdot\nabla)v^{\infty}+P_S(v^{\infty}\cdot\nabla)w_0 \\ &\quad= -P_S(w_0\cdot\nabla)w_0\,, \\ &v^{\infty}(0)=P_S v_0^{\infty}=v_0^{\infty}\,. \end{aligned} \tag{4.6}$$

It israther easy to show that the above equation (4.6) has a unique solution which shows that $f^{\lambda}(t)$ itself converges to ${}^t(0, v^{\infty}(t))$ in $H^{N-1}_{\mathrm{loc}}(\bar{\Omega})$ as $\lambda\to\infty$ without passing to a subsequence.

Finally, letting $u^{\infty}(t)=v^{\infty}(t)+w_0$ and introducing $q^{\infty}(t)$ satisfying $\nabla q^{\infty}(t)=-P_G(u^{\infty}(t)\cdot\nabla)u^{\infty}(t)$, we have

$$\begin{aligned} &\partial_t u^{\infty}+(u^{\infty}\cdot\nabla)u^{\infty}+\nabla q^{\infty}=0\,, \\ &\operatorname{div} u^{\infty}=0\,. \end{aligned}$$

We have thus completed the proof of Therorem B.

5. Derivation of the equation (1.1)

We discuss in this section the derivation of the equation (1.1). Usually, the compressible Euler equation is written as

$$\begin{cases} \partial_t \rho+\operatorname{div}(\rho v)=0\,, \\ \rho(\partial_t V+(V\cdot\nabla)V)+\nabla P=0\,, \end{cases} \tag{5.1}$$

where ρ denotes the density, V the velocity and P the pressure. If the entropy is assumed to be constant, the equation of state becomes

$$\rho = A\,P^{1/\gamma},\ \gamma>1\,, \tag{5.2}$$

A being a positive constant.

First we look for a stationary solution to the incompressible Euler equation. Let $\xi=(1,0,0)$ and $w_0(x)=\xi+\nabla\varphi(x)$, where φ satisfies (1.3). We introduce a large parameter λ and set $w_0^{\lambda}(x)=\lambda^{-1}w_0(x)$. Then, λ^{-1} can be regarded as the speed at infinity of the flow $w_0^{\lambda}(x)$. Let

$$P_0^{\lambda}(x)=P_0-\frac{1}{2}\,|w_0^{\lambda}(x)|^2\,, \tag{5.3}$$

where P_0 is a positive constant. Noting that curl $w_0^{\lambda}(x)=0$ and hence $(w_0^{\lambda}\cdot\nabla)w_0^{\lambda}=\frac{1}{2}\nabla|w_0^{\lambda}(x)|^2$, we have

$$(w_0^{\lambda}\cdot\nabla)w_0^{\lambda}+\nabla P_0^{\lambda}=0\,, \tag{5.4}$$

which shows that $(P_0^\lambda, w_0^\lambda)$ is a stationary solution to the incompressible Euler equation.

The sound speed at infinity is

$$\left(\frac{d\rho}{dP}\right)^{-1/2}\Bigg|_{P=P_0-\lambda^{-2}/2} = (\gamma/A)^{1/2} P_0^{1/2-1/2\gamma} + O(\lambda^{-2}) .$$

The Mach number is defined by (the flow speed)/(the sound speed). If we adopt the speeds at infinity, the Mach unmber at infinity M_∞ is

$$M_\infty = \lambda^{-1}(A/\gamma)^{1/2} P_0^{1/2\gamma-1/2} + O(\lambda^{-2}) .$$

This shows that λ^{-1} is propotional to M_∞ modulo M_∞^2.

The problem is now evident. How the solution of (5.1) behaves when the intial data is slightly perturbed around $(P_0^\lambda, w_0^\lambda)$? To see this, we replace V by $\lambda^{-1} V$ in (5.1) to get

$$\begin{aligned} &\partial_t P + \lambda^{-1}(V\cdot\nabla) P + \lambda^{-1}\gamma P \nabla\cdot V = 0 , \\ &\lambda^{-1}\partial_t V + \lambda^{-2}(V\cdot\nabla) V + A^{-1} P^{-1/\gamma}\nabla P = 0 . \end{aligned} \tag{5.5}$$

To study the behavior of the solution of (5.5), it is convenient to put $P=P_0+p/\lambda$. Then we have

$$\begin{cases} \partial_t p + \lambda^{-1}(V\cdot\nabla) p + \gamma P \nabla\cdot V = 0 , \\ \partial_t V + \lambda^{-1}(V\cdot\nabla) V + A^{-1} P^{-1/\gamma}\nabla p = 0 . \end{cases} \tag{5.6}$$

If we let $\lambda\to\infty$ in (5.6), we merely obtain the linearized equation of acoustics

$$\begin{cases} \partial_t p + \gamma P_0 \nabla\cdot V = 0 , \\ \partial_t V + A^{-1} P_0^{-1/\gamma}\nabla p = 0 . \end{cases} \tag{5.7}$$

Therefore, to observe the more detailed behavior, we further make a change of variable $t\to\lambda t$ in (5.5) and obtain

$$\begin{cases} \partial_t P + (V\cdot\nabla) P + \gamma P \nabla\cdot V = 0 , \\ \partial_t V + (V\cdot\nabla) V + \lambda^2 A^{-1} P^{-1/\gamma}\nabla P = 0 . \end{cases} \tag{5.8}$$

This is just the equation (1.1).

In summary, what we have shown in this article is that, if the initial data for (5.1) is sufficiently close to $(P_0^\lambda, w_0^\lambda)$, the solution $P^\lambda(t, x)$, $V^\lambda(t, x)$ of (5.1) exists in a time interval $(0, \lambda T)$, T being independent of large λ, and furhter, if we set $V^\lambda(\lambda t, x)=\lambda^{-1} u^\lambda(t, x)$, $u^\lambda(t, x)$ is approximated by the solution of the incompressible Euler equation.

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